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Control of PDE–ODE cascades with Neumann interconnections

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Abstract

We extend several recent results on full-state feedback stabilization and state estimation of PDE–ODE cascades, where the PDEs are either of heat type or of wave type, from the previously considered cases where the interconnections are of Dirichlet type, to interconnections of Neumann type. The Neumann type interconnections constrain the PDE state to be subject to a Dirichlet boundary condition at the PDE–ODE interface, and employ the boundary value of the first spatial derivative of the PDE state to be the input to the ODE. In addition to considering heat-ODE and wave-ODE cascades, we also consider a cascade of a diffusion–convection PDE with an ODE, where the convection direction is "away" from the ODE. We refer to this case as a PDE–ODE cascade with "counter-convection." This case is not only interesting because the PDE subsystem is unstable, but because the control signal is subject to competing effects of diffusion, which is in both directions in the one-dimensional domain, and counter-convection, which is in the direction that is opposite from the propagation direction of the standard delay (transport PDE) process. We rely on the diffusion process to propagate the control signal through the PDE towards the ODE, to stabilize the ODE.

Keywords: Delay systems; Distributed parameter systems; Backstepping; Stability; Cascade systems

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1. Introduction

We consider cascade connections of PDEs, whose states are denoted as $u(x,t)$, where $t \geq 0$ is time and $x \in [0,D]$ is the spatial domain, and ODEs whose states are denoted as $X(t) \in \mathbb{R}^n$. ODE systems of the form

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{1}$$

with PDE systems of the form

$$u_t = u_x, \tag{2}$$

which is a delay/transport PDE, were considered in [3,12,13] and more recently in [9,10]. The PDE system

$$u_t = u_{xx} \tag{3}$$

is a heat PDE and it was considered in [7]. Finally, the PDE system

$$u_{tt} = u_{xx} \tag{4}$$

is a wave PDE and it was considered in [8]. In each of the three PDE models, (2)–(4), the control enters through a boundary condition,

$$u(D,t) = U(t), \tag{5}$$

namely, at the end $x = D$ of the domain $[0,D]$, which is opposite from the end $x = 0$ where the PDE and ODE connect.

The PDE state enters the ODE (1) through the variable $u(0,t)$, which we refer to as a Dirichlet interconnection. Unlike this interconnection, which was studied in [7,8,10], the "Neumann interconnection,"

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{6}$$

has not been studied yet. In this paper we study heat and wave PDEs connected in cascade with an ODE, via a Neumann interconnection. As in [7,8,10], the only assumption we impose is that (A,B) is a controllable pair.

We provide infinite-dimensional full-state feedback laws with explicit gain kernels that compensate the PDE dynamics and achieve stabilization of the PDE–ODE system. The key tool in this work is the continuum version of backstepping method [1,2,4,5,11,16,18] which employs infinite-dimensional transformations for the design of the controller and Lyapunov functions for the stability proof.

Next, we briefly review the existing results with Dirichlet interconnections. ODE systems with input delay were considered in [10]. For the system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{7}$$

where D is an arbitrarily long delay, the backstepping approach yields the controller

$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\sigma)}BU(\sigma)\,d\sigma\right]. \tag{8}$$

This infinite-dimensional feedback law turns out to be identical to the classical predictor and finite-spectrum assignment feedback laws [3,12,13] and it represents a "delay-compensated" version of the nominal controller

$$U(t) = KX(t), \tag{9}$$

where the bracketed term in (8) is a D-seconds ahead predictor of $X(t)$, starting from $X(t)$ as an initial condition, and driven by the input history over the D-second window. While recovering the classical designs [3,12,13], the backstepping approach [10] also provides the first construction of a Lyapunov function for the predictor feedback. The Lyapunov function allows various robustness studies, which were conducted in [6].

In [7] a PDE–ODE cascade structure is studied, but for diffusive dynamics at the input of the ODE, namely, for the heat PDE in cascade with an ODE, with a Dirichlet interconnection:

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{10}$$

$$u_t(x,t) = u_{xx}(x,t), \tag{11}$$

$$u_x(0,t) = 0, \tag{12}$$

$$u(D,t) = U(t). \tag{13}$$

In this case, the backstepping approach yields the controller

$$U(t) = K[I \ \ 0]\left\{ e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} D} \begin{bmatrix} I \\ 0 \end{bmatrix} X(t) + \int_0^D \left(\int_0^{D-y} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} \sigma} d\xi \right) \begin{bmatrix} B \\ 0 \end{bmatrix} u(y)\, dy \right\}. \tag{14}$$

Finally, in [8], a wave PDE in cascade with an ODE, namely, the system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{15}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16}$$

$$u_x(0,t) = 0, \tag{17}$$

$$u(D,t) = U(t), \tag{18}$$

is studied, with a Dirichlet interconnection at the PDE–ODE interface. The controller for the system (15)–(18) is given explicitly in [8], however, we do not repeat it here because of its complexity. The difference between the systems (10)–(13) and (15)–(18) may appear subtle, because it is only a difference of one time derivative between the heat equation (11) and the wave equation (16). However, this difference is very significant and it results in vastly different challenges in the control design and in the final feedback formulae.

In this paper we make two significant changes. First, we replace the Neumann boundary conditions (12) and (17) by the Dirichlet boundary condition

$$u(0,t) = 0. \tag{19}$$

Second, we replace the plant (1), which is driven by $u(0,t)$, by the plant (6), which is driven by the non-zero signal $u_x(0,t)$. This is a very significant change in the model and is the basis of the contribution of this paper. Physically, the meaning of this change is the following. For example, while in the case of heat equation dynamics, the problem considered in [7] assumed that the ODE was actuated by temperature, in the present paper we consider actuation by heat flux.

Under these two changes we consider the problems of stabilization of the system (6), (19), (5) with the heat equation (11) and the wave equation (16). In addition, we study a special problem of a diffusion–convection PDE

$$u_t = u_{xx} - bu_x, \tag{20}$$

where for $b>0$ the term $-bu_x$ has the effect of "counter-convection," namely an effect which opposes the propagation of the control signal $U(t)$ from $x = D$ to 0. We rely on the presence of diffusion in (20) to achieve stabilization of the (u, X) system in the presence of counter-convection.

This paper employs the PDE backstepping method, which was initially developed in a spatially discretized setting [4,5] and has since evolved in a spatially continuous setting [14,11] for various applications, including fluid flows [1,2].

The backstepping method provides feedback transformations which convert the closed-loop system into a cascade of an exponentially stable PDE and an exponentially stable ODE. Cascades of asymptotically stable systems have been the focus of much research in nonlinear systems and control since the proof that a cascade of a globally asymptotically stable system and an input-to-state stable (ISS) system is globally asymptotically stable [17, Proposition 7.2]. Such results unfortunately cannot be generalized to PDEs because stability of PDE–ODE and PDE–PDE cascades depends on the types of interconnections and boundary conditions and on the norms used in the stability study. It is for this reason that all the stability results are developed without reliance on off-the-shelf stability theorems.

The control problems for PDE–ODE cascades considered in the paper are motivated by various applications in chemical process control, combustion, and other areas. The motivation for the diffusion–counterconvection PDE (20) can be provided in the context of water channel flows actuated downhill from the area where one wants to influence the flow. However, a motivation can also be provided from vehicle traffic flow, where the convection is the result of the vehicle motion, diffusion is the result of individual drivers maintaining a safe distance with the cars immediately ahead and behind them, and the main control action is applied by speed control (for example by variable speed limit signs) rather than by the more conventional ramp traffic lights, which allows the control action to propagate in the counter-convective direction, affecting the "upstream" traffic.

The paper is organized as follows. In Section 2 we consider a cascade with a heat PDE and present a compensation design that guarantees exponential stability for the closed-loop system. A simulation example is also presented. We then proceed with a more complex design which yields an arbitrarily fast stabilization rate for the closed-loop system. In Section 3 we deal with the problem of robustness of our feedback law for the heat PDE with respect to uncertainty in the diffusion coefficient and we establish a robustness margin for small (positive or negative) perturbations in the diffusion coefficient. In Section 4 we consider the dual problem of infinite-dimensional diffusive dynamics in the sensor instead of the actuator, under Neumann boundary measurement. We design an observer that compensates these dynamics and guarantees stability for the error system. A simulation example is presented with an observer given explicitly. In Section 5 we consider a diffusion–convection PDE and derive a controller and an observer for the respective PDE–ODE and ODE–PDE cascades. The cascade studied in Section 5 is a generalization of the case analyzed in [7]. Finally, in Section 6 we consider a cascade with a wave equation at the input and design an exponentially stabilizing controller.

2. The heat-ODE cascade: full-state feedback under Neumann interconnection

Consider the cascade of a heat equation and an LTI finite-dimensional system

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{21}$$

$$u_t(x,t) = u_{xx}(x,t), \tag{22}$$

$$u(0,t) = 0, \tag{23}$$

$$u(D,t) = U(t), \tag{24}$$

where $X(t) \in \mathbb{R}^n$ is the ODE state, $u(x,t)$ the state of the diffusive dynamics of the actuator, $U(t)$ is the scalar input of the entire system and $D>0$ is the length of the PDE domain. The cascade (21)–(24) is depicted in Fig. 1.

