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CONTROL OF THE LOW ENERGY CHARACTERISTICS OF THE LSR ELECTRON RING USING WIGGLER MAGNETS

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In an electron storage ring the emittance, the polarization time and the betatron and energy damping times are strongly dependent on the energy. In electron–proton (e–p) rings where bunched electrons collide with coasting proton beams, the emittance variation is of secondary importance, while considerable emphasis is placed on polarization phenomena.

A general theory of wiggler magnets designed to control the polarization time at low energies is developed with all the parameters normalized to the standard lattice. The equations are of general validity and are applied to the e–p option of the CERN LSR (Large Storage Rings) project, as an example. It is demonstrated that the polarization vector may be reversed over a considerable range of energies being limited only by the available rf voltage. An exact formula for the emittance is given taking into account the variation of the dispersion function in the wiggler magnets. By judicious placing of the wiggler in the ring and a suitable choice of the wiggler magnet length, the emittance and polarization time may both be maintained approximately constant over a wide energy range while the increase in damping times at low energy is reduced but cannot be maintained constant.

PART 1 NORMAL POLARIZATION

1.1 *Introduction*

In an electron storage ring, the emission of synchrotron radiation produces three main effects: it defines the beam emittance, it produces vertical polarization of the electrons and it provides damping of betatron and energy oscillations (or more precisely, the rf power supplied to compensate the synchrotron radiation loss, provides the damping).[1] In electron–positron storage rings where both beams are bunched the first effect is critical and control of the beam size at all energies is the major function of the wiggler magnets used in these machines.[2] In electron–proton (e–p) rings, where bunched electrons collide with coasting protons, the electron beam size does not enter directly into the luminosity equation. However, considerable emphasis is attached to the provision of longitudinal electron polarization at the intersection region.[3] As is well known, the polarization rate is proportional to the fifth power of the energy and at lower energies becomes very small in the normal lattice of the storage ring. Thus, wiggler magnets would be required in this case primarily for reducing the polarization time. In this paper, the design of a wiggler is given which maintains the polarization time constant over a wide range of energies. It will be shown that the beam emittance can also be controlled at low energy but while the damping rates are improved they will still decrease with decreasing energy. It is assumed that the wiggler produces no net bending.

1.2 *Definition of wiggler length*

The polarization rate is given by[4]

$$\alpha_{\text{pol}} = \frac{1}{\tau_{\text{pol}}} = C_p E^5 \frac{\oint \left(\frac{1}{\rho^3}\right) \mathrm{d}s}{\oint \mathrm{d}s}$$

where

$$C_p = \frac{5\sqrt{3}}{8} \frac{\hbar c^2 r_e}{(m_e c^2)^6} = \frac{1}{98}\,\mathrm{s}^{-1}\,\mathrm{GeV}^{-5}\,\mathrm{m}^2.$$

The denominator of this equation is simply $\bar{L}$, the circumference of the ring. The numerator must be evaluated over the lattice bending magnets of radius of curvature ρ_0 and total length $L_0 = 2\pi\rho_0$, over the positive bends of the wiggler of radius of curvature ρ_1 and total length L_1, and over the reverse bends of the wiggler of radius of curvature ρ_2 and total length L_2. It will be assumed that at maximum energy E_0 the wiggler is switched off ($\rho_1 = \rho_2 = \infty$) and the polarization rate is $\bar{\alpha}_{\text{pol}}$.

Then at any lower energy E, where the wiggler magnets are excited

$$\frac{\alpha_{\text{pol}}}{\bar{\alpha}_{\text{pol}}} = \left(\frac{E}{E_0}\right)^5 \left[\frac{L_0}{\rho_0^3} + \frac{L_1}{\rho_1^3} + \frac{L_2}{\rho_2^3}\right] \Big/ \frac{L_0}{\rho_0^3}.$$

