Control of a Multilevel Converter Using Resultant Theory

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***Abstract*—In this work, a method is given to compute the switching angles in a multilevel converter to produce the required fundamental voltage while at the same time cancel out specified higher order harmonics. Specifically, a complete analysis is given for a seven-level converter (three dc sources), where it is shown that for a range of the modulation index m_I, the switching angles can be chosen to produce the desired fundamental $V_1 = m_I(s4V_{\mathrm{dc}}/\pi)$ while making the fifth and seventh harmonics identically zero.**

***Index Terms*—Cascade inverter, harmonic elimination, multilevel converter, resultants.**

I. Introduction

A MULTILEVEL converter is a power electronic system that synthesizes a desired voltage output from several levels of dc voltages as inputs. In a distributed energy system consisting of fuel cells, wind turbines, solar cells, etc., the multilevel converter can provide a mechanism to feed these sources into an existing three phase power grid [21], [23]. Another area of application interest is heavy duty hybrid-electric vehicles (HEVs) such as tractor trailers, transfer trucks, or military vehicles. Development of large electric drive trains for these vehicles will result in increased fuel efficiency, lower emissions, and likely better vehicle performance (acceleration and braking). For parallel-configured HEVs, a cascaded H-bridges multilevel inverter can be used to drive the traction motor from a set of batteries, ultracapacitors, or fuel cells.

Multilevel inverters also have several advantages with respect to hard-switched two-level pulse width modulation (PWM) adjustable-speed drives (ASDs). Motor damage and failure have been reported by industry as a result of some ASD inverters' high-voltage change rates (dV/dt), which produced a common-mode voltage across the motor windings. High-frequency switching can exacerbate the problem because of the numerous times this common mode voltage is impressed upon the motor each cycle. The main problems reported have been "motor bearing failure" and "motor winding insulation breakdown" because of circulating currents, dielectric stresses, voltage surge, and corona discharge [1], [2], [7].

Multilevel inverters overcome these problems because their individual devices have a much lower dV/dt per switching, and they operate at high efficiencies because they can switch at a much lower frequency than PWM-controlled inverters. Three-, four-, and five-level rectifier-inverter drive systems that have used some form of multilevel PWM as a means to control the switching of the rectifier and inverter sections have been investigated in the literature [10], [12], [15], [16], [25]. Multilevel PWM has lower dV/dt than that experienced in some two-level PWM drives because switching is between several smaller voltage levels. However, switching losses and voltage total harmonic distortion (THD) are still relatively high for some of these proposed schemes.

In this work, a method is given to compute the switching angles in a multilevel converter so as to produce the required fundamental voltage while at the same time cancel out specified higher order harmonics. In particular, a complete analysis is given for a seven-level converter (three dc sources) where it is shown that for a range of the modulation index m_a, the switching angles can be chosen to produce the desired fundamental $V_1 = m_a(s4V_{\mathrm{dc}}/\pi)$ while making the fifth and seventh harmonics identically zero. In contrast to previous work in which iterative numerical techniques were employed [14], [19], [20], the approach here gives the *exact* range of the modulation index for which solutions exist and gives *all* possible solutions. A preliminary version of this paper appeared in [3].

II. Cascaded H-bridges

The cascade multilevel inverter consists of a series of H-bridge (single-phase full-bridge) inverter units. The general function of this multilevel inverter is to synthesize a desired voltage from several separate dc sources (SDCSs), which may be obtained from batteries, fuel cells, or ultracapacitors in an HEV. Fig. 1 shows a single-phase structure of a cascade inverter with SDCSs [11]. Each SDCS is connected to a single-phase full-bridge inverter. Each inverter level can generate three different voltage outputs ($+V_{\mathrm{dc}}$, zero, and $-V_{\mathrm{dc}}$) by connecting the dc source to the ac output side by different combinations of the four switches: S_1, S_2, S_3, and S_4. The ac output of each level's full-bridge inverter is connected in series such that the synthesized voltage waveform is the sum of all of the individual inverter outputs. The number of output phase voltage levels in a cascade mulitilevel inverter is then $2s+1$, where s is the number of dc sources. An example phase voltage waveform

Manuscript received April 9, 2002. Manuscript received in final form November 4, 2002. Recommended by Associate Editor S. Peresada. The work of L. M. Tolbert was supported in part by the National Science Foundation under Contract NSF ECS-0093884 and in part by the Oak Ridge National Laboratory under UT/Battelle Contract 4000007596. The work of J. N. Chiasson was supported in part by the Oak Ridge National Laboratory under UT/Battelle Contract 4000007596.

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Digital Object Identifier 10.1109/TCST.2003.810382

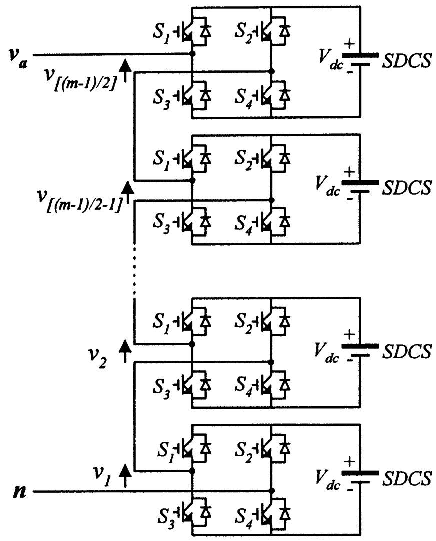

Fig. 1. Single-phase structure of a multilevel cascaded H-bridges inverter.

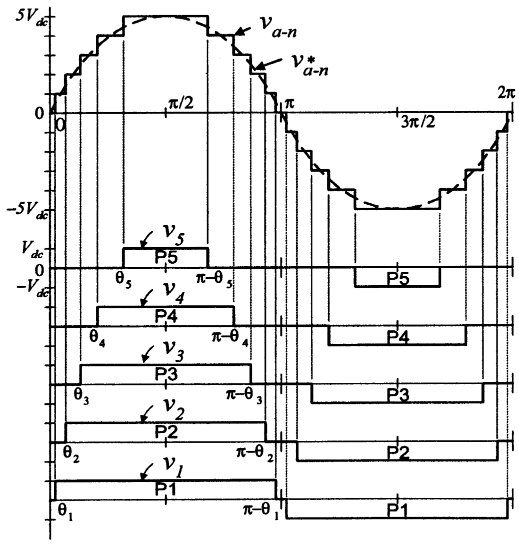

Fig. 2. Output waveform of an 11-level cascade multilevel inverter.

for an 11-level cascaded multilevel inverter with five SDSCs $(s = 5)$ and five full bridges is shown in Fig. 2. The output phase voltage is given by $v_{an} = v_{a1} + v_{a2} + v_{a3} + v_{a4} + v_{a5}$.

With enough levels and an *appropriate* switching algorithm, the multilevel inverter results in an output voltage that is almost sinusoidal. For the 11-level example shown in Fig. 2, the waveform has less than 5% THD with each of the active devices of the H-bridges active devices switching only at the fundamental frequency. Each H-bridge unit generates a quasisquare waveform by phase-shifting its positive and negative phase legs' switching timings. Each switching device always conducts for 180° (or 1/2 cycle) regardless of the pulse width of the quasisquare wave so that this switching method results in equalizing the current stress in each active device.

