CONTRIBUTIONS TO THE THEORY OF RANK ORDER STATISTICS—THE TWO-SAMPLE CASE

By I. Richard Savage

National Bureau of Standards and *Stanford University*

1. Introduction. The idea of a statistical test of a hypothesis and the related concepts introduced by Neyman and Pearson have served as a model for much of modern statistics. In nonparametric work it is seldom possible to apply all of these concepts. This results from the fact that for most of the alternatives that have been considered there do not exist optimum critical regions or analytic tools for finding power functions. The sign test gives an illustration where it is possible to find the exact power function; on the other hand, this procedure is seldom optimum. The c_1 test [11] has optimum limiting properties but little is known about its power function for small samples. The Kolmogorov and Smirnov tests [6] have a certain intuitive appeal but their only justification is consistency. The Wilcoxon test [9] is justified on the basis that it is analogous to a good parametric procedure but has little direct justification.

In the course of this paper we will consider several nonparametric hypotheses that have been treated previously. In Section 5 it will be indicated that for the two-sample problem with such alternatives as slippage, there do not exist optimum nonparametric tests. In particular, we show that the class of admissible tests is too large to be of use. In Section 6 alternatives are considered involving monotone likelihood ratios and a necessary criterion for admissibility is given. In particular, two normal populations differing only in mean value are considered. It is shown that several of the previously proposed tests of this hypothesis satisfy this criterion. Section 7 deals with a special subclass of the alternatives used in Section 6. Members of this subclass are the extreme-value distribution and the exponential distribution. For these alternatives we not only have the results of the previous section on the construction of admissible tests, but also are able to carry out the construction of optimum nonparametric tests for small samples and to evaluate the operating characteristics of these tests. These small-sample tests are uniformly most powerful rank order tests and most stringent rank order tests. Also the limiting optimum test is given.

2. Notation. The main concern in the following will be the situation where there are random variables $X_1, \cdots, X_m$ independently distributed, each with continuous distribution function $F(x)$, and random variables $Y_1, \cdots, Y_n$ which are independent of the X's and are independently distributed, each with continuous distribution $G(x)$, i.e., two independent samples.

The observed values $x_1, \cdots, x_m$ of the random variables $X_1, \cdots, X_m$ will be called the first sample and the observed values $y_1, \cdots, y_n$ of the random variables $Y_1, \cdots, Y_n$ will be called the second sample. When all of the observed values are ordered from smallest to largest, they form a sequence which

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will be denoted by $w_1, \cdots, w_{m+n}$. A new sequence $z_1, \cdots, z_{m+n}$ can be formed from the w sequence by letting $z_i = 0$ if w_i comes from the first sample and by letting $z_i = 1$ if w_i comes from the second sample ($i = 1, \cdots, m + n$). From the z sequence two other sequences are defined by the following formulas:

$$(2.1) \qquad \begin{cases} v_i = \sum_{j=1}^{i} z_j \\ u_i = i - v_i \end{cases} \qquad (i = 1, \cdots, m + n).$$

The ranks of the observations from the first (second) sample, denoted by $r_1, \cdots, r_m$ ($s_1, \cdots, s_n$), are the subscripts of those $z_i = 0(1)$ arranged in increasing order. Corresponding to the observed values w_i, z_i, u_i, v_i, r_i, and s_i are the random variables W_i, Z_i, U_1, V_i, R_i, and S_i. An entire sequence such as $u_1, \cdots, u_{m+n}$ will be denoted by the corresponding letter u without a subscript. It should be noted that any one of z, u, v, r, or s determines the others, and, in general, these sequences will be referred to as rank orders. All of the above quantities are uniquely defined with probability one as a result of the assumption of continuity of the original distribution functions.

The following symbols will be used to denote special rank orders:

$I < II$: To be read as "Sample I is less than Sample II," i.e., all of the x's are less than all of the y's.

I a$<$ II: To be read as "Sample I is almost less than Sample II," i.e., all of the x's are less than all of the y's, except that there is one x larger than one y.

The symbols II < I and II a< I are defined analogously. Thus, when $m = n = 3$, there are among others the following representations for some of the rank orders:

	z	v
$I < II$	000111	000123
I a$<$ II	001011	001123
II a$<$ I	110100	122333
$II < I$	111000	123333

When a distribution function $F(x)$ has a density function, it will be denoted by the corresponding lower case letter $f(x)$.

3. Hypotheses. For all testing situations considered, the following basic assumption will be made.

Basic Assumption. The random variables $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are mutually independent. The X's have a common continuous cumulative distribution function $F(x)$ and the Y's have a common continuous cumulative distribution function $G(x)$.

The null hypothesis will be

$$H_0: \ F(x) \equiv G(x).$$

The following alternatives will be treated:

H_S (*S*lippage): $F(x) \geqq G(x)$, where the inequality holds for some x.

H_T (*T*ranslation): $G(x) \equiv F(x - \theta)$, where $\theta > 0$.

H_{TS} (*T*ranslation and *S*ymmetry)[1]: $G(x) \equiv F(x - \theta)$, where $\theta > 0$ and $F(x) + F(-x) \equiv 1$.

H_{TSU} (*T*ranslation, *S*ymmetry, and *U*nimodal): $G(x) \equiv F(x - \theta)$, where $\theta > 0$, $F(x) + F(-x) \equiv 1$ and where $b > a > 0$ and $c > 0$ implies $F(a + c) - F(a) \geqq F(b + c) - F(b)$.

H_M (*M*onotone likelihood ratio): $F(x)$ and $G(x)$ have density functions $f(x) = h(x, \theta_1)$ and $g(x) = h(x, \theta_2)$, where if $x_1 < x_2$ and $\theta_1 < \theta_2$, then

$$h(x_1, \theta_1)h(x_2, \theta_2) - h(x_1, \theta_2)h(x_2, \theta_1) \geqq 0.$$

H_L (*L*ehmann): $F(x) = [H(x)]^{\Delta_1}$ and $G(x) = [H(x)]^{\Delta_2}$, where $\Delta_2 > \Delta_1 > 0$ and $H(x)$ is a continuous cumulative distribution function.

H_E (*E*xponential): $F(x) = \Theta(x, \Delta_1)$ and $G(x) = \Theta(x, \Delta_2)$, where $\Delta_2 > \Delta_1 > 0$ and

$$\Theta(x, \Delta) = \begin{cases} e^{\Delta x} & \text{if } x < 0 \\ 1 & \phantom{\text{if }} x \geqq 0. \end{cases}$$

H_{EV} (*E*xtreme *V*alue): $F(x) = \Omega(x, \Delta_1)$ and $G(x) = \Omega(x, \Delta_2)$, where $\Delta_2 > \Delta_1$ and

$$\Omega(x, \Delta) = \exp[-e^{-(x-\Delta)}].$$

H_N (*N*ormal): $F(x)$ and $G(x)$ have the density functions $f(x) = N(x, \theta_1)$ and $g(x) = N(x, \theta_2)$, where $\theta_2 > \theta_1$ and

$$N(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp[-(x - \theta)^2/2].$$

The basic assumption of continuity of the distribution functions implies that the occurrence of equal observations is an event with zero probability. In practice, ties will occur and the methods of this paper will need to be modified to accommodate this situation. The choice of the constants in the alternative hypotheses is made so that we need only consider one-sided tests. However, the methods of this paper can be adapted to consider the two-sided cases.

The distribution of rank orders under H_0 is not affected by the underlying distribution function. Therefore, from the distribution theory standpoint, as far as rank order tests are concerned H_0 may be considered a simple hypothesis. The alternative hypotheses can be thought of as either simple or composite. The interpretation used will be clear from the text. Thus, in the alternative H_T

[1] The point of symmetry has been picked as the origin simply as a matter of convenience.

we have a simple hypothesis if $F(x)$ and θ are held fixed; a composite hypothesis if $F(x)$ is held fixed and all $\theta > 0$ are considered; a composite hypothesis if we consider arbitrary $F(x)$ and all $\theta > 0$.

The alternative hypotheses are related in the following ways:

1. All of the alternatives are special cases of H_S.
2. H_{TSU} is a special case of H_{TS} which is a special case of H_T.
3. When $H(x)$ in H_L has a density function, H_L is a special case of H_M.
4. H_E and H_{EV} are special cases of H_L and of H_M.
5. H_N is a special case of H_M.

Nonparametric tests of H_L against H_0 will be introduced in Section 7. As a basis for determining the effectiveness of these procedures, their operating characteristics will be compared with those of the best parametric test of H_E against H_0. Since H_L is a nonparametric alternative, there is no best parametric procedure. However, if $H(x)$ is known and an observation x is replaced by $\ln H(x)$, the testing situation becomes the parametric one just described. Thus the parametric situation serves as a basis of comparison.