If we define the following dimensionless parameters

$$\xi = \frac{E}{E_0} \tag{1}$$

$$b_1 = \frac{\text{bending field in wiggler positive bend}}{\text{bending field in lattice bends at } E_0} \tag{2}$$

$$-rb_1 = \frac{\text{bending field in wiggler reverse bend}}{\text{bending field in lattice bends at } E_0} \tag{3}$$

then

$$\frac{\rho_0}{\rho_1} = \frac{b_1}{\xi} \quad \text{and} \quad \frac{\rho_0}{\rho_2} = -\frac{rb_1}{\xi} \tag{4}$$

so finally

$$\frac{\alpha_{\text{pol}}}{\bar{\alpha}_{\text{pol}}} = \xi^5 \left[1 + \frac{L_1}{L_0}\frac{b_1^3}{\xi^3} - \frac{L_2}{L_0}\frac{r^3 b_1^3}{\xi^3}\right] \tag{5}$$

and the requirement of no net bending gives the relation

$$\frac{L_1}{L_2} = -\frac{\rho_1}{\rho_2} = r \tag{6}$$

and defining L_w as the total length of the wiggler we have

$$L_1 = \frac{r}{1+r} L_w$$

and

$$L_2 = \frac{1}{1+r} L_w$$

so from Eq. (5)

$$\frac{\alpha_{\text{pol}}}{\bar{\alpha}_{\text{pol}}} = \xi^5 \left[1 + \frac{L_w}{L_0}\frac{b_1^3}{\xi^3} r(1-r)\right]. \tag{7}$$

It will be assumed that we require a constant polarization time over the range $\xi = 1$ to $\xi = \xi_1$ and that at ξ_1, $b_1 = B_1$ the maximum allowable value. Then it is possible to express L_w as a function of r, B_1 and ξ_1

$$\frac{L_w}{L_0} = \frac{(1-\xi_1^5)}{B_1^3 \xi_1^2} \frac{1}{r(1-r)}.$$

The minimum length wiggler is given when $r = \frac{1}{2}$ and so the wiggler is now completely determined as a function of B_1 and ξ_1

$$\frac{L_w}{L_0} = \frac{4}{B_1^3 \xi_1^2}(1 - \xi_1^5) \tag{8}$$

and

$$L_1 = \frac{L_w}{3}, \qquad L_2 = \frac{2L_w}{3}.$$

In the following sections the detailed properties of a wiggler as defined by Eq. (8) will be considered.

1.4 *Field required for constant polarization rate as a function of energy*

From Eq. (7) putting $r = \frac{1}{2}$ and $\alpha_{\text{pol}} = \bar{\alpha}_{\text{pol}}$

$$1 = \xi^5 \left[1 + \frac{L_w}{L_0}\frac{b_1^3}{4\xi^3}\right]$$

so

$$b_1^3 = \frac{4L_0}{L_w}\frac{1-\xi^5}{\xi^2}, \qquad b_2 = -\frac{b_1}{2}. \tag{9}$$

1.5 *Damping rates as a function of energy*

The damping rates for the betatron (x, z) and energy (ε) oscillations are given by[1]

$$\alpha_{\text{damp}} = \frac{1}{\tau_{\text{damp}}} = \frac{J}{2}\frac{U}{E} f_0$$

where U is the synchrotron radiation loss per turn, f_0 is the revolution frequency and J the damping partition coefficient. (For the separated function, isomagnetic case $J_x \approx 1$, $J_z \approx 1$ and $J_\varepsilon \approx 2$.) The synchrotron radiation loss per turn is given by

$$U = C_\alpha \frac{E^4}{2\pi} \oint \frac{1}{\rho^2}\, ds, \tag{10}$$

where

$$C_\alpha = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3} = 8.85 \times 10^{-5} \text{ m GeV}^{-3}$$

so

$$\alpha_{\text{damp}} = \frac{C_\alpha}{2\pi}\frac{J}{2} f_0 E^3 \oint \frac{1}{\rho^2}\, ds.$$

Using the definitions in Eqs. (1)–(4) and defining $\bar{\alpha}_{\text{damp}}$ as the damping rate at E_0 when the wiggler is switched off we obtain

$$\frac{\alpha_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \xi^3\left[1 + \frac{1}{2}\frac{L_w}{L_0}\frac{b_1^2}{\xi^2}\right] \tag{11}$$

and substituting Eq. (9) gives

$$\frac{\alpha_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \xi^3\left[1 + \left(\frac{2L_w}{L_0}\right)^{1/3}\left(\frac{1-\xi^5}{\xi^3}\right)^{2/3}\right]. \tag{12}$$

1.6 *Damping rates at injection energy*

So far we have only considered the energy range over which we require constant polarization time, i.e. the energy range over which the experiments will be carried out. However, it is most likely that injection will take place at an even lower energy and the damping rate at injection is an important parameter. Two cases will be considered. If the aperture is limited in the wiggler, then we can assume that at lower energies we maintain the minimum radius of curvature and reduce the magnetic field from B_1. Alternatively, if there is no aperture limitation we can keep B_1 constant at lower energies and this mode provides more damping. The choice between these two modes will be decided by the particular injection scheme selected. Define the injection energy as ξ_2.