III. Switching Algorithm for the Multilevel Converter

The Fourier series expansion of the (stepped) output voltage waveform of the multilevel inverter as shown in Fig. 2 is [18], [20], [22]

$$V(\omega t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{\mathrm{dc}}}{n\pi} \left(\cos(n\theta_1) + \cos(n\theta_2) + \cdots + \cos(n\theta_s)\right) \sin(n\omega t) \quad (1)$$

where s is the number of dc sources. Ideally, given a desired fundamental voltage V_1, one wants to determine the switching angles $\theta_1, \ldots, \theta_s$ so that (1) becomes $V(\omega t) = V_1 \sin(\omega t)$. In practice, one is left with trying to do this approximately. Two predominate methods in choosing the switching angles $\theta_1, \ldots \theta_s$ are 1) eliminate the lower frequency dominant harmonics or 2) minimize the total harmonic distortion. The more popular and straightforward of the two techniques is the first, that is, eliminate the lower dominant harmonics and filter the output to remove the higher residual frequencies. Here, the choice is also to eliminate the lower frequency harmonics.

The goal here is to choose the switching angles $0 \leq \theta_1 < \theta_2 < \cdots < \theta_s \leq \pi/2$ to make the first harmonic equal to the desired fundamental voltage V_1 and specific higher harmonics of $V(\omega t)$ equal to zero. As the application of interest here is a three-phase motor drive, the triplen harmonics in each phase need not be canceled as they automatically cancel in the line-to-line voltages. Specifically, in case of $s = 5$ dc sources, the desire is to cancel the fifth-, seventh-, 11th-, and 13th-order harmonics as they dominate the total harmonic distortion. The mathematical statement of these conditions is then

$$\begin{aligned} \frac{4V_{\mathrm{dc}}}{\pi}\left(\cos(\theta_1) + \cos(\theta_2) + \cdots + \cos(\theta_5)\right) &= V_1 \\ \cos(5\theta_1) + \cos(5\theta_2) + \cdots + \cos(5\theta_5) &= 0 \\ \cos(7\theta_1) + \cos(7\theta_2) + \cdots + \cos(7\theta_5) &= 0 \\ \cos(11\theta_1) + \cos(11\theta_2) + \cdots + \cos(11\theta_5) &= 0 \\ \cos(13\theta_1) + \cos(13\theta_2) + \cdots + \cos(13\theta_5) &= 0. \end{aligned} \quad (2)$$

This is a system of five transcendental equations in the five unknowns $\theta_1, \theta_2, \theta_3, \theta_4,$ and θ_5. One approach to solving this set of nonlinear transcendental (2) is to use an iterative method such as the Newton-Raphson method [6], [18], [20], [22]. The correct solution to the conditions (2) would mean that the output voltage of the 11-level inverter would not contain the fifth-, seventh-, 11th-, and 13th-order harmonic components.

The fundamental question is "When does the set of (2) have a solution?" As will be shown below, it turns out that a solution exists for only specific ranges of the modulation index[1] $m_a \triangleq V_1/(s4V_{\mathrm{dc}}/\pi)$. This range does not include the low end or the high end of the modulation index. A method is now presented to find the solutions when they exist. This method is based on the theory of resultants of polynomials [9]. To proceed, let $s =$

[1]Each inverter has a dc source of V_{dc} so that the maximum output voltage of the multilevel inverter is sV_{dc}. A square wave of amplitude sV_{dc} results in the maximum fundamental output possible of $V_{1\,\max} = 4sV_{\mathrm{dc}}/\pi$. The modulation index is, therefore, $m_I \triangleq V_1/V_{1\,\max} = V_1/(s4V_{\mathrm{dc}}/\pi)$.

5, and define $x_i = \cos(\theta_i)$ for $i = 1, \ldots, 5$. Then, using the trigonometric identities

$$\begin{aligned}\cos(5\theta) =& 5\cos(\theta) - 20\cos^3(\theta) + 16\cos^5(\theta)\\ \cos(7\theta) =& -7\cos(\theta) + 56\cos^3(\theta) - 112\cos^5(\theta)\\ &+ 64\cos^7(\theta)\\ \cos(11\theta) =& -11\cos(\theta) + 220\cos^3(\theta) - 1232\cos^5(\theta)\\ &+ 2816\cos^7(\theta) - 2816\cos^9(\theta)\\ &+ 1024\cos^{11}(\theta)\\ \cos(13\theta) =& 13\cos(\theta) - 364\cos^3(\theta) + 2912\cos^5(\theta)\\ &- 9984\cos^7(\theta) + 16\,640\cos^9(\theta)\\ &- 13\,312\cos^{11}(\theta) + 4096\cos^{13}(\theta)\end{aligned}$$

the conditions (2) become

$$\begin{aligned}p_1(x) &\triangleq x_1 + x_2 + x_3 + x_4 + x_5 - m = 0\\ p_5(x) &\triangleq \sum_{i=1}^{5}\left(5x_i - 20x_i^3 + 16x_i^5\right) = 0\\ p_7(x) &\triangleq \sum_{i=1}^{5}\left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0\\ p_{11}(x) &\triangleq \sum_{i=1}^{5}\left(-11x_i + 220x_i^3 - 1232x_i^5\right.\\ &\qquad \left. +2816x_i^7 - 2816x_i^9 + 1024x_i^{11}\right) = 0\\ p_{13}(x) &\triangleq \sum_{i=1}^{5}\left(13x_i - 364x_i^3 + 2912x_i^5 - 9984x_i^7\right.\\ &\qquad + 16\,640x_i^9 - 13\,312x_i^{11}\\ &\qquad \left. +4096x_i^{13}\right) = 0\end{aligned} \tag{3}$$

where $x = (x_1, x_2, x_3, x_4, x_5)$, and $m \triangleq V_1/\left(4V_{\mathrm{dc}}/\pi\right)$. This is a set of five equations in the five unknowns x_1, x_2, x_3, x_4, x_5. Further, the solutions must satisfy $0 \le x_5 < \cdots < x_2 < x_1 \le 1$. This development has resulted in a set of polynomial equations rather than trigonometric equations. Though the degree is high, there is a well-known theory to solve sets of polynomial equations, which is described in Section III-A.

A. Resultants

Given two polynomials $a(x_1, x_2)$ and $b(x_1, x_2)$, how does one find their common zeros? That is, the values (x_{10}, x_{20}) such that

$$a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0.$$

Consider $a(x_1, x_2)$ and $b(x_1, x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. For example, let $a(x_1, x_2)$ and $b(x_1, x_2)$ have degrees 3 and 2, respectively, in x_2 so that they may be written in the form

$$\begin{aligned}a(x_1, x_2) =& a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2\\ &+ a_0(x_1)\\ b(x_1, x_2) =& b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1).\end{aligned}$$

A consequence of the famed *Nullstellensatz* theorem of Hilbert [4], [5], [8] is that the polynomials $a(x_1, x_2)$ and $b(x_1, x_2)$ do *not* have a common zero if and only if there exists another pair of polynomials

$$\begin{aligned}\alpha(x_1, x_2) =& \alpha_1(x_1)x_2 + \alpha_0(x_1)\\ \beta(x_1, x_2) =& \beta_2(x_1)x_2^2 + \beta_1(x_1)x_2 + \beta_0(x_1)\end{aligned}$$

such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = 1.$$

($\deg_{x2}\{\alpha(x_1, x_2)\} = \deg_{x2}\{b(x_1, x_2)\} - 1$ and $\deg_{x2}\{\beta(x_1, x_2)\} = \deg_{x2}\{a(x_1, x_2)\} - 1$).