4. Construction of rank order tests for small samples. In the two-sample rank order case the sample space consists of the $J = \binom{m+n}{n}$ points or rank orders z^i. A test consists of a sequence of numbers $a_1, \cdots, a_J$ and the rule that if the rank order z^i occurs the null hypothesis should be rejected with probability a_i. Since the rank orders are equally likely under the null hypothesis, the size of the critical region will be $\sum a_i/J$. If for each alternative hypothesis under consideration the rank order z^i is at least as probable as the rank order z^j, then a necessary condition for a test to be admissible is that $a_i \geqq a_j$. Using this as a criterion, it is often possible to ascertain the values of at least some of the a_i's in a specific problem. Unfortunately, the probabilities of the rank orders are seldom uniformly ordered and hence uniformly most powerful rank order tests seldom occur. However, the following situation does occur in practice, and we shall see examples of it in Sections 6 and 7.

Let us assume that a test with level of significance K/J (where for the sake of simplicity we shall assume that K is an integer) is desired. Then, it is clear that the following rules must be followed in constructing admissible tests: If there are K or more rank orders always more probable than z^i, then $a_i = 0$. If there are $J - K$ or more rank orders always less probable than z^i, then $a_i = 1$. In general, it is not possible to determine from the criterion of admissibility alone the values of the remaining a's.

5. Slippage alternatives. In this section we consider the alternatives H_S, H_T, H_{TS}, and H_{TSU} introduced in Section 3. Admissible and other optimum tests will not be constructed. Instead, several examples will be given indicating that the class of admissible tests is so large it is unlikely that uniformly most powerful or related optimum tests exist. This does not mean that there do not exist tests of these hypotheses with some optimum properties. For instance, there exist unbiased tests of these hypotheses (Lehmann [7]). However, there is no evidence that Lehmann's procedure is the best unbiased test.

A reasonable conjecture appears to be that $I < II$ (the first sample is less than the second) is the most probable rank order under $H_S : F(x) \geqq G(x)$. In Section 6 it will be shown that $I < II$ is the most probable rank order when two samples are taken from normal populations which are the same except that the mean of the second is larger than that of the first. Other statistically important examples will be given showing that H_S is compatible with $I < II$ being the most probable rank order.

However, H_S is not sufficient to insure that $I < II$ is the most probable rank order. In fact, it will be shown by Example 1 that even under H_{TSU}, $I < II$ need not be the most probable rank order. Here it should be recalled that H_{TSU} is $G(x) \equiv F(x - \theta)$, where $\theta > 0$, $F(x) + F(-x) \equiv 1$, and $F(a + c) - F(a) \geqq F(b + c) - F(b)$, where $b > a > 0$ and $c > 0$.

Example 1. Let

$$f(x) = \begin{cases} 0, & x < -\frac{5}{2}, \\ \gamma/2, & -\frac{5}{2} \leqq x < -\frac{1}{2}, \\ 1 - 2\gamma, & -\frac{1}{2} \leqq x < \frac{1}{2} \\ \gamma/2, & \frac{1}{2} \leqq x < \frac{5}{2}, \\ 0, & \frac{5}{2} \leqq x, \end{cases} \qquad (\gamma < \tfrac{2}{5}), \tag{5.1}$$

and $g(x) \equiv f(x - 1)$.

Let A be the rank order in which all of the observations from the first sample are less than all of the observations from the second sample, except that there is one observation from the first sample larger than all of the other observations. Thus, in the case that $m = 4$, $n = 2$, A is the rank order 000110. The result will be proved by showing that for some γ, m, and n,

$$P(A) > P(I < II). \tag{5.2}$$

Let B be the event that all of the observations from the second sample are in the interval $(\frac{1}{2}, \frac{3}{2})$, and let $\bar{B}$ be the complement of this event. Let C_i be the event that $m - i$ observations from the first sample are less than $\frac{1}{2}$ and the remaining i observations from the first sample are in the interval $(\frac{1}{2}, \frac{3}{2})$. Let D_i be the event that $m - i$ observations from the first sample are less than $\frac{1}{2}$, that $i - 1$ observations from the first sample are in the interval $(\frac{1}{2}, \frac{3}{2})$, and that one observation from the first sample is in the interval $(\frac{3}{2}, \frac{5}{2})$. Then,

$$\begin{aligned} &P(A) - P(I < II) \\ &= P(B)[P(A \mid B) - P(I < II \mid B)] + P(\bar{B})[P(A \mid \bar{B}) - P(I < II \mid \bar{B})] \\ &= P(B)\left\{\sum_{i=1}^{m} [P(AC_i \mid B) + P(AD_i \mid B)] - \sum_{i=0}^{m} P(I < II \cdot C_i \mid B)\right\} \\ &\qquad + P(\bar{B})[P(A \mid \bar{B}) - P(I < II \mid \bar{B})]. \end{aligned} \tag{5.3}$$

It is clear that

(a) $P(B) = (1 - 2\gamma)^n$,

(b) $P(\bar{B}) = 1 - (1 - 2\gamma)^n$,

(c) $P(AC_i \mid B) = P(I < II \cdot C_i \mid B), i = 1, \cdots, m$,

(d) $P(I < II \cdot C_0 \mid B) = (1 - \gamma)^m$,

(e) $P(AD_1 \mid B) = m\gamma(1 - \gamma)^{m-1}/2$,

(f) $P(AD_i \mid B) > 0, i = 2, \cdots, m$.

Hence,

$$P(A) - P(I < II) > (1 - 2\gamma)^n \left[\frac{m\gamma}{2}(1 - \gamma)^{m-1} - (1 - \gamma)^m\right] + [1 - (1 - 2\gamma)^n][P(A \mid \bar{B}) - P(I < II \mid \bar{B})]. \tag{5.4}$$

Let $\gamma = k/m$, hold n and k fixed, and let $m \to \infty$, then for sufficiently large m and $k > 2$,

$$P(A) - P(I < II) > e^{-k}\left(\frac{k}{2} - 1\right). \tag{5.5}$$

Hence the desired result is obtained.

While the above example is for the most restricted of the slippage alternatives it is only for large m. A counter example against H_S which holds for small m and n is

EXAMPLE 2. Let

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 \leqq x < \epsilon, \\ 0, & \epsilon \leqq x < 2\epsilon \ (0 < \epsilon < 1), \\ 1, & 2\epsilon \leqq x < 1 + \epsilon, \\ 0, & 1 + \epsilon \leqq x, \end{cases} \tag{5.6}$$

and

$$g(x) = \begin{cases} 0, & x < \epsilon, \\ 1, & \epsilon \leqq x < 1 + \epsilon, \\ 0, & 1 + \epsilon \leqq x. \end{cases} \tag{5.7}$$

Then, so long as $\epsilon < 1 - m^{-1/n}$,

$$P(I \text{ a}< II) - P(I < II) = \epsilon^m(m[1 - \epsilon]^n - 1) > 0. \tag{5.8}$$

When $n = 1$, this difference is maximized if $\epsilon = (m - 1)/(m + 1)$, in which case

$$P(I \text{ a}< II) - P(I < II) = \left(\frac{m - 1}{m + 1}\right)^{m+1}. \tag{5.9}$$

(Note: Theorem 5.1.A_1 states that this last result is actually the best possible.)

Using the same distributions and letting $m = n = 2$, the following complete set of probabilities of rank orders is obtained:

$$(5.10)\quad \begin{cases} P(I < II) &= P(0011) = \epsilon^4 + 2\,\epsilon^3(1-\epsilon) + \epsilon^2(1-\epsilon)^2 + R_\epsilon\,, \\ P(I \text{ a}< II) &= P(0101) = 2\,\epsilon^2(1-\epsilon)^2 + R_\epsilon\,, \\ P(0110) &= 2\,\epsilon^3(1-\epsilon) + 2\,\epsilon^2(1-\epsilon)^2 + R_\epsilon\,, \\ P(1001) &= R_\epsilon\,, \\ P(II \text{ a}< I) &= P(1010) = R_\epsilon\,, \\ P(II < I) &= P(1100) = \epsilon^2(1-\epsilon)^2 + R_\epsilon\,, \end{cases}$$

where $R_\epsilon = 2\epsilon(1-\epsilon)^3/3 + (1-\epsilon)^4/6$.

Now then, for all ϵ, the intuitively least probable rank order $II < I$ has a greater probability than the rank orders 1001 and 1010. However, each rank order beginning with 1 is less probable than any rank order beginning with 0. Also $P(0110) > P(0101)$ for all ϵ. Finally,

$$(5.11)\quad \begin{aligned} P(0110) - P(0011) &= \epsilon^2(1-2\epsilon), \\ P(0101) - P(0011) &= \epsilon^2(2\epsilon^2 - 4\epsilon + 1). \end{aligned}$$

The first of these differences is greater than 0, provided $\epsilon < \frac{1}{2}$, and the second difference is greater than 0, provided $\epsilon < 1 - 1/\sqrt{2}$.