Theorem 2.1. *Consider a closed-loop system consisting of the plant* (21)–(24) *and the control law*

$$U(t) = K[0_n \;\; I_n]\left\{ e^{\begin{bmatrix} 0_n & A \\ I_n & 0_n \end{bmatrix} D} \begin{bmatrix} I_n \\ 0_n \end{bmatrix} X(t) + \int_0^D e^{\begin{bmatrix} 0_n & I_n \\ A & 0_n \end{bmatrix}(D-y)} \begin{bmatrix} I_n \\ 0_n \end{bmatrix} Bu(y)\,dy \right\}. \tag{25}$$

For any initial condition such that $u_x(x,0)$ is square integrable in x and compatible with the control law (25), *the closed-loop system has a unique classical solution and is exponentially stable in the sense of the norm*

$$\left(|X(t)|^2 + \int_0^D u_x(x,t)^2\,dx \right)^{1/2}. \tag{26}$$

Proof. We postulate an infinite-dimensional backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)\,dy - \gamma(x)X(t), \tag{27}$$

with kernels $q(x,y)$ and $\gamma(x)$ to be derived, which should transform (21)–(24) into the "target system"

$$\dot{X}(t) = (A+BK)X(t) + Bw_x(0,t), \tag{28}$$

$$w_t(x,t) = w_{xx}(x,t), \tag{29}$$

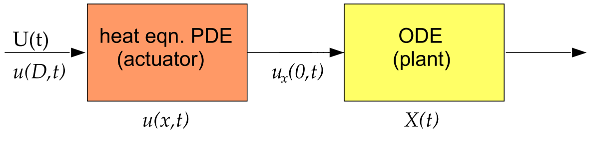

Fig. 1. The cascade of the heat PDE dynamics and the ODE plant.

$$w(0,t) = 0, \tag{30}$$

$$w(D,t) = 0. \tag{31}$$

The nominal control gain K is chosen to make $A+BK$ Hurwitz. This may be done with the LQR/Riccati approach, pole placement, or some other method. We first derive the kernels and then show that the target system is exponentially stable. In order to derive the unknown functions we differentiate $w(x,t)$ as defined in (27) twice with respect to x,

$$w_x(x,t) = u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)\,dy - \gamma'(x)X(t), \tag{32}$$

$$\begin{aligned} w_{xx}(x,t) = u_{xx}(x,t) &- (q(x,x))'u(x,t) - q(x,x)u_x(x,t) - q_x(x,x)u(x,t) \\ &- \int_0^x q_{xx}(x,y)u(y,t)\,dy - \gamma''(x)X(t), \end{aligned} \tag{33}$$

and once with respect to t,

$$w_t(x,t) = u_t(x,t) - \int_0^x q(x,y)u_t(y,t)\,dy - \gamma(x)(AX(t)+Bu_x(0,t)) \tag{34}$$

$$= u_{xx}(x,t) - \int_0^x q(x,y)u_{yy}(y,t)\,dy - \gamma(x)(AX(t)+Bu_x(0,t)) \tag{35}$$

$$\begin{aligned} &= u_{xx}(x) - q(x,x)u_x(x) + [q(x,0)-\gamma(x)B]u_x(0) + q_y(x,x)u(x) \\ &\quad - \int_0^x q_{yy}(x,y)u(y)\,dy - \gamma(x)AX(t). \end{aligned} \tag{36}$$

Evaluating the backstepping transformation (27) and (32) in $x=0$ and exploiting the diffusion equation (29) we get

$$w(0,t) = -\gamma(0)X(t), \tag{37}$$

$$w_x(0,t) = u_x(0,t) - \gamma'(0)X(t), \tag{38}$$

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) = 2(q(x,x))'u(x,t) &+ [q(x,0)-\gamma(x)B]u_x(0) \\ &+ \int_0^x [q_{xx}(x,y)-q_{yy}(x,y)]u(y)\,dy + [\gamma(x)''-A\gamma(x)]X(t), \end{aligned} \tag{39}$$

where we had employed the Dirichlet boundary condition $u(0,t)=0$. A sufficient condition for (29)–(31) to hold for any continuous function $u(x,t)$ and $X(t)$ is that the unknown functions satisfy the following set of conditions:

$$\gamma(x)'' = A\gamma(x), \tag{40}$$

$$\gamma(0) = 0, \tag{41}$$

$$\gamma(0)' = K, \tag{42}$$

$$q_{xx}(x,y) = q_{yy}(x,y), \tag{43}$$

$$q(x,x) = 0, \tag{44}$$

$$q(x,0) = \gamma(x)B, \tag{45}$$

which are a second order ODE in x and a hyperbolic PDE of second order. The solution to the ODE (40)–(42) is

$$\gamma(x) = KM(x) \tag{46}$$

$$= K[0 \;\; I]e^{\left[\begin{smallmatrix} 0 & A \\ I & 0 \end{smallmatrix}\right]x}\begin{bmatrix} I \\ 0 \end{bmatrix} \tag{47}$$

while the solution to the PDE (43)–(45) is

$$q(x,y) = m(x-y) \tag{48}$$

$$= KM(x-y)B, \tag{49}$$

where we have introduced the functions $m(\cdot)$ and $M(\cdot)$ in order to a have more compact notation in the continuation of the proof. It is straightforward to prove that the backstepping transformation is invertible. In a manner similar to the construction of the direct backstepping transformation, we obtain the inverse change of variables

$$u(x,t) = w(x,t) + \int_0^x n(x-y)w(y,t)\,dy + KN(x)X(t), \tag{50}$$

where

$$N(\xi) = [0 \;\; I]e^{\left[\begin{smallmatrix} 0 & A+BK \\ I & 0 \end{smallmatrix}\right]\xi}\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{51}$$

$$n(\xi) = KN(\xi)B. \tag{52}$$

Now we proceed with proving exponential stability. We consider the Lyapunov function

$$V = X^T PX + \frac{a}{2}\|w\|^2 + \frac{b}{2}\|w_x\|^2, \tag{53}$$

where the quantities $\|w(t)\|^2$ and $\|w_x(t)\|^2$ represent compact notation for the L_2 norms $\int_0^D w(x,t)^2\,dx$ and $\int_0^D w_(x,t)^2\,dx$, respectively, the matrix $P = P^T > 0$ is the solution of the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q \tag{54}$$

for some $Q = Q^T > 0$, and the parameters $a > 0$ and $b > 0$ will be chosen later. From

$$w_x(x) = u_x(x) - \int_0^x m_x(x-y)u(y)\,dy - KM'(x)X(t), \tag{55}$$

$$u_x(x) = w_x(x) + \int_0^x n_x(x-y)w(y)\,dy + KN'(x)X(t) \tag{56}$$

it can be shown that

$$\|w_x\|^2 \leq \alpha_1\|u_x\|^2 + \alpha_2\|u\|^2 + \alpha_3|X|^2, \tag{57}$$

$$\|u_x\|^2 \leq \beta_1\|w_x\|^2 + \beta_2\|w\|^2 + \beta_3|X|^2, \tag{58}$$

where

$$\alpha_1 = 3, \tag{59}$$

$$\alpha_2 = 3D\|m_x\|^2, \tag{60}$$

$$\alpha_3 = 3\|KM'\|^2, \tag{61}$$

$$\beta_1 = 3, \tag{62}$$

$$\beta_2 = 3D\|n_x\|^2, \tag{63}$$

$$\beta_3 = 3\|KN'\|^2. \tag{64}$$

In the same manner from the direct (27) and the inverse (50) backstepping transformations we get that

$$\|w\|^2 \leq \alpha_4\|u\|^2 + \alpha_5|X|^2, \tag{65}$$

$$\|u\|^2 \leq \beta_4\|w\|^2 + \beta_5|X|^2, \tag{66}$$

where

$$\alpha_4 = 3(1 + D\|m\|^2), \tag{67}$$

$$\alpha_5 = 3\|KM\|^2, \tag{68}$$

$$\beta_4 = 3(1 + D\|n\|^2), \tag{69}$$

$$\beta_5 = 3\|KN\|^2. \tag{70}$$

From (57) and (65) we get that

$$V \leq \lambda_{\max}(P)|X|^2 + \frac{a}{2}(\alpha_4\|u\|^2 + \alpha_5|X|^2) + \frac{b}{2}(\alpha_1\|u_x\|^2 + \alpha_2\|u\|^2 + \alpha_3|X|^2). \tag{71}$$

Applying Poincaré's inequality we get

$$V \leq \overline{\delta}(\|X\|^2 + \|u_x\|^2), \tag{72}$$

where

$$\overline{\delta} = \max\left\{\lambda_{\max}(P) + \frac{a\alpha_5}{2} + \frac{b\alpha_3}{2}, \frac{b}{2}\alpha_1 + 2D^2(\alpha_4 a + \alpha_2 b)\right\}. \tag{73}$$

From (58) and (66) we get that

$$\begin{aligned} V &\geq \lambda_{\min}(P)\|X\|^2 + \frac{a}{2}\|w\|^2 \\ &\quad + \frac{b}{2}\|w_x\|^2 \geq \frac{\min\{\lambda_{\min}(P), a, b\}}{2\max\{\beta_1, \beta_2, \beta_3\}}(\beta_1\|X\|^2 + \beta_2\|w\|^2 + \beta_3\|w_x\|^2) \\ &\quad + \frac{\lambda_{\min}(P)}{2}\|X\|^2 \geq \underline{\delta}(\|u_x\|^2 + \|X\|^2), \end{aligned} \tag{74}$$

where

$$\underline{\delta} = \frac{1}{2}\min\left\{\frac{\min\{\lambda_{\min}(P), a, b\}}{\max\{\beta_1, \beta_2, \beta_3\}}, \lambda_{\min}(P)\right\}. \tag{75}$$

