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# CONTROL OF DIFFUSION PROCESSES IN $R^d$ AND BELLMAN EQUATION WITH DEGENERATION

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## 0. Introduction

In this paper we consider the existence and uniqueness of solutions of the following Bellman equation:

$$(0.1)\qquad \begin{cases} \inf_{\alpha\in A}\{\partial v/\partial s+1/2\sum_{1\leqq i,j\leqq\nu}a_{ij}(\alpha,s,x)\partial^2 v/\partial x_i\partial x_j+ \\ \sum_{i=1}^{d}b_i(\alpha,s,x)\partial v/\partial x_i-c(\alpha,s,x)v+L(\alpha,s,x)\}=0\,, \\ v(T,x)=h(x) \end{cases}$$

where $1\leqq\nu<d$, $A$ is a separable metric space and $(a_{ij})$, $1\leqq i, j\leqq\nu$, is a positive definite matrix.

W.H. Fleming already considered in [1] the following equation which is more restrictive than Eq. (0.1):

$$(0.2)\qquad \begin{cases} \partial v/\partial s+1/2\sum_{1\leqq i,j\leqq\nu}a_{ij}(s,x)\partial^2 v/\partial x_i\partial x_j+\sum_{i=\nu+1}^{d}b_i(s,x)\partial v/\partial x_i \\ +\inf_{\alpha\in A}\{\sum_{i=1}^{\nu}b_i(\alpha,s,x)\partial v/\partial x_i-c(\alpha,s,x)v+L(\alpha,s,x)\}=0\,, \\ v(T,x)=h(x)\,. \end{cases}$$

In [1] he also considered the deterministic case that $\nu=0$ in Eq. (0.2). His approach to this equation is as follows; consider stochastic control problem for a system described by the following stochastic differential equation:

$$(0.3)\qquad dX_t=b(\alpha_t,t,X_t)dt+\sigma(t,X_t)dB_t,\quad X_s=x\,,$$

where $b_i(\alpha,s,x)=b_i(s,x)$ for all $i=\nu+1,\cdots,d$, $\sigma=\begin{pmatrix}\bar\sigma & 0\\ 0 & 0\end{pmatrix}$, $\bar\sigma$ is a nonsingular $(\nu,\nu)$-matrix, $(B_t)$ is a vector valued Brownian motion, $x$ is a vector of $R^d$ and $(\alpha_t)$ is a non-anticipative control variable having values in $A$. Define the cost $v$ by the following formula:

$$(0.4)\qquad v(s,x)=\inf E[\int_s^T L(\alpha_t,t,X_t^{\alpha,s,x})\exp\{-\int_s^t c(\alpha_r,r,X_r^{\alpha,s,x})dr\}dt$$
$$+h(X_T^{\alpha,s,x})\exp\{-\int_s^T c(\alpha_t,t,X_t^{\alpha,s,x})dt\}]\,,$$

where inf is taken over all non-anticipative control variables and $(X_t^{\alpha,s,x})$ is a solution of Eq. (0.3) associated with $(\alpha, s, x)$. Then W.H. Fleming proved that the function $v$ is continuous in $(s, x)$ and it has generalized derivatives (in distribution sense) $\partial v/\partial s$, $\partial v/\partial x_i\,(1\leqq i\leqq d)$ and $\partial^2 v/\partial x_i \partial x_j\,(1\leqq i, j\leqq \nu)$ belonging to $L_{\lambda,\mathrm{loc}}((0, T)\times R^d)$ for all $1\leqq\lambda<\infty$ and, furthermore, it satisfies Eq. (0.2) $(a=\sigma\sigma^*)$ at almost all points of $(0, T)\times R^d$.

We shall extend his results to Eq. (0.1) in which all the coefficients including their derivatives are assumed to satisfy the polynomial growth conditions. In §2 we shall treat the existence problem of solutions of Eq. (0.1) and we shall obtain the results similar to those in [1]. The method of [1] essentially depends on general theory of linear differential equations of parabolic type, but this method cannot be applied to Eq. (0.1). Instead, we depend on some estimates due to N.V. Krylov [6] in the theory of stochastic control and on some properties of convex functions. In §3 and §4 we shall discuss the uniqueness problem of solutions of Eq. (0.1) under more additional assumptions, which has been treated also in [6] (Chap. 5, §4) in a restrictive sense (see Remark 4.2). Moreover, in §5 we shall extend the results obtained in §1~§4 to the case where the coefficients are unbounded with respect to $\alpha$.

Besides [1], [6], there are several results concerning these problems. In [3] the author studied the same problem as in [1] with respect to Eq. (0.2) in which $b_i(\nu+1\leqq i\leqq d)$ also depend on $\alpha$. [3] is a special case of Eq. (0.1), nevertheless the results in [3] are not necessarily contained in the present paper (cf. Remark 1.1). In [5] N.V. Krylov considered Eq. (0.1) ($\nu$=d) in elliptic case with degeneration in such a way that there is an admissible control variable for which controlled process does not degenerate. Note that in [5] the coefficients are unbounded with respect to $\alpha$(cf. §5 below). P.L. Lions also solved in [7], [8] Eq. (0.1) $(\nu=d)$ in elliptic case, in which the matrix a may degenerate but the coefficients including their derivatives are bounded. Especially, in [8] he discussed about "viscosity solution" of differential equation and its application to Eq. (0.1) $(\nu=d)$.

As an application of our results, we can show the existence of generalized solutions for some nonlinear differential equations of parabolic type with degeneration, which will be explained in detail in §2 and §5. For example, the following three differential equations have generalized solutions: $d=2$, $\nu=1$.

$$v_s+1/2\,v_{xx}-\{(v_x)^2+(v_y)^2\}^{1/2}+L(s, x, y)=0\,, \tag{0.5}$$

$$v_s+v_{xx}-\{(v_{xx})^2/2+v_{xx}v_x+v_{xx}v_y+(v_x)^2+(v_y)^2\}^{1/2}+L(s, x, y)=0\,, \tag{0.6}$$

$$v_s+1/2\,v_{xx}-\{(v_x)^2+(v_y)^2\}+L(s, x, y)=0\,, \tag{0.7}$$

where $v_s$, $v_{xx}$, $v_x$ and $v_y$ denote $\partial v/\partial s$, $\partial^2 v/\partial x^2$, $\partial v/\partial x$ and $\partial v/\partial y$ respectively. We can consider more examples different from (0.5)~(0.7).

## 1. Formulations and preliminaries

Let $T$ be a finite positive number (fixed). Let $A$ be a separable metric space and $\mathcal{A}$ be the Borel subsets of $A$. Put $Q_T=(0, T)\times R^d$ and $\bar{Q}_T=[0, T]\times R^d$. Consider the following control problem for a system described by stochastic differential equations of the type:

$$(1.1)\qquad \begin{cases} dX_t = b(\alpha_t, s+t, X_t)dt+\sigma(\alpha_t, s+t, X_t)dB_t, & 0<t\leqq T-s\,, \\ X_0 = x \end{cases}$$

where $0\leqq s\leqq T$, $x$ is a $R^d$-vector and $(B_t)$, $0\leqq t\leqq T$, is a $R^d$-valued process of independent Brownian motions. Suppose that the coefficients $b$ and $\sigma$ satisfy the following conditions throughout in §1~§4.

(A.1) $b(\alpha, t, x)$: $A\times\bar{Q}_T \rightsquigarrow R^d$ and $\sigma(\alpha, t, x)$: $A\times\bar{Q}_T \rightsquigarrow R^d\otimes R^d$ ($(d, d)$-matrix). We assume that they are continuous with respect to $(\alpha, t, x)$. Furthermore, for some constant $k\geqq 0$, for all $\alpha\in A$, $t\in[0,T]$, $x, y\in R^d$, let

$$(1.2)\qquad |b(\alpha, t, x)-b(\alpha, t, y)|+||\sigma(\alpha, t, x)-\sigma(\alpha, t, y)||\leqq k|x-y|\,,$$

$$(1.3)\qquad |b(\alpha, t, x)|+||\sigma(\alpha, t, x)||\leqq k(1+|x|)\,,$$

where $||\cdot||$ denotes the norm of matrix. For all $\alpha\in A$ and $l\in R^d$, let the derivatives $\gamma_{(l)}(\alpha, t, x)$, $\gamma_{(l)(l)}(\alpha, t, x)$(1), $(\partial/\partial t)\gamma\ (\alpha, t, x)$ exist and be continuous with respect to $(t, x)$ on $\bar{Q}_T$ uniformly over $\alpha$ $(\gamma=b, \sigma)$. Assume that the foregoing derivatives do not exceed $k(1+|x|)^m$ in norm for all $\alpha\in A$, $l\in R^d$, $(t, x)\in\bar{Q}_T$, where $m$ is a nonnegative constant. □

We introduce here the concept of strategy by the following way.

DEFINITION 1.1. By a *strategy* we mean a process $\alpha_t(\omega)$, $0\leqq t\leqq T$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t; P)$ satisfying the standard conditions(2), which is progressively measurable with respect to $(\mathcal{F}_t)$, having values in $A$. We denote by $\mathfrak{A}$ the set of all strategies. □

Remark that from the assumption (A.1) about $b$ and $\sigma$, for each strategy $\alpha\in\mathfrak{A}$ and $(s, x)\in\bar{Q}_T$, there is a unique solution of Eq. (1.1). Then we denote it by $(X_t^{\alpha,s,x})$. We often write $b^\alpha(t, x)$ and $\sigma^\alpha(t, x)$ instead of $b(\alpha, t, x)$ and $\sigma(\alpha, t, x)$ respectively. Next, define the cost by the following formula: for each $\alpha\in\mathfrak{A}$, $(s, x)\in\bar{Q}_T$, let

(1) $\gamma_{(l)}(t, x)$ and $\gamma_{(l)(l)}(t, x)$ denote first and second derivatives of $\gamma(t, x)$ along spacial direction $l$ respectively. Generalized derivatives of $v$ in the $l$-direction, $v_{(l)}$ and $v_{(l)(l)}$, are also defined in a usual way (see [6], p. 47~50).

(2) $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_t)$ is a nondecreasing family of sub $\sigma$-fields of $\mathcal{F}$ and right continuous in $t$, and $\mathcal{F}_0$ is trivial with respect to $P$. Here after we always assume these conditions for all probability spaces unless otherwise mentioned.

$$(1.4)\qquad v^{\alpha}(s, x) = E[\int_0^{T-s} L(\alpha_t, s+t, X_t^{\alpha,s,x})\exp(-\phi_t^{\alpha,s,x})dt + h(X_{T-s}^{\alpha,s,x})\exp(-\phi_{T-s}^{\alpha,s,x})]\,,$$

where $\phi_t^{\alpha,s,x}=\int_0^t c(\alpha_r, s+r, X_r^{\alpha,s,x})dr$, and $(X_t^{\alpha,s,x})$ is the solution of Eq. (1.1) associated with $(\alpha, s, x)$. Suppose that the functions $L$, $c$ and $h$ in (1.4) always satisfy the following conditions in §1~§4:

(A.2) $L(\alpha, t, x)$ $(c(\alpha, t, x))$: $A\times\bar{Q}_T \rightsquigarrow R(R_+)$, $h(x)$: $R^d \rightsquigarrow R$. We assume that the function $L(c)$ is continuous with respect to $(\alpha, t, x)$. For all $\alpha\in A$, $l\in R^d$, let the derivatives $\gamma_{(l)}(\alpha, t, x)$, $\gamma_{(l)(l)}(\alpha, t, x)$ and $(\partial/\partial t)\gamma(\alpha, t, x)$ exist and be continuous with respect to $(t, x)\in\bar{Q}_T$ $(\gamma=L, c, h)$. Furthermore, assume that $\gamma(=L, c, h)$ itself and the foregoing derivatives satisfy the following conditions: for all $(\alpha, t, x)\in A\times\bar{Q}_T$, $l\in R^d$,

$$(1.5)\qquad |\gamma^{\alpha}(t, x)|+|(\partial/\partial t)\gamma^{\alpha}(t, x)|+|\gamma^{\alpha}_{(l)}(t, x)|+|\gamma^{\alpha}_{(l)(l)}(t, x)| \leqq k(1+|x|)^m\,. \quad \square$$

Then we have the following well known result which will be often used in this paper (for the proof, see, for example, [2], Chap. 5, §4, Theorem 4.2).

**Proposition 1.1.** (a) *For each* $\lambda=1, 2, \cdots$, *there is a constant* $k_\lambda\geqq 0$ *such that for all* $(\alpha, s, x)\in\mathfrak{A}\times\bar{Q}_T$,

$$(1.6)\qquad E[\sup_{0\leqq t\leqq T-s}|X_t^{\alpha,s,x}|^\lambda]\leqq k_\lambda(1+|x|)^\lambda\,,$$

*where* $k_\lambda$ *depends on* $(k, \lambda)$.
(b) *There is a constant* $N=N(k, m)\geqq 0$ *such that for all* $(\alpha, s, x)\in\mathfrak{A}\times\bar{Q}_T$,

$$(1.7)\qquad |v^{\alpha}(s, x)|\leqq N(1+|x|)^m\,. \quad \square$$

For each $(s, x)\in\bar{Q}_T$, put

$$(1.8)\qquad v(s, x) = \inf_{\alpha\in\mathfrak{A}} v^{\alpha}(s, x)\,.$$

Then the finiteness of $v$ will be given in Proposition 1.2 (see below).

Let $1\leqq\nu<d$ and let $\bar{\sigma}$ be a $(\nu, \nu)$-matrix such that for all $(t, x)\in\bar{Q}_T$ and $\xi\in R^\nu$ such that $\xi_1^2+\cdots+\xi_\nu^2=1$,

$$(1.9)\qquad \sup_{\alpha\in A}(\xi, \bar{\sigma}^{\alpha}(t, x)\bar{\sigma}^{\alpha}(t, x)^*\xi)>0\,,$$

where $\bar{\sigma}^*$ denotes a transposed matrix of $\bar{\sigma}$. From now on we assume that the $(d, d)$-matrix $\sigma$ in Eq. (1.1) is written as follows:

$$(1.10)\qquad \sigma = \begin{pmatrix}\bar{\sigma} & 0\\ 0 & 0\end{pmatrix}.$$

Then our object in §1∼§2 is to show that $v$ of (1.8) is a "generalized solution" of Eq. (0.1) with $a=\sigma\sigma^*$, whose meanings will be given rigorously in the next paragraph. The rest of this one is devoted to state preparatory results about $v$, which are mainly due to W.H. Fleming and Rishel [2] (Chap. VI, §8), and N.V. Krylov [6] (Chap. 3∼4). The following results are fundamental for our discussions, some of which are easily verified by hand.

**Proposition 1.2.** (a) *The function $v(s, x)$ is continuous on $\bar{Q}_T$ and $v(T, x)=h(x)$. Moreover, there exists a constant $N=N(k, m)$ such that for all $(s, x)\in\bar{Q}_T$,*

$$(1.11)\qquad |v(s, x)|\leqq N(1+|x|)^m .$$

(b) *For each $s\in[0, T]$, the function $v$ has first-order generalized derivatives with respect to $x_i$ $(1\leqq i\leqq d)$, and for each $x\in R^d$ it is absolutely continuous with respect to $s\in[0, T]$, has also on this interval a generalized derivative with respect to $s$. Furthermore, there is a constant $N=N(k, m)$ such that for almost all $(s, x)\in\bar{Q}_T$,*

$$(1.12)\qquad |(\partial/\partial s)v(s, x)|+|\nabla_x v(s, x)|\leqq N(1+|x|)^{2m} ,$$

*where $\nabla_x v(s, x)=(\partial v/\partial x_1, \cdots, \partial v/\partial x_d)(s, x)$.*
(c) *There is a constant $N=N(k, m)$ such that for each $s\in[0, T]$*

$$(1.13)\qquad v(s, x)-N(1+|x|^2)^{(3m/2)+1}$$

*is a concave function with respect to $x$.*
(d) *Especially, in $Q_T$, $v(s, x)$ has second-order generalized derivatives with respect to $x_i x_j$ $(1\leqq i, j\leqq \nu)$ and, further, the foregoing derivatives are locally bounded in $Q_T$.*
(e) *For each $\alpha\in A$, define operators $F^\alpha$ and $F$ by the following:*

$$(1.14)\qquad \begin{aligned} F^\alpha(u_0, u_{ij}, u_i, u, s, x) &= u_0+1/2\sum_{1\leqq i,j\leqq\nu} a^\alpha_{ij}(s, x)u_{ij} \\ &+\sum_{1\leqq i\leqq d} b^\alpha_i(s, x)u_i-c^\alpha(s, x)u+L^\alpha(s, x) , \end{aligned}$$

*and*

$$(1.15)\qquad F(u_0, u_{ij}, u_i, u, s, x)\equiv\inf_{\alpha\in A} F^\alpha(u_0, u_{ij}, u_i, u, s, x) .$$

*Then it holds that*

$$(1.16)\qquad F[v](s, x)\equiv F(v_s, v_{x_i x_j}, v_{x_i}, v, s, x)\geqq 0 \quad a.e.\ (Q_T) ,$$

*where $v_s$, $v_{x_i x_j}$ and $v_{x_i}$ denote $\partial v/\partial s$, $\partial^2 v/\partial x_i\partial x_j$ and $\partial v/\partial x_i$, respectively.* □

Proof. (a) and (b) are due to [6] (Theorem 3.1.5 and 4.4.3). (c) is also due to [6] (Theorem 4.2.3) but it is also verified directly by hand. (d) and (e) are just simple applications of N.V. Krylov [6] (Theorem 4.3.5 and 4.7.4).