As a result of the preceding examples it is clear that under alternatives such as slippage the probabilities of the rank orders will not be uniformly ordered. The following theorem summarizes the information regarding uniform ordering for these alternatives. The results are meager since they are mostly for sample sizes that do not occur in practice.

THEOREM 5.1.

A_1: *If* $n = 1$ *and* H_S, *then* $P(I\text{a} < II) - P(I < II) \leqq [(m-1)/(m+1)]^{m+1}$.

A_2: *If* H_S, *then* $P(I < II) > \binom{m+n}{n}^{-1}$.

B: *If* $m = 2, n = 1,$ *and* H_{TS}, *then* $P(I < II) > P(I\text{a} < II) > P(II < I)$.

C: *If* $m = 3, n = 1,$ *and* H_{TSU}, *then* $P(I < II) > P(I\text{a} < II) > P(II\text{a} < I) > P(II < I)$.

PROOF. These results are obtained by elementary manipulation from the definitions of the probabilities involved. The fact that all of the probabilities can be expressed as single integrals involving the c.d.f.'s is the unifying and simplifying feature of the statement and proof of the theorem.

Example 3 below illustrates a situation under H_S allowing a uniform ordering of the probabilities of the rank orders and thus the construction of uniformly most powerful rank order tests is possible for all combinations of sample sizes.

EXAMPLE 3. Let $X_1, \cdots, X_m$ be a sample from the rectangular distribution with range from 0 to 1, and let $Y_1, \cdots, Y_n$ be an independent sample from a rectangular distribution with range from 0 to L (where $L > 1$). Then,

1. The probability of a rank order depends on the length of the last run of 1's only.

2. The longer the last run of 1's the more probable is the rank order.

Let A stand for a specific rank order. Then,

$$P(A) = \sum_{i=0}^{n} P(A \mid i \text{ of the } Y\text{'s} > 1)\, P(i \text{ of the } Y\text{'s} > 1)$$
$$(5.12) \qquad = \sum_{i=0}^{n} \binom{n}{i} L^{-n}(L-1)^{i} \binom{m+n-i}{m}^{-1} G_i(A),$$

where $G_i(A) = 1$ if A can occur when there are as many as i of the Y's $>$ than all of the X's, and otherwise $G_i(A) = 0$. From this the results are immediate.

Example 2, with Theorem 5.1.A_2, shows that H_S is sufficient for $I < II$ to be the most probable rank order only when $m = n = 1$. Also, in this example, when $m = n = 2$, we have further evidence that for these alternatives the criterion of Section 4 for constructing optimum tests is inadequate.

Example 1, with Theorem 5.1.C, shows that H_{TSU} implies that $I < II$ is the most probable rank order only for certain m and n. The example could also be used for showing that there are rank orders, other than the one treated, that are sometimes more probable than $I < II$. Thus, even for this more restrictive alternative, it does not appear possible to apply the methods of Section 4.

Example 3 is a situation under slippage where it is actually possible to construct the best test. The more common statistical situations will be discussed in the next two sections. For these cases, it will turn out that the hypotheses induce a partial ordering of the probabilities of rank orders which are intermediate between the orderings given by the examples of this section. For the alternatives discussed in these latter sections, the partial ordering will be adequate to give a useful criterion for the construction of admissible tests. Finally, in Section 7 a case is treated where it is possible to construct various types of best tests and their operating characteristics are given.

6. Monotone likelihood ratio alternatives. In the following theorem it is shown that for alternatives of the monotone likelihood ratio type it is possible to give an easily applied necessary criterion for the admissibility of rank order tests.

THEOREM 6.1. *If the random variables* $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ *are mutually independent and the* X*'s have the density function* $h(x, \theta_1)$ *and the* Y*'s have the density function* $h(x, \theta_2)$*, where* $h(x_1, \theta_1)h(x_2, \theta_2) - h(x_1, \theta_2)h(x_2, \theta_1) \geqq 0$ *if* $x_2 > x_1$*, then the rank order* z *is more probable than* z' *when the two rank orders are identical except for their ith and jth elements* $(i < j)$*, which are* $(0, 1)$ *for* z *and* $(1, 0)$ *for* z'.

PROOF. We have

$$(6.1) \qquad P(Z = z) - P(Z = z') = m!n! \int_{-\infty<x_1<\cdots<x_{m+n}<\infty} \cdots \int \prod_{\substack{k=1 \\ i\neq k\neq j}}^{m+n} h(x_k, \theta_{1+z_k}) \times [h(x_i, \theta_1)h(x_j, \theta_2) - h(x_i, \theta_2)h(x_j, \theta_1)] \prod_{k=1}^{m+n} dx_k.$$

By assumption (remember, $x_i < x_j$) the integrand is nonnegative and actually positive on a set of positive measure (except for the case $h(x, \theta_1) = h(x, \theta_2)$ almost everywhere). Hence, the desired result is obtained.

Thus, when $m = n = 2$, the rank order $z = (0101)$ must be put into the critical region with probability one before the rank order $z' = (1001)$ is put into the critical region with nonzero probability. In the equal sample case, the one-sided Smirnov test [6] is based on large values of the statistic

$$\max_{1 \leq i \leq m+n} (i - 2v_i), \tag{6.2}$$

where it should be recalled that $v_i = \sum_{j=1}^{i} z_j$. However, for the two rank orders just mentioned the Smirnov statistic has the same value, i.e., 1. Thus, the Smirnov procedure could lead to the use of inadmissible tests of H_0 against H_M.

Many procedures proposed for testing H_0 against H_M are based on statistics of the form

$$\sum_{i=1}^{m+n} c_i z_i, \tag{6.3}$$

where the c_i's are an increasing (decreasing) sequence and large (small) values of (6.3) are critical. Some typical examples of this are

1. The Wilcoxon statistic [9], where $c_i = i$ is an increasing sequence.
2. The c_1 statistic [11], where the coefficients $c_i =$ the expected value of the ith order statistic in a sample of $m + n$ observations from the standardized normal distribution form an increasing sequence.
3. The T statistic (introduced in Section 7), where $c_i = \sum_{j=i}^{m+n} 1/j$ is a decreasing sequence.

Statistics of the form (6.3) satisfy the admissibility criterion of Theorem 6.1, for if rank orders z and z' are in the desired relationship, the difference in the corresponding values of the statistic will be $c_j - c_i$ which is positive (negative) when large (small) values are critical. It should be noted that (6.3) is *not* a sufficient condition for admissibility.

7. Lehmann alternatives. Alternatives of the form H_L were introduced by Lehmann [8] in order to study nonparametric procedures when the alternatives themselves are given in a nonparametric form. In this section we continue the study of these alternatives and show that for them it is possible to construct optimum critical regions of various types. The H_L alternatives are of statistical interest since they include the extreme-value and exponential distributions as was pointed out in Section 3.

7. a. *General formulas.* One of the reasons why the nonparametric treatment of the H_L alternatives can be so complete from the Neyman-Pearson point of view is that it is possible to give in explicit form the probabilities of the rank orders. This will be done in Corollary 7.a.1.

THEOREM 7.a.1. *If the random variables $X_1, \cdots, X_N$ are mutually independent* *and X_i has the cumulative distribution function $[H(x)]^{\Delta_i}$, where $\Delta_i > 0$ and $H(x)$ is a continuous distribution function, then*

$$P(X_1 \leqq X_2 \leqq \cdots \leqq X_{N-1} \leqq X_N) = \left(\prod_{i=1}^{N} \Delta_i\right) \Big/ \prod_{i=1}^{N} \left(\sum_{j=1}^{i} \Delta_j\right).$$

By a proper numbering of the X's the probability of any ordering can be found.

PROOF. Let

$$P = P(X_1 \leqq X_2 < \cdots \leqq X_{N-1} \leqq X_N). \tag{7.a.1}$$

Then,

$$P = \int_{-\infty \leqq x_1 \leqq \cdots \leqq x_N < \infty} \cdots \int \prod_{i=1}^{N} d[H(x_i)]^{\Delta_i}. \tag{7.a.2}$$

Making the change of variables

$$y_i = H(x_i) \qquad (i = 1, \cdots, N), \tag{7.a.3}$$

we have

$$\begin{aligned} P &= \int_{0 \leqq y_1 \leqq \cdots \leqq y_N \leqq 1} \cdots \int \prod_{i=1}^{N} d(y_i)^{\Delta_i} \\ &= \left(\prod_{i=1}^{N} \Delta_i\right) \int_{0 \leqq y_1 \leqq \cdots \leqq y_N \leqq 1} \cdots \int \prod_{i=1}^{N} (y_i^{\Delta_i - 1}\, dy_i) \\ &= \left(\prod_{i=1}^{N} \Delta_i\right) \Big/ \prod_{i=1}^{N} \left(\sum_{j=1}^{i} \Delta_j\right). \end{aligned} \tag{7.a.4}$$

The following corollary is equivalent to Equation (4.5) of Lehmann [8].