So we have that

$$\underline{\delta}(\|u_x\|^2+\|X\|^2)\leq V\leq\overline{\delta}(\|u_x\|^2+\|X\|^2). \tag{76}$$

Taking the time derivative of the Lyapunov function along the solution of the PDE–ODE system (28)–(31), we get

$$\dot{V}=-X^TQX+2X^TPBw_x(0)-a\|w_x\|^2-b\|w_{xx}\|^2\leq-\frac{\lambda_{\min}(Q)}{2}|X|^2+\frac{2|PB|^2}{\lambda_{\min}(Q)}w_x(0)^2-a\|w_x\|^2-b\|w_{xx}\|^2. \tag{77}$$

With Agmon's inequality it can be proved that for the system (28)–(31) the following inequality holds:

$$-\|w_{xx}\|^2\leq\frac{1+D}{D}\|w_x\|^2-w_x(0)^2, \tag{78}$$

hence we have that

$$\dot{V}\leq-\frac{\lambda_{\min}(Q)}{2}|X|^2-\left(b-\frac{2|PB|^2}{\lambda_{\min}(Q)}\right)w_x(0)^2-\left(a-b\frac{1+D}{D}\right)\|w_x\|^2. \tag{79}$$

Taking now

$$b\geq\frac{2|PB|^2}{\lambda_{\min}(Q)}, \tag{80}$$

$$a\geq b\frac{1+D}{D} \tag{81}$$

we get that $\dot{V}<0$. From (79)

$$\dot{V}\leq-\frac{\lambda_{\min}(Q)}{2}|X|^2-\left(a-b\frac{1+D}{D}\right)\|w_x\|^2\leq-\frac{\lambda_{\min}(Q)}{2}|X|^2-\frac{4D^2}{1+4D^2}\left(a-b\frac{1+D}{D}\right)\|w_x\|^2 \tag{82}$$

$$-\frac{1}{1+4D^2}\left(a-b\frac{1+D}{D}\right)\|w_x\|^2. \tag{83}$$

Applying again Poincaré's inequality we obtain

$$\dot{V}\leq-\frac{\lambda_{\min}(Q)}{2}|X|^2-\frac{1}{1+4D^2}\left(a-b\frac{1+D}{D}\right)(\|w\|^2+\|w_x\|^2)\leq-\gamma V, \tag{84}$$

where, exploiting the fact that we had chosen (81),

$$\gamma=\min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)},\frac{2}{1+4D^2}\left(1-\frac{b}{a}\frac{1+D}{D}\right)\right\}. \tag{85}$$

Hence,

$$|X(t)|^2+\|u_x(t)\|^2\leq\frac{\overline{\delta}}{\underline{\delta}}e^{-\gamma t}(|X(0)|^2+\|u_x(0)\|^2) \tag{86}$$

for all $t \geq 0$, where $\overline{\delta}, \underline{\delta}$ are defined in (73), (75) and their role is displayed in (76). This completes the proof. □

Example 2.1. We now give an example where the controller has an explicit form. Consider the scalar unstable system

$$\dot{X}(t) = X(t) + u_x(0,t),$$

with a heat equation actuator dynamics:

$$u_t(x,t) = u_{xx}(x,t),$$

$$u(0,t) = 0,$$

$$u(D) = U(t).$$

We choose the feedback gain $K = -(1+h)$ with $h > 0$, in order to have $A + BK$ Hurwitz, and obtain the backstepping controller (25) in the form

$$U(t) = -(1+h)\left[\cosh(D)X(t) + \int_0^D \cosh(D-y)u(y)\,dy\right]. \tag{87}$$

In Fig. 2 we show the simulation results with $h = 3$, $D = 1$ and with initial conditions $u(x,0) = 0$ and $X(0) = 1$.

The convergence rate for the closed-loop system is determined by the union of the eigenvalues of the ODE plant and of the eigenvalues of the heat equation (29)–(31). While the controllable eigenvalues of the ODE can be placed at desirable locations by the vector K, the heat equation, though exponentially stable, may not necessarily have a fast decay. Its decay rate is limited by its first eigenvalue, $-(\pi/D)^2$. The PDE actuator dynamics can be sped up by a modified controller given in the next theorem.

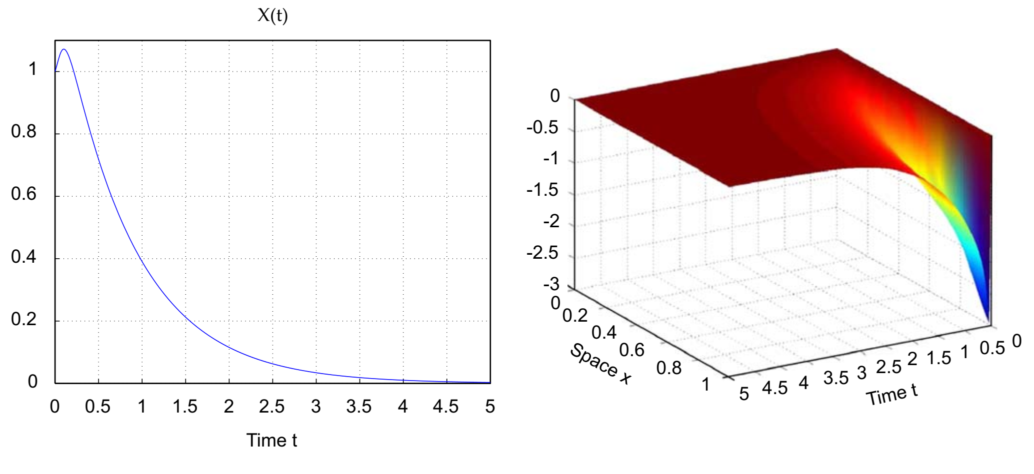

Fig. 2. The closed-loop solutions for the heat-ODE cascade in Example 2.1.

Theorem 2.2. *Consider a closed-loop system consisting of the plant* (21)–(24) *and the control law*

$$U(t) = \phi(D)X(t) + \int_0^D \Psi(D,y)u(y,t)\,dy, \tag{88}$$

where

$$\phi(x) = KM(x) + \int_0^D k(x,y)KM(y)\,dy, \tag{89}$$

$$\Psi(x,y) = k(x,y) + KM(x-y)B - \int_y^x k(x,\xi)\phi(\xi - y)B\,d\xi, \tag{90}$$

$$k(x,y) = -cy\frac{I_1(\sqrt{c(x^2-y^2)})}{\sqrt{c(x^2-y^2)}}, \quad c>0, \tag{91}$$

and I_1 denotes the modified Bessel function of order one. For all initial conditions such that $u_x(x,0)$ is square integrable in x and compatible with the control law (88), *the closed-loop system has a unique classical solution and its eigenvalues are given by*

$$\mathrm{eig}\{A+BK\} \cup \mathrm{eig}\left\{-c - \frac{\pi^2 n^2}{D^2},\ n = 1,2,\ldots\right\}. \tag{92}$$

Proof. The proof is developed in the same manner as the correspondent proof in [7]. Consider a second, invertible change of variables

$$z(x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)\,dy, \tag{93}$$

which aims to map system (28)–(31) into a new target system:

$$\dot{X}(t) = (A+BK)X(t) + Bz_x(0,t), \tag{94}$$

$$z_t(x,t) = z_{xx}(x,t) - cz(x,t), \tag{95}$$

$$z(0,t) = 0, \tag{96}$$

$$z(D,t) = 0. \tag{97}$$

It was shown in [14,11] that the kernel function $k(x,y)$ must satisfy the PDE:

$$k_{xx}(x,y) - k_{yy}(x,y) = ck(x,y), \tag{98}$$

$$k(x,x) = -\frac{c}{2}x, \tag{99}$$

$$k(x,0) = 0. \tag{100}$$

The solution to Eq. (98)–(100) is given by Eq. (91). Taking the composition of the two backstepping transformation (27) and (93), we obtain that

$$z(x) = u(x) - \int_0^x KM(x-y)Bu(y)\,dy - KM(x)X - \int_0^x k(x,y)u(y)\,dy + \int_0^x k(x,y)KM(y)\,dyX + \int_0^x k(x,y)\int_0^y KM(y-\xi)u(\xi)\,d\xi\,dy. \tag{101}$$

Setting $x = D$, employing the boundary conditions $u(D,t) = U(t)$, $z(D,t) = 0$, and the calculus formula for changing the order of integration

$$\int_0^x \int_0^y f(x,y,\xi)\,d\xi\,dy = \int_0^x \int_\xi^x f(x,y,\xi)\,dy\,d\xi, \tag{102}$$

yields to the controller (88)–(90). With a calculation of the eigenvalues of Eqs. (94)–(97), we get the set (92). □

3. Robustness to diffusion uncertainty

We study the robustness of the controller (25) to perturbations in the diffusion coefficient. Consider the following system:

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{103}$$

$$u_t(x,t) = (1+\varepsilon)u_{xx}(x,t), \tag{104}$$

$$u_x(0,t) = 0, \tag{105}$$

$$u(D,t) = \int_0^D m(D-y)u(y,t)\,dy + KM(D)X(t), \tag{106}$$

where ε represents a perturbation in the diffusion coefficient of the heat PDE. The perturbation ε can be either positive or negative, but it has to be small.