Case 1 Aperture limitation in the wiggler.

The magnetic field in the wiggler varies linearly with energy so

$$b_1 = B_1\frac{\xi_2}{\xi_1}.$$

Substituting this value in Eq. (11) gives

$$\frac{\alpha_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \xi_2^3\left[1 + \frac{1}{2}\frac{L_w}{L_0}\frac{B_1^2}{\xi_1^2}\right]. \tag{13}$$

Case 2 No aperture limitation in the wiggler.

The magnetic field in the wiggler is equal to B_1 and substituting in Eq. (11) gives

$$\frac{\alpha_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \xi_2^3\left[1 + \frac{1}{2}\frac{L_w}{L_0}\frac{B_1^2}{\xi_2^2}\right]. \tag{14}$$

1.7 *Synchrotron radiation loss in the wiggler*

The function of the wiggler is to produce synchrotron radiation but it is obvious that the energy produced must be dissipated in some fashion. In fact, this is no small problem and it is likely that the value of B_1 will be determined more by the allowable power dissipation in the wiggler than by saturation of the magnet yokes. The radiation loss in the wiggler is closely connected to the damping rates as was seen above. If we define the radiation loss in the wiggler as U_w and the energy loss at maximum energy due to the lattice bends as U_0, we have from Eq. (10)

$$\frac{U_w}{U_0} = \frac{\xi^2}{2}\frac{L_w}{L_0}b_1^2 \tag{15}$$

and substituting for b_1 from Eq. (9)

$$\frac{U_w}{U_0} = \left(\frac{2L_w}{L_0}\right)^{1/3}(\xi - \xi^6)^{2/3}. \tag{16}$$

It is useful to notice that this expression has a maximum value at $\xi = (1/6)^{1/5} = 0.7$ when

$$\left(\frac{U_w}{U_0}\right)_{\max} = 0.7\left(\frac{2L_w}{L_0}\right)^{1/3}. \tag{17}$$

The problem of dissipation of the synchrotron radiation is most serious in the positive bend of the wiggler where it is 4 times that of the reverse bend. The maximum synchrotron radiation power loss per unit length of the positive bend is given by

$$(P_1)_{\max} = \left(\frac{U_1 I}{L_1}\right)_{\max} = \frac{2U_w I}{L_w} = 2.8\left(\frac{L_0}{2L_w}\right)^{2/3}\frac{U_0 I}{L_0}. \tag{18}$$

1.8 *Beam size control*

The equation for the emittance of the electron beam is given by[1]

$$\varepsilon_x = \frac{\sigma_x^2}{\beta_x} = \frac{C_q E^2 \oint \left|\frac{1}{\rho^3}\right| W\,\mathrm{d}s}{J_x \oint \frac{1}{\rho^2} X\,\mathrm{d}s}, \tag{19}$$

where

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar c}{(m_0 c^2)^3} = 1.47 \times 10^{-6}\ \text{m}\,(\text{GeV})^{-2}$$

and W is the Courant and Snyder "Invariant" (also denoted by $\mathscr{H}$ in the literature) defined as

$$W = \frac{\alpha_p^2 + I^2}{\beta_x}$$

with

$$I = \alpha_p' \beta_x + \alpha_x \alpha_p.$$

Equation (19) may be expressed in dimensionless form by defining $\bar{\varepsilon}_x$ as the emittance at maximum energy with the wiggler not excited

$$\frac{\varepsilon_x}{\bar{\varepsilon}_x} = \xi^2 \frac{1 + \dfrac{L_1}{L_0}\left|\left(\dfrac{\rho_0}{\rho_1}\right)^3\right|\dfrac{\langle W_1\rangle}{\langle W_0\rangle} + \dfrac{L_2}{L_0}\left|\left(\dfrac{\rho_0}{\rho_2}\right)^3\right|\dfrac{\langle W_2\rangle}{\langle W_0\rangle}}{1 + \dfrac{L_1}{L_0}\left(\dfrac{\rho_0}{\rho_1}\right)^2 + \dfrac{L_2}{L_0}\left(\dfrac{\rho_0}{\rho_2}\right)^2} \tag{20}$$

where $\langle W_0\rangle$ is the average value of W over the lattice bends, $\langle W_1\rangle$ is the average value of W over the positive wiggler magnets and $\langle W_2\rangle$ is the average value of W over the reverse wiggler magnets. Note that if $\langle W_1\rangle = \langle W_2\rangle$ this expression reduces to that in Ref. 2.