To prove these, we introduce some notations follow him. For any $\alpha\in A$, $(t, x)\in\bar{Q}_T$, $l\in R^d$, put

$$(1.17)\qquad n^\alpha(t, x) = \{1+tr\, a^\alpha(t, x)+|b^\alpha(t, x)|+c^\alpha(t, x)+|L^\alpha(t, x)|\}^{-1},$$

$$(1.18)\qquad \mu(l)\equiv\mu(t, x, l) = \inf_{\lambda\in R^d,(l,\lambda)=1}\sup_{\alpha\in A} n^\alpha(t, x)\,(a^\alpha(t, x)\lambda, \lambda),$$

$$(1.19)\qquad Q(l) = \{(t, x)\in\bar{Q}_T;\ \mu(t, x, l)>0\}.$$

Notice that $n^\alpha(t, x)$ is continuous in $(t, x)$ and $0<n^\alpha(t, x)\leqq 1$ from the assumptions (A.1) and (A.2), and notice also that $Q(l)$ is a Borel set with respect to $(t, x)$. We can show easily that if $\sigma^\alpha$ is of the form (1.10) with (1.9), then $Q(l)=Q_T$ for all $l\in R^d$, $|l|=1$, $l_i=0(\nu+1\leqq i\leqq d)$. Indeed, to prove this, remark first the following equivalent relation: for such $l$, $\mu(t, x, l)>0\Leftrightarrow \inf_{\lambda\in R^d,(l,\lambda)=1}\sup_{\alpha\in A}(a^\alpha(t, x)\lambda, \lambda)>0$, by the fact that $0<n^\alpha(t, x)\leqq 1$ for all $(t, x)\in\bar{Q}_T$ (note that for such $l$, if $(l, \lambda)=1$, then there is a number $i$ $(1\leqq i\leqq\nu)$ such that $\lambda_i\neq 0$). Next, put $\mu(t, x)=\inf_{\xi\in R^\nu,|\xi|=1}\sup_{\alpha\in A}(\bar{a}^\alpha(t, x)\xi, \xi)$ (where $\bar{a}=\bar{\sigma}\bar{\sigma}^*$). Then, $\inf_{(l,\lambda)=1}\sup_{\alpha\in A}(a^\alpha(t, x)\lambda, \lambda)\geqq\mu(t, x)>0$ from the assumption (1.9). The rest of proof for (d) and (e) is quite the same as N.V. Krylov ([6], Theorem, 4.3.5, p. 187). □

REMARK 1.1. In (d) of Proposition 1.2, it is shown actually that for any unit vector $l\in R^d$ such that $l_{\lambda+1}=\cdots=l_d=0$, the second-order generalized derivatives $v_{(l)(l)}$ satisfy the inequality: there exist a constant $N=N(k, m)$ such that

$$(1.20)\qquad -N(1+|x|)^{3m}/\mu(t, x, l)\leqq v_{(l)(l)}(t, x)\leqq N(1+|x|)^{3m}\quad \text{a.e. } (\bar{Q}_T).$$

Suppose, further, that $\bar{\sigma}^\alpha$ satisfies the following stronger condition than (1.9): there exists a constant $\mu>0$

$$(1.21)\qquad (\bar{a}^\alpha(t, x)\xi, \xi)\geqq\mu|\xi|^2$$

for all $\alpha\in A$, $(t, x)\in\bar{Q}_T$ and $\xi\in R^\nu$. Then it holds from (1.20) that $v_{(l)(l)}$ satisfy also the polynomial growth condition as $v_{(l)}$ (see, [6], Theorem 4.7.4, cf. [1] and [3]). □

## 2. Bellman equation

In this section we shall show that the function $v$ of (1.8) satisfies the Bellman equation (0.1), i.e. we have the following.

**Theorem 2.1.** *For almost all* $(s, x)\in Q_T$, $F[v](s, x)=0$. □

According to Proposition 1.2 (e), it suffices to verify that for a.e. $(Q_T)$, $F[v](s, x)\leqq 0$, which will be shown by perturbation method. Let $1>\varepsilon>0$, and for any $(\alpha, t, x)\in A\times\bar{Q}_T$, put

$$(2.1)\qquad \sigma^{\alpha,\varepsilon}(t, x) = \begin{pmatrix} \bar{\sigma}^{\alpha}(t, x) & 0 \\ 0 & \begin{matrix} \varepsilon & & 0 \\ & \ddots & \\ 0 & & \varepsilon \end{matrix} \end{pmatrix} \begin{matrix} \}\nu \\ \}d-\nu \end{matrix}$$

$$\underbrace{\qquad}_{\nu}\ \underbrace{\qquad}_{d-\nu}$$

Then clearly $\sigma^{\alpha,\varepsilon}$ is $(d, d)$-matrix such that $a^{\alpha,\varepsilon}=\sigma^{\alpha,\varepsilon}(\sigma^{\alpha,\varepsilon})^*$ satisfies that for each $\varepsilon>0$, for any unit vector $l\in R^d$, $\inf_{(l,\lambda)=1}\sup_{\alpha\in A} n^{\alpha,\varepsilon}(t, x)\,(a^{\alpha,\varepsilon}(t, x)\lambda, \lambda)>0$ for ${}^{\forall}(t, x)\in\bar{Q}_T$, by (1.9), where $n^{\alpha,\varepsilon}(t, x)=\{1+tr\ a^{\alpha,\varepsilon}(t, x)+|b^{\alpha}(t, x)|+c^{\alpha}(t, x)+|L^{\alpha}(t, x)|\}^{-1}$. For any $\varepsilon>0$, $(\alpha, s, x)\in\mathfrak{A}\times\bar{Q}_T$, consider the following stochastic differential equation associated with $\sigma^{\alpha,\varepsilon}$:

$$(2.2)\qquad \begin{cases} dX_t = b^{\alpha_t}(s+t, X_t)dt+\sigma^{\alpha_t,\varepsilon}(s+t, X_t)dB_t, & 0<t\leqq T-s\,, \\ X_0 = x \end{cases}$$

and denote by $(X_t^{\alpha,s,x,\varepsilon})$ a unique solution of Eq. (2.2) associated with $(\alpha, s, x, \varepsilon)$. In the same way as (1.4) and (1.8), define $v^{\alpha,\varepsilon}$ and $v^{\varepsilon}$ by the formulas:

$$(2.3)\qquad v^{\alpha,\varepsilon}(s, x) = E\Big[\int_0^{T-s} L^{\alpha_t}(s+t, X_t^{\alpha,s,x,\varepsilon})\exp(-\phi_t^{\alpha,s,x,\varepsilon})dt+ h(X_{T-s}^{\alpha,s,x,\varepsilon})\exp(-\phi_{T-s}^{\alpha,s,x,\varepsilon})\Big]\,,$$

where $\phi_t^{\alpha,s,x,\varepsilon}=\int_0^t c^{\alpha_r}(s+r, X_r^{\alpha,s,x,\varepsilon})dr$, and

$$(2.4)\qquad v^{\varepsilon}(s, x) = \inf_{\alpha\in\mathfrak{U}} v^{\alpha,\varepsilon}(s, x)\,.$$

Then, like the function $v$, it is easily shown that $v^{\varepsilon}$ is also continuous with respect to $(s, x)\in\bar{Q}_T$, $v^{\varepsilon}(T, x)=h(x)$ and, in addition, it satisfies the same inequality as (1.11), i.e. there is a constant $N=N(k, m)$ such that

$$(2.5)\qquad |v^{\varepsilon}(s, x)|\leqq N(1+|x|)^m \qquad \text{for all } (\varepsilon, s, x)\,.$$

We introduce here the following notations: for each $1\leqq\gamma\leqq d$, $p\geqq1$ we say that a function $u(s, x)$ over $\bar{Q}_T$ belongs to $W_{p,\mathrm{loc}}^{1,2,\gamma}(Q_T)$ if $u$ has a first order generalized derivative with respect to $s$, first order generalized derivatives with respect to $x_i\,(1\leqq i\leqq d)$ and second order generalized derivatives with respect to $x_i\,(1\leqq i\leqq\gamma)$ on $Q_T$ and, moreover, for any bounded open set $Q\subset Q_T$ the foregoing derivatives belong to $L_p(Q)$. Put

$$(2.6)\qquad ||u||_{W_p^{1,2,\gamma}(Q)}\equiv||u||_{p,Q}+||u_s||_{p,Q}+\sum_{i=1}^d ||u_{x_i}||_{p,Q}+\sum_{1\leqq i,j\leqq\gamma}||u_{x_ix_j}||_{p,Q}\,,$$

where $||u||_{p,Q}=(\iint_Q|u(s, x)|^p dsdx)^{1/p}$. We write $W_{p,\mathrm{loc}}^{1,2}(Q_T)$ when $\gamma=d$. It is well known that the function $v^{\varepsilon}$ has the following properties ([6], Chap. 4).

**Lemma 2.2.** (a) $\lim_{\varepsilon\to0} v^{\varepsilon}(s, x)=v(s, x)$ *and the convergence is uniform in*

*each cylinder* $\bar{Q}_{T,R}$, *where* $\bar{Q}_{T,R}=[0, T]\times\{x\in R^d;\ |x|\leqq R\}$.
(b) *For all* $p\geqq 1$, $v^\varepsilon\in W^{1,2}_{p,\mathrm{loc}}(Q_T)$ *and, moreover, there is a number* $N=N(k, m)\geqq 0$ *such that for all* $(s, x)\in Q_T$, $\varepsilon>0$,

$$|(\partial/\partial s)v^\varepsilon(s, x)|+|\nabla_x v^\varepsilon(s, x)|\leqq N(1+|x|)^{2m}. \tag{2.7}$$

(c) *For all* $\alpha\in A$, $0<\varepsilon<1$, *define* $F^{\alpha,\varepsilon}$ *and* $F^\varepsilon$ *by the following formulas*:

$$F^{\alpha,\varepsilon}(u_0, u_{ij}, u_i, u, s, x) = F^\alpha(u_0, u_{ij}, u_i, u, s, x)+(\varepsilon^2/2)\sum_{i=\nu+1}^{d} u_{ii}, \tag{2.8}$$

*and*

$$F^\varepsilon(u_0, u_{ij}, u_i, u, s, x) = \inf_{\alpha\in A} F^{\alpha,\varepsilon}(u_0, u_{ij}, u_i, u, s, x). \tag{2.9}$$

*Then it holds that for each* $\varepsilon(0<\varepsilon<1)$

$$F^\varepsilon[v^\varepsilon](s, x)\equiv F^\varepsilon(v^\varepsilon_s, v^\varepsilon_{x_ix_j}, v^\varepsilon_{x_i}, v^\varepsilon, s, x) = 0 \quad \text{a.e. } (Q_T). \quad \square \tag{2.10}$$

For all $(\alpha, s, x)\in A\times\bar{Q}_T$, $\varepsilon>0$, set

$$\chi^{\alpha,\varepsilon}(s, x) = \sum_{i=1}^{d} b_i^\alpha(s, x)v^\varepsilon_{x_i}(s, x)+L^\alpha(s, x), \quad \text{and} \tag{2.11}$$

$$\chi^\alpha(s, x) = \sum_{i=1}^{d} b_i^\alpha(s, x)v_{x_i}(s, x)+L^\alpha(s, x). \tag{2.12}$$

Then we can get easily the following inequality by means of (1.3), (1.5), (1.11) and (2.7): for all $(\alpha, s, x)\in A\times\bar{Q}_T$, $0<\varepsilon<1$,

$$|\chi^{\alpha,\varepsilon}(s, x)|+|\chi^\alpha(s, x)|\leqq N(1+|x|)^{3m}, \tag{2.13}$$

where $N=N(k, m)\geqq 0$ is a constant.

Now we consider the following transformation of variables $(s, x)\in\bar{Q}_T$, due to W.H. Fleming and R. Rishel ([2], Chap. 6, §8): let fix arbitrary $(\xi, \varepsilon)$ such that $\xi=(\xi_{\nu+1}, \cdots, \xi_d)\in R^{d-\nu}$ and $0<\varepsilon<1$, and define new variables $(s, y)\in\bar{Q}_T$ by the following way.

$$\begin{cases} s = s \\ y_i = x_i, \quad 1\leqq i\leqq \nu, \\ \varepsilon y_i = x_i-\xi_i, \quad \nu+1\leqq i\leqq d. \end{cases} \tag{2.14}$$

For any $y\in R^d$, let $\bar{y}$ and $\hat{y}$ denote the first $\nu$ and the last $d-\nu$ components of $y$ respectively (similar for $x=(\bar{x}, \hat{x})$). Then (2.14) is equivalent to say that $\bar{y}=\bar{x}$ and $\varepsilon\hat{y}=\hat{x}-\xi$. Let also $\psi^\varepsilon(s, y)=v^\varepsilon(s, x)$. Note that the function $\psi^\varepsilon$ depends on $(\xi, \varepsilon)$. For simplicity, for the moment, suppose that $a^{\alpha,\varepsilon}$ does not depend on $x$. From (2.10) it is easy to verify that $\psi^\varepsilon$ satisfies the following equation: for a.e. $(s, y)\in Q_T$,

$$\inf_{\alpha\in A}\{(\partial/\partial s)\psi^\varepsilon+\sum_{1\leqq i,j\leqq\nu} a_{ij}^\alpha(s)\,(\partial^2/\partial y_i\partial y_j)\psi^\varepsilon/2+ (1/2)\sum_{\nu+1\leqq i\leqq d}(\partial^2/\partial y_i^2)\psi^\varepsilon-\tilde{c}^{\alpha,\varepsilon}(s, y)\psi^\varepsilon+\tilde{\chi}^{\alpha,\varepsilon}(s, y)\} = 0, \tag{2.15}$$

where

$$\text{(2.16)} \qquad \begin{cases} \tilde{\chi}^{\alpha,\varepsilon}(s, y) = \chi^{\alpha,\varepsilon}(s, x) \ (= \chi^{\alpha,\varepsilon}(s, \bar{y}, \varepsilon\hat{y}+\xi)), \\ \tilde{c}^{\alpha,\varepsilon}(s, y) = c^{\alpha}(s, x) \ (= c^{\alpha}(s, \bar{y}, \varepsilon\hat{y}+\xi)). \end{cases}$$

We have the following result about the function $\psi^{\varepsilon}$.

**Lemma 2.3.** (a) *For each $\varepsilon>0$, for any $p\geqq 1$, $\psi^{\varepsilon}\in W^{1,2}_{p,\mathrm{loc}}(Q_T)$.*
(b) *There is a number $N=N(k, m, \xi)\geqq 0$ such that for all $(\varepsilon, s, y)$,*

$$\text{(2.17)} \qquad |\psi^{\varepsilon}(s, y)|+|(\partial/\partial s)\psi^{\varepsilon}(s, y)|+\sum_{i=1}^{d}|(\partial/\partial y_i)\psi^{\varepsilon}(s, y)|\leqq N(1+|y|)^{2m},$$

*and, furthermore, $\psi^{\varepsilon}_{y_iy_j}$ $(1\leqq i, j\leqq d)$ are locally bounded in $(s, y)$ uniformly with respect to $\varepsilon$.*
(c) *$\lim_{\varepsilon\to 0}\psi^{\varepsilon}(s, y)=v(s, \bar{y}, \xi)$ and its convergence is uniform in each cylinder $\bar{Q}_{T,R}$.*

Proof. Since (b) implies (a), we shall prove only (b) and (c). First, it is clear that for all $(s, y)\in\bar{Q}_T$, $|\psi^{\varepsilon}(s, y)|\leqq N(1+|y|)^m$ for a constant $N=N(k, m, \xi)$. Indeed, by means of (2.5) and (2.14), it is sufficient to remark the inequality: $|x|^2\leqq N(1+|y|)^2$, where $N=N(\xi)\geqq 0$ is a constant so long as $\xi(\in R^{d-\nu})$ is fixed. Since $\psi^{\varepsilon}_s(s, y)=v^{\varepsilon}_s(s, x)$ and $|v^{\varepsilon}_s(s, x)|\leqq N(1+|x|)^{2m}$, it holds that $|\psi^{\varepsilon}_s(s, y)|\leqq N(1+|y|)^{2m}$ for a constant $N=N(k, m, \xi)\geqq 0$. Concerning $\psi^{\varepsilon}_{y_i}$ $(1\leqq i\leqq \nu)$, $|\psi^{\varepsilon}_{y_i}(s, y)|=|v^{\varepsilon}_{x_i}(s, x)|\leqq N(1+|y|)^{2m}$, while, for $\nu+1\leqq i\leqq d$, $|\psi^{\varepsilon}_{y_i}(s, y)|=|\varepsilon v^{\varepsilon}_{x_i}(s, x)|\leqq N(1+|y|)^{2m}$, where $N=N(k, m, \xi)\geqq 0$ is a constant. Finally, since $|\tilde{c}^{\alpha,\varepsilon}(s, y)|+|\tilde{\chi}^{\alpha,\varepsilon}(s, y)|\leqq N(1+|y|)^{3m}$ by means of (2.13), it follows from (2.15) and N.V. Krylov ([6], Theorem 4.3.5, p. 187) that $\psi^{\varepsilon}_{y_iy_j}(s, y)$ $(1\leqq i, j\leqq d)$ are locally bounded in $(s, y)$ uniformly with respect to $\varepsilon$. As to (c), suppose that $|\xi|\leqq R$. Note that if $(s, y)\in\bar{Q}_{T,R}$, then $(s, \bar{y}, \xi)\in\bar{Q}_{T,\sqrt{2}R}$. Therefore, it holds that

$$\begin{aligned} |\psi^{\varepsilon}(s, y)-v(s, \bar{y}, \xi)| &\leqq |v^{\varepsilon}(s, \bar{y}, \varepsilon\hat{y}+\xi)-v^{\varepsilon}(s, \bar{y}, \xi)|+|v^{\varepsilon}(s, \bar{y}, \xi) \\ &\quad -v(s, \bar{y}, \xi)|\leqq \varepsilon|\hat{y}|N_R+\sup_{(s,x)\in\bar{Q}_{T,\sqrt{2}R}}|v^{\varepsilon}(s, x)-v(s, x)|\to 0 \quad \text{as } \varepsilon\to 0, \end{aligned}$$

by Lemma 2.2 (a) and (b). Here we used the estimate that there is a constant $N_R=N(k, m, R)\geqq 0$ such that for all $\varepsilon>0$, $(s, x)$, $(s, x')\in\bar{Q}_{T,R}$, $|v^{\varepsilon}(s, x)-v^{\varepsilon}(s, x')|\leqq N_R|x-x'|$. □

Proof of Theorem 2.1. Since $v^{\varepsilon}$ satisfies Eq. (2.10) associated with $\sigma^{\varepsilon}$ $(a=\sigma\sigma^*)$, it holds that for almost all $(s, x)\in Q_T$,

$$\begin{aligned} 0 = F^{\varepsilon}[v^{\varepsilon}](s, x) = \inf_{\alpha\in A}\{&v^{\varepsilon}_s+(1/2)\sum_{1\leqq i,j\leqq\nu}a^{\alpha}_{ij}(s)v^{\varepsilon}_{x_ix_j} \\ &+(\varepsilon^2/2)\sum_{\nu+1\leqq i\leqq d}v^{\varepsilon}_{x_ix_i}+\chi^{\alpha,\varepsilon}(s, x)-c^{\alpha}(s, x)v^{\varepsilon}\}, \end{aligned}$$

and then we have the following:

$$(2.18)\qquad 0\geqq\inf_{\alpha\in A}\{v_s^{\varepsilon}+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^{\alpha}(s)v_{x_ix_j}^{\varepsilon}+(\varepsilon^2/2)\sum_{\nu+1\leqq j\leqq d}v_{x_ix_i}^{\varepsilon}$$
$$+\chi^{\alpha}(s,x)-c^{\alpha}(s,x)v^{\varepsilon}\}+\inf_{\alpha\in A}\{\chi^{\alpha,\varepsilon}(s,x)-\chi^{\alpha}(s,x)\}\ ,$$

and we denote by $f_1^{\varepsilon}(s, x)$ the second term of the right side:

$$(2.19)\qquad f_1^{\varepsilon}(s,x)=\inf_{\alpha\in A}\{\chi^{\alpha,\varepsilon}(s,x)-\chi^{\alpha}(s,x)\}\ .$$