Theorem 3.1. *Consider the closed-loop system consisting of the plant* (103)–(106) *and the control law* (25). *There exists a sufficiently small* $\varepsilon^* > 0$ *such that, for all* $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$, *the closed-loop system has a unique classical solution for all compatible initial conditions* $u(x,0)$ *such that* $u_x(x,0)$ *is square integrable in* x, *and is exponentially stable in the sense of the norm*

$$(|X(t)|^2 + \|u(t)\|^2 + \|u_x(t)\|^2)^{1/2}. \tag{107}$$

Proof. By differentiating the transformation

$$w(x,t) = u(x,t) - \int_0^x m(x-y)u(y,t)\,dy - KM(x)X(t), \tag{108}$$

and substituting into Eqs. (103)–(106), it can be verified that

$$\dot{X}(t) = (A+BK)X(t) + Bw_x(0,t), \tag{109}$$

$$w_t(x,t) = (1+\varepsilon)w_{xx}(x,t) + \varepsilon KM(x)[(A+BK)X(t) + Bw_x(0,t)], \tag{110}$$

$$w(0,t)=0, \tag{111}$$

$$w(D,t)=0. \tag{112}$$

Consider again the Lyapunov function (53). The derivative of V along the solutions of system (109)–(112) is

$$\begin{aligned}\dot V \le & -\frac{\lambda_{\min}(Q)}{2}|X|^2 \\ & +\frac{2|PB|^2}{\lambda_{\min}(Q)}w_x(0)^2 - a(1-|\varepsilon|)\|w_x\|^2 - b(1-|\varepsilon|)\|w_{xx}\|^2 \\ & +\varepsilon a\int_0^D w(x)KM(x)\,dx[(A+BK)X(t)+Bw_x(0)] \\ & -\varepsilon b\int_0^D w_{xx}(x)KM(x)\,dx[(A+BK)X(t)+Bw_x(0)]\end{aligned} \tag{113}$$

$$\begin{aligned}\le & -\frac{\lambda_{\min}(Q)}{4}|X|^2 - \left(a-\frac{1+D}{D}\frac{b}{2}\right)\|w_x\|^2 \\ & -\left[\frac{b}{2}-\frac{2|PB|^2}{\lambda_{\min}(Q)}-|\varepsilon|(a+b)\right]w_x(0)^2-\frac{b}{2}\|w_{xx}\|^2 \\ & +a|\varepsilon|\left\{1+4\|\mu_1\|+a|\varepsilon|\frac{8\|\mu_2\|^2}{\lambda_{\min}(Q)}\right\}\|w_x\|^2 \\ & +b|\varepsilon|\left\{1+\|\mu_1\|+b|\varepsilon|\frac{2\|\mu_2\|^2}{\lambda_{\min}(Q)}\right\}\|w_{xx}\|^2,\end{aligned} \tag{114}$$

where

$$\mu_1(x)=KM(x)B, \tag{115}$$

$$\mu_2(x)=|KM(x)|. \tag{116}$$

In the second inequality we have split the term $b\|w_{xx}\|^2$ in half, used the inequality (78), and employed Young's and Poincaré inequalities. Choosing for example

$$a=\frac{1+D}{D}\frac{8|PB|^2}{\lambda_{\min}(Q)}, \tag{117}$$

$$b=\frac{8|PB|^2}{\lambda_{\min}(Q)} \tag{118}$$

it is possible to select $|\varepsilon|$ sufficiently small to achieve negative definiteness of $\dot V$. □

4. Observer for the ODE-heat cascade with Neumann measurement

Consider the cascade depicted in Fig. 3 and governed by the equations

$$Y(t)=u_x(0,t), \tag{119}$$

$$u_t(x,t)=u_{xx}(x,t), \tag{120}$$

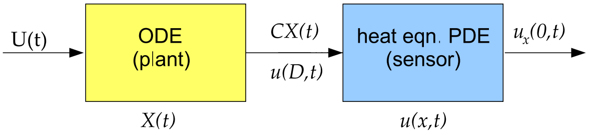

Fig. 3. The cascade of the ODE plant and the heat equation PDE dynamics.

$$u(0,t)=0, \tag{121}$$

$$u(D,t)=CX(t) \tag{122}$$

$$\dot{X}(t)=AX(t)+BU(t), \tag{123}$$

that is, the case where an ODE system has a diffusive sensor acting on the output map $CX(t)$, with Neumann measurement $u_x(0,t)$. We now present an observer, inspired by [15], which compensates the sensor dynamics and guarantees exponential stability of the error system.

Theorem 4.1. *The observer*

$$\hat{u}_t(x,t)=\hat{u}_{xx}(x,t)+CM(x)[Y(t)-\hat{u}_x(0,t)], \tag{124}$$

$$\hat{u}(0,t)=0, \tag{125}$$

$$\hat{u}(D,t)=C\hat{X}(t), \tag{126}$$

$$\dot{\hat{X}}(t)=A\hat{X}(t)+BU(t)+M(D)L[Y(t)-\hat{u}_x(0,t)], \tag{127}$$

where L is chosen such that $(A-LC)$ is Hurwitz, guarantees that the error system

$$\tilde{u}_t(x,t)=\tilde{u}_{xx}(x,t)-CM(x)L\tilde{u}_x(0,t), \tag{128}$$

$$\tilde{u}(0,t)=0, \tag{129}$$

$$\tilde{u}(D,t)=C\tilde{X}(t), \tag{130}$$

$$\dot{\tilde{X}}(t)=A\tilde{X}(t)-M(D)L\tilde{u}_x(0,t), \tag{131}$$

where $\tilde{X}=X-\hat{X}$ and $\tilde{u}=u-\hat{u}$, is exponentially stable in the sense of the norm

$$\left(|\tilde{X}(t)|^2+\int_0^D \tilde{u}_x(x,t)^2\,dx\right)^{1/2}. \tag{132}$$

Proof. Consider the backstepping transformation

$$\tilde{w}(x)=\tilde{u}(x)-CM(x)M(D)^{-1}\tilde{X} \tag{133}$$

to map the error system into a new PDE system. Differentiating the transformation (133) with respect to time and space

$$\tilde{w}_t(x)=\tilde{u}_t(x)-CM(x)M(D)^{-1}A\tilde{X}(t)+CM(x)L\tilde{u}_x(0), \tag{134}$$

$$\tilde{w}_x(x) = \tilde{u}_x(x) - CM'(x)M(D)^{-1}\tilde{X}(t), \tag{135}$$

$$\tilde{w}_{xx}(x) = \tilde{u}_{xx}(x) - CM(x)AM(D)^{-1}\tilde{X}(t). \tag{136}$$

Then evaluating (135) in $x = 0$ and using the initial condition

$$M'(0) = I_n \tag{137}$$

and the plant equation (131), we obtain

$$\dot{\tilde{X}}(t) = (A - M(D)LCM(D)^{-1})\tilde{X}(t) - M(D)L\tilde{w}_x(0,t). \tag{138}$$

Summarizing we get the following PDE:

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t), \tag{139}$$

$$\tilde{w}(0,t) = 0, \tag{140}$$

$$\tilde{w}(D,t) = 0, \tag{141}$$

$$\dot{\tilde{X}}(t) = (A - M(D)LCM(D)^{-1})\tilde{X}(t) + M(D)L\tilde{w}_x(0,t). \tag{142}$$

The matrix $[A - M(D)LCM(D)^{-1}]$ is Hurwitz, which can be seen with a similarity transformation $M(D)$, which commutes with A thanks to the properties of the matrix exponential. Now consider the Lyapunov function

$$V = \tilde{X}^T M(D)^{-T} P M(D)^{-1}\tilde{X} + \frac{a}{2}\|\tilde{w}\|^2 + \frac{b}{2}\|\tilde{w}\|^2, \tag{143}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A - LC) + (A - LC)^T P = -Q \tag{144}$$

for some $Q = Q^T > 0$, and the parameters $a > 0$ and $b > 0$ will be chosen later. From this point on, we develop the proof using similar ideas as in the proof of Theorem 2.1. We first compute the time-derivative of function V,

$$\dot{V} = -\tilde{X}^T M(D)^{-T} Q M(D)^{-1}\tilde{X} + 2\tilde{X}^T PL\tilde{w}_x(0) + \frac{a}{2}\int_0^D \frac{d}{dt}(\tilde{w}(x)^2)\,dx + \frac{b}{2}\int_0^D \frac{d}{dt}(\tilde{w}_x(x)^2)\,dx. \tag{145}$$