COROLLARY 7.a.1. *Under H_L the probability of a rank order z is*

$$m!n!\Delta_1^m \Delta_2^n \Big/ \prod_{i=1}^{m+n} \left(\sum_{j=1}^{i} [(1 - z_j)\Delta_1 + z_j \Delta_2]\right)$$

or

$$m!n!\delta^n \Big/ \prod_{i=1}^{m+n} (u_i + v_i \delta),$$

where $\delta = \Delta_2 / \Delta_1$.

The quantity $\prod_{i=1}^{m+n} (u_i + \delta v_i)$ occurring in Corollary 7.a.1 is a polynomial in δ, whose coefficients depend on the rank order z. For convenience, denote this polynomial by $f_z(\delta)$. The nonzero coefficients of $f_z(\delta)$ are positive integers. Using $u_i + v_i = i$ and setting $\delta = 1$, the sum of the coefficients is found to be $(m + n)!$. If $\mathfrak{r} = \min (r_1, \cdots, r_m)$, i.e., if $\mathfrak{r}$ is the rank of the smallest observation from the first sample, then the smallest power of δ with a nonzero coefficient is $\mathfrak{r} - 1$. In particular, if $z_1 = 0$ the polynomial has a constant term. If

$$\mathfrak{s} = \min (s_1, \cdots, s_n),$$

i.e., if $\mathfrak{s}$ is the rank of the smallest observation from the second sample, then the largest power of δ with a nonzero coefficient is $m + n - \mathfrak{s} + 1$. In particular, if $z_1 = 1$, the polynomial is of degree $m + n$. If $I = \max (\mathfrak{r}, \mathfrak{s})$, the coefficient of $\delta^{\mathfrak{r}-1}$ is $(I - 1)! \prod_{i=I}^{m+n} u_i$ and the coefficient of $\delta^{m+n-\mathfrak{s}+1}$ is $(I - 1)! \prod_{i=I}^{m+n} v_i$. All of the nonzero coefficients of $f_z(\delta)$ are $\geq m!$.

Let z be a rank order for sample sizes m and n, and let $z^0(z^1)$ be a rank order for sample sizes $m + 1$ and $n(m$ and $n + 1)$ such that the first $m + n$ elements of $z^0(z^1)$ are the same as the elements of z and the $(m + n + 1)$-st element of $z^0(z^1)$ is a $0(1)$. Then,

$$\begin{cases} f_{z^0}(\delta) = [(m + 1) + n\delta]f_z(\delta), \\ (f_{z^1}(\delta) = [m + (n + 1)\delta]f_z(\delta)). \end{cases} \tag{7.a.5}$$

When two rank orders z and z' are identical except in their kth and $k + 1$-st elements, which are (0, 1) for z and (1, 0) for z', then we have the following relationship between their probabilities:

$$P(Z = z) = \frac{(u_i + \delta v_i + \delta - 1)}{(u_i + \delta v_i)} P(Z = z'), \tag{7.a.6}$$

where u_i and v_i are computed for z. The probability of $I < II$, all of the first sample less than the second, is

$$P(I < II) = n!\delta^n \Big/ \prod_{i=1}^{n} (m + i\delta) \tag{7.a.7}$$

and the probability of $II < I$, all of the second sample less than the first, is

$$P(II < I) = m! \Big/ \prod_{i=1}^{m} (i + n\delta). \tag{7.a.8}$$

7.b. *Composite alternatives.* In this section optimum tests of H_0 against H_L, where Δ_1 and Δ_2 are restricted only by $\delta = \Delta_2 / \Delta_1 > 1$, are considered. Theorem 6.1 gives an easily applied necessary criterion for admissibility. It will be possible in this section to go farther and find more details about the structure of optimal tests than was possible in Section 4.

The statistic $T(z)$, or simply T, defined as

$$T(z) = \sum_{i=1}^{m+n} v_i/i \tag{7.b.1}$$

will be used in the next theorem. $T(z)$ will be the center of discussion of the remaining subsections.

THEOREM 7.b.1. *Under H_L, if $T(z) < T(z')$, then there exists a δ, say δ^*, such* *that $\delta^* > 1$ and for δ in the interval $(1, \delta^*)$ the probability of z is greater than the probability of z'. In fact, the δ^* may be chosen independently of z and z'.*

PROOF. From Corollary 7.a.1, we have

$$P(Z = z) = \frac{m!n!}{(m+n)!}\delta^n[1 - (\delta - 1)T(z) + 0(\delta - 1)^2]. \tag{7.b.2}$$

Hence,

$$\begin{aligned} P(Z = z) &- P(Z = z') \\ &= \frac{m!n!}{(m+n)!}\delta^n(\delta - 1)[T(z') - T(z) + 0(\delta - 1)]. \end{aligned} \tag{7.b.3}$$

Thus, for any z and z' such that $T(z) < T(z')$, there exists a $\delta^* > 1$ such that $P(Z = z) > P(Z = z')$ for $1 < \delta < \delta^*$; and since the number of rank orders is finite, δ^* can be chosen independently of z and z'. This implies the theorem.

THEOREM 7.b.2. *Under H_L, the rank order z will be more probable than the rank order z' for sufficiently large δ if $\mathcal{S} > \mathcal{S}'$ or if $\mathcal{S} = \mathcal{S}'$ and*

$$(I - 1)! \prod_{i=I}^{m+n} v_i < (I' - 1)! \prod_{i=I'}^{m+n} v_i'.$$

PROOF. The conclusion follows immediately from the discussion after Corollary 7.a.1, since the coefficient of the term of highest degree of a polynomial dominates its behavior for large values of the argument.

Thus, in order for the rank order z to be always more probable than the rank order z' under H_L, it is necessary that $\mathcal{S}$ and $\mathcal{S}'$ satisfy the conditions of Theorem 7.b.2. When this is the case, the necessary and sufficient condition for z to be more probable than z' is that the polynomial

$$f_{z,z'}(\delta) = f_{z'}(\delta) - f_z(\delta) \tag{7.b.4}$$

has no (real) roots larger than 1. This results from the fact that the condition on (7.b.4) is equivalent to the denominator of the formula for the probability of z being less than the denominator of the formula for the probability of z', where these formulas are given in Corollary 7.a.1.

Figure 1 gives relationships between probabilities of rank orders. The numbers in the figure are the numbers assigned to the rank orders in Table I, printed at the end of the text. If for $i < j$ it is possible to connect i and j by a sequence of ascending segments, i.e., segments connecting a smaller number to a larger, then the rank order with number i is always (under H_L) more probable than the rank order with the number j. If this is not possible, rank order i is more probable than j for some δ's, and rank order i is less probable than j for other δ's.

The diagrams in the figure were drawn using the criteria given by Theorem 6.1 and (7.b.4).

When the diagram corresponding to a particular combination of sample sizes is in the form of a simple chain, it is possible to construct a uniformly most