Substituting for L_1, L_2, ρ_1, ρ_2 from Eqs. (1), (2), (3), (4), (8) and (9) we obtain

$$\frac{\varepsilon_x}{\bar{\varepsilon}_x} = \xi^2 \frac{1 + \dfrac{1-\xi^5}{\xi^5}\,\dfrac{4\langle W_1\rangle + \langle W_2\rangle}{3\langle W_0\rangle}}{1 + \left(\dfrac{2L_w}{L_0}\right)^{1/3}\left(\dfrac{1-\xi^5}{\xi^5}\right)^{2/3}}. \tag{21}$$

It is now necessary to consider the detailed characteristics of the Courant and Snyder "Invariant" W. In actual fact W is only invariant in regions with no bending, i.e. drift spaces or focussing elements, and changes in bending magnets. We thus have the situation where $\langle W_1\rangle$ and $\langle W_2\rangle$ are not constants in Eq. (21) but depend on the field in the magnets. This effect may or may not be small depending on the lengths of the magnets (or more precisely the total bending angle per magnet) and the values of the betatron functions at the wiggler magnets. A simple derivation of the size of the effect is given here, where the magnet lengths are assumed to be sufficiently short to be treated in a "thin magnet" approximation (i.e. magnet lengths are short enough so that the change in the β_x value over the three wiggler magnets may be ignored). An exact derivation is given in the Appendix.

The value of $W(W^*)$ after a bending angle θ is given by[5]

$$W^* = W_w + 2I\theta + \beta_x\theta^2 \quad (W_w \text{ is input value}).$$

If we require an average value of W, then this expression may be integrated with respect to the bending angle and divided by the total bending angle. Thus

$$\langle W\rangle = \frac{1}{\theta_B - \theta_A}\int_{\theta_A}^{\theta_B} W^*\,d\theta$$

$$\langle W\rangle = W_w + I(\theta_B + \theta_A) + \frac{\beta x}{3}(\theta_B^2 + \theta_A\theta_B + \theta_A^2). \tag{22}$$

The assumption will be made that the wiggler magnets are all grouped in blocks of three identical magnets, the centre magnet powered positively and the two outer magnets powered in the reverse direction and with half the field—i.e. the bending angle of the reverse magnets θ_2 is equal to $-\theta_1/2$ where θ_1 is the bending angle of the centre magnet. So, using Eq. (22) for the central magnet, we obtain by integrating from $-\theta_1/2$ to $+\theta_1/2$

$$\langle W_1\rangle = W_w + I\left(\frac{\theta_1}{2} - \frac{\theta_1}{2}\right) + \frac{\beta x}{3}\left(\frac{\theta_1^2}{4} - \frac{\theta_1^2}{4} + \frac{\theta_1^2}{4}\right)$$

or simply

$$\langle W_1\rangle = W_w + \frac{\beta_x\theta_1^2}{12}. \tag{23}$$

For the outer magnets the average value of W is given by the arithmetic mean of the two magnets, i.e. the mean of the integral from 0 to $-\theta_1/2$ and the integral from $\theta_1/2$ to 0

$$\langle W_2\rangle = \frac{1}{2}\left\{W_w + I\left(-\frac{\theta_1}{2} + 0\right) + \frac{\beta x}{3}\left(\frac{\theta_1^2}{4} + 0 + 0\right) + W_w + I\left(0 + \frac{\theta_1}{2}\right) + \frac{\beta x}{3}\left(0 + 0 + \frac{\theta_1^2}{4}\right)\right\}$$

or simply

$$\langle W_2\rangle = W_w + \frac{\beta_x\theta_1^2}{12}. \tag{24}$$

Thus we obtain the simple result that $\langle W_1\rangle = \langle W_2\rangle$ and β_x takes the meaning of the average horizontal β function over all the wiggler magnets in the ring. θ is related to the length of the wiggler magnets and is conveniently expressed in terms of N, the number of wiggler blocks in the ring (i.e. there are a total of $3N$ wiggler magnets). Then, if the wiggler magnet length is L_m, we have

$$\theta_1 = \frac{L_m}{\rho_1} = \frac{\pi}{3N}\frac{2L_w}{L_0}\frac{b_1}{\xi}. \tag{25}$$