If we change the variables $(s, x)$ into $(s, y)$ as we have seen in (2.14), then it holds that for a.e. $(s, y)\in Q_T$,

$$(2.20)\qquad 0\geqq\inf_{\alpha\in A}\{\psi_s^{\varepsilon}(s,y)+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^{\alpha}(s)\psi_{y_iy_j}^{\varepsilon}(s,y)$$
$$+(1/2)\sum_{\nu+1\leqq i\leqq d}\psi_{y_iy_i}^{\varepsilon}(s,y)-\tilde{c}^{\alpha,\varepsilon}(s,y)\psi^{\varepsilon}(s,y)+\tilde{\chi}^{\alpha,\varepsilon}(s,y)\}+\tilde{f}_1^{\varepsilon}(s,y)\ ,$$

where

$$(2.21)\qquad \tilde{f}_1^{\varepsilon}(s,y)=f_1^{\varepsilon}(s,\bar{y},\varepsilon\hat{y}+\xi)\text{ and }\tilde{\chi}^{\alpha,\varepsilon}(s,y)=\chi^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi)\ .$$

Define $\tilde{F}$ by the following:

$$(2.22)\qquad \tilde{F}(u_o,u_{ij},u,s,y)=\inf_{\alpha\in A}\{u_o+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^{\varepsilon}(s)u_{ij}$$
$$+(1/2)\sum_{\nu+1\leqq i\leqq d}u_{ii}-\tilde{c}^{\alpha}(s,y)u+\tilde{\chi}^{\alpha}(s,y)\}\ ,$$

where

$$(2.23)\qquad \tilde{c}^{\alpha}(s,y)=c^{\alpha}(s,\bar{y},\xi)\text{ and }\tilde{\chi}^{\alpha}(s,y)=\chi^{\alpha}(s,\bar{y},\xi)\ .$$

For all $(s, y)\in\bar{Q}_T$, $0<\varepsilon<1$, put

$$(2.24)\qquad \begin{cases}\tilde{f}_2^{\varepsilon}(s,y)=\inf\limits_{\alpha\in A}\{-\tilde{c}^{\alpha,\varepsilon}(s,y)+\tilde{c}^{\alpha}(s,y)\}\psi^{\varepsilon}(s,y)\ ,\\ \tilde{f}_3^{\varepsilon}(s,y)=\inf\limits_{\alpha\in A}\{\tilde{\chi}^{\alpha,\varepsilon}(s,y)-\tilde{\chi}^{\alpha}(s,y)\}\ .\end{cases}$$

Then it is easily seen from (2.20) that for a.e. $(s, y)\in Q_T$,

$$(2.25)\qquad 0\geqq\tilde{F}[\psi^{\varepsilon}](s,y)+\sum_{i=1}^{3}\tilde{f}_i^{\varepsilon}(s,y)\ ,$$

where $\tilde{F}[\psi^{\varepsilon}](s,y)=\tilde{F}(\psi_s^{\varepsilon},\psi_{y_iy_j}^{\varepsilon},\psi^{\varepsilon},s,y)$.

In order to complete the proof of Theorem 2.1, we have to show further the following two lemmas.

**Lemma 2.4.** *For any $i$ $(=1, 2, 3)$ and $\{\varepsilon_n\}$ such that $\varepsilon_n\to 0$, $\lim_{\varepsilon_n\to 0}\tilde{f}_i^{\varepsilon_n}(s,y)=0$ a.e. $(Q_T)$.*

Proof. First, it is easy to verify that $\lim_{\varepsilon\to 0}\tilde{f}_2^{\varepsilon}(s,y)=0$ ${}^{\forall}(s,y)\in Q_T$. Indeed, $|\tilde{f}_2^{\varepsilon}(s,y)|\leqq\sup_{\alpha\in A}|\tilde{c}^{\alpha,\varepsilon}(s,y)-\tilde{c}^{\alpha}(s,y)|\,|\psi^{\varepsilon}(s,y)|\leqq N(1+|y|)^{3m}\varepsilon|\hat{y}|\to 0$ $(\varepsilon\to 0)$, where $N=N(k, m, \xi)$ is a constant$\geqq 0$ (see (1.5), (2.16), (2.17), (2.23) and (2.24)).

Second, for $\tilde{f}_3^{\varepsilon}$, $|\tilde{f}_3^{\varepsilon}(s,y)|\leqq\sup_{\alpha}|\chi^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi)-\chi^{\alpha}(s,\bar{y},\xi)|\leqq\sup_{\alpha}|(b^{\alpha}(s,y,$

$\bar{y}, \varepsilon\hat{y}+\xi), \nabla v(s, \bar{y}, \varepsilon\hat{y}+\xi))-(b^{\alpha}(s, \bar{y}, \xi), \nabla v(s, \bar{y}, \xi))|+\sup_{\alpha}|L^{\alpha}(s, \bar{y}, \varepsilon\hat{y}+\xi)-L^{\alpha}(s, \bar{y}, \xi)|\leqq\sup_{\alpha}|b^{\alpha}(s, \bar{y}, \varepsilon\hat{y}+\xi)||\nabla v(s, \bar{y}, \varepsilon\hat{y}+\xi)-\nabla v(s, \bar{y}, \xi)|+\sup_{\alpha}|b^{\alpha}(s, \bar{y}, \varepsilon\hat{y}+\xi)-b^{\alpha}(s, \bar{y}, \xi)||\nabla v(s, \bar{y}, \xi)|+\sup_{\alpha}|L^{\alpha}(s, \bar{y}, \varepsilon\hat{y}+\xi)-L^{\alpha}(s, \bar{y}, \xi)|\equiv I_1+I_2+I_3.$ It is easily shown from (A.1), (1.5) and (1.11) that the last two terms $I_2$ and $I_3$ tend to 0 as $\varepsilon\to 0$ for ${}^{\forall}(s, y)\in Q_T$. As for $I_1$, we have the following: $|I_1|\leqq N(1+|y|)|\nabla v(s, \bar{y}, \varepsilon\hat{y}+\xi)-\nabla v(s, \bar{y}, \xi)|$, where $N=N(k, \xi)$ is a constant$\geqq 0$. Here we can assume that $v$ is differentiable with respect to $x$ at $(s, \bar{x}, \xi)$ and also at $(s, \bar{y}, \varepsilon_n\hat{y}+\xi)$ for all $n\geqq 1$. Indeed, from Proposition 1.2 (b) and simple observations, for each $s$ there are such points almost everywhere in $R^d$. We denote by $[v_{x_i}]$, $1\leqq i\leqq d$, first-order (ordinary) derivatives with respect to $x$. Then it is well known ([10], Theorem 25.4, p. 244) that $[v_{x_i}]$ is continuous at $(s, \bar{x}, \xi)$, from which it follows that $\lim_{\varepsilon_n\to 0}|[v_{x_i}](s, \bar{y}, \varepsilon_n\hat{y}+\xi)-[v_{x_i}](s, \bar{y}, \xi)|=0$, $1\leqq i\leqq d$, and this implies that $|I_1|\to 0$ as $\varepsilon_n\to 0$ because $v_{x_i}=[v_{x_i}]$.

Finally, as for $\tilde{f}^{\varepsilon}_1$, $|\tilde{f}^{\varepsilon}_1(s, y)|\leqq\sup_{\alpha}|\chi^{\alpha,\varepsilon}(s, x)-\chi^{\alpha}(s, x)|\leqq\sup_{\alpha}|b^{\alpha}(s, \bar{y}, \varepsilon\hat{y}+\xi)||\nabla v^{\varepsilon}(s, \bar{y}, \varepsilon\hat{y}+\xi)-\nabla v(s, \bar{y}, \varepsilon\hat{y}+\xi)|\leqq N(1+|y|)|\nabla v^{\varepsilon}(s, \bar{y}, \varepsilon\hat{y}+\xi)-\nabla v(s, \bar{y}, \varepsilon\hat{y}+\xi)|$, due to (A.1), where $N=N(k, \xi)$ is a constant$\geqq 0$. On the other hand, remark that $[v^{\varepsilon}_{x_i}]$, $1\leqq i\leqq d$, exist on the whole space from Sobolev's Lemma and that there exist a constant $N=N(k, m)\geqq 0$ such that for each $s$, for any $\varepsilon>0$, the function $v^{\varepsilon}(s, x)-N(1+|x|^2)^{(3m/2)+1}$ is concave with respect to $x$, which can be shown by the way similar to Proposition 1.2 (c) (cf. [6], Chap. 4, §6). Since $\lim_{\varepsilon\to 0} v^{\varepsilon}(s, x)=v(s, x)$ uniformly in each cylinder $\bar{Q}_{T,R}$, it follows from the well known result (see, for example, [10], Theorem 25.7, p. 248) that for each $s$, $i(1\leqq i\leqq d)$, for a.e. $y$ and $\xi$, $\lim_{\varepsilon_n\to 0}|[v^{\varepsilon_n}_{x_i}](s, \bar{y}, \varepsilon_n\hat{y}+\xi)-[v_{x_i}](s, \bar{y}, \varepsilon_n\hat{y}+\xi)|=0$, from which follows immediately $\lim_{\varepsilon_n\to 0}\tilde{f}^{\varepsilon_n}_1(s, y)=0$ a.e. $(s, y)\in Q_T$. □

**Lemma 2.5.** *Let $Q\subset Q_T$ be a bounded open set. Then it holds that for a.e. $(s, y)\in Q$,*

$$\varlimsup_{\varepsilon\to 0}\tilde{F}[\psi^{\varepsilon}](s, y)\geqq\tilde{F}[\psi](s, y), \tag{2.26}$$

*where $\psi(s, y)\equiv\lim_{\varepsilon\to 0}\psi^{\varepsilon}(s, y)$.*

Proof. According to Lemma 2.3 (c), $\psi(s, y)=v(s, \bar{y}, \xi)$ for all $(s, y)\in\bar{Q}_T$. Since $v\in W^{1,2,\nu}_{\lambda,\mathrm{loc}}(Q_T)$ and $\psi_{y_iy_j}(s, y)=0$ $(\nu+1\leqq i, j\leqq d)$, $\psi\in W^{1,2}_{\lambda}(Q)$ for any $\lambda\geqq 1$. On the other hand, for any $\varepsilon>0$, clearly $\psi^{\varepsilon}\in W^{1,2}_{\lambda}(Q)$, from Lemma 2.3 (a). Moreover, it is also shown that $\sup_{\varepsilon\geqq 0}\sup_{(s,y)\in\bar{Q}}|\psi^{\varepsilon}(s, y)|<\infty$ and $\lim_{\varepsilon\to 0}\|\psi^{\varepsilon}-\psi\|_{d+1,Q}=0$, where $\psi^{0}=\psi$, by the fact that $\lim_{\varepsilon\to 0}\psi^{\varepsilon}(s, y)=\psi(s, y)$ uniformly in each cylinder $\bar{Q}_{T,R}$. Due to N.V. Krylov ([6], Theorem 4.5.1, p. 193), in order to obtain the inequality (2.26), it is sufficients to show that $\psi^{\varepsilon}$ and $\psi$ satisfy the following two conditions:

(a) $\sup_{\varepsilon \geqq 0} \tilde{F}[-\psi^{\varepsilon}] \in L_{d+1}(Q)$, and

(b) $\sup_{\varepsilon \geqq 0} \{-\psi^{\varepsilon}_s - \sum_{i=1}^{d} \psi^{\varepsilon}_{y_i y_i}\} \in L_{d+1}(Q)$.

But remark that the foregoing functions $\psi$, $\psi_s$, $\psi_{y_i}$ $(1 \leqq i \leqq d)$ and $\psi_{y_i y_j}(1 \leqq i, j \leqq d)$ are locally bounded in $(s, y)$, while it follows from Lemma 2.3 (b) that $\psi^{\varepsilon}$, $\psi^{\varepsilon}_s$, $\psi^{\varepsilon}_{y_i}(1 \leqq i \leqq d)$ and $\psi^{\varepsilon}_{y_i y_j}(1 \leqq i, j \leqq d)$ are also locally bounded in $(s, y)$ uniformly with respect to $\varepsilon (0 < \varepsilon < 1)$. Recalling that the coefficients $a$, $\tilde{c}$ and $\tilde{\chi}$ in the formula (2.22) of $\tilde{F}$ are also locally bounded in $(s, y)$ uniformly with respect to $\alpha$, therefore, we get the assertions (a) and (b) immediately. $\square$

Let complete the proof of Theorem 2.1. For any $\{\varepsilon_n\}$, letting $\varepsilon_n \to 0$ in (2.25), then it follows from Lemma 2.4 that we have the following:

$$(2.27) \qquad 0 \geqq \overline{\lim_{\varepsilon_n \to 0}}\, \tilde{F}[\psi^{\varepsilon_n}]\,(s, y) \quad \text{a.e. } (Q_T)\,.$$

By means of (2.26) and (2.27) we have the following:

$$(2.28) \qquad 0 \geqq \tilde{F}[\psi]\,(s, y) \quad \text{a.e. } (Q_T)\,.$$

Since $\psi(s, y) = v(s, \bar{y}, \xi)$ for all $(s, y) \in \bar{Q}_T$, it holds that $\tilde{F}[\psi](s, y) = F[v](s, \bar{y}, \xi)$ (note the equalities such that $\psi_s = v_s$, $\psi_{y_i y_j} = v_{y_i y_j}$, $1 \leqq i, j \leqq \nu$, $\psi_{y_i y_i} = 0$, $\nu + 1 \leqq i \leqq d$) for all $(s, y) \in \bar{Q}_T$. Then we have the following inequality relative to $v$:

$$(2.29) \qquad 0 \geqq F[v](s, \bar{y}, \xi) \quad \text{a.e. } (s, \bar{y}) \in (0, T) \times R^{\nu}\,.$$

Since $\xi \in R^{d-\nu}$ is arbitrary, we can conclude from (2.29) that for a.e. $(s, x) \in Q_T$,

$$(2.30) \qquad 0 \geqq F[v](s, x)\,. \quad \square$$

REMARK 2.1. It is not essential that $a^{\alpha,\varepsilon}$ is assumed to be independent of $x$, made in the proof of Theorem 2.1. Indeed, we have to show in this case that Lemma 2.3 (b) still holds and that for any $\xi \in R^{d-\nu}$, $(s, y) \in \bar{Q}_T$, $\liminf_{\varepsilon \to 0} \sum_{\alpha} \sum_{1 \leqq i, j \leqq \nu} \{a^{\alpha}_{ij}(s, \bar{y}, \xi) - a^{\alpha}_{ij}(s, \bar{y}, \varepsilon\hat{y} + \xi)\} \times \psi^{\varepsilon}_{y_i y_j}(s, y) = 0$. Note, however, that to prove the former, only important thing is that $a^{\alpha,\varepsilon} = \sigma^{\alpha,\varepsilon}(\sigma^{\alpha,\varepsilon})^*$ does not degenerate and, furthermore, it is locally bounded uniformly with respect to $\alpha$, which are easily derived by virtue of (1.3) and (1.9). On the other hand, the latter follows from (1.2) and Lemma 2.3 (b). $\square$

We can now apply the above results to some interesting partial differential equations of parabolic type whose examples are given in the following:

EXAMPLE 2.1. Let $d=2$, $\nu=1$ and $A = \{(\alpha, \beta) \in R^2;\ \alpha^2 + \beta^2 \leqq 1\}$, and consider the following Bellman equation:

$$(2.31)\qquad \inf_{(\alpha,\beta)\in A}\{v_s+(1/2)v_{xx}+\alpha v_x+\beta v_y+L(s,x,y)\}=0,$$

where $L(s, x, y)$ is independent of $\alpha$. It is easy to verify that $\inf_{(\alpha,\beta)\in A}\{\alpha v_x+\beta v_y\}$ $=-\sqrt{(v_x)^2+(v_y)^2}$ (take $(\alpha,\beta)=-(v_x, v_y)/\sqrt{(v_x)^2+(v_y)^2}$). Then it follows from Proposition 1.2 and Theorem 2.1 that there exist a generalized solution of Eq. (0.5) satisfying the conditions such as (1.7), (1.12) and Proposition 1.2 (d). □

EXAMPLE 2.2. Let $(d, \nu, A)$ be the same as Example 2.1, and consider the following Bellman equation:

$$(2.32)\qquad \inf_{(\alpha,\beta)\in A}\{v_s+(1/2)(\alpha+\beta+2)v_{xx}+\alpha v_x+\beta v_y+L(s,x,y)\}=0.$$

Then it is easily shown like as Example 2.1 that the above equation is equivalent to Eq. (0.6). Since $\alpha+\beta+2>0$ for all $(\alpha,\beta)\in A$, there is a generalized solution of Eq. (0.6) possessing the properties stated in Proposition 1.2. □

Thus we can consider many other examples analogous to Examples 2.1 and 2.2. See also Example 5.2 in the case where $A$ is not bounded.

REMARK 2.2. It is not difficult to consider examples of such stochastic control problem as §1 whose state and cost are given by Eq. (1.1) and (1.4) respectively. For instance, let consider a movement of an object in which only some components (or directions) receive random disturbances but not so are others. Then the state may be written as Eq. (1.1). Moreover, actually we know an example of stochastic control model lying in our framework, which is induced by linear partially observable one with non-gaussian initial distribution (see [4] and also [3]). □

## 3. Superharmonic function

In the following two paragraphs we shall consider the uniqueness of solutions of Eq. (0.1). From the preceding discussions in §1 and §2 we already know that the cost function $v$, defined by (1.8), is a generalized solution of Eq. (0.1) where $a=\sigma\sigma^*$ and $\sigma$ is of the form (1.9). Therefore, it is sufficient to show that any "solution" $u$ is equal to $v$.

Let $C(\bar{Q}_T)$ be a space of real-valued continuous functions defined over $\bar{Q}_T$. We define a superharmonic function on $\bar{Q}_T$ by the following manner, similar to one due to N.V. Krylov ([6], p. 229).