More generally, there is always a polynomial $r(x_1)$ (called the *resultant polynomial*) such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1).$$

So if $a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0$, then $r(x_{10}) = 0$, that is, if (x_{10}, x_{20}) is a common zero of the pair $\{a(x_1, x_2), b(x_1, x_2)\}$, then the first coordinate x_{10} is a zero of $r(x_1) = 0$. The roots of $r(x_1)$ are easy to find (numerically) as it is a polynomial in one variable. To find the common zeros of $\{a(x_1, x_2), b(x_1, x_2)\}$, one computes the n_1 roots x_{1i} $i = 1, \ldots, n_1$ of $r(x_1)$. Next, for each such x_{1i}, one (numerically) computes the roots of

$$a(x_{1i}, x_2) = 0 \tag{4}$$

and the roots of

$$b(x_{1i}, x_2) = 0. \tag{5}$$

Any root x_{2j} that is in the solution set of both (4) and (5) for a given x_{1i} results in the pair (x_{1i}, x_{2j}) being a common zero of $a(x_1, x_2)$ and $b(x_1, x_2)$. Thus, this gives a method of solving polynomials in one variable to compute the common zeros of $\{a(x_1, x_2), b(x_1, x_2)\}$.

To see how one obtains $r(x_1)$, let

$$\begin{aligned}a(x_1, x_2) =& a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1)\\ b(x_1, x_2) =& b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1).\end{aligned}$$

Next, see if polynomials of the form

$$\begin{aligned}\alpha(x_1, x_2) =& \alpha_1(x_1)x_2 + \alpha_0(x_1)\\ \beta(x_1, x_2) =& \beta_2(x_1)x_2^2 + \beta_1(x_1)x_2 + \beta_0(x_1)\end{aligned}$$

can be found such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1). \tag{6}$$

Equating powers of x_2, this equation may be rewritten in matrix form as

$$\begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0\\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0\\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0\\ a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1\\ 0 & a_3(x_1) & 0 & 0 & b_2 \end{bmatrix}\begin{bmatrix}\alpha_0(x_1)\\ \alpha_1(x_1)\\ \beta_0(x_1)\\ \beta_1(x_1)\\ \beta_2(x_1)\end{bmatrix} = \begin{bmatrix} r(x_1)\\ 0\\ 0\\ 0\\ 0\end{bmatrix}.$$

The matrix on the left-hand side is called the *Sylvester* matrix and is denoted here by $S_{a,b}(x_1)$. The inverse of $S_{a,b}(x_1)$ has the form

$$S_{a,b}^{-1}(x_1) = \frac{1}{\det S_{a,b}(x_1)} \mathrm{adj}\,(S_{a,b}(x_1))$$

where $\mathrm{adj}\,(S_{a,b}(x_1))$ is the adjoint matrix and is a 5×5 *polynomial* matrix in x_1. Solving for $\alpha_i(x_1)$, $\beta_i(x_1)$ gives

$$\begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \frac{\mathrm{adj} S_{a,b}(x_1)}{\det S_{a,b}(x_1)} \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Choosing $r(x_1) = \det S_{a,b}(x_1)$ this becomes

$$\begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \mathrm{adj} S_{a,b}(x_1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and guarantees that $\alpha_0(x_1), \alpha_1(x_1), \beta_0(x_1), \beta_1(x_1), \beta_2(x_1)$ are polynomials in x_1. That is, the *resultant polynomial* defined by $r(x_1) = \det S_{a,b}(x_1)$ is the polynomial required for (6).

In short, the polynomials $\{a(x_1,x_2), b(x_1,x_2)\}$ have a common zero at (x_{10}, x_{20}) only if $r(x_{10}) \triangleq \det S_{a,b}(x_{10}) = 0$. For an arbitrary pair of polynomials $\{a(x), b(x)\}$ of degrees n_a, n_b in x respectively, the matrix $S_{a,b}$ is of dimension $(n_a + n_b) \times (n_a + n_b)$ (see [4], [5], and [8]).

Remark: It was pointed out that if $a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0$, then $r(x_{10}) \triangleq \det S_{a,b}(x_{10}) = 0$ as a simple consequence of (6). Does $r(x_{10}) \triangleq \det S_{a,b}(x_{10}) = 0$ imply that there exists x_{20} such that

$$a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0?$$

The answer is yes if either of the leading coefficients in x_2 of $a(x_1,x_2)$, $b(x_1,x_2)$ are not zero at x_{10}, i.e., $a_3(x_{10}) \neq 0$ *or* $b_2(x_{10}) \neq 0$ (See [4], [5], and [8] for a detailed explanation).

Procedure to compute the common zeros:

1) Compute the roots x_{1k}, $k = 1, \ldots, n_{r_1} = \deg_{x_1}\{r_1(x_1)\}$ of $r(x_1) = 0$.
2) Substitute these roots into $a(x_1, x_2)$.
3) For $k = 1, \ldots, n_{r_1}$ solve $a(x_{1k}, x_2) = 0$ to get the roots $x_{2k\ell}$ for $\ell = 1, \ldots, n_{a2} = \deg_{x_2}\{a(x_{1k}, x_2)\}$.
4) The common zeros of $\{a(x_1,x_2), b(x_1,x_2)\}$ are then those values of $(x_{1k}, x_{2k\ell})$ that satisfy $b(x_{1k}, x_{2k\ell}) = 0$.

B. Seven-Level Case

To illustrate the procedure of using the theory of resultants to solve the system (3), consider the seven-level case (three dc sources). The conditions are

$$\begin{aligned} p_1(x) &\triangleq x_1 + x_2 + x_3 - m = 0 \\ m &\triangleq V_1 / \frac{4V_{\mathrm{dc}}}{\pi} = s m_a \\ p_5(x) &\triangleq \sum_{i=1}^{3} \left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \\ p_7(x) &\triangleq \sum_{i=1}^{3} \left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0. \end{aligned} \tag{7}$$

Substitute $x_3 = m - (x_1 + x_2)$ into p_5, p_7 to get

$$\begin{aligned} p_5(x_1,x_2) =& 5x_1 - 20x_1^3 + 16x_1^5 + 5x_2 - 20x_2^2 + 16x_2^5 \\ &+ 5(m - x_1 - x_2) - 20(m - x_1 - x_2)^3 \\ &+ 16(m - x_1 - x_2)^5 \\ p_7(x_1,x_2) =& -7x_1 + 56x_1^3 - 112x_1^5 + 64x_1^7 - 7x_2 \\ &+ 56x_2^3 - 112x_2^5 + 64x_2^7 - 7(m - x_1 - x_2) \\ &+ 56(m - x_1 - x_2)^3 - 112(m - x_1 - x_2)^5 \\ &+ 64(m - x_1 - x_2)^7. \end{aligned}$$