Using the boundary conditions (140)–(141) we get

$$\dot{V} = -\tilde{X}^T M(D)^{-T} Q M(D)^{-1}\tilde{X} + 2\tilde{X}^T PL\tilde{w}_x(0) - a\|\tilde{w}_x\|^2 - b\|\tilde{w}_{xx}\|^2 \leq -\frac{\lambda_{\min}(Q)}{2}|\tilde{X}|^2 + \frac{2|M(D)^{-T}PL|}{\lambda_{\min}(Q)}\tilde{w}_x(0)^2 - a\|\tilde{w}_x\|^2 - b\|\tilde{w}_{xx}\|^2, \tag{146}$$

where we had applied Young's Inequality. We now use inequality (78), which gives us

$$-\|\tilde{w}_{xx}\|^2 \leq \frac{1+D}{D}\|\tilde{w}_x\|^2 - \tilde{w}_x(0)^2, \tag{147}$$

hence we get

$$\dot{V} \leq -\frac{\lambda_{\min}(Q)}{2}|\tilde{X}|^2 - (b - \frac{2|M(D)^{-T}PL|}{\lambda_{\min}(Q)})\tilde{w}_x(0)^2 - \left(a - b\frac{1+D}{D}\right)\|\tilde{w}_x\|^2. \tag{148}$$

As was done in the proof of Theorem 2.1, taking a and b sufficiently large and then applying again Poincaré's inequality, we get that $\dot{V} \leq -\nu V$ for some $\nu > 0$. This implies that the target system $(\tilde{X}, \tilde{w})$ is exponentially stable at the origin. From the backstepping transformation (133) we get exponential stability of the error system (128)–(131) in the sense of the norm

$$|\tilde{X}(t)|^2 + \|\tilde{u}_x(t)\|^2 \leq \frac{\overline{\delta}}{\underline{\delta}} e^{-bt}(|\tilde{X}(0)|^2 + \|\tilde{u}_x(0)\|^2) \tag{149}$$

for all $t \geq 0$. □

The observer presented in this section has the same structure of the observer proposed in [7] for the cascade

$$Y(t) = u(0,t), \tag{150}$$

$$u_t(x,t) = u_{xx}(x,t), \tag{151}$$

$$u_x(0,t) = 0, \tag{152}$$

$$u(D,t) = CX(t), \tag{153}$$

$$\dot{X}(t) = AX(t) + BU(t). \tag{154}$$

The observer proposed in [7] is actually

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + CM_1(x)L[Y(t) - \hat{u}(0,t)], \tag{155}$$

$$\hat{u}_x(0,t) = 0, \tag{156}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{157}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M_1(D)L[Y(t) - \hat{u}(0,t)], \tag{158}$$

where

$$M_1(x) = [I \;\; 0]e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}x}\begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{159}$$

The transformation (133) used to study the two observers is the same in both proofs of stability; this similarity is possible thanks to the swapping between the initial conditions of function M and M_1 in the two problems. Note that $M(0) = M_1'(0) = I$ and $M'(0) = M_1(0) = 0$.

Example 4.1. Consider the scalar unstable system

$$\dot{X}(t) = X(t) + U(t), \tag{160}$$

with a heat equation sensor dynamics:

$$Y(t) = u(0,t), \tag{161}$$

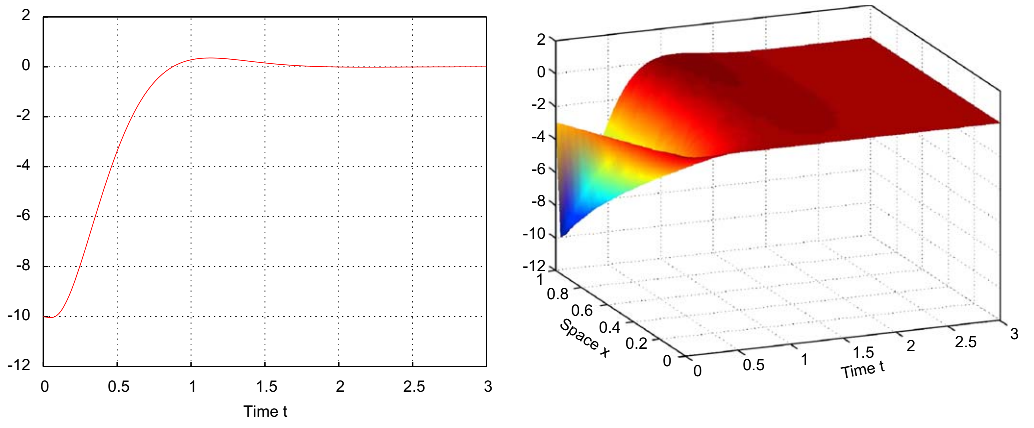

Fig. 4. The error system evolution for Example 4.1. Left: $\tilde{X}(t)$. Right: $\tilde{u}(t)$.

$$u_t(x,t) = u_{xx}(x,t), \tag{162}$$

$$u_x(0,t) = 0, \tag{163}$$

$$u(D,t) = X(t). \tag{164}$$

In this case we have that $M(x) = \cosh x$ and the backstepping observer proposed is

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + (1+g)\cosh x[Y(t) - \hat{u}_x(0,t)], \tag{165}$$

$$\hat{u}(0,t) = 0, \tag{166}$$

$$\hat{u}(D,t) = \hat{X}(t), \tag{167}$$

$$\dot{\hat{X}}(t) = \hat{X}(t) + U(t) + (1+g)\cosh D[Y(t) - \hat{u}_x(0,t)], \tag{168}$$

where we choose $L = 1+g$, $g > 0$, in order to have $A - LC$ Hurwitz. In Fig. 4 we show simulation results for $g = 1$, $D = 1$, $U(t) = 0.05\sin(15t)$ and with initial conditions $u(x,0) = -(3/D)x$ and $X(0) = -10$, whereas the observer initial conditions are $\hat{X}(0) = 0$, $\hat{u}(x,0) \equiv 0$.

5. The diffusion–counterconvection PDE: controller and observer design

We now study a more complicated parabolic PDE, incorporating both the effects of diffusion and of convection,

$$\dot{X}(t) = AX(t) + Bu(0), \tag{169}$$

$$u_t = u_{xx} - bu_x, \tag{170}$$

$$u_x(0) = 0, \tag{171}$$

$$u(D) = U(t). \tag{172}$$

For $b > 0$ the convection effect opposes the propagation of the control input $U(t)$ towards the ODE plant. We refer to this effect as *counter-convection*. The problem considered in this section is a generalization of the problem considered in [7] in which the system (169)–(172) was studied for $b=0$. We provide results which recover those presented in [7] for $b=0$.

Theorem 5.1. *Consider a closed-loop system consisting of the plant* (169)–(172) *and the control law*

$$U(t)=\int_0^D Km(D-y)u(y,t)\,dy+KM(D)X(t), \tag{173}$$

where

$$m(s)=-\frac{b}{2}e^{bs}+\int_0^s e^{b\sigma}KM(s-\sigma)B\,d\sigma, \tag{174}$$

$$M(x)=\begin{bmatrix} I & \frac{b}{2}I\end{bmatrix}e^{\begin{bmatrix}0 & A\\ I & bI\end{bmatrix}x}\begin{bmatrix} I\\ 0\end{bmatrix}. \tag{175}$$

For any initial condition such that $u(x,0)$ *is square integrable in* x *and compatible with the control law* (173), *the closed-loop system has a unique classical solution and is exponentially stable in the sense of the norm*

$$\left(|X(t)|^2+\int_0^D u(x,t)^2\,dx\right)^{1/2}. \tag{176}$$

Proof. We employ again the backstepping transformation (27) to map the original system (169)–(172) into the target one

$$\dot{X}(t)=(A+BK)X(t)+Bw(0), \tag{177}$$

$$w_t=w_{xx}-bw_x, \tag{178}$$

$$w_x(0)=\frac{b}{2}w(0), \tag{179}$$

$$w(D)=0. \tag{180}$$

As done in the previous proof of stabilization, we first compute the derivatives w_x, w_{xx} and w_t in order to derive the kernel functions $q(x,y)$ and $\gamma(x)$; comparing the expressions obtained with Eqs. (177)–(179) we get the following sets of conditions to be satisfied:

$$\gamma(0)=0, \tag{181}$$

$$\gamma'(0)=K\frac{b}{2}, \tag{182}$$

$$\gamma''(x)=b\gamma'(x)+\gamma(x)A, \tag{183}$$

that is, a second order ODE in x, and

$$k_{xx}-k_{yy}=b[k_x+k_y], \tag{184}$$

$$k_y(x,0)=-bk(x,0)-\gamma(x)B, \tag{185}$$

$$k(x,x)=-\frac{b}{2}, \tag{186}$$

which is a hyperbolic second order PDE. The ODE (181)–(183) is solved by

$$\gamma(x) = KM(x), \tag{187}$$

$$M(x) = \begin{bmatrix} I & \frac{b}{2}I \end{bmatrix} e^{\left[\begin{smallmatrix} 0 & A \\ I & bI \end{smallmatrix}\right]x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{188}$$