DEFINITION 3.1. We say that a real-valued function $u$ given on $\bar{Q}_T$ is *superharmonic* if there exist constants $p$, $\lambda$, $k\geqq 0$ such that $u\in W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T)\cap C(\bar{Q}_T)$ and

$$(3.1)\qquad |u(t,x)|\leqq k(1+|x|)^{\lambda},$$

and, furthermore, $F[u](t, x)\geqq 0$ a.e. $(Q_T)$ with $u(T, x)\leqq h(x)$ for all $x\in R^d$. □

Our main concern is the following:

**Theorem 3.1.** *Let $u$ be a superharmonic function on $\bar{Q}_T$. Moreover, suppose that $\partial u/\partial x_i$ $(1\leqq i\leqq d)$ and $\partial^2 u/\partial x_i\partial x_j$ $(1\leqq i, j\leqq \nu)$ are locally bounded. Then,* for all $(s, x)\in\bar{Q}_T$, $u(s, x)\leqq v(s, x)$. □

Proof. Let $\varepsilon>0$ and let $\rho_\varepsilon\geqq 0$ be a molifier function defined over $R^{d+1}$ such that $\rho_\varepsilon\in C_0^\infty(R^{d+1})$ and $\int_{R^{d+1}}\rho_\varepsilon(t, y)dtdy=1$. Let $\varepsilon_n\to 0$ as $n\to\infty$, and for any $n=1, 2, \cdots$, define $u_n$ by the formula: $u_n(s, x)=u^*\rho_{\varepsilon_n}(s, x)\equiv\int_{R^{d+1}}u(t, y)\rho_{\varepsilon_n}(s-t, x-y)dtdy$, here we extend $u$ to $R^{d+1}$ by the way that $u\equiv 0$ outside of $\bar{Q}_T$ and we write it again $u$. For any $(\alpha, s, x)\in\mathfrak{A}\times\bar{Q}_T$, let $(X_t^{\alpha,s,x})$ be a unique solution of Eq. (1.1) on a probability space $(\Omega, \mathcal{F}, P)$ and define a process $(Y_t^{\alpha,s,x})$, $s\leqq t\leqq T$, by the formula:

$$Y_t^{\alpha,s,x}=X_{t-s}^{\alpha,s,x}. \tag{3.2}$$

Then $Y_t^{\alpha,s,x}$ is $\mathcal{F}_t^s$-adapted process, where $\mathcal{F}_t^s$ stands for the $\sigma$-field $\mathcal{F}_{t-s}$ $(t\geqq s)$, and it also satisfies the following equality for all $t\geqq s$,

$$Y_t=x+\int_s^t b(\tilde{\alpha}_r, r, Y_r)dr+\int_s^t\sigma(\tilde{\alpha}_r, r, Y_r)d\tilde{\xi}_r \quad \text{a.e.}, \tag{3.3}$$

where $\tilde{\alpha}_t(\omega)=\alpha_{t-s}(\omega)$ and $(\tilde{\xi}_s)$, $s\leqq t\leqq T$, is a vector valued Brownian motion with respect to $\mathcal{F}_t^s$, due to Eq. (1.1). Notice also that $(Y_t^{\alpha,s,x})$ is a unique solution of Eq. (3.3) with initial condition $Y_s=x$ associated with $(\alpha, s, x)$, and it follows from (1.6) that for any $\lambda\geqq 0$ there is a constant $k_\lambda$ such that

$$E[\sup_{s\leqq t\leqq T}|Y_t^{\alpha,s,x}|^\lambda]\leqq k_\lambda(1+|x|)^\lambda \qquad \text{for all } (s, x, \alpha). \tag{3.4}$$

For any sufficiently large number $R$, let

$$\tau_R^{\alpha,s,x}=\inf\{t\geqq s;\ |Y_t^{\alpha,s,x}|>R\},\ \tau_R^{\alpha,s,x}=T \quad \text{if } \{\ \}=\phi. \tag{3.5}$$

Then it holds that $P(\tau_R<T)\leqq P(\sup_{s\leqq t\leqq T}|Y_t^\alpha|\geqq R)\leqq E[\sup_{s\leqq t\leqq T}|Y_t^\alpha|]/R$ (we omit from now on supersuffices $(s, x)$ because they are fixed).

Since $u_n$ is a smooth function over $R^{d+1}$, applying Ito's formula to the function $u_n(s, x)e^{-y}$, for any $n$, $R$ and $s\leqq t\leqq T$, we have the following:

$$\begin{aligned} u_n(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})-u_n(s, x) &= \int_s^{t\wedge\tau_R^\alpha} e^{-\phi_r^{\tilde{\alpha}}}\{\partial_s u_n \\ &+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^{\tilde{\alpha}_r}\partial_i\partial_j u_n+\sum_{i=1}^d b_{ij}^{\tilde{\alpha}_r}\partial_i u_n-c^{\tilde{\alpha}_r}u_n\}(r, Y_r)dr \\ &+\int_s^{t\wedge\tau_R^\alpha}e^{-\phi_r^{\tilde{\alpha}}}(\nabla u_n(r, Y_r^\alpha), \sigma^{\tilde{\alpha}_r}(r, Y_r^\alpha)d\tilde{\xi}_r), \end{aligned} \tag{3.6}$$

where $\phi_t^{\tilde{\alpha}}=\int_s^t c(\tilde{\alpha}_r,\ r,\ Y_r^{\alpha})dr\equiv\int_0^{t-s} c(\alpha_r,\ s+r,\ X_r^{\alpha})dr$ and $\partial_s u_n=\partial u_n/\partial s$, $\partial_i\partial_j u_n=\partial^2 u_n/\partial x_i\partial x_j$, $\partial_i u_n=\partial u_n/\partial x_i$. Adding $\int_s^{t\wedge\tau_R^{\alpha}} L(\tilde{\alpha}_r,\ r,\ Y_r^{\alpha})e^{-\phi_r^{\tilde{\alpha}}}dr$ to the both terms of (3.6), and, further, taking the mathematical expectation with respect to $P$, we have the following:

$$\text{(3.7)}\qquad E[u_n(t\wedge\tau_R^{\alpha},\ Y_{t\wedge\tau_R^{\alpha}}^{\alpha})\exp(-\phi_{t\wedge\tau_R^{\alpha}}^{\tilde{\alpha}})-u_n(s,\ x)+\int_s^{t\wedge\tau_R^{\alpha}} L(\tilde{\alpha}_r,\ r,\ Y_r^{\alpha})e^{-\phi_r^{\tilde{\alpha}}}dr]$$
$$=E[\int_s^{t\wedge\tau_R^{\alpha}} e^{-\phi_r^{\tilde{\alpha}}}F^{\tilde{\alpha}_r}[u_n]\,(r,\ Y_r^{\alpha})dr]\,.$$

Indeed the second term of the right side of (3.6) is a square integrable martingale with respect to $(\mathcal{F}_t^s,\ P)$, by the assumption (A.1). It follows easily from (3.7) and (1.15) that we have the following inequality:

$$\text{(3.8)}\qquad E[u_n(t\wedge\tau_R^{\alpha},\ Y_{t\wedge\tau_R^{\alpha}}^{\alpha})\exp(-\phi_{t\wedge\tau_R^{\alpha}}^{\tilde{\alpha}})-u_n(s,\ x)+\int_s^{t\wedge\tau_R^{\alpha}} L(\tilde{\alpha}_r,\ r,\ Y_r^{\alpha})e^{-\phi_r^{\tilde{\alpha}}}dr]$$
$$\geqq E[\int_s^{t\wedge\tau_R^{\alpha}} e^{-\phi_r^{\tilde{\alpha}}}F[u_n]\,(r,\ Y_r^{\alpha})dr]\,.$$

On the other hand, since $u$ is a superharmonic function on $\bar{Q}_T$ from the assumption,

$$\text{(3.9)}\qquad F[u]\,(s,\ x)\geqq 0\quad \text{a.e.}$$

Taking the convolution of (3.9) and $\rho_{\varepsilon_n}$ (we extend $a$, $b$, $c$ and $L$ on $R^{d+1}$), we have:

$$\text{(3.10)}\qquad 0\leqq\partial_s u_n+F_1[u]^*\rho_{\varepsilon_n}\qquad \text{on}\quad Q_T\,,$$

where for $u\in W_{p,\mathrm{loc}}^{1,2,\nu}(Q_T)$,

$$\text{(3.11)}\qquad F_1[u]\,(t,\ x)\equiv\inf_{\alpha\in A}\{(1/2)\sum_{1\leq i,j\leq\nu} a_{ij}^{\alpha}\partial_i\partial_j u+(b^{\alpha},\ \nabla u)-c^{\alpha}u+L^{\alpha}\}\,(t,\ x)\,,$$

and for convenience sake let's write $F_1^{\alpha}[u]$ the parenthesis of (3.11). Note also that $(\partial_s u)^*\rho_{\varepsilon_n}=\partial_s u_n$ and moreover we shall use the well known relations that for each $n$, $(\nabla u)^*\rho_{\varepsilon_n}=\nabla u_n$ and $(\partial_i\partial_j u)^*\rho_{\varepsilon_n}=\partial_i\partial_j u_n$ $1\leq i,\ j\leq\nu$. It holds from (3.10) and (3.11) that for all $(s,\ x)$,

$$\text{(3.12)}\qquad F[u_n]\,(s,\ x)=\partial_s u_n+F_1[u_n]\,(s,\ x)\geqq -F_1[u]^*\rho_{\varepsilon_n}(s,\ x)+$$
$$F_1[u_n]\,(s,\ x)\geqq -\inf_{\alpha}(F_1^{\alpha}[u]^*\rho_{\varepsilon_n})\,(s,\ x)+F_1[u_n]\,(s,\ x)$$
$$\geqq -\sup_{\alpha}|F_1^{\alpha}[u_n]\,(s,\ x)-F_1^{\alpha}[u]^*\rho_{\varepsilon_n}(s,\ x)|\,.$$

The second inequality is due to the fact that $(\inf_{\alpha} F_1^{\alpha}[u])^*\rho_{\varepsilon_n}\leqq\inf_{\alpha} F_1^{\alpha}[u]^*\rho_{\varepsilon_n}$, and the third one is due to the well known inequality that $|\inf_{\alpha} f^{\alpha}-\inf_{\alpha} g^{\alpha}|\leqq\sup_{\alpha}|f^{\alpha}-g^{\alpha}|$. We need the following result which is rather general and it is verified easily by hand.

For each $M=1, 2, \cdots$, let $S_M$ be a ball of radius $M$ in $R^{d+1}$.

**Lemma 3.2.** *Let* $1>\varepsilon>0$. *Let* $f$ *be a real valued locally Lipshitz continuous function on* $R^{d+1}$, *i.e., for each* $M$ *there exists a constant* $k_M\geqq 0$ *such that for all* $(s, x)$ *and* $(s', x')$ *in* $S_M$, $|f(s, x)-f(s', x')|\leqq k_M(|s-s'|+|x-x'|)$. *Furthermore, let* $g$ *be a real valued function on* $R^{d+1}$ *which is locally bounded. Then we have the following inequality: for each* $M$ *there exists a constant* $N=N(M)\geqq 0$ *such that*

$$\sup_{(s,x)\in S_M}|(fg)^*\rho_\varepsilon(s, x)-f(g^*\rho_\varepsilon)(s, x)|\leqq\varepsilon N\|g\|_{S_{M+1}}, \tag{3.13}$$

*where* $\|g\|_{S_{M+1}}=\sup_{S_{M+1}}|g(s, x)|$. $\square$

It follows from the formula (3.11) that we have the following inequality: for all $(\alpha, s, x)\in A\times Q_T$

$$\begin{aligned}(3.14)\qquad &|F_1^\alpha[u_n](s, x)-F_1^\alpha[u]^*\rho_{\varepsilon_n}(s, x)|\\ &\leqq(1/2)\sum_{1\leqq i,j\leqq\nu}|a_{ij}^\alpha(s, x)\partial_i\partial_j u_n(s, x)-(a_{ij}^\alpha\partial_i\partial_j u)^*\rho_{\varepsilon_n}(s, x)|\\ &\quad+\sum_{i=1}^{d}|b_i^\alpha(s, x)\partial_i u_n(s, x)-(b_i^\alpha\partial_i u)^*\rho_{\varepsilon_n}(s, x)|+\\ &\quad|c^\alpha u_n(s, x)-(c^\alpha u)^*\rho_{\varepsilon_n}(s, x)|+|L^\alpha(s, x)-L^{\alpha *}\rho_{\varepsilon_n}(s, x)|\,.\end{aligned}$$

Since for all $i$ and $j$, $a_{ij}$ is locally Lipshitz on $R^{d+1}$ uniformly with respect to $\alpha$, from the assumption (A.1) and, furthermore, $\partial_i\partial_j u$ is locally bounded on $R^{d+1}$ by the assumption of this theorem, $(a_{ij}, \partial_i\partial_j u)$ satisfy the assumptions of Lemma 3.2 above. Therefore it holds that for any $M\geqq 0$, $|a_{ij}^\alpha(s, x)\partial_i\partial_j u_n(s, x)-(a_{ij}^\alpha\partial_i\partial_j u)^*\rho_{\varepsilon_n}(s, x)|\leqq\varepsilon_n N\|\partial_i\partial_j u\|_{S_{M+1}}$ for all $\alpha\in A$, $(s, x)\in S_M$, all $n\in\boldsymbol{N}$, where $N=N(M)$ is a constant$\geqq 0$. Since $(b_i^\alpha, \partial_i u)\,(1\leqq i\leqq d)$, $(c^\alpha, u)$ and $L^\alpha$ also satisfy the assumptions of Lemma 3.2, we have the same estimate as $(a_{ij}^\alpha, \partial_i\partial_j u)$ to the rest terms of the right side of (3.14). Consequently, we can conclude that for each $M\geqq 0$ there is a constant $N=N(M)\geqq 0$ such that for all $n=1, 2, \cdots$, $(s, x)\in S_M$ and $\alpha\in A$

$$|[F_1^\alpha[u_n](s, x)-F_1^\alpha[u]^*\rho_{\varepsilon_n}(s, x)|\leqq\varepsilon_n N\,. \tag{3.15}$$

It follows from (3.12) and (3.15) that for all $(t, y)\in S_M$ and $n\geqq 1$

$$F[u_n](t, y)\geqq-\varepsilon_n N \qquad \text{where } N=N(M)\geqq 0\,. \tag{3.16}$$

Note that if $s\leqq r\leqq t\wedge\tau_R$, then $(r, Y_r^\alpha)\in S_M$ for some $M=M(R)\geqq 0$. Then it follows from (3.8) and (3.16) that for all $R\geqq 0$, $n\geqq 1$, $(\alpha, s, x)\in\mathfrak{A}\times Q_T$, $s\leqq t\leqq T$,

$$\begin{aligned}(3.17)\qquad &E[u_n(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde\alpha})-u_n(s, x)\\ &+\int_s^{t\wedge\tau_R^\alpha}L(\tilde\alpha_r, r, Y_r^\alpha)e^{-\phi_r^{\tilde\alpha}}dr]\geqq-\varepsilon_n N\,E[\int_s^{t\wedge\tau_R^\alpha}e^{-\phi_r^{\tilde\alpha}}dr]\end{aligned}$$

where $N=N(R)$ is a constant $\geqq 0$.

Let $n\to\infty$ in (3.17). It is well known that if $u$ is continuous on $\bar{Q}_T$, then $u_n$ converges to $u$ as $n\to\infty$ uniformly in each cylinder $\bar{Q}_{T,R}$, so that for any $R$, $|u_n(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)-u(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)|\to 0$ as $n\to\infty$ for a.a. $(P)$. Next, it is easily shown that

$$|u_n(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})|\leqq|u_n(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)|\leqq\sup_{S_M}|u_n(t, y)|\leqq N$$

a.e. $(P)$, where $M=M(R)$ and $N=N(R)$ are nonnegative constants. Therefore, by virtue of Lebesgue's bounded convergence theorem, for any $R\geqq 0$, $\lim_{n\to\infty} E[u_n(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})]=E[u(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})]$. On the other hand, since $\varepsilon_n\to 0$ as $n\to\infty$ and $N\,E[\int_s^{t\wedge\tau_R^\alpha} e^{-\phi_r^{\tilde{\alpha}}}dr]$ is independent of $n$, the right side of (3.17) $\to 0$ as $n\to\infty$. It follows from (3.17) that for any $R\geqq 0$

$$E[u(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})+\int_s^{t\wedge\tau_R^\alpha} L(\tilde{\alpha}_r, r, Y_r^\alpha)e^{-\phi_r^{\tilde{\alpha}}}dr]\geqq u(s, x)\,. \tag{3.18}$$

Next, let $R\to\infty$ in (3.18). First, it is not difficult to show that for $s\leqq t\leqq T$, $\lim_{R\to\infty} u(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)=u(t, Y_t^\alpha)$ a.e. because of (3.5) (and the note following it) and the continuity of $u$. Similarly, it holds that $\lim_{R\to\infty}\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})=\exp(-\phi_t^\alpha)$ a.e. Now it follows from (3.1) that $|u(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})|\leqq k(1+|Y_{t\wedge\tau_R^\alpha}^\alpha|)^\lambda\leqq k(1+\sup_{s\leqq t\leqq T}|Y_t^\alpha|)^\lambda$ and, moreover, $E[(1+\sup_{s\leqq t\leqq T}|Y_t^\alpha|)^\lambda]<\infty$, because of (3.4). Using again Lebesgue's theorem, we can prove that $\lim_{R\to\infty} E[u(t\wedge\tau_R^\alpha, Y_{t\wedge\tau_R^\alpha}^\alpha)\exp(-\phi_{t\wedge\tau_R^\alpha}^{\tilde{\alpha}})]=E[u(t, Y_t^\alpha)\exp(-\phi_t^{\tilde{\alpha}})]$ for all $(\alpha, s, x)\in\mathfrak{A}\times Q_T$ and $t\in[s, T]$. Similarly, we can show that

$$E[\int_s^{t\wedge\tau_R^\alpha} L(\tilde{\alpha}_r, r, Y_r^\alpha)e^{-\phi_r^{\tilde{\alpha}}}dr]\to E[\int_s^t L(\tilde{\alpha}_r, r, Y_r^\alpha)e^{-\phi_r^{\tilde{\alpha}}}dr] \quad \text{as } R\to\infty\,.$$

Thus, letting $R\to\infty$ in (3.18), we have the following:

$$E[u(t, Y_t^\alpha)e^{-\phi_t^{\tilde{\alpha}}}+\int_s^t L(\tilde{\alpha}_r, r, Y_r^\alpha)e^{-\phi_r^{\tilde{\alpha}}}dr]\geqq u(s, x)\,. \tag{3.19}$$

Since $u(T, x)\leqq h(x)$ for all $x\in R^d$, by the assumption, letting $t\uparrow T$ in (3.19), we get finally the following inequality: for all $(\alpha, s, x)\in\mathfrak{A}\times Q_T$

$$E[h(Y_T^{\alpha,s,x})e^{-\phi_T^{\tilde{\alpha},s,x}}+\int_s^T L(\tilde{\alpha}_r, r, Y_r^{\alpha,s,x})e^{-\phi_r^{\tilde{\alpha},s,x}}dr]\geqq u(s, x)\,. \tag{3.20}$$

Recall that $Y_t^{\alpha,s,x}=X_{t-s}^{\alpha,s,x}$, $s\leqq t\leqq T$, (and also the definitions of $\tilde{\alpha}$ and $\phi^\alpha$). Then it is clear that (3.20) is equivalent to say that $v^\alpha(s, x)\geqq u(s, x)$ for all $(\alpha, s, x)$, so that $v(s, x)\geqq u(s, x)$ for all $(s, x)\in\bar{Q}_T$, which is what we wanted to show. □

REMARK 3.1. It is obvious that the function $v$ of (1.8) satisfies the assumptions of Theorem 3.1, by virtue of Proposition 1.2 (a), (b), (d) and (e) (cf. [6], Theorem 5.3.11, p. 237).