Substituting Eqs. (23) and (24) into Eq. (21) gives

$$\frac{\varepsilon_x}{\bar{\varepsilon}_x} = \xi^2 \frac{1 + \frac{5}{3}\frac{1-\xi^5}{\xi^5}\left(\frac{\langle W_w\rangle}{\langle W_0\rangle} + \frac{\beta_x \theta_1^2}{12\langle W_0\rangle}\right)}{1 + \left(\frac{2L_w}{L_0}\right)^{1/3}\left(\frac{1-\xi^5}{\xi^5}\right)^{2/3}} \tag{26}$$

and substituting Eqs. (25) and (9) gives finally

$$\frac{\varepsilon_x}{\bar{\varepsilon}_x} = \xi^2 \frac{1 + \frac{5}{3}\frac{1-\xi^5}{\xi^5}\left\{\frac{\langle W_w\rangle}{\langle W_0\rangle} + \frac{\pi^2}{27}\frac{\beta_x}{N^2\langle W_0\rangle}\left(\frac{2L_w}{L_0}\right)^{4/3}\left(\frac{1-\xi^5}{\xi^5}\right)^{2/3}\right\}}{1 + \left(\frac{2L_w}{L_0}\right)^{1/3}\left(\frac{1-\xi^5}{\xi^5}\right)^{2/3}} \tag{27}$$

We can distinguish two extreme cases; the first when θ_1 is small and W_w large so the second term may be ignored; the second case, where the wiggler blocks are in dispersion-free zones where W_w is zero and the θ term predominates.

To get some idea of the relative importance of the two terms we can take the values of Ref. 2 at the lowest energy. Then at 5 GeV, strong field = 2 T, magnet length = 1.5 m we obtain $\theta_1 = 0.18$ and taking β as 17.7 we have $\beta\theta_1^2/12 = 0.048$ while $\langle W_0\rangle = 0.059$. Thus the "self produced" W is about 80% of the input value and ignoring this effect leads to an over estimate of the field required for constant emittance and hence an over estimate of the improvement in polarization and damping times.

1.9 *Critical energy of the synchrotron radiation*

The critical energy of the synchrotron radiation ε_c is an important parameter since it is a measure of the hardness of the radiation. It will therefore determine the thickness of the shielding required to stop synchrotron radiation. The critical energy is a function of the bending field and the beam energy and is given by[6]

$$\varepsilon_c = \frac{3}{2}\frac{\hbar c}{(m_e c^2)^3}\frac{E^3}{\rho} = 2218\frac{E^3}{\rho}\ \text{m GeV}^{-3}\ \text{eV}. \tag{28}$$

The critical energy will therefore be different in the positive and negative bends denoted by suffixes 1 and 2 respectively. $\bar{\varepsilon}_c$ is the critical energy of synchrotron radiation in the lattice at E_0. From Eqs. (4) and (9)

$$\rho_1 = \rho_0\left(\frac{L_w}{4L_0}\right)^{1/3}\left(\frac{\xi^5}{1-\xi^5}\right)^{1/3}$$

so

$$\frac{\varepsilon_{c1}}{\bar{\varepsilon}_c} = 2\left(\frac{L_0}{2L_w}\right)^{1/3}(\xi^4 - \xi^9)^{1/3} \tag{29}$$

and

$$\varepsilon_{c2} = \frac{\varepsilon_{c1}}{2}. \tag{30}$$

Both ε_{c1} and ε_{c2} have maxima at $\xi = (4/9)^{1/5} = 0.85$ so

$$\frac{(\varepsilon_{c1})_{\max}}{\bar{\varepsilon}_c} = 1.324\left(\frac{L_0}{2L_w}\right)^{1/3}. \tag{31}$$

The spectrum of the synchrotron radiation is easily calculated from the tables given in Ref. 6.