The goal here is to find solutions of

$$\begin{aligned} p_5(x_1,x_2) =& 0 \\ p_7(x_1,x_2) =& 0. \end{aligned}$$

For each fixed x_1, $p_5(x_1,x_2)$ can be viewed as a polynomial of (at most) degree 5 in x_2 whose coefficients are polynomials of (at most) degree 5 in x_1. For example[2]

$$\begin{aligned} p_5(x_1,x_2) =& 5m - 20m^3 + 16m^5 + 60m^2x_1 - 80m^4x_1 \\ &- 60mx_1^2 + 160m^3x_1^2 - 160m^2x_1^3 \\ &+ 80mx_1^4 \\ &+ \big[60m^2 - 80m^4 - 120mx_1 + 320m^3x_1 \\ &\quad +60x_1^2 - 480m^2x_1^2 + 320mx_1^3 - 80x_1^4\big] \\ &\times x_2 \\ &+ \big[-60m + 160m^3 + 60x_1 \\ &\quad -480m^2x_1 + 480mx_1^2 - 160x_1^3\big]\, x_2^2 \\ &+ \left[-160m^2 + 320mx_1 - 160x_1^2\right] x_2^3 \\ &+ [80m - 80x_1]x_2^4. \end{aligned}$$

This is often written as $p_5(x_1,x_2) \in \Re[x_1](x_2)$ to emphasize that p_5 is being viewed as a polynomial in x_2 whose coefficients are in the ring of polynomials $\Re[x_1]$. Similarly, $p_7(x_1,x_2) \in \Re[x_1](x_2)$ is a polynomial in x_2 whose coefficients are polynomials in x_1.

For each fixed x_1, the pair of polynomials $p_5(x_1,x_2) = 0$, $p_7(x_1,x_2) = 0$ has a solution x_2 if and only if the corresponding resultant matrix $S_{p_5,p_7}(x_1)$ is singular. Here, $\deg_{x_2}\{p_5(x_1,x_2)\} = 4$ and $\deg_{x_2}\{p_7(x_1,x_2)\} = 6$ so that the

[2]In this case, it turns out that the coefficient of the x_2^5 is zero so that $p_5(x_1,x_2)$ has degree 4 in x_2.

resultant matrix $S_{p_5,p_7}(x_1)$ is an element of $\Re^{10\times 10}[x_1]$, that is, it is a 10×10 matrix whose elements are polynomials in x_1. The determinant of this matrix $r(x_1)\triangleq \det S_{p_5,p_7}(x_1)$ is a polynomial in x_1. For any (x_{10},x_{20}) which is a simultaneous solution of $p_5(x_1,x_2)=0$, $p_7(x_1,x_2)=0$, it must follow that $r(x_{10})=0$. Consequently, finding the roots $r(x_1)$ gives candidate solutions for x_1 to check for common zeros of $p_5(x_1,x_2)=p_7(x_1,x_2)=0$.

In more detail, the resultant polynomial $r(x_1)$ of the pair $\{p_5(x_1,x_2),p_7(x_1,x_2)\}$ was found using the software package MATHEMATICA with the `Resultant` command [24]. The polynomial $r(x_2)$ turned out to be a 22nd-order polynomial given by

$$r(x_1)=6\,777\,216m^4(m-x_1)^4r_1^2(x_1)$$

where $r_1(x_1)$ is a ninth-order polynomial as given in the Appendix. Note that $x_1=m$ is a zero of $r(x_1)$. However

$$\begin{aligned}p_5(m,x_2)=&5m-20m^3+16m^5\\ p_7(m,x_2)=&-7m+56m^3-112m^5+64m^7\end{aligned}$$

and, as these two polynomials have a common zero only for $m=0$, only the roots of $r_1(x_1)=0$ need be checked.

The algorithm is as follows.

Algorithm for the Seven-Level Case

1) Given m, find the roots of $r_1(x_1)=0$.
2) Discard any roots that are less than zero, greater than one, or that are complex. Denote the remaining roots as $\{x_{1i}\}$.
3) For each fixed zero x_{1i} in the set $\{x_{1i}\}$, substitute it into p_5 and solve for the roots of $p_5(x_{1i},x_2)=0$.
4) Discard any roots (in x_2) that are complex, less than zero or greater than one. Denote the pairs of remaining roots as $\{(x_{1j},x_{2j})\}$.
5) Compute $m-x_{1j}-x_{2j}$ and discard any pair (x_{1j},x_{2j}) that makes this quantity negative or greater than one. Denote the triples of remaining roots as $\{(x_{1k},x_{2k},x_{3k})\}$.
6) Discard any triple for which $x_{3k}<x_{2k}<x_{1k}$ does not hold. Denote the remaining triples as $\{(x_{1l},x_{2l},x_{3l})\}$. The switching angles that are a solution to the three level system (7) are

$$\begin{aligned}&\{(\theta_{1l},\theta_{2l},\theta_{3l})\}\\ &\qquad=\{(\cos^{-1}(x_{1l}),\cos^{-1}(x_{2l}),\cos^{-1}(x_{3l}))\}.\end{aligned}$$

1) Minimization of the Fifth and Seventh Harmonic Components: For those values of m for which $p_5(x_1,x_2)$, $p_7(x_1,x_2)$ do not have common zeros satisfying $0\le x_1\le 1, 0\le x_2\le 1$, the next best thing is to minimize the error

$$c(x_1,x_2)=\frac{p_5^2(x_1,x_2)}{25}+\frac{p_7^2(x_1,x_2)}{49}.$$

This was accomplished by simply computing the values of $c(j\Delta x,k\Delta y)$ for j, $k=0,1,2,\cdots,1000$ with $\Delta x=0.001$, $\Delta y=0.001$ and then choosing the minimum value.

2) Results for the Seven-Level Inverter: The results are summarized in Fig. 3 which shows the switching angles θ_1, θ_2, and θ_3 versus m for those values of m in which the system (7) has a solution. The parameter m was incremented in steps of 0.01. Note that for m in the range from approximately 1.49 to 1.85, there are two different sets of solutions that solve (7). On the other hand, for $m\in[0,0.8]$, $m\in[0.83,1.15]$, and $m\in[2.52,2.77]$, there are *no* solutions to (7). Interestingly, for $m\approx 0.8$, $m\approx 0.82$ and $m\approx 2.76$ there are (isolated) solutions.

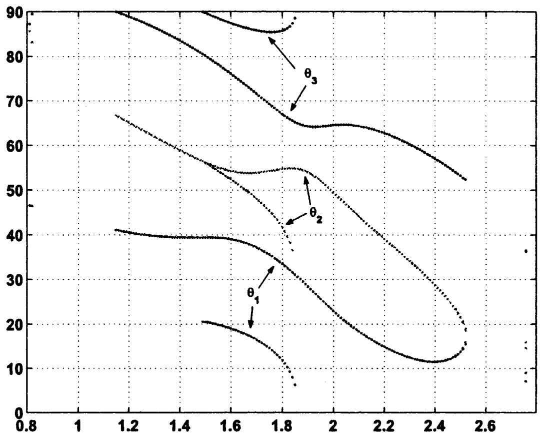

Fig. 3. Switching angles θ_1, θ_2, θ_3 in degrees versus m.