## 4. Uniqueness

In §3 we proved that if $u$ is a superharmonic function satisfying the auxiliary conditions relative to its derivatives, then $u\leqq v$ on $\bar{Q}_T$. Let's show the inverse relation. For any function $f$ over $\bar{Q}_T$, $(s, x)\in\bar{Q}_T$, $l\in R^d$ such that $|l|=1$ and $\delta\in(0, 1)$, define $D^2_{l,\delta}f(s, x)$ by the formula:

$$(4.1)\qquad D^2_{l,\delta}f(s, x)=\{f(s, x+\delta l)+f(s, x-\delta l)-2f(s, x)\}/\delta^2 .$$

It is easily seen that if $f$ is twice continuously differentiable with respect to $x$ at $(s, x)$, then $D^2_{l,\delta}f(s, x)\to[f_{(l)(l)}](s, x)$ as $\delta\to 0$. In this paragraph we assume the condition (1.21) instead of (1.9), and assume also that $u$ satisfies (3.1). Then we have:

**Theorem 4.1.** *Suppose that a function $u$ belongs to $W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T)\cap C(\bar{Q}_T)$ $(p\geqq d+1)$, $F[u]\leqq 0$ a.e. $(Q_T)$ and $u(T, x)\geqq h(x)$ for all $x\in R^d$. Moreover, suppose that there are nonnegative constants $k$ and $\lambda$ such that for all $(s, x)\in Q_T$, $0<\delta<1$, $l\in R^d$ such that $|l|=1$,*

$$(4.2)\qquad D^2_{l,\delta}u(s, x)\leqq k(1+|x|)^\lambda .$$

*Then it holds that $u\geqq v$ on $\bar{Q}_T$.* □

Remark that $v$ also satisfies (4.2) by means of Proposition 1.2 (see Remark 4.1 below). In order to prove Theorem 4.1, we need two auxiliary results which are rather general for our purpose.

**Lemma 4.2.** *For some $p$, let $u\in W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T)\cap C(\bar{Q}_T)$. Also, let $F[u]\leqq 0$ a.e. $(Q_T)$. Then, for each $\kappa=1, 2, \cdots$, there is a Borel function $\alpha_k$ over $\bar{Q}_T$ with values in $A$ such that*

$$(4.3)\qquad 1/\kappa>F^{\alpha_\kappa}[u](s, x)\quad a.e.\ (Q_T),$$

*where*

$$(4.4)\qquad F^{\alpha_\kappa}[u](s, x)=u_s+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}(\alpha_\kappa(s, x), s, x)u_{x_ix_j}+ \\ (b(\alpha_\kappa(s, x), s, x), \nabla u(s, x))-c(\alpha_\kappa(s, x), s, x)u+L(\alpha_\kappa(s, x), s, x) .\quad\square$$

Proof. By virtue of the assumption that $A$ is separable metric space, there is a countable subset $\{\alpha(i)\}$, $i=1, 2, \cdots$, dense everywhere in $A$. Since, for each $(s, x)$, $F^\alpha[u](s, x)$ is continuous with respect to $\alpha$, it holds that for all $(s, x)$

(4.5) $\quad F[u](s, x) = \inf_{1 \leqq i < \infty} F^{\alpha_i}[u](s, x)$.

Therefore, for all $(s, x)$, there is a function $\alpha_\kappa(s, x) = \alpha(i_\kappa(s, x))$, Borel measurable with respect to $(s, x)$, where $i_\kappa(s, x)$ is the minimum number such that

(4.6) $\quad 1/\kappa > F^{\alpha(i)}[u](s, x)$ a.e. $(Q_T)$.

The measurability of the function $\alpha_\kappa$ follows from the fact that for any $\Gamma \in \mathcal{A}$, $\{(s, x); \alpha_\kappa(s, x) \in \Gamma\} = \bigcup_{\alpha_i \in \Gamma} \{(s, x); 1/\kappa > F^{\alpha(i)}[u](s, x)$ and $1/\kappa \leqq F^{\alpha(j)}[u](s, x), 1 \leq j < i\}$ and that $F^{\alpha}[u](s, x)$ is Borel measurable with respect to $(s, x)$. □

For each $\gamma = 1, 2, \cdots$, let $\eta_\gamma(x)$ be a $C^\infty$-function over $R^d$ such that $0 \leqq \eta_\gamma(x) \leqq 1$, $\eta_\gamma(x) = 1$ if $|x| \leqq \gamma$, $\eta_\gamma(x) = 0$ if $|x| \geqq \gamma + 1$, and, moreover, $|\nabla \eta_\gamma(x)| \leqq 2$ for all $\gamma$ and $x$. For each $(\alpha, t, x) \in A \times \bar{Q}_T$, $\gamma = 1, 2, \cdots$, put

(4.7) $\quad \sigma^{\alpha,\gamma}_{ij}(t, x) = \sigma^{\alpha}_{ij}(t, \eta_\gamma(x)x)$ and $b^{\alpha,\gamma}_i(t, x) = b^{\alpha}_i(t, \eta_\gamma(x)x)$, $1 \leqq i, j \leqq d$.

Then it follows from (A.1) that for each $\gamma$, $\sigma^{\alpha,\gamma}_{ij}(b^{\alpha,\gamma}_i)$ is continuous with respect to $(\alpha, t, x)$. Furthermore, it holds that for each $\gamma = 1, 2, \cdots$, there is a constant $k_\gamma$ such that for all $\alpha \in A$ $t \in [0, T]$, $x, x' \in R^d$,

(4.8) $\quad \|\sigma^{\alpha,\gamma}(t, x) - \sigma^{\alpha,\gamma}(t, x')\| + |b^{\alpha,\gamma}(t, x) - b^{\alpha,\gamma}(t, x')| \leqq k_\gamma |x - x'|$, and

(4.9) $\quad \|\sigma^{\alpha,\gamma}(t, x)\| + |b^{\alpha,\gamma}(t, x)| \leqq k_\gamma$.

Note also that $\sigma^{\alpha,\gamma}_{ij}(b^{\alpha,\gamma}_i)$ converges to $\sigma^{\alpha}_{ij}(b^{\alpha}_i)$ as $\gamma \to \infty$ uniformly on any compact set of $x$ and uniformly over $(\alpha, t)$, from the fact that $\sigma^{\alpha}_{ij}(b^{\alpha}_i)$ is Lipshitz continuous with respect to $x$ uniformly over $(\alpha, t)$.

For each $\varepsilon (0 < \varepsilon < 1)$ and $(\alpha, t, x) \in A \times \bar{Q}_T$, define $\sigma^{\alpha,\gamma,\varepsilon}(t, x)$ by the formula: $\sigma^{\alpha,\gamma,\varepsilon}_{ij}(t, x) = \sigma^{\alpha,\varepsilon}_{ij}(t, \eta_\gamma(x)x)$, $1 \leqq i, j \leqq d$ (see (2.1)). Let $\{\alpha_\kappa(s, x)\}$ be the sequence obtained in Lemma 4,2 and consider the following stochastic differential equation associated with $(\alpha_\kappa, s, x, \gamma, \varepsilon)$:

$$\text{(4.10)} \quad \begin{cases} dX_t = b^\gamma(\alpha_\kappa(s+t, X_t), s+t, X_t)dt + \sigma^{\gamma,\varepsilon}(\alpha_\kappa(s+t, X_t), s+t, X_t)dB_t, \\ X_0 = x. \end{cases}$$

Remark that for each $\gamma$, $b^\gamma(\alpha_\kappa(t, x), t, x)$ and $\sigma^{\gamma,\varepsilon}(\alpha_\kappa(t, x), t, x)$ are bounded and Borel measurable with respect to $(t, x)$ from (4.9) and Lemma 4.2. Moreover, the matrix $a^{\gamma,\varepsilon} = \sigma^{\gamma,\varepsilon}(\sigma^{\gamma,\varepsilon})^*$ is uniformly positive definite, because for each $\varepsilon > 0$ there is a constant $\mu_\varepsilon > 0$ such that $(a^{\gamma,\varepsilon}(t, x)\xi, \xi) \geqq \mu_\varepsilon |\xi|^2$ for all $\gamma = 1, 2, \cdots$, $(t, x) \in \bar{Q}_T$, $\xi \in R^d$ (in fact, take $\min(\mu, \varepsilon^2)$ as $\mu_\varepsilon$, where $\mu$ is given in (1.21)). Then the following result is well known ([6], Theorem 2.6.1, p. 87).

**Lemma 4.3.** *There exists a solution of* Eq. (4.10) *on a probability space satisfying the standard conditions on which is given a Brownian motion* $(B_t, \mathcal{F}_t)$. *Furthermore, this solution is progressively measurable with respect to* $(\mathcal{F}_t)$. □

Denote by $(X_t^{\kappa,s,x,\gamma,\varepsilon})$ a solution of Eq. (4.10) associated with $(\alpha_\kappa, s, x, \gamma, \varepsilon)$ on a probability space $(\Omega, \mathcal{F}, P)$.

Proof of Theorem 4.1. As in §3, define a process $(Y_t^{\kappa,s,x,\gamma,\varepsilon})$, $s\leqq t\leqq T$, and a stopping time $\tau_R^{\kappa,s,x,\gamma,\varepsilon}$ by the formulas:

(4.11) $\quad Y_t^{\kappa,s,x,\gamma,\varepsilon} = X_{t-s}^{\kappa,s,x,\gamma,\varepsilon},\ s\leqq t\leqq T$, and

(4.12) $\quad \tau_R^{\kappa,s,x,\gamma,\varepsilon} = \inf\{t\geqq s;\ |Y_t^{\kappa,s,x,\gamma,\varepsilon}|>R\},\ \tau_R^{\kappa,s,x,\gamma,\varepsilon} = T \qquad \text{if } \{\ \ \} = \phi$.

Then, by the same reason as §3, $(Y_t^{\kappa,\gamma,\varepsilon})$ is a solution of the following stochastic differential equation:

(4.13) $\quad dY_t = b^{\kappa,\gamma}(t, Y_t)dt+\sigma^{\kappa,\gamma,\varepsilon}(t, Y_t)d\xi_t,\ Y_s = t$,

where $b^{\kappa,\gamma}(t, y)=b^\gamma(\alpha_\kappa(t, y), t, y)$, $\sigma^{\kappa,\gamma,\varepsilon}(t, y)=\sigma^{\gamma,\varepsilon}(\alpha_\kappa(t, y), t, y)$ and $(\xi_t)$ is a vector valued Brownian motion with respect to $\mathcal{F}_t^s$ (we omit supersuffices $(s, x)$ like as §3). Define $u_n$ by the formula: $u_n=u*\rho_{\varepsilon_n}$ $(n=1, 2, \cdots)$, where $\rho_{\varepsilon_n}$ is the same as one given in §3. Then it is well known that for any bounded region $Q\subset Q_T$, $\|u_n-u\|_{W_p^{1,2,\nu}(Q)}\to 0$ as $n\to\infty$. Fix an arbitrary number $R>0$ such that $(s, x)\in\bar{Q}_{T,R}$. Then it is easily seen that $P(\tau_R^{\kappa,\gamma,\varepsilon}<T)\leqq(N/R)(1+|x|)$, where $N=N(k)\geqq 0$ is a constant, (see (3.5) and the remark following it). Applying Ito's formula to the function $u_n(t, x)e^{-y}$, we have the following: for any $t\in[s, T]$,

$$
\begin{aligned}
(4.14)\quad & u_n(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})-u_n(s, x)\\
&= \int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}\{\partial_s u_n+(1/2)\sum_{1\leqq i,j\leqq\nu} a_{ij}^{\kappa,\gamma}\partial_i\partial_j u_n+\\
&\quad (\varepsilon^2/2)\sum_{\nu+1\leqq i\leqq d}\partial_i^2 u_n+(b^{\kappa,\gamma}, \nabla u_n)-c^\kappa u_n\}(r, Y_r^{\kappa,\gamma,\varepsilon})dr\\
&\quad +(\text{square integrable martingale}),
\end{aligned}
$$

where $\tilde{\phi}_t^{\kappa,\gamma,\varepsilon}=\int_s^t c^\kappa(r, Y_r^{\kappa,\gamma,\varepsilon,\kappa})dr$ and $c_\kappa(t,y)=c(\alpha_\kappa(t,y),t,y)$. By the way similar to §3, we have the following:

$$
\begin{aligned}
(4.15)\quad & E[u_n(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})-u_n(s, x)+\\
&\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}L^\kappa(r, Y_r^{\kappa,\gamma,\varepsilon})dr] = E[\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}\times F^{\alpha_\kappa,\gamma}[u_n](r, Y_r^{\kappa,\gamma,\varepsilon})dr]\\
&+(\varepsilon^2/2)E[\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}\sum_{\nu+1\leqq i\leqq d}\partial_i^2 u_n(r, Y_r^{\kappa,\gamma,\varepsilon})dr]\\
&\equiv I_1+I_2,
\end{aligned}
$$

where $L^\kappa(t, y)=L(\alpha_\kappa(t, y), t, y)$ and

(4.16) $\quad F^{\alpha_\kappa,\gamma}[u_n](t, y) = \partial_s u_n+(1/2)\sum_{1\leqq i,j\leqq\nu} a_{ij}^{\kappa,\gamma}(t, y)\partial_i\partial_j u_n$

$$+(b^{\kappa,\gamma}(t, y), \nabla u_n)-c^{\kappa}(t, y)u_n+L^{\kappa}(t, y) .$$

Now remark that there is a constant $N=N(k, \lambda)\geqq 0$ such that for all $(s, x) \in Q_T$, $n\geqq 1$, $\partial_i^2 u_n(s, x)\leqq N(1+|x|)^{\lambda}$. In fact, from the assumption (4.2) it is easily shown that for each $(s, x)$ and for any $\delta(0<\delta<1)$, $l\in R^d$ such that $|l|=1$, $D_{l,\delta}^2 u_n(s, x)\leqq N(1+|x|)^{\lambda}$, from which the assertion follows immediately by the fact that $u_n$ is smooth. Then we can conclude that

$$(4.17)\qquad I_2\leqq(N\varepsilon^2/2)E[\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}(1+|Y_r^{\kappa,\gamma,\varepsilon}|)^{\lambda}dr]\leqq\varepsilon^2 N\times(1+|x|)^{\lambda},$$

where $N=N(k, \lambda)\geqq 0$ is a constant.

Letting $n\to\infty$ in (4.15), then by virtue of the same reason as in the proof of Theorem 3.1, $u_n(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})\to u(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})$ $(n\to\infty)$ in $L_1(\Omega, \mathcal{F}, P)$. On the other hand, as for $I_1$, it is shown that $F^{\alpha_\kappa,\gamma}[u_n]\to F^{\alpha_\kappa,\gamma}[u]$ $(n\to\infty)$ in $L_{d+1}(Q)$, where $Q$ is any bounded region $Q_T$. Indeed, since $u\in W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T)$ $(p\geqq d+1)$, it is easily seen that $\phi_n(=u_n, \partial_s u_n, \partial_i u_n, 1\leqq i\leqq d, \partial_i\partial_j u_n, 1\leqq i, j\leqq\nu)$ converges to $\phi(=u, \partial_s u, \partial_i u, \partial_i\partial_j u$, respectively) in $L_{p,\mathrm{loc}}(Q_T)$. Since $(Y_t^{\kappa,\gamma,\varepsilon})$ is nondegenerate process and $b^{\kappa,\gamma}$ is bounded, it follows from N.V. Krylov ([6], Theorem 2.2.4, p. 54) that

$$E[\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}|F^{\alpha_\kappa,\gamma}[u_n]-F^{\alpha_\kappa,\gamma}[u]|(r, Y_r^{\kappa,\gamma,\varepsilon})dr]\leqq N\|F^{\alpha_\kappa,\gamma}[u_n]-F^{\alpha_\kappa,\gamma}[u]\|_{L_{d+1}(Q_{T,R})}$$

$\to 0$ $(n\to\infty)$, so long as $R$ is fixed, where $N=N(\gamma, \varepsilon, R)$ is a constant $\geqq 0$. Therefore, letting $n\to\infty$ in (4.15), we have the following:

$$(4.18)\qquad \begin{aligned}&E[u(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})\exp(-\tilde{\phi}^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})-u(s, x)+\\ &\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}L^{\kappa}(r, Y_r^{\kappa,\gamma,\varepsilon})dr]\leqq\\ &E[\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}F^{\alpha_\kappa,\gamma}[u]\,(r, Y_r^{\kappa,\gamma,\varepsilon})dr]+(\varepsilon^2 N)\,(1+|x|)^{\lambda}.\end{aligned}$$

Now remark that for each $R\geqq 0$ if $|x|\leqq R$, then $F^{\alpha_\kappa,\gamma}[u]\,(t, x)=F^{\alpha_\kappa}[u](t, x)$ for all $(t, x)\in Q_{T,R}$ for sufficiently large $\gamma$. From this we deduce the fact that $F^{\alpha_\kappa,\gamma}[u]\,(r, Y_r^{\kappa,\gamma,\varepsilon})=F^{\alpha_\kappa}[u]\,(r, Y_r^{\kappa,\gamma,\varepsilon})$ a.e. on $\{s<r<\tau_R^{\kappa,\gamma,\varepsilon}\}$. While, using again the result due to N.V. Krylov (see above), we can show from Lemma 4.2 that $F^{\alpha}[u]\,(r, Y_r^{\kappa,\gamma,\varepsilon})\leqq 1/\kappa$ a.e. on $\{s<r<\tau_R^{\kappa,\gamma,\varepsilon}\}$. Then by means of (4.18) we have the following: for sufficient large $\gamma$,

$$(4.19)\qquad \begin{aligned}&E[u(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})\exp(-\tilde{\phi}^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})+\\ &\int_s^{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}} e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}L^{\kappa}(r, Y_r^{\kappa,\gamma,\varepsilon})dr]\leqq u(s, x)+(T-s)/\kappa+(\varepsilon^2 N)\,(1+|x|)^{\lambda}.\end{aligned}$$