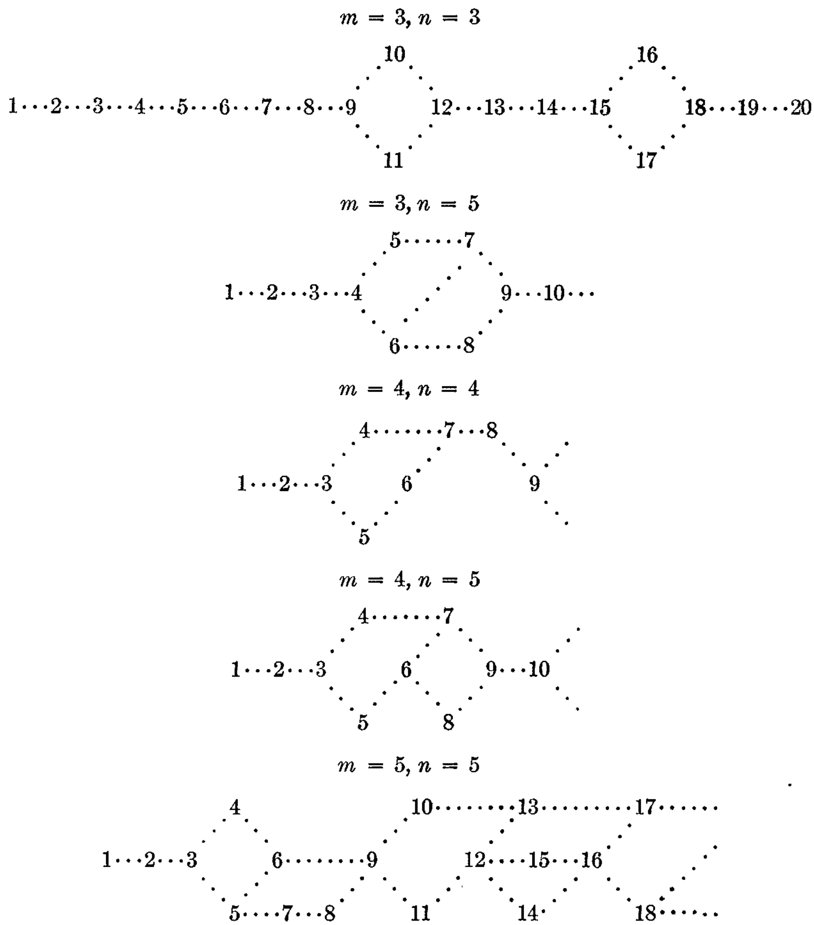

FIG. 1

powerful rank order test for every level of significance. When $m = 1$, or $m = 2$, and $n = 2, 3, 4$, or 5, uniformly most powerful rank order tests of H_0 against H_L can be formed for every level of significance. Not all cases with $m = 2$ give a simple ordering, for instance, $m = 2, n = 6$.

The diagram for $m = n = 3$ is the least complicated one where there is not a simple ordering. In this case it would not be possible to construct uniformly most powerful rank order procedures for levels of significance in the intervals (0.45, 0.55) and (0.75, 0.85). Since these are unusual levels, there would be no practical difficulty. The diagram for $m = 3$ and $n = 4$ is like the above in that there does not exist a simple ordering, and for all of the usual levels of significance there are uniformly most powerful procedures.

The case of $m = 3, n = 5$ illustrates where the lack of simple ordering causes difficulty in finding optimum procedures for a reasonable level of significance, i.e., 0.10. Since there are 56 rank orders, a randomized test procedure at the 0.10

level involves the choice of probabilities $a_1, \cdots, a_{56}$ such that their sum is 5.6. Using the results of Section 4, we have $a_1 = \cdots = a_4 = 1$, $a_7 = a_9 = \cdots = a_{56} = 0$, $a_5 + a_6 + a_8 = 1.6$, $0 \leqq a_i \leqq 1$. If $a_5 = 0.15$, $a_6 = 1$, $a_8 = 0.45$, the most stringent rank order test is obtained. The maximum difference between the envelope power function of all rank order tests and this test, which has been minimized, is 0.0021. This maximum difference occurs when $\delta = 2.2$ or 16. The numerical work to carry out such an analysis is so large as to make it prohibitive except for very small samples.

When $m = n = 4$ it is possible to construct uniformly most powerful rank order tests with levels of significance in the intervals (0, 0.043) and (0.086, 0.129). To obtain a test at the exact 0.05 level, we use the criterion of Section 4. We have then

$$a_1 = a_2 = a_3 = 1, \qquad a_6 = \cdots = a_{70} = 0, \qquad a_4 + a_5 = 0.5, \qquad 0 \leqq a_i \leqq 1.$$

The most stringent procedure is given by $a_4 = 0.00156$, and $a_5 = 0.49844$. The maximum deviation from the envelope power function is 0.00005, which occurs at $\delta = 15$.

When $m = 4$, $n = 5$, it is possible to construct uniformly most powerful rank order tests for levels of significance in the intervals (0, 0.024) and (0.063, 0.079). If a test at the exact 0.05 level is desired, we have, using the results of Section 4, $a_1 = a_2 = a_3 = a_5 = a_6 = 1$, $a_9 = \cdots = a_{126} = 0$, $a_4 + a_7 + a_8 = 1.3$, $0 \leqq a_i \leqq 1$.

When $m = n = 5$, there exist uniformly most powerful rank order tests with levels of significance in the intervals (0, 0.012) and (0.032, 0.036). If a test at the 0.05 level is desired, we have from Section 4, $a_1 = \cdots = a_9 = a_{11} = a_{12} = 1$, $a_{16} = \cdots = a_{252} = 0$, $a_{10} + a_{13} + a_{14} + a_{15} = 1.6$, $0 \leqq a_i \leqq 1$.

It is interesting to note that we can obtain a test near the 0.05 level which would have only half as many rank orders, whose a_i's are not determined by the criterion of admissibility alone, as a test exactly at the 0.05 level. Thus, in order to construct a test at the $11/252 = 0.044$ level, we have $a_1 = \cdots = a_9 = a_{11} = 1$, $a_{13} = \cdots = a_{252} = 0$, $a_{10} + a_{12} = 1$, $0 \leqq a_i \leqq 1$.

This, then, completes the discussion of the construction of exact optimum rank order tests of H_0 against H_L. We have seen that for small sample sizes it is possible to construct the uniformly most powerful rank order tests or most stringent rank order tests. However, the amount of computing becomes much larger as the sample sizes increase, and these exact methods will not be applicable for most of the situations arising in practice. The fact (see Table II) that most stringent tests for the cases examined are never much more powerful than *any* admissible test would lead to the conjecture that it is not necessary to find the best test but some reasonable substitute. In the next subsections we develop the theory of such a test.

7.c. *Exact distribution of the limiting statistic.* Using the notation

$$D_{Ni} = \sum_{j=i}^{N} j^{-1}, \tag{7.c.1}$$

we have the following methods for expressing the statistic introduced before Theorem 7.b.1:

$$T(z) = \sum_{i=1}^{m+n} v_i/i = \sum_{i=1}^{N} z_i D_{Ni} = \sum_{i=1}^{n} D_{Ns_i}. \tag{7.c.2}$$

A reinterpretation of Theorem 7.b.1 shows that the locally most powerful rank order test of H_0 against H_L is based on small values of $T(z)$. Using this as a motivation, the exact distribution of $T(z)$ under H_0 will be examined in this subsection. In the next subsection we shall examine its limiting distribution for large samples.

LEMMA 7.c.1. *Let* $U = \sum_{i=1}^{m+n} a_i Z_i$ *and* $V = \sum_{i=1}^{m+n} b_i Z_i$. *Then, under* H_0, $EU = n/N \sum_{i=1}^{N} a_i$ *and*

$$\text{cov}\,(U, V) = \frac{mn}{N^2(N-1)}\left(\sum_{i=1}^{N} a_i b_i - \frac{\sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i}{N}\right).$$

PROOF. The proof is routine, using the facts that

$$\text{cov}\,(U, V) = \sum_{i=1}^{N} a_i b_i \,\text{var}\,(Z_i) + \sum_{i \neq j}\sum a_i b_j \,\text{cov}\,(Z_i Z_j)$$

and that under H_0,

$$EZ_i = EZ_i^2 = \frac{n}{N}, \qquad \text{var}\, Z_i = \frac{nm}{N^2}, \qquad \text{and} \qquad \text{cov}\, Z_i Z_j = -\frac{mn}{N^2(N-1)}.$$

THEOREM 7.c.1. *Under* H_0 *the mean and variance of* T *are* $ET = n$,

$$\sigma^2 = mn / N - 1(1 - D_{N1} / N).$$

PROOF. In Lemma 7.c.1, let $a_i = b_i = D_{Ni}$ and note that $\sum_{i=1}^{N} D_{Ni} = N$ and $\sum_{i=1}^{N} D_{Ni}^2 = 2N - D_{N1}$.

The Wilcoxon statistic [9], which can be written as

$$W = \sum_{i=1}^{N} i z_i, \tag{7.c.3}$$

is used as a test of the hypothesis that two samples come from populations differing only in location. H_{EV}, a special case of H_L, is a hypothesis of this type. Thus T and W will sometimes be used for the same purpose. Therefore, it is interesting to have some information about their joint distribution.

THEOREM 7.c.2. *Under* H_0, *the covariance of* T *and* W *is* $-mn / 4$.

PROOF. In Lemma 7.c.1, let $a_i = D_{Ni}$ and $b_i = i$ and note that $\sum_{i=1}^{N} iD_{Ni} = N(N + 3) / 4$.

COROLLARY 7.c.2. *Under* H_0, *the correlation between* T *and* W *is*

$$-\tfrac{1}{2}\sqrt{\frac{3(N-1)}{(N+1)(1 - D_{N1} / N)}},$$

or approximately $-\sqrt{3/2} = -0.8660 \cdots$.

The above work is similar to the study made by Terry ([11], Section 9), where he gives the correlation between W and c_1. The limiting correlation in the case considered by Terry is $\sqrt{3/\pi}$ (=0.9772) which is somewhat larger (in absolute value) than $-\sqrt{3/2}(= -0.8660)$ found in the above case.