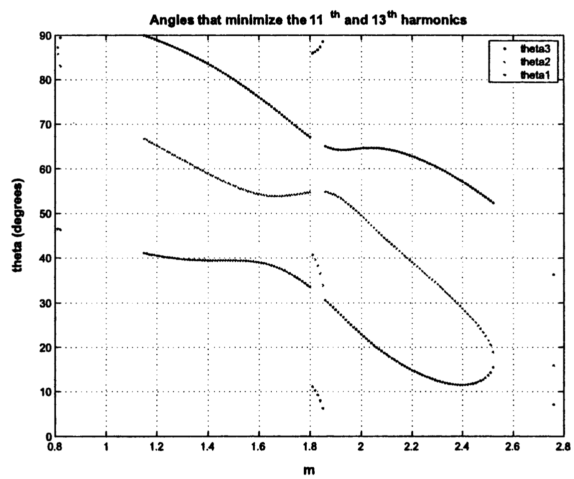

Fig. 4. Angles that give zero third and fifth harmonics and the smallest 11th and 13th harmonics.

In the range $m\in[1.49,1.85]$ for which there are two sets of solutions, the solution which gives the smallest distortion due to the 11th and 13th harmonics is a good choice. This set of angles is given in Fig. 4.

As pointed out above, for $m\in[0,0.8]$, $m\in[0.83,1.15]$, $m\in[2.52,2.77]$, and $m\in[2.78,3]$, there are no solutions satisfying the conditions (7). Consequently, for these ranges of m, the switching angles were determined by minimizing $\sqrt{(p_5/5)^2+(p_7/7)^2}$. Fig. 5 shows a plot of the resulting mimimum error $\sqrt{(p_5/5)^2+(p_7/7)^2}$ versus m for these values of m.

As Fig. 5 shows, when $m\approx 0.81$ and $m\approx 2.76$, the error is zero corresponding to the isolated solutions to (7) for those values of m. For $m=1.15$ and $m=2.52$, the error goes to zero because these values correspond to the boundary of the exact solutions of (6). However, note, e.g., when $m=0.25$, the error is about 0.25, that is, the error is the same size as m. Other

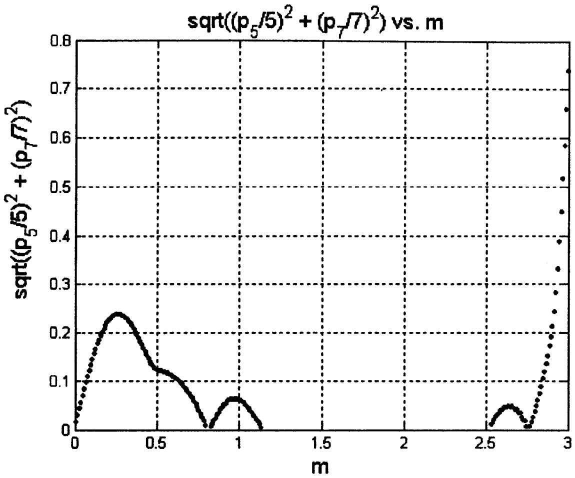

Fig. 5. Error $= \sqrt{(p_5/5)^2 + (p_7/7)^2}$ versus m.

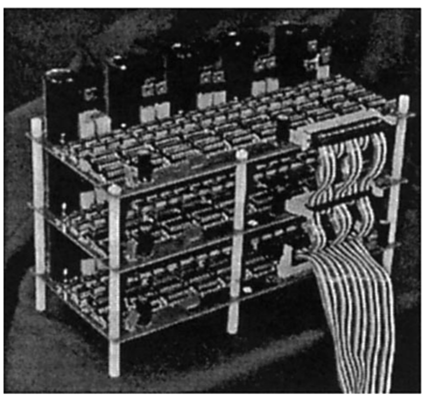

Fig. 6. Gate driver boards and MOSFETs for the mulitlevel inverter.

than close to the endpoints of the two intervals $[0, 0.8]$, $[2.78, 3]$ the minimum error $\sqrt{(p_5/5)^2 + (p_7/7)^2}$ is too large to make the corresponding switching angles for this interval of any use. Consequently, for m in this interval, one must use some other approach (e.g., PWM) in order to get reduced harmonics. For the other two intervals $[0.83, 1.15]$, $[2.52, 2.77]$, the minimum error $\sqrt{(p_5/5)^2 + (p_7/7)^2}$ is around 5% or less so that it might be satisfactory to use the corresponding switching angles for these intervals.

IV. Experimental Work

A prototype three-phase 11-level wye-connected cascaded inverter has been built using 100 V, 70 A MOSFETs as the switching devices [1]. The gate driver boards and MOSFETs are shown in Fig. 6. A battery bank of 15 SDCSs of 48 Vs dc (not shown) each feed the inverter (five SDCSs per phase). In the experimental study here, this prototype system was configured to be a seven-level (three SDCSs per phase) converter with each level being 12 V.The ribbon cable shown in the figure provides the communication link between the gate driver board and the real-time processor. In this work, the RT-LAB real-time computing platform from *Opal-RT-Technologies Inc.* [13] was used to interface the computer (which generates the logic signals) to this cable. The RT-LAB system allows one to write the switching algorithm in SIMULINK which is then converted to *C* code using RTW (real-time workshop) from *Mathworks*. The RT-LAB software provides icons to interface the SIMULINK model to the digital I/O board and converts the *C* code into executables. The time resolution (the precision for the time at which a switch is turned on or off) was chosen to be 1/1000 of an electrical cycle. For the 60-Hz frequency results reported here, this comes to $(1/60)/1000 = 16.7\mu$s. Using the XHP (extra high performance) option in RT-LAB as well as the multiprocessor option to spread the computation between two processors, an execution time of 16 μs was achieved.

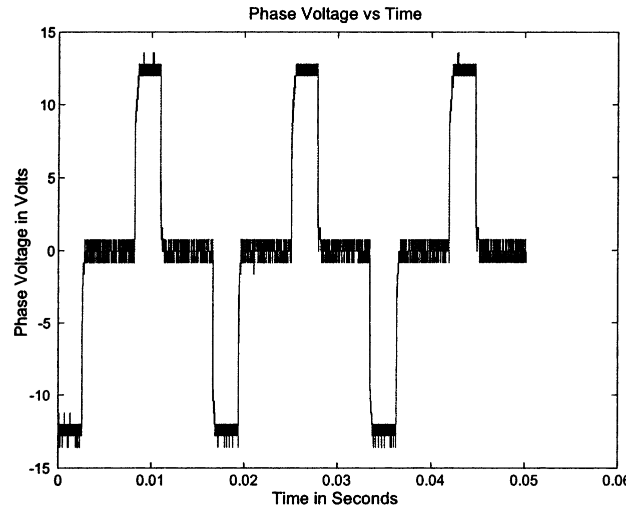

Fig. 7. Phase voltage when $m = 0.5$.