Since $P(\tau_R^{\kappa,\gamma,\varepsilon}\leqq T)=1$, the left side of (4.19) is equal to $E[u(t, Y_t^{\kappa,\gamma,\varepsilon})\times\exp(-\tilde{\phi}_t^{\kappa,\gamma,\varepsilon})+\int_s^t e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}L^{\kappa}(r, Y_r^{\kappa,\gamma,\varepsilon})\,dr]+E[u(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})\exp(-\tilde{\phi}^{\kappa,\gamma,\varepsilon}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}})$

$-u(t, Y_t^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_t^{\kappa,\gamma,\varepsilon});\ \tau_R^{\kappa,\gamma,\varepsilon}<T]+E[-\int_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^t e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}L^\kappa(r, Y_r^{\kappa,\gamma,\varepsilon})dr;\tau_R^{\kappa,\gamma,\varepsilon}<T]$

$\equiv I_1+I_2+I_3$. In the following we shall estimate two terms $I_2$ and $I_3$. For $I_2$, by virtue of Schwartz's inequality, we have the following:
$|I_2|\leqq\{E[|u(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})-u(t, Y_t^{\kappa,\gamma,\varepsilon})\times\exp(-\tilde{\phi}_t^{\kappa,\gamma,\varepsilon})|^2\}^{1/2}$ $P(\tau_R^{\kappa,\gamma,\varepsilon}<T)^{1/2}$. Note that $P(\tau_R^{\kappa,\gamma,\varepsilon}<T)\leqq k(1+|x|)/R$ and, furthermore, $|u(t\wedge\tau_R^{\kappa,\gamma,\varepsilon}, Y_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_{t\wedge\tau_R^{\kappa,\gamma,\varepsilon}}^{\kappa,\gamma,\varepsilon})-u(t,Y_t^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_t^{\kappa,\gamma,\varepsilon})|\leqq N(1+\sup_{s\leqq t\leqq T}|Y_t^{\kappa,\gamma,\varepsilon}|)^\lambda$,
due to (3.1), where $N=N(k,\lambda)$ is a constant$\geqq 0$. It follows from the above estimations and (3.5) that $|I_2|\leqq N(1+|x|)^{\lambda+(1/2)}/(R)^{1/2}$, where $N=N(k,\lambda)\geqq 0$. Because $L^\kappa(t, x)$ also satisfies (1.5), by the way similar to $I_2$ we can obtain the similar inequality relative to $I_3$ as $I_2$, so that we have the following:

$$\begin{aligned}(4.20)\qquad & E[u(t, Y_t^{\kappa,\gamma,\varepsilon})\exp(-\tilde{\phi}_t^{\kappa,\gamma,\varepsilon})+\int_s^t e^{-\tilde{\phi}_r^{\kappa,\gamma,\varepsilon}}L(r, Y_r^{\kappa,\gamma,\varepsilon})dr]\\ &\leqq u(s, x)+(T-s)/\kappa+\varepsilon^2N(1+|x|)^\lambda+N'(1+|x|)^{\lambda+(1/2)}/(R)^{1/2},\end{aligned}$$

for all $s\leqq t\leqq T$, $R>0$, sufficiently large $\gamma$, and $N'=N'(k,\lambda)$ is a constant$\geqq 0$. Letting $t=T$ and taking the infimum with respect to $\alpha\in\mathfrak{A}$ in (4.20), we have the following:

$$\begin{aligned}(4.21)\qquad & \inf_{\alpha\in\mathfrak{A}} E[h(X_{T-s}^{\alpha,\gamma,\varepsilon})\exp(-\phi_{T-s}^{\alpha,\gamma,\varepsilon})+\int_0^{T-s} e^{-\phi_r^{\alpha,\gamma,\varepsilon}}L(\alpha_r, s+r, X_r^{\alpha,\gamma,\varepsilon})dr]\\ &\leqq u(s, x)+(T-s)/\kappa+\varepsilon^2N(1+|x|)^\lambda+N'(1+|x|)^{\lambda+(1/2)}/(R)^{1/2},\end{aligned}$$

(recall that $u(T, x)\geqq h(x)$ for all $x\in R^d$), where $\phi_t^{\alpha,\gamma,\varepsilon}=\int_0^t c(\alpha_r, s+r, X_r^{\alpha,\gamma,\varepsilon})dr$, and $(X_t^{\alpha,\gamma,\varepsilon})$ is a solution of the following stochastic differential equation:

$$(4.22)\qquad dX_t=b^\gamma(\alpha_t, s+t, X_t)dt+\sigma^{\gamma,\varepsilon}(\alpha_t, s+t, X_t)dB_t,\ X_0=x\,.$$

In order to prove the theorem, we need further an auxiliary lemma:

**Lemma 4.4.** *Let $\varepsilon(0<\varepsilon<1)$ be fixed. For any $\alpha\in\mathfrak{A}, \gamma=1,2,\cdots$, let $(X_t^{\alpha,\gamma,\varepsilon})$ and $(X_t^{\alpha,\varepsilon})$ be unique solutions of* Eq. (4.22) *and* Eq. (2.2) *respectively. Then it holds that for each $\varepsilon>0$*

$$\begin{aligned}(4.23)\qquad & \lim_{\gamma\to\infty}\sup_{\alpha\in\mathfrak{A}} E[|h(X_{T-s}^{\alpha,\gamma,\varepsilon})\exp(-\phi_{T-s}^{\alpha,\gamma,\varepsilon})+\int_0^{T-s} e^{-\phi_r^{\alpha,\gamma,\varepsilon}}\times L(\alpha_r, s+r, X_r^{\alpha,\gamma,\varepsilon})dr\\ &-h(X_{T-s}^{\alpha,\varepsilon})\exp(-\phi_{T-s}^{\alpha,\varepsilon})-\int_0^{T-s} e^{-\phi_r^{\alpha,\varepsilon}}L(\alpha_r, s+r, X_r^{\alpha,\varepsilon})dr|]=0\,.\end{aligned}$$

Proof. Let's show at first that for each $\varepsilon>0$

$$(4.24)\qquad \lim_{\gamma\to\infty}\sup_{\alpha\in\mathfrak{A}} E[\sup_{0\leqq t\leqq T-s}|X_t^{\alpha,\gamma,\varepsilon}-X_t^{\alpha,\varepsilon}|^2]=0$$

For any $(\alpha, s, x)\in\mathfrak{A}\times\bar{Q}_T$ the difference $X_t^{\alpha,\gamma,\varepsilon}-X_t^{\alpha,\varepsilon}$ can be written as follows:

$$(4.25)\qquad X_t^{\alpha,\gamma,\varepsilon}-X_t^{\alpha,\varepsilon}=\int_0^t\{b^\gamma(\alpha_r,s+r,X_r^{\alpha,\gamma,\varepsilon})-b(\alpha_r,s+r,X_r^{\alpha,\varepsilon})dr\}+\int_0^t\{\sigma^\gamma(\alpha_r,s+r,X_r^{\alpha,\gamma,\varepsilon})-\sigma(\alpha_r,s+r,X_r^{\alpha,\varepsilon})\}dB_r,\ X_0^{\alpha,\gamma,\varepsilon}=X_0^{\alpha,\varepsilon}=x\,.$$

Since $|b^\gamma(\alpha,t,x)-b(\alpha,t,y)|+\|\sigma^\gamma(\alpha,t,x)-\sigma(\alpha,t,y)\|\leqq k\{|x-y|+|y||\eta_\gamma(x)-1|\}(\because|\eta_\gamma(x)|\leqq 1)$ for all $(\alpha,s,x,y)\in A\times[0,T]\times R^{2d}$ and $\gamma=1,2,\cdots$, it follows from (4.25) and the martingale inequality that

$$(4.26)\qquad E[\sup_{0\leqq r\leqq t}|X_r^{\alpha,\gamma,\varepsilon}-X_r^{\alpha,\varepsilon}|^2]\leqq N\{E[\int_0^t\sup_{0\leqq r'\leqq r}|X_{r'}^{\alpha,\gamma,\varepsilon}-X_{r'}^{\alpha,\varepsilon}|^2dr]+E[\int_0^T|X_r^{\alpha,\gamma,\varepsilon}|^2|\eta_\gamma(X_r^{\alpha,\varepsilon})-1|^2dr]\}\,,$$

where $N=N(k)\geqq 0$ is a constant. By means of Grownwell's inequality, we get the following:

$$(4.27)\qquad E[\sup_{0\leqq r\leqq t}|X_r^{\alpha,\gamma,\varepsilon}-X_r^{\alpha,\varepsilon}|^2]\leqq Ne^{N(T-s)}E[\int_0^{T-s}|X_r^{\alpha,\gamma,\varepsilon}|^2\times|\eta_\gamma(X_r^{\alpha,\varepsilon})-1|^2dr]\,.$$

Now by virtue of Schwarz's inequality and (1.6), it holds that $E[\int_0^{T-s}|X_r^{\alpha,\gamma,\varepsilon}|^2|\eta_\gamma(X_r^{\alpha,\varepsilon})-1|^2dr]\leqq N\{E[\int_0^{T-s}|X_r^{\alpha,\gamma,\varepsilon}|^4dr]\}^{1/2}\times\{\int_0^{T-s}P(|X_r^{\alpha,\varepsilon}|>\gamma)dr\}^{1/2}\leqq N(1+|x|)^3/\gamma^{1/2}$, where $N=N(k)$ is a constant. This implies immediately (4.24) because the right side of (4.27) is dominated by a function independent of $\alpha$. The assertion (4.23) follows immediately from (4.24) and the assumption (A.1) relative to $(L, c, h)$ (see (1.5) especially), whose details are routine works and omitted here (see, for examples, [2] or [6]). □

Letting $\gamma\to\infty$, $\kappa\to\infty$ and $R\to\infty$ in (4.21), and in view of Lemma 4.4 just proved above, we have the following:

$$(4.28)\qquad \inf_{\alpha\in\mathfrak{A}}E[h(X_{T-s}^{\alpha,\varepsilon})\exp(-\phi_{T-s}^{\alpha,\varepsilon})+\int_0^{T-s}e^{-\phi_r^{\alpha,\varepsilon}}L(\alpha_r,s+r,X_r^{\alpha,\varepsilon})dr]\leqq u(s,x)+\varepsilon^2N(1+|x|)^\lambda\,,$$

where $N=N(k,\lambda)\geqq 0$ is a constant. But if we recall that the left side of (4.28) is equal to $v^\varepsilon(s,x)$ (see (2.4)), then it holds that for all $(s,x)\in\bar{Q}_T$

$$(4.29)\qquad v^\varepsilon(s,x)\leqq u(s,x)+\varepsilon^2N(1+|x|)^\lambda\,.$$

Let $\varepsilon\to 0$ in (4.29). Then $v^\varepsilon(s,x)\to v(s,x)$, by virtue of Lemma 2.2 (a), and, on the other hand, the right side tends to $u(s,x)$. Thus the proof of the theorem is completed. □

By combining Theorem 3.1 with Theorem 4.1, we have the following uniqueness result concerning the Bellman equation (0.1), where $a=\sigma\sigma^*$ and $\sigma$ satisfies (1.21).

**Corollary 4.5.** *Let $u \in W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T) \cap C(\bar{Q}_T)$ $(p \geqq d+1)$ be such that $F[u](s,x)=0$ a.e. $(Q_T)$ and $u(T,x)=h(x)$ for all $x \in R^d$. Suppose that the function $u$ satisfies* (3.1), (4.2) *and that, furthermore, $\partial u/\partial x_i (1 \leqq i \leqq d)$ and $\partial^2 u/\partial x_i \partial x_j (1 \leqq i, j \leqq \nu)$ are locally bounded. Then $u=v$ on $\bar{Q}_T$.* □

Remark 4.1. The function $v$ also satisfies (4.2). Let's show this. Notice first the fact that for all $(s, x) \in \bar{Q}_T$, $0<\delta<1$, $l \in R^d$ such that $|l|=1$, $D^2_{l,\delta} v(s, x) \leqq \sup_{\alpha \in \mathfrak{A}} D^2_{l,\delta} v^{\alpha}(s, x)$. Since, for each $\alpha \in \mathfrak{A}$, $s \in [0, T]$, $v^{\alpha}(s, \cdot) \in C^2(Q_T)$ and there exists a constant $N=N(k, m)$ such that $v^{\alpha}_{(l)(l)}(s, x) \leqq N(1+|x|)^{3m}$ ([6], Lemma 4.2.2, p. 176), it follows from Taylor's expansion that $D^2_{l,\delta} v^{\alpha}(s, x)=\{v^{\alpha}_{(l)(l)}(s, \xi)+v^{\alpha}_{(l)(l)}(s, \xi')\}/2$, where $\xi=x+\theta l(0<\theta<\delta)$ and $\xi'=x-\theta' l(0<\theta'<\delta)$. From this it holds that $D^2_{l,\delta} v(s, x) \leqq N(1+|x|)^{3m}$, where $N=N(k, m)$ is a constant $\geqq 0$. In general, if $D^2_{l,\delta} u(s, x) \leqq 0$, then $u$ is a concave function with respect to $x$ ([9], p. 15), therefore we can also say that $v$ satisfies (4.2) if and only if Proposition 1.2 (c) is valid. □

Remark 4.2. In [6] (Theorem 5.3.14, p. 239), N.V. Krylov proved the uniqueness of solutions of the Bellman equation in the case where $u \in W^{1,2}_{p,\mathrm{loc}}(Q_T) \cap C(\bar{Q}_T)$. Although his method of proof is different from ours, it is applicable to our case if we modify it slightly so as to approximate $u$ by a sequence of smooth functions (c.f. [6]). □

## 5. The normed Bellman equation

In §1∼4 we studied about controlled processes on a finite interval under the conditions that the coefficients $\sigma^{\alpha}(t, x)$, $b^{\alpha}(t, x)$, $c^{\alpha}(t, x)$ and $L^{\alpha}(t, x)$ are bounded functions of $\alpha$ for each $(t, x)$. The objective of this paragraph is to carry the results obtained in §1∼4 over to controlled processes with coefficients unbounded with respect to $\alpha$. Although processes and cost functions of which we treat below are quite simple, it is not difficult to extend the results to general cases (see Remark 5.1 below). More general results about controlled processes with unbounded coefficients on an infinite interval will appear in a forthcoming paper.

Let $A$ be a separable metric space which is a countable union of non-empty increasing sets $A_n$: $A=\bigcup_{n=1}^{\infty} A_n$, $A_{n+1} \supset A_n$ (possively, $A_1=A_2= \cdots =A$) and we fix this representation throughout this section. For each $(t, x) \in \bar{Q}_T$ and $\alpha \in A$, we assume that the functions $\sigma(\alpha, t, x)$, $b(\alpha, t, x)$, $c(\alpha, t, x)$ and $L(\alpha, t, x)$ (but $h(x) \equiv 0$) have the same meanings as the functions given in §1 have. We always assume (1.9) and, also, assume that the functions $\sigma$ and $b$ are continuous with respect to $(\alpha, t, x)$ and, further, $\sigma(\alpha, t, x)$ does not depend on $x$. Moreover, let there exist a sequence of nonnegative constants $\{k_n\}$, $n=1, 2, \cdots$, such that for each $n \in \boldsymbol{N}$

$$
\text{(A.3)} \qquad \begin{cases} (5.1) & |b^{\alpha}(t, x)-b^{\alpha}(t, y)| \leqq k_n |x-y| , \\ (5.2) & \|\sigma^{\alpha}(t)\|+|b^{\alpha}(t, x)| \leqq k_n(1+|x|) , \end{cases}
$$

for all $x, y \in R^d$, $t \in [0, T]$ and $\alpha \in A_n$.

(A.4) We assume that the functions $L$ and $c$ are nonnegative and continuous with respect to $(\alpha, t, x)$, and, also, assume that $c(\alpha, t, x)$ does not depend on $(t, x)$. Moreover, let there exist a constant $m \geqq 0$ such that for all $n=1, 2, \cdots$,

$$
(5.3) \qquad L^{\alpha}(t, x)+c(\alpha) \leqq k_n(1+|x|)^m, \qquad \text{for all } (\alpha, t, x) \in A_n \times \bar{Q}_T .
$$

We also assume that for each $\alpha \in A$ the foregoing functions are continuously (in $(t, x)$) differentiable with respect to $t$, and twice continuously (in $(t, x)$) differentiable with respect to $x$, and, in addition, they are all bounded functions of $(t, x)$, that is, for all $(\alpha, t, x) \in A_n \times \bar{Q}_T$, $l \in R^d$, ($\gamma^{\alpha}=\sigma^{\alpha}$, $b^{\alpha}$, or $L^{\alpha}$)

$$
(5.4) \qquad |(\partial\gamma^{\alpha}/\partial t)(t, x)|+|\gamma^{\alpha}_{(l)}(t, x)|+|\gamma^{\alpha}_{(l)(l)}(t, x)| \leqq k_n .
$$

Finally, we assume that for all $(\alpha, t, x) \in A \times \bar{Q}_T$,

$$
(5.5) \qquad c^{\alpha} \geqq 8|\nabla_x b^{\alpha}(t, x)|+|\nabla_x L^{\alpha}(t, x)|^2 ,
$$

and, further, assume that there is a function $u(t, x) \geqq 0$, $\in C^{1,2}(\bar{Q}_T)$ such that

$$
(5.6) \qquad \sum_{i=1}^{d} |\partial_i L^{\alpha}(t, x)|^2+|\partial_s L^{\alpha}(t, x)|^2+|\partial_s b^{\alpha}(t, x)|^2+|\partial_s \sigma^{\alpha}(t)|^2+ \sum_{i,j=1}^{d} |\partial_i\partial_j b^{\alpha}(t, x)|^4+\sum_{i,j=1}^{d} |\partial_i\partial_j L^{\alpha}(t, x)|^2 \leqq -\mathcal{L}^{\alpha,\varepsilon}u(t, x) ,
$$

for all $(\alpha, t, x) \in A \times \bar{Q}_T$, $0<\varepsilon<1$, where $\mathcal{L}^{\alpha,\varepsilon}u(t, x) \equiv u_t+(1/2)\sum_{1 \leq i,j \leq \nu} a^{\alpha}_{ij}(t)u_{x_i x_j}+(\varepsilon^2/2)\sum_{i=\nu+1}^{d} u_{x_i x_i}+\sum_{i=1}^{d} b^{\alpha}_i(t, x)u_{x_i}-c^{\alpha}u$ .