For each rank order z, we can form its complement rank order z^c, i.e., if an element of z is 0(1), the corresponding element of z^c is 1(0). Using $\sum D_{Ni} = N$, we obtain $T(z) + T(z^c) = N$. Also available are the recursion formulas

$$T(z^0) = T(z) + n / (m + n + 1) \tag{7.c.4}$$

and

$$T(z^1) = T(z) + (n + 1) / (m + n + 1). \tag{7.c.5}$$

The rank orders z^0 and z^1 (used at the end of Section 7.a) are formed from z by placing an additional element, 0 for z^0 and 1 for z^1, at the extreme right of z. These results are useful in preparing tables of the distribution of T under the null hypothesis.

7.d. *Large sample distribution of the limiting statistic.* We first show that under H_0 the statistic T has a limiting normal distribution and then indicate that under H_L it also has a normal distribution and is asymptotically most powerful.

We need the result of Epstein and Sobel ([4], Appendix A) that if $X_1, \cdots, X_N$ are independently distributed, and each X has the density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ e^{-x} & \phantom{\text{if }} x \geqq 0, \end{cases} \tag{7.d.1}$$

then

$$EX_{Ni} = \sum_{j=i'}^{N} j^{-1} = D_{Ni'}, \qquad i' = N - i + 1, \tag{7.d.2}$$

where X_{Ni} is the ith order statistic in a sample of N and D_{Ni} was introduced in (7.c.1). This result, combined with a theorem of Dwass [2], yields

THEOREM 7.d.1. *Under H_0, when $N \to \infty$ in such a way that $n / (m + n)$ tends to a constant λ different from 0 or 1, the random variable*

$$t = \frac{T - \lambda N}{\sqrt{\lambda(1 - \lambda)N}}$$

has a distribution which approaches a normal distribution with zero mean and unit variance.

A rigorous treatment of the limiting distribution of T under H_L would be complicated and will not be given here in view of the fact that we are primarily interested in exact, instead of limiting, properties. However, it is reasonable to conjecture (see Dwass [1], Hoeffding [5] and Lehmann [7]) that T, when properly normalized, has a limiting distribution which is Gaussian and yields an asymptotically most powerful test under alternatives of the form H_L.

8. Acknowledgment. For the continuing encouragement and guidance of Professor Howard Levene of Columbia University from the initiation to the completion of this research, I wish to express sincere appreciation.

TABLE I

Distribution of Rank Orders under H_L

This table gives the probabilities of some of the rank orders (see Section 2) for all combinations of sample sizes $1 \leqq m \leqq n \leqq 5$ and alternatives of the form H_L (see Section 3). The rank orders have been arranged in order of increasing values of the statistic T and, hence, for values of δ slightly greater than 1, the rank orders are arranged from most probable toward least probable. The value of δ in the column headed P_β^α is that value required to obtain a test with power $1 - \beta$ at the α level of significance when the best similar region test of H_E is used (see Eisenhart [3], Chapter 8, Sections 4 and 6.2., and Tables 8.3 and 8.4). The values of the probabilities of the rank orders were computed using (7.a.5), (7.a.6) and (7.a.7).

It should be noted that this table is not symmetric in m and n, but see the remarks at the end of Section 7.c. It was decided to present the results for $n \geqq m$, since in this situation the rank order procedures make a more favorable comparison to the parametric procedures for comparable alternatives (see Table 2).

$N = 2,\ m = 1,\ n = 1$

i	R.O.	T	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			9.0000	27.0000	57.0000	361.0000
1	01	0.5000	.9000	.9643	.9828	.9972
2	10	1.5000	.1000	.0357	.0172	.0048

$N = 3,\ m = 1,\ n = 2$

			δ			
			7.6575	18.4868	38.4940	133.6569
1	011	1.1667	.8303	.9237	.9622	.9889
2	101	2.1667	.1084	.0500	.0250	.0074
3	110	2.6667	.0613	.0264	.0128	.0037

$N = 4,\ m = 1,\ n = 3$

			δ			
			7.2717	16.4334	34.0633	99.4200
1	0111	1.9167	.7865	.8966	.9482	.9818
2	1011	2.9167	.1082	.0546	.0278	.0101
3	1101	3.4167	.0615	.0290	.0143	.0051
4	1110	3.7500	.0438	.0199	.0097	.0034

$N = 4,\ m = 2,\ n = 2$

			δ			
			4.1073	8.4783	13.1867	40.8104
1	0011	.8333	.5408	.7238	.8071	.9305
2	0101	1.3333	.2118	.1527	.1138	.0445
3	0110	1.6667	.1404	.0891	.0631	.0231
4	1001	2.3333	.0516	.0180	.0086	.0011
5	1010	2.6667	.0342	.0105	.0048	.0006
6	1100	3.1667	.0213	.0059	.0026	.0003

TABLE I—*Continued*

$N = 5,\ m = 1,\ n = 4$

i	R.O.	T	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			7.0891	15.5199	32.0958	86.3753
1	01111	2.7167	.7552	.8769	.9378	.9763
2	10111	3.7167	.1065	.0565	.0292	.0113
3	11011	4.2167	.0608	.0301	.0151	.0057
4	11101	4.5500	.0434	.0208	.0104	.0038
5	11110	4.8000	.0341	.0159	.0079	.0029

$N = 5,\ m = 2,\ n = 3$

			δ			
			3.7769	7.1663	11.0147	27.9416
1	00111	1.4333	.4394	.6277	.7316	.8800
2	01011	1.9333	.1840	.1537	.1218	.0608
3	01101	2.2667	.1242	.0919	.0688	.0320
4	01110	2.5167	.0963	.0667	.0486	.0218
5	10011	2.9333	.0487	.0214	.0111	.0022
6	10101	3.2667	.0329	.0128	.0063	.0011
7	10110	3.5167	.0255	.0093	.0044	.0008
8	11001	3.7667	.0208	.0073	.0034	.0006
9	11010	4.0167	.0161	.0054	.0024	.0004
10	11100	4.3500	.0122	.0038	.0017	.0003

$N = 6,\ m = 1,\ n = 5$

			δ			
			6.9826	15.0031	30.9851	79.5779
1	011111	3.5500	.7312	.8615	.9297	.9718
2	101111	4.5500	.1047	.0574	.0300	.0122
3	110111	5.0500	.0598	.0306	.0155	.0062
4	111011	5.3833	.0428	.0211	.0105	.0041
5	111101	5.6333	.0336	.0162	.0080	.0031
6	111110	5.8333	.0278	.0132	.0064	.0025

$N = 6,\ m = 2,\ n = 4$

			δ			
			3.6173	6.5817	10.0534	23.1841
1	001111	2.1000	.3743	.5619	.6777	.8397
2	010111	2.6000	.1621	.1482	.1226	.0694
3	011011	2.9333	.1106	.0898	.0700	.0369
4	011101	3.1833	.0862	.0656	.0497	.0253
5	011110	3.3833	.0716	.0522	.0388	.0193
6	100111	3.6000	.0448	.0225	.0122	.0030
7	101011	3.9333	.0305	.0136	.0070	.0016
8	101101	4.1833	.0238	.0100	.0050	.0011
9	101110	4.3833	.0198	.0079	.0039	.0008
10	110011	4.4333	.0195	.0078	.0038	.0008
11	110101	4.6833	.0152	.0058	.0027	.0006
12	110110	4.8833	.0126	.0046	.0021	.0004
13	111001	5.0167	.0115	.0042	.0019	.0004
14	111010	5.2167	.0096	.0033	.0015	.0003
15	111100	5.4667	.0078	.0026	.0011	.0002

TABLE I—*Continued*

$N = 6,\ m = 3,\ n = 3$

i	R.O.	T	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			3.0546	5.4436	7.6343	18.3518
1	000111	1.1500	.2549	.4270	.5305	.7535
2	001011	1.4833	.1513	.1721	.1652	.1111
3	001101	1.7333	.1130	.1128	.1017	.0613
4	001110	1.9333	.0922	.0854	.0746	.0426
5	010011	1.9833	.0746	.0534	.0383	.0115
6	010101	2.2333	.0557	.0350	.0236	.0063
7	010110	2.4333	.0455	.0265	.0173	.0044
8	011001	2.5667	.0396	.0219	.0140	.0034
9	011010	2.7667	.0324	.0166	.0102	.0024
10	100011	2.9833	.0244	.0098	.0050	.0006
11	011100	3.0167	.0259	.0123	.0074	.0017
12	100101	3.2333	.0182	.0064	.0031	.0003
13	100110	3.4333	.0149	.0049	.0023	.0002
14	101001	3.5667	.0129	.0040	.0018	.0002
15	101010	3.7667	.0106	.0031	.0014	.0001
16	101100	4.0167	.0085	.0023	.0010	.0001
17	110001	4.0667	.0086	.0024	.0010	.0001
18	110010	4.2667	.0070	.0018	.0007	.0001
19	110100	4.5167	.0056	.0013	.0005	.0001
20	111000	4.8500	.0043	.0010	.0004	.0000