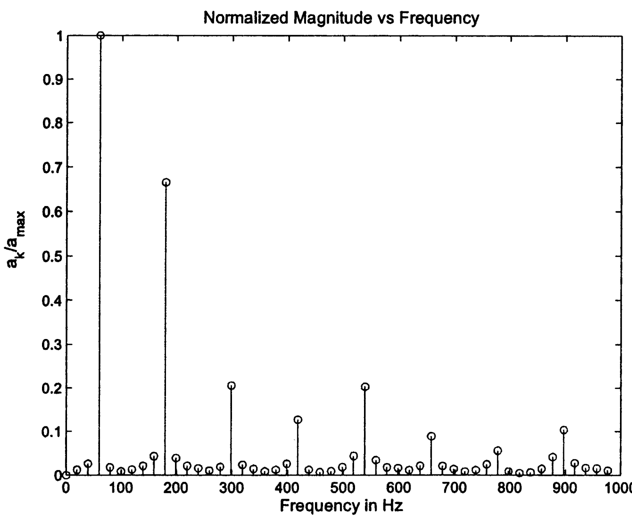

Fig. 8. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 0.5$.

Note that while the computation of the lookup table of Figs. 3 and 4 require some offline computational effort, the real-time implementation is accomplished by putting the data (i.e., Figs. 3 and 4) in a lookup table and therefore does not require high computational power for implementation.

Experiments were performed to validate the theoretical results of Section III-B2. That is, the elimination of the fifth and seventh harmonics (at 300 and 420 Hz, respectively) in the three phase output of a mutlilevel inverter. Recall, from Section III, that the triplen harmonics (180, 360, 540, etc.) in

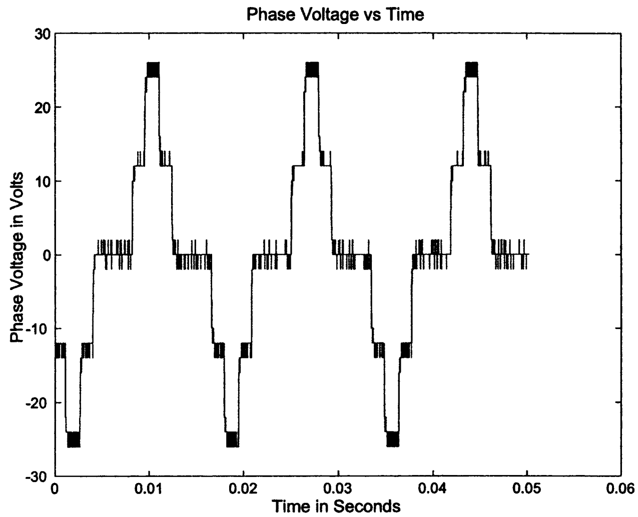

Fig. 9. Phase voltage when $m = 1$.

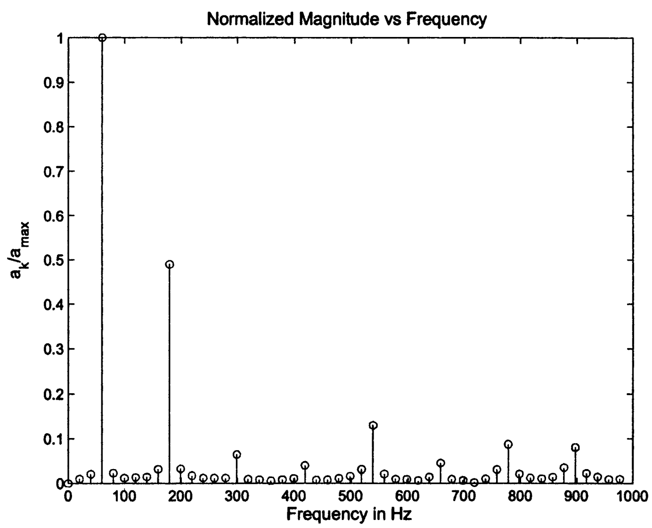

Fig. 10. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 1$.

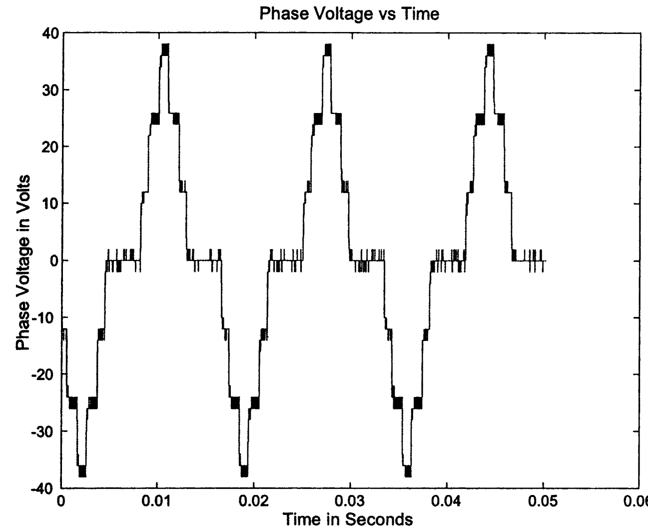

Fig. 11. Phase voltage when $m = 1.5$.

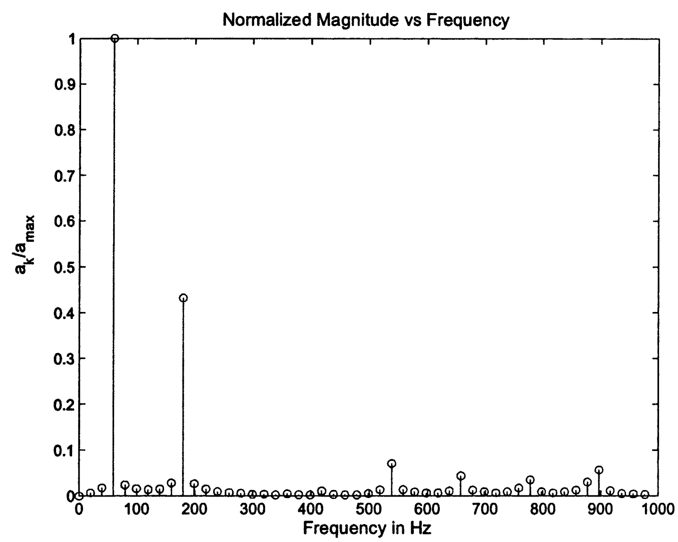

Fig. 12. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 1.5$.

each phase need not be canceled as they automatically cancel in the line-to-line voltages. Experimental data was taken for the parameter m having the values $m = 0.5$, 1.0, 1.5, 2.0, and 2.5. In this set of data, the angles were chosen according to Fig. 4. The frequency was set to 60 Hz in each case, and the program was run in real time with a 16-μs sample period, i.e., the logic signals were updated to the gate driver board every 16 μs.