DEFINITILN 5.1. Let $n \geqq 1$. We denote by $\mathfrak{A}_n$ a set of all strategies (in the sense of Definition 1.1) having values in $A_n$. Let $\mathfrak{A}=\bigcup_n \mathfrak{A}_n$ and the elements of a set $\mathfrak{A}$ are said to be *strategies*. □

Using the usual notations given in §1, now we put

$$
(5.7) \qquad v_n(s, x)=\inf_{\alpha \in \mathfrak{A}_n} v^{\alpha}(s, x) ,
$$

and

$$
(5.8) \qquad v(s, x)=\inf_{\alpha \in \mathfrak{A}} v^{\alpha}(s, x) \quad \text{(see (1.4))} .
$$

Note that we already studied about $v_n$, $n \geqq 1$, in §1~4, from the assumptions (A.3) and (A.4). It is easily shown that $v(s, x)$ is locally bounded over $\bar{Q}_T$ by the fact that $0 \leqq v(s, x) \leqq v_1(s, x) \leqq N(1+|x|)^m$ (see Proposition 1.2 (a)). Now we summarize some properties with respect to the functions $v_n$ and $v$.

**Proposition 5.1.** (a) $v_n(s, x)$ *is uniformly bounded, and also* (b) *equicontinuous in each cylinder* $\bar{Q}_{T,R}$. (c) $\lim_{n\to\infty} v_n(s, x)=v(s, x)$ *uniformly in each cylinder* $\bar{Q}_{T,R}$. (d) $v$ *is absolutely continuous in* $(s, x)$, *hence there are first-order generalized derivatives with respect to* $(s, x)$; $\partial v/\partial s$ *and* $\partial v/\partial x_i$, $1\leqq i\leqq d$. *Furthermore, the foregoing derivatives are bounded in each* $\bar{Q}_{T,R}$. (e) *There are second-order generalized derivatives*; $\partial^2 v/\partial x_i\partial x_j$, $1\leqq i, j\leqq \nu$, *which are also bounded in each* $\bar{Q}_{T,R}$. (f) *For all* $\alpha\in A$, $F^\alpha[v](s, x)\geqq 0$ *for a.e.* $(s, x)\in Q_T$.

Proof. (a) Since there are constants $N$, $m\geqq 0$ such that for all $(s, x)$ and $n\geqq 1$, $0\leqq v_n(s,x)\leqq v_1(s,x)\leqq N(1+|x|)^m$, $v_n(s,x)$ is clearly uniformly bounded in $\bar{Q}_{T,R}$. (b) In order that $v_n$ is equicontinuous in each $\bar{Q}_{T,R}$, it is sufficient to show that $\partial v_n/\partial x_i$, $1\leqq i\leqq d$, and $\partial v_n/\partial s$ are bounded uniformly with respect to $n\in \boldsymbol{N}$ and $(s, x)\in\bar{Q}_{T,R}$. Since $L^\alpha(t, x)$ and $X_t^{\alpha,s,x}$ are twice continuously differentiable with respect to $x$, for each $\alpha\in\mathfrak{A}_n$, $(s, x)\in\bar{Q}_{T,R}$, $1\leqq i\leqq d$,

$$\partial_i v^\alpha(s, x) = E\Big[\int_0^{T-s} e^{-\phi_t^\alpha}\sum_{j=1}^d \partial_j L^{\alpha_t}(s+t, X_t^{\alpha,s,x})\partial_i X_{t;j}^{\alpha;s,x}dt\Big]\,,$$

where $\phi_t^\alpha=\int_0^t c(\alpha_r)dr$, $\partial_j L^\alpha(s, \xi)=(\partial L^\alpha/\partial\xi_j)(s, \xi)$ and $\partial_i X_{t;j}^{\alpha;s,x}=\partial X_{t;j}^{\alpha;s,x}/\partial x_i$. By the assumption (A.4), we can obtain easily the inequality: $E[\exp(-\phi_t^\alpha)\sum_{i,j=1}^d |\partial_i X_{t;j}^{\alpha;s,x}|^2]\leqq N$, where $N=N(d)$ is a constant. It follows from (A.4) (5.5), (5.6) that for all $(\alpha, s, x)\in\mathfrak{A}_n\times Q_T$,

$$\begin{aligned}(5.9)\qquad &\sum_{i=1}^d |\partial_i v^\alpha(s, x)|^2\leqq E\Big[\int_0^{T-s} e^{-\phi_t^\alpha}\sum_{j=1}^d |\partial_j L^{\alpha_t}(s+t, X_t^{\alpha,s,x})|^2 dt\Big]\times\\ &E\Big[\int_0^{T-s} e^{-\phi_t^\alpha}\sum_{i,j=1}^d |\partial_i X_{t;j}^{\alpha;s,x}|^2 dt\Big]\leqq N\, E\Big[\int_0^{T-s} e^{-\phi_t^\alpha}\sum_{j=1}^d |\partial_j L^{\alpha_t}(s+t, X_t^{\alpha,s,x})|^2 dt\Big]\\ &\leqq N\, E[-u(T-s, X_{T-s}^{\alpha,s,x})\exp(-\phi_{T-s}^\alpha)+u(s, x)]\leqq Nu(s, x)\,.\end{aligned}$$

Further, since $L(\alpha, s, x)$ and $X_t^{\alpha,s,x}$ are continuously differentiable with respect to $s$, we have the following:

$$\begin{aligned}&\partial_s v^\alpha(s, x) = E\Big[\int_0^{T-s} e^{-\phi_t^\alpha}\{\partial_s L^{\alpha_t}(s+t, X_t^{\alpha,s,x})+\sum_{j=1}^d \partial_j L^{\alpha_t}(s+t, X_t^{\alpha,s,x})\times\\ &\partial_s X_{t;j}^{\alpha;s,x}\}dt-L^{\alpha_{T-s}}(T, X_{T-s}^{\alpha,s,x})\exp(-\phi_{T-s}^\alpha)\Big]\\ &\leqq E\Big[\int_0^{T-s} e^{-\phi_t^\alpha}\{\partial_s L^{\alpha_t}(s+t, X_t^{\alpha,s,x})+\sum_{j=1}^d \partial_j L^{\alpha_t}(s+t, X_t^{\alpha,s,x})\times\partial_s X_{t;j}^{\alpha;s,x}\}dt\Big]\,,\end{aligned}$$

from the assumption that $L\geqq 0$. By the way similar to $\nabla_x v$, it is shown from (A.4) that there is a function $u_1(s, x)\geqq 0$, $\in C^{1,2}(\bar{Q}_T)$ such that for all $(\alpha, s, x)\in\mathfrak{A}_n\times\bar{Q}_T$,

$$(5.10)\qquad \partial_s v^\alpha(s, x)\leqq u_1(s, x)\,.$$

Let's prove that $\partial_s v_n$ is also bounded from below. For each $n \geqq 1$ we proved that the following equation holds:

$$(5.11) \quad \inf_{\alpha \in A_n} \{\partial_s v_n + (1/2) \sum_{1 \leqq i,j \leqq \nu} a_{ij}^{\alpha}(s) \partial_i \partial_j v_n + \sum_{i=1}^{d} b_i^{\alpha}(s, x) \partial_i v_n - c^{\alpha} v_n + L^{\alpha}(s, x)\} = 0 \quad \text{a.e.} \ (Q_T),$$

(see Theorem 2.1). If we fix $n_0$ and $\alpha^* \in A_{n_0}$ arbitrarily, then $\alpha^* \in A_n$ for all $n \geqq n_0$, and it follows from (5.11) that for all $n \geqq n_0$,

$$(5.12) \quad \partial_s v_n \geqq -\{(1/2) \sum_{1 \leqq i,j \leqq \nu} a_{ij}^{\alpha^*}(s) \partial_i \partial_j v_n + \sum_{i=1}^{d} b_i^{\alpha^*}(s, x) \partial_i v_n - c^{\alpha^*} v_n + L^{\alpha^*}(s, x)\} \quad \text{a.e.} \ (Q_T).$$

Here note that $|v_n|$ and $|\partial_i v_n| (1 \leqq i \leqq d)$ are dominated by a locally bounded function uniformly with respect to $n$, and also that $\partial_i \partial_j v_n (1 \leqq i, j \leqq \nu)$ are dominated from above by a locally bounded function uniformly with respect to $n$. In fact, the first assertion has been proved in (a) above and (5.9). Next, it is shown from the assumptions (A.3) and (A.4) that there exists a function $u \geqq 0$, $\in C^{1,2}(\bar{Q}_T)$ such that for all $(\alpha, s, x) \in \mathfrak{A} \times Q_T$, $l \in R^d (|l| = 1)$,

$$(5.13) \quad |v_{(l)(l)}^{\alpha}(s, x)| \leqq u(s, x),$$

from which $\partial_i \partial_j v_n (1 \leqq i, j \leqq \nu)$ are dominated by the function $u$ from above. Hence, by (5.12), $\partial_s v_n$ is bounded from below uniformly with respect to $n$ and the assertion (b) is proved completely. (c) and (d) follow immediately from (b). Finally, we can show (e) and (f) simultaneously like Proposition 1.2 by taking into account the inequality (5.13) just mentioned above. That is, it suffices to only apply the method used for $v_n$ in Proposition 1.2 to $v$ on $Q_{T,R}$ for each $R > 0$. □

Let's consider whether the inverse inequality of ($f$) of Proposition 5.1 is also valid. For this purpose, let us introduce some notations used by N.V. Krylov ([6], Chap. 6, §3, p. 267). Let $m_\alpha(t, x)$ be a nonnegative function given for $\alpha \in A$, $0 \leqq t \leqq T$, $x \in R^d$, and define $G^{m_\alpha}$ by the formula:

$$(5.14) \quad G^{m_\alpha}(u_0, u_{ij}, u_i, u, t, x) = \inf_{\alpha \in A} m_\alpha(t, x) \{u_0 + (1/2) \sum_{1 \leqq i,j \leqq \nu} a_{ij}^{\alpha}(t) u_{ij} + \sum_{i=1}^{d} b_i^{\alpha}(t, x) u_i - c^{\alpha} u + L^{\alpha}(t,x)\}.$$

DEFINITION 5.2. A nonnegative function $m_\alpha(t, x)$ over $A \times [0, T] \times R^d$ is said to be a *normalizing multiplier* if for all $u_0$, $u_{ij}$, $u_i$, $u$, $t \in [0, T]$, $x \in R^d$,

$$(5.15) \quad G^{m_\alpha}(u_0, u_{ij}, u_i, u, t, x) > -\infty. \quad □$$

The normalizing multiplier $m_\alpha(t, x)$ is called *regular* if there exists a function $N(t, x) < \infty$ such that for all $(\alpha, t, x)$ the inequality $m_{\alpha 0}(t, x) \leqq N(t, x) m_\alpha(t, x)$

holds, where

$$(5.16)\qquad m_{\alpha 0}(t, x) = \{1+(1/2)\sum_{1\leqq i,j\leqq \nu}|a_{ij}^{\alpha}(t)|^2+\sum_{i=1}^{d}|b_i(t, x)|^2+ |c^{\alpha}|^2+|L^{\alpha}(t, x)|^2\}^{-1/2}.$$

Our object of this paragraph is to prove the following result on the normed Bellman equation.

**Theorem 5.2.** *Let $m_{\alpha}(t, x)$ be a regular normalizing multiplier. Then it holds that*

$$(5.17)\qquad G^{m_{\alpha}}[v](t, x) = 0 \quad \text{a.e.}\ (Q_T). \quad \square$$

Note, however, that Lemmas 6.3.6~8 of [6] still hold in our case. Therefore, in order to obtain Theorem 5.2, it suffices to show the following:

**Lemma 5.3.** *If $m_{\alpha}(t, x)=m_{\alpha 0}(t, x)$, then the equality* (5.17) *is valid.* $\square$

To prove this we shall use the usual notations given in §2 for controlled processes and cost functions. Let $\varepsilon$ be an arbitrary number such that $0<\varepsilon<1$, and for each $\varepsilon$, $(\alpha, s, x)\in A\times\bar{Q}_T$, let $\sigma^{\alpha,\varepsilon}$ be the same meanings as the function given in (2.1). For each $\alpha\in\mathfrak{A}$, $(s, x)\in\bar{Q}_T$, let $(X_t^{\alpha,s,x,\varepsilon})$ and $(v^{\alpha,\varepsilon})$ be given in (2.2) and (2.3) respectively. Remark that for each $(\alpha, s, x, \varepsilon)$ the existence of the process $(X_t^{\alpha,s,x,\varepsilon})$ follows from (A.3), and the finiteness of $v^{\alpha,\varepsilon}$ follows from (A.4). For each $n\geqq 1$, define $v_n^{\varepsilon}$ and $v^{\varepsilon}$ by the following:

$$(5.18)\qquad \begin{cases} v_n^{\varepsilon}(s, x) = \inf\limits_{\alpha\in\mathfrak{A}_n} v^{\alpha,\varepsilon}(s, x), \quad \text{and} \\ v^{\varepsilon}(s, x) = \inf\limits_{n\geqq 1} v_n^{\varepsilon}(s, x). \end{cases}$$

Note that $v_n^{\varepsilon}$ and $v^{\varepsilon}$ are obtained if we only replace $\mathfrak{A}$ by $\mathfrak{A}_n$ and $\mathfrak{A}=\bigcup_{n\geqq 1}\mathfrak{A}_n$ in (2.4) respectively. In this connection, we have the following:

**Proposition 5.4.** (a) *$v_n^{\varepsilon}$ is uniformly (in $(\varepsilon, n)$) bounded, and also equicontinuous in $(s, x)$ uniformly with respect to $\varepsilon$ in each cylinder $\bar{Q}_{T,R}$.* (b) *For each $\varepsilon>0$, $\lim_{n\to\infty} v_n^{\varepsilon}(s, x)=v^{\varepsilon}(s, x)$ uniformly in each $\bar{Q}_{T,R}$ and $v^{\varepsilon}$ is continuous in $(s, x)$.* (c) *For each $\varepsilon>0$, $n\geqq 1$, $v_n^{\varepsilon}\in W_p^{1,2}(Q)$ and $v^{\varepsilon}\in W_p^{1,2}(Q)$ for any bounded subregion $Q\subset Q_T$, $p\geqq 1$. Moreover, their first-order generalized derivatives with respect to $s$ and $x_i$ $(1\leqq i\leqq d)$, and second-order generalized derivatives with respect to $x_ix_j$ $(1\leqq i, j\leqq \nu)$ are locally bounded in $Q_T$ uniformly with respect to $(\varepsilon, n)$.* (d) *$\lim_{\varepsilon\to 0} v^{\varepsilon}(s, x)=v(s, x)$, whose convergence is uniform in each cylinder $\bar{Q}_{T,R}$.* $\square$

Proof. (a) The first assertion is obvious from the fact that $0\leqq v_n^{\varepsilon}(s, x)\leqq v_1^{\varepsilon}(s, x)\leqq N(1+|x|)^m$, where $N=N(k, m)$ is a constant $\geqq 0$ (see (2.5)), for all $(s, x, \varepsilon, n)$. The second one can be obtained from the assumptions (5.5) and (5.6) by the same method as Proposition 5.1. (b) and (c) are shown similarly. Let us prove (d). In the equality:

$$(5.19) \qquad v^{\varepsilon}(s, x)-v(s, x) = \{v^{\varepsilon}(s, x)-v_n^{\varepsilon}(s, x)\}+\{v_n^{\varepsilon}(s, x)-v_n(s, x)\} \\ +\{v_n(s, x)-v(s, x)\} = I_1+I_2+I_3 ,$$

we want to estimate $I_2$.

$$|I_2| \leqq \sup_{\alpha\in\mathfrak{A}_n} |v^{\alpha,\varepsilon}(s, x)-v^{\alpha}(s, x)| = \sup_{\alpha\in\mathfrak{A}_n} |E[\int_0^{T-s} e^{-\phi_t^{\alpha}}\{L^{\alpha_t}(s+t, X_t^{\alpha,s,x,\varepsilon}) \\ -L^{\alpha_t}(s+t, X_t^{\alpha,s,x})\}dt]| \\ = \sup_{\alpha\in\mathfrak{A}_n} |E[\int_0^{T-s} e^{-\phi_t^{\alpha}}dt\int_0^1(\nabla_x L^{\alpha_t}(s+t, \tilde{X}_t^{\lambda}), X_t^{\alpha,s,x,\varepsilon}-X_t^{\alpha,s,x})d\lambda]| \\ \leqq \sup_{\alpha\in\mathfrak{A}_n} \{E[\int_0^{T-s} e^{-\phi_t^{\alpha}}c(\alpha_t)dt]\}^{1/2}\{E[\int_0^{T-s} e^{-\phi_t^{\alpha}}|X_t^{\alpha,s,x,\varepsilon}-X_t^{\alpha,s,x}|^2dt]\}^{1/2} ,$$

where $\tilde{X}_t^{\lambda}=\lambda X_t^{\alpha,s,x,\varepsilon}+(1-\lambda)X_t^{\alpha,s,x}$. Here we used Hadamard's inequality, Schwarz's inequality and (5.5). Clearly the first part of the right side is bounded uniformly with respect to $n$, while for the second one, by using also (5.5), we can show that there is a constant $N=N(d, T)\geqq 0$ such that for all $(\alpha, s, x)\in \mathfrak{A}\times Q_T$, $E[\int_0^{T-s}\exp(-\phi_t^{\alpha})|X_t^{\alpha,s,x,\varepsilon}-X_t^{\alpha,s,x}|^2dt]\leqq N\varepsilon^2$. From these results we can conclude that for some constant $N=N(d, T)$, $|I_2|\leqq\varepsilon N$, for all $(s, x)$. Let $\varepsilon>0$ fix and $n\to\infty$. Then $I_1$ and $I_3$ converge to 0 uniformly in each cylinder $\bar{Q}_{T,R}$, by means of (b) just mentioned above, and Proposition 5.1 (c) respectively. Next, letting $\varepsilon\to 0$ in (5.19), we get the assertion (d). □

We denote by $F_n^{\varepsilon}(u_o, u_{ij}, u_i, s, x)$ and $G_n^{m_\alpha}(u_o, u_{ij}, u_i, u, s, x)$ the right side in (2.9) and (5.14) respectively if we replace $A$ by $A_n$. Moreover, define $G^{m_\alpha,\varepsilon}$ by the formula:

$$(5.20) \qquad G^{m_\alpha,\varepsilon}(u_o, u_{ij}, u_i, u, s, x) = \inf_{\alpha\in A} m_\alpha(s, x)\ \{u_o+(1/2)\sum_{1\leqq i,j\leqq\nu} a_{ij}^{\alpha}(s)u_{ij} \\ +(\varepsilon^2/2)\sum_{i=\nu+1}^{d} u_{ii}+\sum_{i=1}^{d} b_i^{\alpha}(s, x)u_i-c^{\alpha}u+L^{\alpha}(s, x)\} .$$