In order to solve the PDE (184)–(186) we introduce a change of variables

$$k(x,y) = p(x,y)e^{(b/2)(x-y)}. \tag{189}$$

Differentiating (189) twice with respect to x and twice with respect to y we obtain a new PDE

$$p_{xx} = p_{yy}, \tag{190}$$

$$p_y(x,0) = -\frac{b}{2}p(x,0) - \gamma(x)Be^{-(b/2)x}, \tag{191}$$

$$p(x,x) = -\frac{b}{2}. \tag{192}$$

Eq. (190) has a general solution of the form

$$p(x,y) = \phi(x-y) + \psi(x+y). \tag{193}$$

Using the boundary condition $p(x,x) = -b/2$ we get

$$\phi(0) + \psi(2x) = -\frac{b}{2}. \tag{194}$$

Without loss of generality we can set $\psi \equiv 0$ and $\phi(0) = -b/2$. Hence we have that

$$p(x,y) = \phi(x-y). \tag{195}$$

Substituting this into the PDE (190)–(192) we get the following differential equation:

$$-\phi'(x) = -\frac{b}{2}\phi(x) - KM(x)Be^{-(b/2)x}. \tag{196}$$

Applying to (196) the Laplace transform with respect to x we obtain

$$\phi(s) = \frac{1}{s - \frac{b}{2}}\left[-\frac{b}{2} + KM\left(s + \frac{b}{2}\right)B\right]. \tag{197}$$

Anti-transforming this expression we get

$$\phi(z) = -\frac{b}{2}e^{(b/2)z} + (f_* g)(z), \tag{198}$$

where

$$f(z) = e^{(b/2)z}, \tag{199}$$

$$g(z) = KM(z)Be^{-(b/2)z}, \tag{200}$$

and hence

$$\phi(z) = -\frac{b}{2}e^{(b/2)z} + \int_0^z e^{b\sigma}KM(z-\sigma)Be^{-(b/2)z}\, d\sigma. \tag{201}$$

We finally obtain the solution for $k(x,y)$ as

$$k(x,y)=\phi(x-y)e^{(b/2)(x-y)}=-\frac{b}{2}e^{b(x-y)}+\int_0^{x-y}e^{b\sigma}KM(x-y-\sigma)B\,d\sigma \tag{202}$$

$$=m(x-y). \tag{203}$$

In the same manner we get the inverse transformation

$$u(x)=w(x)+\int_0^x Kn(x-y)w(y)\,dy+KN(x)X(t), \tag{204}$$

where

$$N(s)=[I\ \ 0]e^{\begin{bmatrix}0 & A+BK\\ I & bI\end{bmatrix}s}\begin{bmatrix}I\\0\end{bmatrix}, \tag{205}$$

$$n(s)=-\frac{b}{2}e^{2bs}\int_0^s N(\sigma)Be^{2b\sigma}e^{(b/2)(s-\sigma)}\,d\sigma. \tag{206}$$

Consider now the Lyapunov function

$$V=X^TPX+\tfrac{1}{2}\|w\|^2, \tag{207}$$

where $P=P^T>0$ is the solution of the Lyapunov equation

$$P(A+BK)+(A+BK)^TP=-Q \tag{208}$$

for some $Q=Q^T>0$. Differentiating (207) with respect to time, exploiting the boundary conditions (179) and (180), and applying Poincaré's inequality we get

$$\dot V\leq-\lambda_{\min}(Q)|X|^2-\frac{1}{4D^2}\|w\|^2\leq-bV, \tag{209}$$

where

$$b=\min\left\{\lambda_{\min}(Q),\frac{1}{4D^2}\right\}. \tag{210}$$

Applying Cauchy–Schwartz and Young's inequalities we obtain the bounds

$$\|w\|^2\leq\alpha_1\|u\|^2+\alpha_2|X|^2, \tag{211}$$

$$\|u\|^2\leq\beta_1\|w\|^2+\beta_2|X|^2, \tag{212}$$

where

$$\alpha_1=3(1+D\|m\|^2), \tag{213}$$

$$\alpha_2=3\|KM\|^2, \tag{214}$$

$$\beta_1=3(1+D\|n\|^2), \tag{215}$$

$$\beta_2=3\|KN\|^2. \tag{216}$$

Exploiting (211) and (212) we get

$$\underline{\delta}(\|u\|^2+\|X\|^2)\leq V\leq\overline{\delta}(\|u\|^2+\|X\|^2), \tag{217}$$

where

$$\underline{\delta} = \max\left\{\frac{\alpha_1}{2}, \lambda_{\max}(P) + \frac{\alpha_2}{2}\right\}, \tag{218}$$

$$\overline{\delta} = \frac{\max\{\beta_1, \beta_2 + 1\}}{\min\left\{\frac{1}{2}, \lambda_{\min}(P)\right\}}. \tag{219}$$

Combining the last inequality with (210) yields to

$$|X(t)|^2 + \|u(t)\|^2 \leq \frac{\overline{\delta}}{\underline{\delta}} e^{-bt}(|X(0)|^2 + \|u(0)\|^2) \tag{220}$$

for all $t \geq 0$. □

It is interesting to analyze the results we had achieved in this section by comparison with the case without counter-convection, which was studied in [7]. With $b = 0$ we get

$$M_{[b=0]}(s) = [I \;\; 0] e^{\left[\begin{smallmatrix} 0 & A \\ I & 0 \end{smallmatrix}\right]s} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{221}$$

$$N_{[b=0]}(s) = [I \;\; 0] e^{\left[\begin{smallmatrix} 0 & A+BK \\ I & 0 \end{smallmatrix}\right]s} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{222}$$

$$m_{[b=0]}(s) = \int_0^s KM_{[b=0]}(s - \sigma)B \, d\sigma, \tag{223}$$

$$n_{[b=0]}(s) = \int_0^s KM_{[b=0]}(s - \sigma)B \, d\sigma, \tag{224}$$

which are exactly the same functions used in the controller presented in [7]. The generalization that we have obtained here is thus consistent with [7].

For the diffusion–counterconvection PDE we study the dual case where the PDE is in the sensor dynamics,

$$Y(t) = u(0, t), \tag{225}$$

$$u_t(x, t) = u_{xx}(x, t) - bu_x(x, t), \tag{226}$$

$$u_x(0, t) = 0, \tag{227}$$

$$u(D, t) = CX(t), \tag{228}$$

$$\dot{X}(t) = AX(t) + BU(t). \tag{229}$$

Theorem 5.2. *The observer*

$$\hat{u}_t(x, t) = \hat{u}_{xx}(x, t) - b\hat{u}_x(x, t) + CM(x)L[Y(t) - \hat{u}(0, t)], \tag{230}$$

$$\hat{u}_x(0, t) = -\frac{b}{2}[Y(t) - \hat{u}(0, t)], \tag{231}$$

$$\hat{u}(D, t) = C\hat{X}(t), \tag{232}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M(D)L[Y(t) - \hat{u}(0,t)], \tag{233}$$

where L is chosen such that $A - LC$ *is Hurwitz and matrix function* $M(\cdot)$ *is defined by* (188), *guarantees that the error system*

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) - b\tilde{u}_x(x,t) - CM(x)L\tilde{u}(0,t), \tag{234}$$

$$\tilde{u}_x(0,t) = -\frac{b}{2}\tilde{u}(0,t), \tag{235}$$

$$\tilde{u}(D,t) = C\tilde{X}(t), \tag{236}$$

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - M(D)L\tilde{u}(0,t), \tag{237}$$

where $\tilde{X} = X - \hat{X}$ *and* $\tilde{u} = u - \hat{u}$, *is exponentially stable in the sense of the norm*

$$\left(|\tilde{X}(t)|^2 + \int_0^D \tilde{u}(x,t)^2\,dx\right)^{1/2}. \tag{238}$$

Proof. We introduce the transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) - CM(x)M(D)^{-1}\tilde{X}(t), \tag{239}$$

which is actually the same the one used in proof of Theorem 4.1, but with a different function $M(x)$. Evaluating (239) at $x=0$ and D we get

$$\tilde{w}(0,t) = \tilde{u}(0,t) - CM(D)^{-1}\tilde{X}(t), \tag{240}$$

$$\tilde{w}(D,t) = 0. \tag{241}$$

Exploiting (240) and the ODE system equation (237) we obtain

$$\dot{\tilde{X}}(t) = [A - M(D)LCM(D)^{-1}]\tilde{X}(t) - M(D)L\tilde{w}(0). \tag{242}$$

We now proceed to differentiate (239) with respect to x:

$$\tilde{w}_x(x,t) = \tilde{u}_x(x,t) - CM'(x)M(D)^{-1}\tilde{X}(t). \tag{243}$$

Evaluating (243) at the boundary $x=0$ and making use of the initial condition $M(0) = (b/2)I$ and the boundary condition (235), we get

$$\tilde{w}_x(0,t) = \frac{b}{2}\tilde{u}(0,t). \tag{244}$$

Differentiating again with respect to x we get:

$$\tilde{w}_{xx}(x,t) = \tilde{u}_{xx}(x,t) - bCM'(x)M(D)^{-1}\tilde{X}(t) - CM(x)AM(D)^{-1}\tilde{X}(t), \tag{245}$$

where we had used the equation $M''(x) = bM'(x) + M(x)A$. Differentiating the transformation (239) with respect to time we obtain

$$\begin{aligned}\tilde{w}_t(x,t) &= \tilde{u}_t(x,t) \\ &\quad - CM(x)M(D)^{-1}\dot{\tilde{X}}(t) \\ &= \tilde{u}_{xx}(x,t) - b\tilde{u}_x(x,t) - CM(x)L\tilde{u}(0,t) \\ &\quad - CM(x)M(D)^{-1}A\tilde{X}(t) + CM(x)L\tilde{u}(0,t) \\ &= \tilde{w}_{xx}(x,t) - b\tilde{w}_x(x,t),\end{aligned} \tag{246}$$

where we had used Eqs. (237), (234), (243) and (245). Summarizing, we have obtained the following PDE–ODE system:

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - b\tilde{w}_x(x,t), \tag{247}$$

$$\tilde{w}_x(0,t) = \frac{b}{2}\tilde{w}(0,t), \tag{248}$$

$$\tilde{w}(D,t) = 0, \tag{249}$$

$$\dot{\tilde{X}}(t) = [A - M(D)LCM(D)^{-1}]\tilde{X}(t) - M(D)L\tilde{w}(0,t). \tag{250}$$

We now consider the Lyapunov function

$$V = \tilde{X}^T M(D)^{-T} P M(D)^{-1}\tilde{X} + \tfrac{1}{2}\|\tilde{w}\|^2, \tag{251}$$

where $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A - LC) + (A - LC)^T P = -Q \tag{252}$$

for some $Q = Q^T > 0$. Differentiating (251) with respect to time and exploiting the boundary conditions (247) and (248) we get

$$\begin{aligned}\dot{V} &= -\tilde{X}^T M(D)^{-T} Q M(D)^{-1}\tilde{X} - \tilde{w}(0)\tilde{w}_x(0) - \|w_x\|^2 \\ &\quad - 2\tilde{X}^T M(D)^{-T} Q M(D)^{-1}\tilde{X} + \frac{b}{2}\tilde{w}(0)^2.\end{aligned} \tag{253}$$

Using the Poincaré inequality and the boundary condition $w(D) = 0$ we get

$$\dot{V} \leq -\lambda_{\min}(Q)|\tilde{X}|^2 - \frac{1}{4D^2}\|\tilde{w}\|^2 \leq -\eta V, \tag{254}$$

where

$$\eta = \min\left\{\lambda_{\min}(Q), \frac{1}{4D^2}\right\}. \tag{255}$$

From the transformation (239) we derive the following inequalities:

$$\|\tilde{w}\|^2 \leq \|\tilde{u}\|^2 + \beta|\tilde{X}|^2, \tag{256}$$

$$\|\tilde{u}\|^2 \leq \|\tilde{w}\|^2 + \beta|\tilde{X}|^2, \tag{257}$$

with

$$\beta = \|CMM(D)^{-1}\|^2. \tag{258}$$

Thus

$$\underline{\delta}(|\tilde{X}|^2 + \|\tilde{u}\|^2) \quad \leq V \leq \quad \overline{\delta}(|\tilde{X}|^2 + \|\tilde{u}\|^2), \tag{259}$$

where

$$\overline{\delta} = \max\{\lambda_{\max}(P) + \beta, \tfrac{1}{2}\}, \tag{260}$$

$$\underline{\delta} = \frac{\min\{\frac{1}{2}, \lambda_{\min}(P)\}}{\max\left\{1, \beta + 1\right\}}. \tag{261}$$

Combining (259) with (254) yields

$$|\tilde{X}(t)|^2 + \|\tilde{u}_x(t)\|^2 \leq \frac{\overline{\delta}}{\underline{\delta}} e^{-\eta t}(|\tilde{X}(0)|^2 + \|\tilde{u}_x(0)\|^2) \tag{262}$$

for all $t \geq 0$. □

As we did for the full-state control problem, it is interesting to compare the results that we have obtained for to the corresponding case with $b = 0$, that is, the purely diffusive PDE, without counter-convection. For $b = 0$ the observer (230)–(233) becomes

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + CM(x)L[Y(t) - \hat{u}(0,t)], \tag{263}$$

$$\hat{u}_x(0,t) = 0, \tag{264}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{265}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M(D)L[Y(t) - \hat{u}(0,t)], \tag{266}$$

which is exactly the same observer presented in [7] for the following problem:

$$Y(t) = u(0,t), \tag{267}$$

$$u_t(x,t) = u_{xx}(x,t), \tag{268}$$

$$u_x(0,t) = 0, \tag{269}$$

$$u(D,t) = CX(t), \tag{270}$$

$$\dot{X}(t) = AX(t) + BU(t). \tag{271}$$

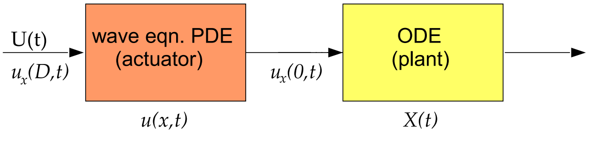

Fig. 5. The cascade of a wave PDE with an ODE plant.

6. The wave-ODE cascade with Neumann interconnection

Consider the following system:

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{272}$$

$$u(0,t) = 0, \tag{273}$$

$$u_x(D,t) = U(t), \tag{274}$$

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{275}$$

which is depicted in Fig. 5. We are looking for a backstepping transformation that makes the system (272)–(275) behave as the following target system:

$$w_{tt}(x,t) = w_{xx}(x,t), \tag{276}$$

$$w(0,t) = 0, \tag{277}$$

$$w_x(D,t) = -cw_t(D,t), \quad c>0, \tag{278}$$

$$\dot{X}(t) = (A+BK)X(t) + Bw_x(0,t), \tag{279}$$

where K is such that $(A+BK)$ is Hurwitz.

We postulate the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy - \int_0^x l(x,y)u_t(y,t)\,dy - \gamma(x)X(t), \tag{280}$$

which is inspired by the construction in [16]. As done in the previous problems, we differentiate the transformation (280) to gather conditions that the unknown functions must satisfy. In order to verify the wave equation (276) we must differentiate twice with respect to time and to space. We first differentiate with respect to time[1]:

$$\begin{aligned} w_t(x) = u_t(x) - \int_0^x k(x,y)u_t(y)\,dy - l(x,x)u_x(x) + l(x,0)u_x(0) + l_y(x,x)u(x) \\ - \gamma(x)Bu_x(0) - \gamma(x)AX - \int_0^x l_{yy}(x,y)u(y)\,dy, \end{aligned} \tag{281}$$

where we had used (276), the boundary condition (277) and integrated twice by parts. Differentiating again with respect to time:

$$\begin{aligned} w_{tt}(x) = u_{xx}(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) \\ - \int_0^x k_{yy}(x,y)u_t(y)\,dy + l(x,x)u_{xt}(x) + l_y(x,x)u_t(x) - \gamma(x)A\dot{X} \\ + - \int_0^x l_{yy}(x,y)u_t(y)\,dy + [l(x,0) - \gamma(x)B]u_{xt}(0). \end{aligned} \tag{282}$$

[1]We drop the dependence on time of state variables $w(x,t)$ and $u(x,t)$ for the sake of compact notation.

Now we proceed differentiating with respect to space:

$$w_x(x) = u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy - l(x,x)u_t(x) - \int_0^x l_x(x,y)u_t(y)\,dy - \gamma(x)'X(t), \tag{283}$$

$$w_{xx}(x) = u_{xx}(x) - \frac{d}{dx}[k(x,x)]u(x) - k(x,x)u_x(x) - k_x(x,x)u(x) - \int_0^x k_{xx}(x,y)u(y)\,dy - \frac{d}{dx}[l(x,x)]u_t(x) - l(x,x)u_{tx}(x) - l_x(x,x)u_t(x) - \int_0^x l_{xx}(x,y)u_t(y)\,dy - \gamma(x)''X(t). \tag{284}$$

Matching (282)–(284) yields to the following PDEs:

$$l_{xx} = l_{yy}, \tag{285}$$

$$l(x,0) = \gamma(x)B, \tag{286}$$

$$l(x,x) = 0, \tag{287}$$

$$k_{xx} = k_{yy}, \tag{288}$$

$$k(x,0) = \gamma(x)AB, \tag{289}$$

$$k(x,x) = 0, \tag{290}$$

$$\gamma(x)'' = \gamma(x)A^2, \tag{291}$$

$$\gamma(0) = 0, \tag{292}$$

$$\gamma(0)' = K. \tag{293}$$

Exploiting the previous we obtain the following expression for the unknown functions:

$$\gamma(x) = KM(x) \tag{294}$$

$$= K[0 \ \ I]e^{\left[\begin{smallmatrix} 0 & A^2 \\ I & 0 \end{smallmatrix}\right]x}\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{295}$$

$$l(x,y) = m(x-y), \tag{296}$$

$$k(x,y) = \mu(x-y), \tag{297}$$

$$m(s) = \gamma(s)B, \tag{298}$$

$$\mu(s) = \gamma(s)AB. \tag{299}$$

The explicit expression for the controller is derived from (278). Evaluating $w(D,t)$ from (281) and $w_x(D,t)$ and exploiting (283) we get

$$U(t) = c[KBu(D,t) - u_t(D,t)] + \rho(D)X(t) + \int_0^D \rho(D-x)[ABu(x,t) + Bu_t(x,t)]\,dx, \tag{300}$$

where the function $\rho(s)$ is defined by

$$\rho(s) = K[0 \;\; I]e^{\left[\begin{smallmatrix} 0 & A^2 \\ I & 0 \end{smallmatrix}\right]s}\begin{bmatrix} cA \\ I \end{bmatrix}. \tag{301}$$

In the same manner in which we derive the direct backstepping transformation, we also derive the inverse transformation

$$u(x) = w(x) + \int_0^x \phi(x-y)w(y)\,dy + \int_0^x n(x-y)w_t(y)\,dy + \psi(x)X, \tag{302}$$

where

$$\psi(x) = KN(x), \tag{303}$$

$$N(x) = [0 \;\; I]e^{\left[\begin{smallmatrix} 0 & (A+BK)^2 \\ I & 0 \end{smallmatrix}\right]x}\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{304}$$

$$n(s) = \psi(s)B, \tag{305}$$

$$\phi(s) = \psi(s)AB. \tag{306}$$

Having designed the controller we now show that it guarantees exponential stability to the original system.