The voltage was measured using a high-speed data acquisition oscilloscope every $T_s = 5$ μs, resulting in the data $\{v(nT_s), n = 1, \ldots, N\}$ where $N = 3(1/60)/\left(5 \times 10^{-6}\right) = 10\,000$ samples corresponding to three periods of the 60-Hz waveform. A fast Fourier transform was performed on this voltage data to get $\{\hat{v}(k\omega_0), k = 1, \ldots, N\}$ where the frequency increment is $\omega_0 = (2\pi/T_s)/N = 2\pi(20)$ rad/s or 20 Hz. The number $\hat{v}(k\omega_0)$ is simply the Fourier coefficient of the kth harmonic (whose frequency is $k\omega_0$ with $\omega_0 = (2\pi/N)(1/T_s)$) in the Fourier series expansion of the phase voltage signal $v(t)$. With $a_k = |\hat{v}(k\omega_0)|$ and $a_{\max} = \max_k \{|\hat{v}(k\omega_0)|\}$, the data that is plotted is the normalized magnitude $a_k/a_{\max}$.

Fig. 7 is the plot of the phase voltage for $m = 0.5$, and the corresponding FFT of this signal is given in Fig. 8. Fig. 8 shows a 0.225 normalized magnitude of the fifth harmonic and a 0.15 normalized magnitude of the seventh harmonic for a total normalized distortion of $\sqrt{(0.225)^2 + (0.15)^2} = 0.27$ due to these two harmonics. Fig. 5 shows an error of about 0.125 at $m = 0.5$ for a normalized magnitude of $0.125/0.5 = 0.25$ because of these two harmonics, which is in close agreement. At this low value for m, only one SDCS of the converter is used to achieve the fundamental amplitude, and nothing can be done to eliminate any harmonics without additional switching.

Fig. 9 is the plot of the phase voltage for $m = 1$. The corresponding FFT of this signal is given in Fig. 10. Fig. 10 shows a 0.07 normalized magnitude of the fifth harmonic and a 0.05 normalized magnitude of the seventh harmonic for a total normalized distortion of $\sqrt{(0.07)^2 + (0.05)^2} = 0.086$ due to these two harmonics. Fig. 5 predicts an error of about 0.07 at $m = 1.0$ for a normalized magnitude of $0.07/1 = 0.07$ due to these two harmonics which again is in close agreement.

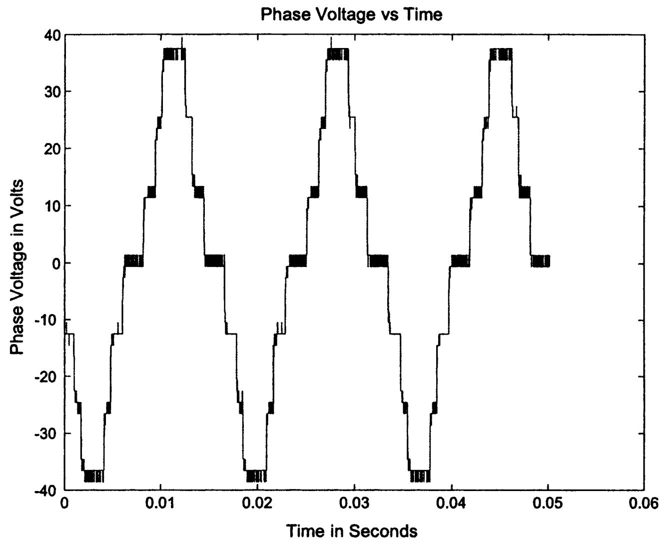

Fig. 13. Phase voltage when $m = 2$.

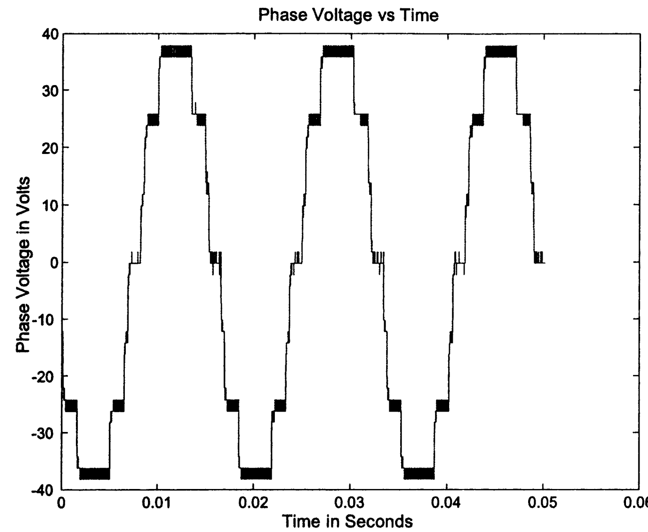

Fig. 15. Phase voltage when $m = 2.5$.

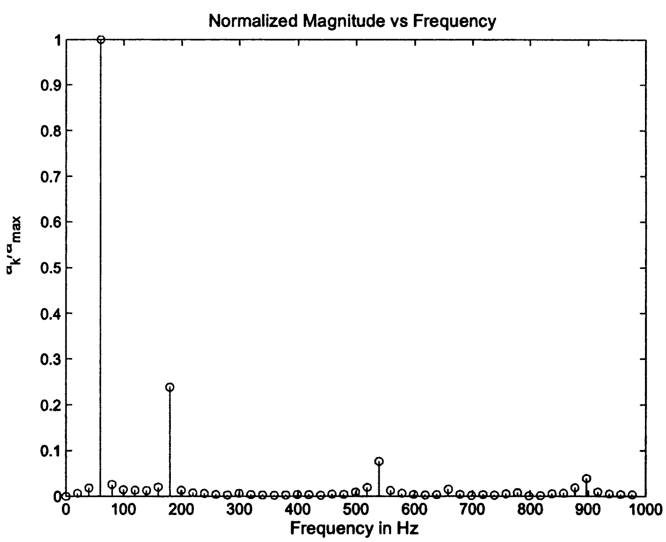

Fig. 14. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 2$.

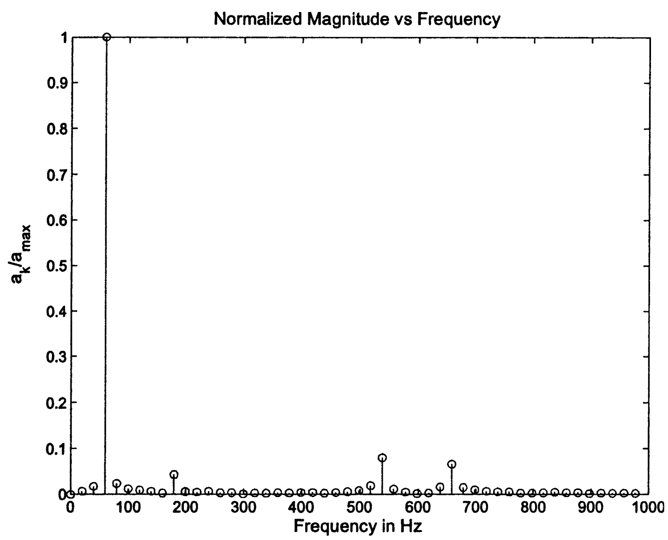

Fig. 16. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 2.5$.