$N = 7,\ m = 2\ n = 5$

			δ			
			3.5233	6.2518	9.5126	20.7442
1	0011111	2.8143	.3286	.5136	.6370	.8076
2	0101111	3.3143	.1453	.1416	.1212	.0743
3	0110111	3.6476	.0997	.0865	.0697	.0398
4	0111011	3.8976	.0780	.0635	.0496	.0274
5	0111101	4.0976	.0650	.0507	.0388	.0210
6	0111110	4.2643	.0562	.0424	.0320	.0170
7	1001111	4.3143	.0412	.0226	.0127	.0036
8	1010111	4.6476	.0283	.0138	.0073	.0019
9	1011011	4.8976	.0221	.0102	.0052	.0013
10	1011101	5.0976	.0184	.0081	.0041	.0010
11	1100111	5.1476	.0182	.0080	.0040	.0010
12	1011110	5.2643	.0160	.0068	.0034	.0008
13	1101011	5.3976	.0142	.0059	.0029	.0007
14	1101101	5.5976	.0118	.0047	.0023	.0005
15	1110011	5.7310	.0108	.0042	.0020	.0005
16	1101110	5.7643	.0103	.0039	.0019	.0004
17	1110101	5.9310	.0090	.0034	.0016	.0003
18	1110110	6.0976	.0078	.0028	.0013	.0003
19	1111001	6.1810	.0074	.0027	.0012	.0002
20	1111010	6.3476	.0064	.0023	.0010	.0002
21	1111100	6.5476	.0055	.0019	.0008	.0002

TABLE I—*Continued*

$N = 7, m = 3, n = 4$

i	R.O.	T	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			2.8968	4.9243	6.8455	14.8480
1	0001111	1.7214	.1911	.3436	.4485	.6739
2	0010111	2.0548	.1171	.1489	.1521	.1200
3	0011011	2.3048	.0886	.0996	.0954	.0676
4	0011101	2.5048	.0729	.0763	.0707	.0475
5	0100111	2.5548	.0601	.0503	.0388	.0151
6	0011110	2.6714	.0627	.0626	.0566	.0368
7	0101011	2.8048	.0455	.0336	.0243	.0085
8	0101101	3.0048	.0374	.0258	.0180	.0060
9	0110011	3.1381	.0328	.0214	.0146	.0046
10	0101110	3.1714	.0322	.0211	.0144	.0046
11	0110101	3.3381	.0270	.0165	.0108	.0033
12	0110110	3.5048	.0232	.0135	.0087	.0025
13	1000111	3.5548	.0207	.0102	.0057	.0010
14	0111001	3.5881	.0217	.0124	.0079	.0023
15	0111010	3.7548	.0187	.0102	.0063	.0018
16	1001011	3.8048	.0157	.0068	.0035	.0006
17	0111100	3.9548	.0159	.0083	.0050	.0014
18	1001101	4.0048	.0129	.0052	.0026	.0004
19	1010011	4.1381	.0113	.0043	.0021	.0003
20	1001110	4.1714	.0112	.0043	.0021	.0003
21	1010101	4.3381	.0093	.0033	.0016	.0002
22	1010110	4.5048	.0080	.0027	.0013	.0002
23	1011001	4.5881	.0075	.0025	.0012	.0001
24	1100011	4.6381	.0076	.0026	.0012	.0002
25	1011010	4.7548	.0065	.0020	.0010	.0001
26	1100101	4.8381	.0063	.0020	.0009	.0001
27	1011100	4.9548	.0055	.0016	.0006	.0001
28	1100110	5.0048	.0054	.0016	.0007	.0001
29	1101001	5.0881	.0051	.0015	.0007	.0001
30	1101010	5.2548	.0044	.0012	.0006	.0001
31	1110001	5.4214	.0040	.0011	.0005	.0001
32	1101100	5.4548	.0037	.0010	.0003	.0001
33	1110010	5.5881	.0034	.0009	.0004	.0001
34	1110100	5.7881	.0029	.0007	.0003	.0001
35	1111000	6.0381	.0024	.0006	.0002	.0000

TABLE I—*Continued*

$N = 8, m = 3, n = 5$

i	R.O.	T	$P^{.10}_{.60}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			2.8029	4.6300	6.4006	13.0618
1	00011111	2.3464	.1507	.2872	.3904	.6126
2	00101111	2.6798	.0941	.1300	.1394	.1220
3	00110111	2.9298	.0718	.0881	.0885	.0697
4	00111011	3.1298	.0594	.0680	.0660	.0493
5	01001111	3.1798	.0495	.0462	.0377	.0174
6	00111101	3.2964	.0513	.0560	.0531	.0383
7	01010111	3.4298	.0378	.0314	.0239	.0099
8	00111110	3.4393	.0455	.0479	.0447	.0314
9	01011011	3.6298	.0312	.0242	.0178	.0070
10	01100111	3.7631	.0275	.0203	.0145	.0055
11	01011101	3.7964	.0270	.0199	.0144	.0054
12	01011110	3.9393	.0239	.0170	.0121	.0045
13	01101011	3.9631	.0227	.0156	.0108	.0039
14	01101101	4.1298	.0196	.0129	.0088	.0030
15	10001111	4.1798	.0177	.0100	.0059	.0013

$N = 8, m = 4, n = 4$

			δ			
			2.5893	4.2454	5.6371	11.8205
1	00001111	1.4619	.1056	.2156	.2966	.5295
2	00010111	1.7119	.0756	.1190	.1374	.1429
3	00011011	1.9119	.0609	.0854	.0928	.0849
4	00100111	2.0452	.0494	.0572	.0540	.0310
5	00011101	2.0786	.0519	.0678	.0712	.0610
6	00011110	2.2214	.0457	.0568	.0583	.0479
7	00101011	2.2452	.0398	.0410	.0365	.0184
8	00101101	2.4119	.0339	.0326	.0280	.0132
9	00110011	2.4952	.0310	.0283	.0237	.0106
10	01000111	2.5452	.0275	.0218	.0163	.0048
11	00101110	2.5548	.0299	.0273	.0229	.0104
12	00110101	2.6619	.0264	.0225	.0182	.0076
13	01001011	2.7452	.0222	.0156	.0110	.0029
14	00110110	2.8048	.0233	.0189	.0149	.0060
15	00111001	2.8619	.0221	.0175	.0137	.0054
16	01001101	2.9119	.0189	.0124	.0084	.0021
17	01010011	2.9952	.0173	.0108	.0071	.0017
18	00111010	3.0048	.0195	.0147	.0112	.0042
19	01001110	3.0548	.0167	.0104	.0069	.0016
20	01010101	3.1619	.0147	.0086	.0054	.0012
21	00111100	3.1714	.0170	.0122	.0091	.0033
22	01010110	3.3048	.0130	.0072	.0045	.0009
23	01100011	3.3286	.0128	.0071	.0044	.0010
24	01011001	3.3619	.0123	.0067	.0041	.0009
25	01100101	3.4952	.0109	.0057	.0034	.0007

TABLE I—*Continued*

$N = 8, m = 4, n = 4$—*Cont.*

i	R.O.	T	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			2.5893	4.2454	5.6371	11.8205
26	01011010	3.5048	.0109	.0056	.0034	.0007
27	10000111	3.5452	.0106	.0051	.0029	.0004
28	01100110	3.6381	.0097	.0047	.0028	.0005
29	01011100	3.6714	.0095	.0047	.0027	.0005
30	01101001	3.6952	.0091	.0044	.0026	.0005