Theorem 6.1. *Consider the closed-loop system consisting of the plant* (272)–(275) *and the control law* (300). *For any initial conditions such that $u_x(x,0)$ and $u_t(x,0)$ are square integrable in x and compatible with* (300), *the closed-loop system has a unique solution that is exponentially stable in the sense of the norm*

$$\left(|X(t)|^2 + \int_0^D u_x(x,t)^2\,dx + \int_0^D u_t(x,t)^2\,dx\right)^{1/2}. \tag{307}$$

Proof. We start by introducing the system norms

$$\Omega(t) = \|u_x(t)\|^2 + \|u_t(t)\|^2 + |X(t)|^2, \tag{308}$$

$$\Xi(t) = \|w_x(t)\|^2 + \|w_t(t)\|^2 + |X(t)|^2. \tag{309}$$

To prove the stability of the closed-loop system (272)–(275) and (300) we employ the Lyapunov function

$$V(t) = X(t)^T PX(t) + aE(t), \tag{310}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A + BK) + (A + BK)^T P = -Q \tag{311}$$

for some $Q = Q^T > 0$, the design parameter $a > 0$ is to be chosen later and the function $E(t)$ is defined by

$$E(t) = \frac{1}{2}(\|w_x(t)\|^2 + \|w_t(t)\|^2) + \delta \int_0^D (1 + y) w_x(y,t) w_t(y,t)\, dy, \tag{312}$$

where $\delta > 0$ is also a parameter to be chosen later. We observe that

$$\theta_1 \Xi \le V \le \theta_2 \Xi, \tag{313}$$

where

$$\theta_1 = \min\left\{\lambda_{\min}(P), \frac{a}{2}[1 - \delta(1 + D)]\right\}, \tag{314}$$

$$\theta_2 = \min\left\{\lambda_{\max}(P), \frac{a}{2}[1 + \delta(1 + D)]\right\}. \tag{315}$$

We choose

$$0 < \delta < \frac{1}{1 + D} \tag{316}$$

in order to ensure that θ_1 and θ_2 are non-negative and so the Lyapunov function V is positive semi-definite. Next, we compute the time derivative of $E(t)$

$$\begin{aligned} \dot{E}(t) = &-\frac{\delta}{2}[\|w_x(t)\|^2 + \|w_t(t)\|^2 + w_x(0,t)^2] \\ &+ \frac{\delta}{2}(1 + D)[w_t(D,t)^2 + w_x(D,t)^2] + w_x(D,t) w_t(D,t). \end{aligned} \tag{317}$$

We substitute the feedback law $w_x(D,t) = -c w_t(D,t)$ and get

$$\dot{E}(t) = -\frac{\delta}{2}[\|w_x(t)\|^2 + \|w_t(t)\|^2 + w_x(0,t)^2] + -\underbrace{\left[c - \delta\frac{1 + D}{2}(1 + c^2)\right]}_{b} w_t(D,t)^2. \tag{318}$$

Choosing now

$$\delta < \frac{2c}{(1 + D)(1 + c^2)} \tag{319}$$

we have that $b > 0$. We now compute the derivative of $V(t)$:

$$\begin{aligned} \dot{V}(t) = &-X(t)^T Q X(t) \\ &+ 2X(t)^T PB w_x(0,t) + a\dot{E}(t) \le -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 \\ &+ \left[\frac{2|PB|^2}{\lambda_{\min}(Q)} - a\frac{\delta}{2}\right] w_x(0,t)^2 - a\frac{\delta}{2}[\|w_x(t)\|^2 + \|w_t(t)\|^2] - ab w_t(D,t)^2. \end{aligned} \tag{320}$$

To have $\dot{V}(t)<0$ we choose

$$a \geq \frac{4|PB|^2}{\delta\lambda_{\min}(Q)}. \tag{321}$$

We now obtain

$$\dot{V}(t) \leq -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - a\frac{\delta}{2}[\|w_x(t)\|^2 + \|w_t(t)\|^2] \tag{322}$$

$$\leq -\eta V(t), \tag{323}$$

where

$$\eta = \frac{\min\left\{\frac{\lambda_{\min}(Q)}{2}, \frac{a\delta}{2}\right\}}{\theta_2}. \tag{324}$$

Thus we arrive at the estimate

$$V(t) \leq e^{-\eta t} V(0). \tag{325}$$

To prove stability of the closed-loop system in its original variables (u, X), from (325) we provide inequalities relating the variables $u(x,t)$ and $w(x,t)$. From the inverse transformation (302) we obtain that

$$u_x(x,t) = w_x(x,t) + \int_0^x \phi'(x-y)w(y)\,dy + \int_0^x n'(x-y)w_t(y)\,dy + \psi(x)'X(t), \tag{326}$$

$$u_t(x,t) = w_t(x,t) + \int_0^x \phi(x-y)w_t(y)\,dy + \int_0^x n'(x-y)w(y)\,dy + \psi(x)(A+BK)X(t). \tag{327}$$

Applying Poincaré, Young's and the Cauchy–Schwartz inequality, we get

$$\|u_x(t)\|^2 \leq \alpha_1\|w_x(t)\|^2 + \alpha_2\|w_t(t)\|^2 + \alpha_3|X(t)|^2, \tag{328}$$

$$\|u_t(t)\|^2 \leq \beta_1\|w_x(t)\|^2 + \beta_2\|w_t(t)\|^2 + \beta_3|X(t)|^2, \tag{329}$$

where

$$\alpha_1 = 4(1 + 4D^3\|\phi'\|^2), \tag{330}$$

$$\alpha_2 = 4D\|n'\|^2, \tag{331}$$

$$\alpha_3 = 4\|\psi'\|^2, \tag{332}$$

$$\beta_1 = 4\|n'\|^2, \tag{333}$$

$$\beta_2 = 4(1 + 4D^3\|\phi\|^2), \tag{334}$$

$$\beta_3 = 4\|\psi(A+BK)\|^2. \tag{335}$$

Applying (328) and (329) we obtain

$$\Omega(t) \le \theta_4 \Xi(t), \tag{336}$$

where

$$\theta_4 = \max\{\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3 + 1\}. \tag{337}$$

With the help of Eqs. (281) and (283) and applying again Poincaré, Young's and the Cauchy–Schwartz inequalities, we obtain the following inequalities:

$$\|w_x(t)\|^2 \le a_1 \|u_x(t)\|^2 + a_2 \|u_x(t)\|^2 + a_3 |X(t)|^2, \tag{338}$$

$$\|w_t(t)\|^2 \le b_1 \|u_x(t)\|^2 + b_2 \|u_x(t)\|^2 + b_3 |X(t)|^2, \tag{339}$$

where

$$a_1 = 4(1 + 4D^3 \|\mu'\|^2), \tag{340}$$

$$a_2 = 4D\|m'\|^2, \tag{341}$$

$$a_3 = 4\|\phi'\|^2, \tag{342}$$

$$b_1 = 4D\|m'\|^2, \tag{343}$$

$$b_2 = 4(1 + 4\|\mu\|^2), \tag{344}$$

$$b_3 = 4\|\phi A\|^2. \tag{345}$$

Applying (338) and (339) we obtain

$$\theta_3 \Xi(t) \le \Omega(t), \tag{346}$$

where

$$\theta_3 = \frac{1}{\max\{a_1 + b_1, a_2 + b_2, a_3 + b_3 + 1\}}. \tag{347}$$

With the help of inequalities (313), (325), (336) and (346) we get

$$\Omega(t) \le \frac{\theta_1 \theta_3}{\theta_2 \theta_4} \Omega(0) e^{-\eta t}, \tag{348}$$

that is,

$$\|u_x(t)\|^2 + \|u_x(t)\|^2 + |X(t)|^2 \le \frac{\theta_1 \theta_3}{\theta_2 \theta_4} e^{-\eta t} [\|u_x(0)\|^2 + \|u_t(0)\|^2 + |X(0)|^2]. \quad \square \tag{349}$$

7. Conclusions

In this article we have developed explicit controllers and observers for PDE–ODE cascades involving heat and wave equation, extending the results in [7,8]. Many open problems in PDE–ODE cascades remain. For example, the system

$$\dot{X}(t) = AX(t) + B_0 u(0,t) + B_1 u_x(0,t), \tag{350}$$

$$u_t(x,t) = u_{xx}(x,t) - bu_x(x,t) + \lambda u(x,t), \tag{351}$$

$$u_x(0,t) = qu(0,t) + HX(t), \tag{352}$$

$$u(D,t) = U(t), \tag{353}$$

contains several interesting problems, including one where $u(0,t)$ and $u_x(0,t)$ simultaneously appear in the ODE, and an even more interesting one where the "interconnection" between the PDE and the ODE is not one-directional (as in this paper and in [7,8]), but where the ODE acts back on the PDE, such as, for example, the term $HX(t)$ in the boundary condition (353).

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