Then we have the following:

**Lemma 5.5.** *For each $\varepsilon>0$ it holds that*

$$(5.21) \qquad G^{m_{\alpha 0},\varepsilon}[v^{\varepsilon}](s, x) = 0 \quad \text{a.e. } (Q_T) .$$

Proof. It follows from Lemma 2.2 (c) that for all $\varepsilon$ and $n$, $F_n^{\varepsilon}[v_n^{\varepsilon}](s, x) = 0$ a.e. $(Q_T)$. Since $A=\bigcup_{n\geqq 1} A_n$ and $A_n\subset A_{n+1}$, for each $n_0$, for all $n\geqq n_0$, $F_{n_0}^{\varepsilon}[v_n^{\varepsilon}](s, x)\geqq 0$ a.e. $(Q_T)$. Moreover, clearly for all $n\geqq 1$, $F^{\varepsilon}[v_n^{\varepsilon}](s, x)\leqq 0$ a.e. $(Q_T)$. From the fact that $m_{\alpha 0}$ is nonnegative and bounded ($\leqq 1$), it also holds that for each $n_0$, for all $n\geqq n_0$, $G_{n_0}^{m_{\alpha 0},\varepsilon}[v_n^\varepsilon]\geqq 0$ a.e., and for all $n\geqq 1$, $G^{m_{\alpha 0},\varepsilon}[v_n^\varepsilon]$ $(s, x)\leqq 0$ a.e. $(Q_T)$. We take now the limit as $n\to\infty$ for each $\varepsilon>0$. Since the functions $m_{\alpha 0}a^\alpha$, $m_{\alpha 0}b^\alpha$, $m_{\alpha 0}c^\alpha$ and $m_{\alpha 0}L^\alpha$ are uniformly bounded, it follows from Theorem 4.5.1 of [6] (see also Lemma 2.5 of §2) that $0\leqq G_{n_0}^{m_{\alpha 0},\varepsilon}[v^\varepsilon](s, x)$ and $G^{m_{\alpha 0},\varepsilon}[v^\varepsilon](s, x)\leqq 0$ a.e. $(Q_T)$. Since $n_0$ was an arbitrary number $\geqq 1$, letting $n_0\uparrow\infty$, we obtain the equality (5.21). □

We further take the limit in (5.21) as $\varepsilon\to 0$. To show Lemma 5.3 we also use the same transformation of variables as (2.14). For each $\varepsilon>0$ and each $\xi\in R^{d-\nu}$, we define new variables $(s, y)\in\bar{Q}_T$ and a function $\psi^\varepsilon(s, y)$ by (2.14), where in this case $v^\varepsilon$ is given by (5.18). If we change the variables $(s, x)$ into $(s, y)$ in (5.21), then it holds that

$$(5.22)\qquad 0=\inf_{\alpha\in A}\tilde{m}_{\alpha 0}^\varepsilon(s, y)\,\{\psi_s^\varepsilon+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^\alpha(s)\psi_{y_iy_j}^\varepsilon+ (1/2)\sum_{i=\nu+1}^d\psi_{y_iy_i}^\varepsilon+\tilde{\mathcal{X}}^{\alpha,\varepsilon}(s, y)-c^\alpha\psi^\varepsilon\}\quad\text{a.e. }(s, y)\in Q_T\,,$$

where $\tilde{\mathcal{X}}^{\alpha,\varepsilon}(s, y)$ is given in (2.16), and

$$(5.23)\qquad \tilde{m}_{\alpha 0}^\varepsilon(s, y)=m_{\alpha 0}(s, x)\,(\equiv m_{\alpha 0}(s, \bar{y}, \varepsilon\hat{y}+\xi))\,.$$

Here it is easy to see that for all $\xi$, $\varepsilon$, $\alpha$ and $(s, y)$, $0<\tilde{m}_{\alpha 0}^\varepsilon(s, y)\leqq 1$, therefore, the value in parentheses of (5.22) is nonnegative for each $\alpha\in A$. By the way similar to Lemma 2.3 we obtain the following:

**Lemma 5.6.** (a) *For each $\varepsilon>0$, $\psi^\varepsilon\in W_{p,\mathrm{loc}}^{1,2}(Q_T)$, $p\geqq 1$.* (b) *The function $\psi^\varepsilon$ itself and its generalized derivatives $\psi_s^\varepsilon$, $\psi_{y_i}^\varepsilon(1\leqq i\leqq d)$, $\psi_{y_iy_j}^\varepsilon(1\leqq i, j\leqq d)$ are locally bounded in each cylinder $\bar{Q}_{T,R}$ uniformly with respect to $\varepsilon$.* (c) *For any $(s, y)\in\bar{Q}_T$, $\lim_{\varepsilon\to 0}\psi^\varepsilon(s, y)=v(s, \bar{y}, \xi)$ $(y=(\bar{y}, \hat{y})\in R^d)$, whose convergence is uniform in each cylinder $\bar{Q}_{T,R}$.* □

Proof of Lemma 5.3. It is sufficient to show that $G^{m_{\alpha 0}}[v]\leqq 0$ a.e., by virtue of Proposition 5.1 (f). Like the proof of Theorem 2.1, we rewrite (5.22) as follows:

$$(5.24)\qquad 0\geqq\tilde{G}^{\tilde{m}_{\alpha 0}}[\psi^\varepsilon](s, y)+\sum_{i=1}^2\tilde{f}_i^\varepsilon(s, y)\quad\text{a.e. }(Q_T)\,,$$

where $\tilde{G}^{\tilde{m}_{\alpha 0}}(u_o, u_{ij}, u, s, y)=\inf_{\alpha\in A}\tilde{m}_{\alpha 0}(s, y)\,\{u_o+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^\alpha(s)u_{ij}+ (1/2)\sum_{i=\nu+1}^d u_{ii}-c^\alpha u+\tilde{\mathcal{X}}^\alpha(s, y)\}$, $\tilde{m}_{\alpha 0}(s, y)=m_{\alpha 0}(s, \bar{y}, \xi)$,

$\tilde{\mathcal{X}}^\alpha(s, y)=\mathcal{X}^\alpha(s, \bar{y}, \xi)$ (see (2.23)),

$\tilde{f}_1^\varepsilon(s, y)=\inf_{\alpha\in B}\{\tilde{m}_{\alpha 0}^\varepsilon(s, y)-\tilde{m}_{\alpha 0}(s, y)\}\,\{\psi_t^\varepsilon+(1/2)\sum_{1\leqq i,j\leqq\nu}a_{ij}^\alpha(s)\psi_{y_iy_j}^\varepsilon+ (1/2)\sum_{i=\nu+1}^d\psi_{y_iy_i}^\varepsilon-c^\alpha\psi^\varepsilon\}$,

and

$$\tilde{f}_2^{\varepsilon}(s, y) = \inf_{\alpha\in A}\{\tilde{m}_{\alpha 0}^{\varepsilon}(s, y)\tilde{X}^{\alpha,\varepsilon}(s, y) - \tilde{m}_{\alpha 0}(s, y)\tilde{X}^{\alpha}(s, y)\} .$$

Let us prove that for any $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ $(n\to\infty)$, $\lim_{\varepsilon_n\to 0} \tilde{f}_i^{\varepsilon_n}(s, y)=0$ $(i=1, 2)$ a.e. in each cylinder $\bar{Q}_{T,R}$. It is easily seen from the definitions of $\tilde{m}_{\alpha 0}^{\varepsilon}$ and $\tilde{m}_{\alpha 0}$ that

$$(5.25)\qquad |\tilde{f}_1^{\varepsilon}(s, y)| \leq \sup_{\alpha\in A}|m_{\alpha 0}(s, \bar{y}, \varepsilon\hat{y}+\xi) - m_{\alpha 0}(s, \bar{y}, \xi)|\{|\psi_s^{\varepsilon}| + (1/2)\sum_{1\leq i,j\leq\nu}|a_{ij}^{\alpha}(s)||\psi_{y_iy_j}^{\varepsilon}| + (1/2)\sum_{i=\nu+1}^{d}|\psi_{y_iy_i}^{\varepsilon}| + |c^{\alpha}\psi^{\varepsilon}|\} .$$

If we note that $\partial_j m_{\alpha 0}(s, x) = -(1/2)\{m_{\alpha 0}(s, x)\}^3\{2\sum_{i=1}^{d} b_i^{\alpha}(s, x)\times\partial_j b_i^{\alpha}(s, x) + 2L^{\alpha}(s,x)\partial_j L^{\alpha}(s,x)\}$, then $|\partial_j m_{\alpha 0}(s,x)| \leq \{m_{\alpha 0}(s,x)\}^2\times\sum_{i=1}^{d}|\partial_j b_i^{\alpha}(s,x)| + |\partial_j L^{\alpha}(s,x)|\}$. Since $m_{\alpha 0}(s, x)|\partial_j b_i^{\alpha}(s, x)|$ and $m_{\alpha 0}(s, x)|\partial_j L^{\alpha}(s, x)|$ are bounded functions($\leq 1$) uniformly with respect to $(\alpha, s, x)$ from the assumption (5.5), we can conclude that

$$(5.26)\qquad |\partial_j m_{\alpha 0}(s, x)| \leq N\, m_{\alpha 0}(s, x), \qquad \text{for all } (\alpha, s, x) ,$$

for some constant $N\geq 0$. Hence from (5.25) and (5.26) it holds that

$$(5.27)\qquad |\tilde{f}_1^{\varepsilon}(s, y)| \leq N\varepsilon|\hat{y}|\sup_{\alpha\in A} m_{\alpha 0}(s, \bar{y}, \theta\hat{y}+\xi)\{|\psi_s^{\varepsilon}| + (1/2)\sum_{1\leq i,j\leq\nu}|a_{ij}^{\alpha}(s)|\times|\psi_{y_iy_j}^{\varepsilon}| + (1/2)\sum_{i=\nu+1}^{d}|\psi_{y_iy_i}^{\varepsilon}| + |c^{\alpha}\psi^{\varepsilon}|\} ,$$

where $0<\theta<\varepsilon<1$. Since $\psi_s^{\varepsilon}$, $\psi_{y_iy_j}^{\varepsilon}$, $\psi_{y_iy_i}^{\varepsilon}$ and $\psi^{\varepsilon}$ are locally bounded functions uniformly with respect to $\varepsilon$, by means of Lemma 5.6 (b), it follows immediately from (5.27) that $\lim_{\varepsilon\to 0}\tilde{f}_1^{\varepsilon}(s, y)=0$ a.e. in each cylinder $\bar{Q}_{T,R}$. From the same reason as $\tilde{f}_1^{\varepsilon}$, in order to prove that $\lim_{\varepsilon_n\to 0}\tilde{f}_2^{\varepsilon_n}(s, y)=0$, it is sufficient to show that $\lim_{\varepsilon_n\to 0}|\nabla v^{\varepsilon_n}(s, \bar{y}, \varepsilon_n\hat{y}+\xi) - \nabla v(s, \bar{y}, \xi)|=0$. But this can be obtained by the way similar to Lemma 2.4, because $v$ is represented as difference of two convex functions (with respect to $x$) and, hence, $v$ is once differentiable with respect to almost all $x$ (c.f. Proposition 1.2 (b) and (c)).

On the other hand, in the same way as we prove Lemma 2.5 of §2, we can show that $\overline{\lim_{\varepsilon\to 0}}\, \tilde{G}^{\tilde{m}_{\alpha 0}}[\psi^{\varepsilon}](s, y) \geq \tilde{G}^{\tilde{m}_{\alpha 0}}[\psi](s, y)$ a.e., where $\psi(s, y)\equiv\lim_{\varepsilon\to 0}\psi^{\varepsilon}(s, y)$ (cf. Theorem 4.5.1, Lemma 6.3.5 of [6]). To complete the proof it remains only to take the limit in (5.24) as $\varepsilon_n\to 0$, then we get the inequality:

$$(5.28)\qquad 0\geq\tilde{G}^{\tilde{m}_{\alpha 0}}[\psi](s, y) \quad \text{a.e.}.$$

Since $\tilde{G}^{\tilde{m}_{\alpha 0}}[\psi](s, y) = G^{m_{\alpha 0}}[v](s, \bar{y}, \xi)$ (see Lemma 5.6 (c)) and $\xi$ was arbitrarily fixed, we can conclude that $G^{m_{\alpha 0}}[v](s, x)\leq 0$. a.e. $(Q_T)$. □

Now we give two examples in which the assumptions of Theorem 5.2 are verified easily.

EXAMPLE 5.1. Let $d\geqq 2$ and let $\gamma_i(\gamma_1=\sigma,\ \gamma_2=b,\ \gamma_3=L)$ be function of the following type: $\gamma_i^\alpha(t, x)=\gamma_i(\alpha)\gamma_i(t, x)$. Assume that $\gamma_i(\alpha)$ $(i=1, 2, 3)$ is continuous function in $A$ and, for any $n\geqq 1$, $|\gamma_i(\alpha)|\leqq k_n$ for all $\alpha\in A_n$ (see (A.3)). As for $\gamma_i(t, x)$, it is assumed that $|\gamma_1(t)|\leqq k$, $|\gamma_2(t, x)|\leqq k(1+|x|)$, and $|\gamma_3(t, x)|\leqq k(1+|x|)^m$ for all $(t, x)\in\bar{Q}_T$, where $k$ and $m$ are nonnegative constants. Moreover, for each $i=1, 2, 3$, $\alpha\in A$, let $\gamma_i^\alpha\in C^{1,2}(\bar{Q}_T)$, and let all of their first (in $(t, x)$) and second (in $x$) derivatives be dominated by a constant $k$ in absolute value when $\alpha$ is fixed. Concerning $c^\alpha$, we assume that for all $\alpha \in A$,

$$(5.29) \qquad c^\alpha\geqq 8k|\gamma_2(\alpha)|+k|\gamma_3(\alpha)|^2,\ (k\geqq 1) \quad \text{and}$$

$$(5.30) \qquad c^\alpha\geqq |\gamma_1(\alpha)|^2+|\gamma_2(\alpha)|^2+|\gamma_2(\alpha)|^4+3|\gamma_3(\alpha)|^2 .$$

It is easy to see that, in this case, (5.5) and (5.6) are derived immediately from (5.29) and (5.30) respectively. In fact, if we put $u(t, x)=N$, where $N$ is a constant such that $N\geqq k^4$, then the left side of (5.6) is less than $Nc^\alpha$, while it holds that $Nc^\alpha=-\mathcal{L}^{\alpha,\varepsilon}u(t, x)$ for all $(\alpha, t, x)\in A\times\bar{Q}_T$, $1>\varepsilon>0$, from which (5.6) follows. □

EXAMPLE 5.2. Let $d\geqq 2$, $A=R^d$ and $A_n=\{\alpha\in R^d;\ |\alpha|\leqq n\}$. Assume that $\sigma^\alpha(t)$ is independent of $\alpha$, $b^\alpha(t, x)=\alpha$ and $L^\alpha(t, x)=|\alpha|^2$. Note that in this case the assumptions of Theorem 5.2 are clearly satisfied and that, further, we can take $c^\alpha=0$. Then it follows from Theorem 5.2 that $v$ of (5.8) satisfies the normed Bellman equation:

$$(5.31) \qquad G^{m_\alpha}[v]\,(t, x) = \inf_{\alpha\in R^d} m_\alpha(t, x)\,\{v_t+(1/2)\sum_{1\leqq i,j\leqq \nu} a_{ij}(t)v_{x_ix_j} + \sum_{i=1}^d \alpha_i v_{x_i}+|\alpha|^2\} = 0 \quad \text{a.e.}.$$

Note also that $m_\alpha(t, x)\equiv 1$ (constant) is regular normalized multiplier so that $G^1[v]\,(t, x)=0$ a.e. Now it is easy to see that for all $\alpha\in R^d$, $\sum_{i=1}^d \alpha_i v_{x_i}+|\alpha|^2\geqq -\sum_{i=1}^d (v_{x_i})^2/4$ and the equality holds if and only if $\alpha_i=-(v_{x_i})/2$, $1\leqq i\leqq d$, which enables us to assure the existence of a generalized solution of Eq. (0.7). Remark that this result is also correct in the case where $L^\alpha(t, x)$ is written as $L^\alpha(t, x)=|\alpha|^2+\tilde{L}(t, x)$, where $\tilde{L}\geqq 0$, $\in C^{1,2}(\bar{Q}_T)$ and its derivatives $\tilde{L}_t$, $\tilde{L}_x$ and $\tilde{L}_{xx}$ are uniformly bounded. □

REMARK 5.1. It is not difficult to extend the results to the case where the coefficients $\sigma^\alpha$, $b^\alpha$, $c^\alpha$, $L^\alpha$ and $h$ satisfy more general conditions than (5.1)$\sim$(5.4). Indeed, for example, assume that $h\not\equiv 0$ and for each $\alpha\in A$, $\gamma^\alpha(=\sigma^\alpha$, $b^\alpha$, $c^\alpha$, $L^\alpha$ and $h)\in C^{1,2}(\bar{Q}_T)$ such that its derivatives $\gamma_t^\alpha$, $\gamma_x^\alpha$, and $\gamma_{xx}^\alpha$ satisfy the polynomial growth condition. Then, in order that Theorem 5.2 holds, it is sufficient to take $c^{\alpha}(t, x)$ and $u(s, x)$ such that (5.5) and (5.6) hold (including the derivatives of $\sigma^{\alpha}$), and that, moreover, $u(t, x)=k(1+|x|)^{2m}$. The computations, however, are quite complicate in that case (cf. Chap. 6, §2 of [6]). □

REMAPK 5.2. Finally, we shall state about the uniqueness of solutions of the normed Bellman equation (5.17). It is easy to obtain the result corresponding to Theorem 3.1, that is, suppose that $u$ is a function in $W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T)$ $\cap C(\bar{Q}_T)$ such that $|u(t, x)|\leqq k(1+|x|)^m$, $u_{x_i}$ and $u_{x_ix_j}$ are locally bounded, and, also, suppose that for all regular normalizing multiplier $m_{\alpha}$, $G^{m_{\alpha}}[u](t, x)\geqq 0$ a.e., $u(T, x)\leqq h(x)$. Then $u(t, x)\leqq v(t, x)$ in $\bar{Q}_T$. On the other hand, in general, it is difficult to show the inverse relation (i.e. corresponding to Theorem 4.1) except the case where 1 is regular normalizing multiplier (see Example 5.2 and also [6], p. 272, Excercise 10), even then it is necessary to assume some suitable conditions such as (5.5) and (5.6). □

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