$N = 9, m = 4, n = 5$

			δ			
			2.4942	3.9646	5.2287	10.2816
1	000011111	2.0175	.0751	.1645	.2377	.4511
2	000101111	2.2675	.0547	.0945	.1155	.1359
3	000110111	2.4675	.0445	.0689	.0792	.0824
4	001001111	2.6008	.0365	.0475	.0479	.0332
5	000111011	2.6341	.0382	.0552	.0613	.0598
6	000111101	2.7770	.0338	.0465	.0505	.0472
7	001010111	2.8008	.0297	.0347	.0329	.0201
8	000111110	2.9020	.0305	.0452	.0432	.0391
9	001011011	2.9675	.0255	.0278	.0254	.0146
10	001100111	3.0508	.0233	.0243	.0217	.0118
11	010001111	3.1008	.0209	.0191	.0154	.0059
12	001011101	3.1103	.0226	.0234	.0210	.0115
13	001101011	3.2175	.0200	.0195	.0168	.0086
14	001011110	3.2353	.0204	.0227	.0179	.0096
15	010010111	3.3008	.0170	.0140	.0106	.0036
16	001101101	3.3603	.0178	.0164	.0139	.0068
17	001110011	3.4175	.0168	.0153	.0128	.0062
18	010011011	3.4675	.0146	.0112	.0082	.0026
19	001101110	3.4853	.0160	.0159	.0118	.0057
20	010100111	3.5508	.0133	.0098	.0070	.0021
21	001110101	3.5603	.0150	.0129	.0106	.0049
22	010011101	3.6103	.0129	.0094	.0067	.0020
23	001110110	3.6853	.0135	.0125	.0090	.0041
24	010101011	3.7175	.0114	.0079	.0054	.0015
25	001111001	3.7270	.0131	.0108	.0086	.0038
26	010011110	3.7353	.0117	.0091	.0057	.0017
27	001111010	3.8520	.0118	.0104	.0073	.0032
28	010101101	3.8603	.0101	.0066	.0044	.0012
29	011000111	3.8841	.0100	.0065	.0044	.0012
30	010110011	3.9175	.0096	.0062	.0041	.0011
31	010101110	3.9853	.0092	.0064	.0038	.0010
32	001111100	3.9948	.0106	.0090	.0062	.0026

TABLE I—*Continued*

$N = 10,\ m = 5,\ n = 5$

i	R.O.	T	$P^{.10}_{.50}$	$P^{.10}_{.25}$	$P^{.05}_{.25}$	$P^{.05}_{.05}$
			δ			
			2.3226	3.6030	4.6201	8.8697
1	0000011111	1.7718	.0404	.0982	.1482	.3308
2	0000101111	1.9718	.0319	.0646	.0860	.1285
3	0000110111	2.1385	.0270	.0496	.0625	.0820
4	0001001111	2.2218	.0240	.0391	.0451	.0433
5	0000111011	2.2813	.0237	.0409	.0498	.0609
6	0001010111	2.3885	.0203	.0300	.0328	.0276
7	0000111101	2.4063	.0213	.0351	.0418	.0488
8	0000111110	2.5175	.0195	.0309	.0362	.0409
9	0001011011	2.5313	.0179	.0247	.0262	.0205
10	0010001111	2.5552	.0167	.0209	.0204	.0120
11	0001100111	2.5885	.0168	.0223	.0231	.0171
12	0001011101	2.6563	.0161	.0212	.0220	.0164
13	0010010111	2.7218	.0141	.0160	.0148	.0077
14	0001101011	2.7313	.0148	.0184	.0185	.0127
15	0001011110	2.7675	.0147	.0187	.0190	.0138
16	0001101101	2.8563	.0133	.0158	.0155	.0102
17	0010011011	2.8647	.0124	.0132	.0119	.0057
18	0001110011	2.8980	.0128	.0149	.0145	.0093
19	0011000111	2.9218	.0117	.0119	.0105	.0047
20	0001101110	2.9675	.0122	0139	.0134	.0086
21	0010011101	2.9897	.0112	.0114	.0100	.0045
22	0001110101	3.0230	.0115	.0128	.0122	.0075
23	0100001111	3.0552	.0101	.0091	.0073	.0024
24	0010101011	3.0647	.0103	.0099	.0084	.0035
25	0010011110	3.1008	.0102	.0100	.0086	.0038
26	0001110110	3.1341	.0106	.0113	.0105	.0063
27	0001111001	3.1659	.0103	.0109	.0101	.0060
28	0011001111	3.1718	.0094	.0085	.0071	.0013
29	0010101101	3.1897	.0092	.0085	.0070	.0028
30	0100010111	3.2218	.0085	.0070	.0053	.0016
31	0010110011	3.2313	.0089	.0080	.0060	.0026
32	0001111010	3.2770	.0095	.0096	.0087	.0050
33	0010101110	3.3008	.0085	.0074	.0061	.0024
34	0011001011	3.3147	.0092	.0071	.0057	.0021
35	0010110101	3.3563	.0080	.0069	.0055	.0021
36	0100011011	3.3647	.0075	.0057	.0042	.0012
37	0001111100	3.4020	.0086	.0084	.0075	.0042
38	0100100111	3.4218	.0070	.0052	.0037	.0010
39	0011001101	3.4397	.0074	.0061	.0047	.0017
40	0010110110	3.4675	.0074	.0061	.0046	.0017
41	0011010011	3.4813	.0071	.0057	.0045	.0016
42	0100011101	3.4897	.0067	.0050	.0036	.0009
43	0010111001	3.4992	.0071	.0058	.0046	.0017
44	0011001110	3.5508	.0068	.0053	.0041	.0014
45	0100101011	3.5647	.0064	.0043	.0030	.0007
46	0100011110	3.6008	.0061	.0043	.0031	.0008
47	0011010101	3.6063	.0064	.0049	.0038	.0013
48	0010111010	3.6103	.0066	.0051	.0039	.0014
49	0101000111	3.6718	.0057	.0037	.0025	.0003

TABLE II

Power Functions of Nonparametric Tests

In each case, the first entry gives the power of the test based on T; the second entry gives the power of the best rank order test; the third entry gives the power of the test based on c_1 .

$\alpha = .10 \quad \beta = .50$

m	n = 1	2	3	4	5
1	.1800	.2771	.3146	.3776	.4387
	.1800	.2771	.3146	.3776	.4387
	.1800	.2771	.3146	.3776	.4387
2	.2624	.3245	.4394	.4553	.4839
	.2624	.3245	.4394	.4553	.4839
	.2624	.3245	.4394	.4553	.4839
3	.2387	.3667	.4062	.4332	.4563
	.2387	.3667	.4062	.4332	.4570
	.2387	.3667	.4062	.4268	.4482
4	.2535	.3494	.3917	.4289	.4398
	.2535	.3494	.3938	.4289	.4402
	.2535	.3494	.3828	.4107	.4272
5	.2642	.3513	.3933	.4195	.4370
	.2642	.3513	.3933	.4199	.4375
	.2642	.3495	.3737	.3948	.4322

$\alpha = .10 \quad \beta = .25$

m	n = 1	2	3	4	5
1	.1929	.2491	.3586	.4384	.5169
	.1929	.2491	.3586	.4384	.5169
	.1929	.2491	.3586	.4384	.5169
2	.2169	.4343	.6377	.6360	.6638
	.2169	.4343	.6377	.6360	.6638
	.2169	.4343	.6377	.6360	.6638
3	.3171	.5604	.5991	.6302	.6531
	.3171	.5604	.5991	.6302	.6570
	.3171	.5604	.5991	.6173	.6383
4	.3614	.5420	.5933	.6428	.6599
	.3614	.5420	.6021	.6428	.6635
	.3614	.5420	.5738	.6078	.6259
5	.3983	.5585	.6122	.6375	.6558
	.3983	.5585	.6180	.6389	.6587
	.3983	.5349	.5693	.6264	.6179

TABLE II—*Continued*

$\alpha = .05 \quad \beta = .25$

m	n = 1	2	3	4	5
1	.0983	.1443	.1896	.2344	.2789
	.0983	.1443	.1896	.2344	.2789
	.0983	.1443	.1896	.2344	.2789
2	.1377	.2421	.3658	.5083	.6431
	.1377	.2421	.3658	.5083	.6431
	.1377	.2421	.3658	.5083	.6431
3	.1701	.3270	.5305	.5626	.6006
	.1701	.3270	.5305	.5626	.6006
	.1701	.3270	.5305	.5626	.6006
4	.1972	.3996	.5098	.5543	.5823
	.1972	.3996	.5098	.5582	.5871
	.1972	.3996	.5098	.5344	.5791
5	.2204	.4788	.5162	.5650	.6030
	.2204	.4788	.5162	.5737	.6055
	.2204	.4788	.4884	.5292	.5612

$\alpha = .05 \quad \beta = .05$

m	n = 1	2	3	4	5
1	.0997	.1483	.1964	.2441	.2915
	.0997	.1483	.1964	.2441	.2915
	.0997	.1483	.1964	.2441	.2915
2	.1478	.2792	.4400	.6298	.8114
	.1478	.2792	.4400	.6298	.8114
	.1478	.2792	.4400	.6298	.8114
3	.1941	.4285	.7535	.7639	.7904
	.1941	.4285	.7535	.7639	.7904
	.1941	.4285	.7535	.7639	.7904
4	.2389	.5888	.7442	.7728	.8140
	.2389	.5888	.7442	.7878	.8213
	.2389	.5888	.7442	.7458	.7843
5	.2823	.7482	.7718	.8044	.8334
	.2823	.7482	.7718	.8252	.8382
	.2823	.7482	.7275	.7528	.7830

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