Fig. 11 is the plot of the phase voltage for $m = 1.5$. The corresponding FFT of this signal is given in Fig. 12. Fig. 12 shows essentially zero for the normalized magnitude of the fifth harmonic and about 0.01 normalized magnitude of the seventh harmonic for a total normalized distortion of 0.01 due to these two harmonics which corresponds well with the predicted error of zero in Fig. 5. Note that there are still large triplen harmonics in the phase voltage, but these cancel in the line-line voltage.

Fig. 13 is the plot of the phase voltage for $m = 2$. The corresponding FFT of this signal is given in Fig. 14. Fig. 14 shows the fifth and seventh harmonics are zero as predicted in Fig. 5.

Fig. 15 is the plot of the phase voltage for $m = 2.5$, and its corresponding FFT is given in Fig. 16. Fig. 16 shows the fifth and seventh harmonics are zero as predicted in Fig. 5.

For $m = 1.83$, there are two possible set of solutions which generate zero fifth and seventh harmonics (See Figs. 3 and 4). To compare the two sets of switching angles, Fig. 17 shows the FFT of the data where the fifth and seventh harmonics are zero, and the normalized 11th and 13th harmonics (at 660 Hz and

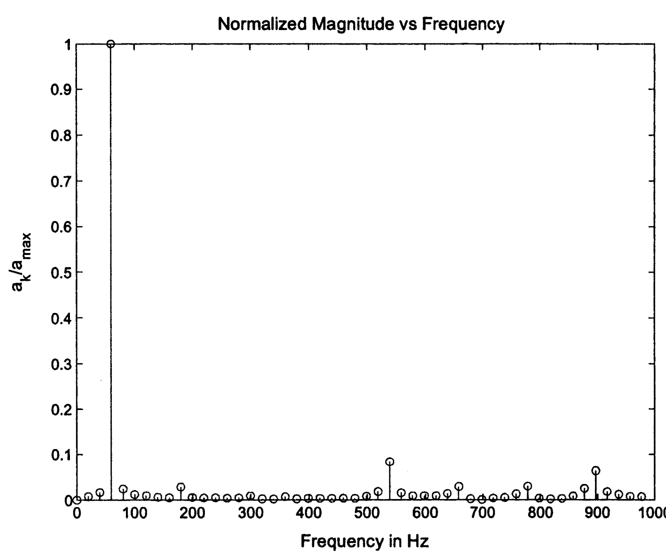

Fig. 17. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 1.82$ and θ_1, θ_2, and θ_3 chosen to give smallest distortion due to the 11th and 13th harmonics.

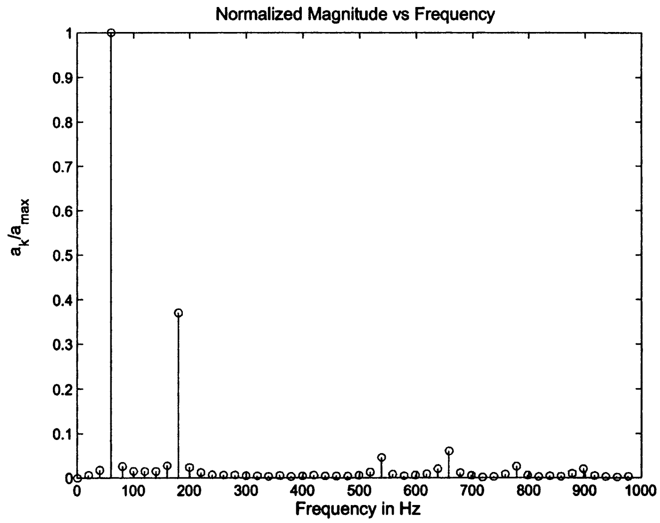

Fig. 18. Normalized FFT $a_k/a_{\max}$ versus frequency for $m = 1.82$ and θ_1, θ_2, and θ_3 chosen to which give a larger distortion generated by the 11th and 13th harmonics.

780 Hz, respectively) are both 0.04 when the switching angles are chosen according to Fig. 4.

In contrast, Fig. 18 is an FFT of the data $(m = 1.83)$ in which the other set of switching angles is chosen. In this case, Fig. 17 shows the fifth and seventh harmonics are zero, but the normalized 11th and 13th harmonics are about 0.06 and 0.04, respectively.

V. Conclusion and Further Work

A full solution to the problem of eliminating the fifth and seventh harmonics in a seven level multilevel inverter has been given. Specifically, resultant theory was used to completely characterize for each m when a solution existed and when it did not (in contrast to numerical techniques such as Newton-Raphson). Futher, it was shown that for a range of values of m, there were two sets of solutions and these values were also completely characterized. For each value of m, the solution set that minimized the 11th and 13th harmonics was chosen. Experimental results were also presented and corresponded well to the theoretically predicted results. In [17], the authors have extended this work to the case studied by Cunnyngham [6] where the separate dc sources do not all provide equal voltages $V_{\rm dc}$.

Appendix
Resultant Polynomial

$$
\begin{aligned}
&r_1(x_1)\\
=&6125m - 49\,000m^3 + 137\,200m^5 - 179\,200m^7\\
&+ 116\,480m^9 - 35\,840m^{11} + 4096m^{13}\\
&- 12\,250x_1 + 220\,500m^2x_1 - 882\,000m^4x_1\\
&+ 1\,512\,000m^6x_1 - 1\,245\,440m^8x_1 + 465\,920m^{10}x_1\\
&- 61\,440m^{12}x_1 - 367\,500mx_1^2 + 2\,352\,000m^3x_1^2\\
&- 5\,644\,800m^5x_1^2 + 6\,048\,000m^7x_1^2 - 2\,795\,520m^9x_1^2\\
&+ 430\,080m^{11}x_1^2 + 269\,500x_1^3 - 3\,430\,000m^2x_1^3\\
&+ 12\,230\,400m^4x_1^3 - 17\,337\,600m^6x_1^3 + 10\,106\,880m^8x_1^3\\
&- 1\,863\,680m^{10}x_1^3 + 2\,940\,000mx_1^4 - 16\,464\,000m^3x_1^4\\
&+ 31\,987\,200m^5x_1^4 - 24\,192\,000m^7x_1^4 + 5\,591\,040m^9x_1^4\\
&- 1\,470\,000x_1^5 + 13\,720\,000m^2x_1^5 - 39\,513\,600m^4x_1^5\\
&+ 39\,782\,400m^6x_1^5 - 12\,185\,600m^8x_1^5 - 7\,056\,000mx_1^6\\
&+ 32\,928\,000m^3x_1^6 - 45\,158\,400m^5x_1^6 + 19\,353\,600m^7x_1^6\\
&+ 2\,744\,000x_1^7 - 17\,248\,000m^2x_1^7 + 35\,123\,200m^4x_1^7\\
&- 21\,504\,000m^6x_1^7 + 4\,704\,000mx_1^8 - 18\,816\,000m^3x_1^8\\
&+ 15\,052\,800m^5x_1^8 - 1\,568\,000x_1^9 + 6\,272\,000m^2x_1^9\\
&- 5\,017\,600m^4x_1^9.
\end{aligned}
$$

Acknowledgment

The authors would like to also thank Opal-RT Technologies for their courteous and professional help in support of this project.

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