Example for the LSR e–p *option*†

The following parameters have been adopted for the LSR e–p option[7]

Maximum energy	E_0	= 20 GeV
Total bending length	L_0	= 3236 m
Lattice bending field at E_0		= 0.13 T
Maximum energy loss per turn	U_0	= 27.5 MeV
Beam current	I	= 0.25 A
Polarization rate at E_0	$\bar{\alpha}_{pol}$	= 1.26×10^{-4} s^{-1}
Betatron damp rate at at E_0	$(\alpha_{damp})_x$	= 33 s^{-1}
Energy damp rate at E_0	$(\alpha_{damp})_\varepsilon$	= 66 s^{-1}

In the definition of the wiggler characteristics only six free parameters are involved and the

† The CERN LSR (Large Storage Rings) project involves two intersecting storage rings for 400-GeV coasting protons and a third ring for 20-GeV bunched electrons colliding with one of the proton rings.

TABLE I

Wiggler characteristics as a function of energy

Energy	E	20	18	16	14	12	10	GeV
Positive bending field b_1		—	0.66	0.84	0.99	1.13	1.30	tesla
Reverse bending field b_2		—	−0.33	−0.42	−0.49	−0.57	−0.65	tesla
Damp rate ratio $\frac{\alpha_{\text{damp}}}{\bar{\alpha}_{\text{damp}}}$		1.0	0.91	0.77	0.66	0.57	0.51	
Radiation loss in wiggler U_w		—	4.44	5.71	6.02	5.82	5.33	MeV/turn
Total radiation loss U_{tot}		27.5	22.48	16.98	12.63	9.39	7.05	MeV/turn
Power in positive bend P_1		—	44.28	56.97	60.06	58.05	53.12	kW/m
Power in reverse bend P_2		—	11.07	14.24	15.02	14.51	13.28	kW/m
Critical energy pos. bend ε_{c1}		—	141.5	142.7	128.2	108.0	86.1	keV
Critical energy rev. bend ε_{c2}		—	70.8	71.4	64.1	54.0	43.1	keV

following values are assumed

Maximum field in wiggler positive bends 1.3 T	i.e $B_1 = 10$
Experimental energy range is 10 – 20 Gev	i.e. $\xi_1 = 0.5$
Injection energy is 5 GeV	i.e. $\xi_2 = 0.25$

The beam size will be tabulated for several values of $\langle W_w \rangle$ and N while β_x will be taken as 60 m.

Wiggler characteristics

Maximum positive bending field B_1	1.3 T	
Maximum reverse bending field B_2	0.65 T	(9)
Total wiggler length L_w	50.16 m	(8)
Length of positive bends L_1	16.72 m	
Length of reverse bends L_2	33.44 m	
Betatron damping rate at injection	6.9 s^{-1} no aperture limit	(14)
Betatron damping rate at injection	2.12 s^{-1} aperture limit at 10 GeV	(13)
Maximum energy loss in the wiggler U_w	6.05 MeV/turn	(17)
Maximum linear power loss $(P_1)_{\max}$	60.3 kW/m constant current 0.25 A	(18)
Maximum critical energy $(\varepsilon_{c1})_{\max}$	145.3 keV	(31)

The detailed characteristics are shown in Tables I and II. The emittance ratio is given by the sum of the two relevant lines. Note that in the present example $N = 8$ implies $L_m \approx 2$ m; $N = 16$, $L_m \approx 1$ m.

TABLE II

Emittance ratio as a function of energy

a) Emittance ratio with $\langle W_w \rangle$ treated as a constant

Energy	E	20	18	16	14	12	10	GeV
$\frac{\langle W_w \rangle}{\langle W_0 \rangle}$	0.0	1.0	0.65	0.43	0.26	0.14	0.06	
	0.1	1.0	0.72	0.57	0.47	0.41	0.38	
	0.2	1.0	0.80	0.72	0.68	0.68	0.69	
	0.3	1.0	0.87	0.86	0.89	0.95	1.01	
	0.4	1.0	0.95	1.01	1.10	1.22	1.32	
	0.5	1.0	1.02	1.15	1.31	1.49	1.64	
	0.6	1.0	1.10	1.30	1.52	1.76	1.95	
	0.7	1.0	1.18	1.45	1.73	2.03	2.26	
	0.8	1.0	1.25	1.59	1.94	2.30	2.58	
	0.9	1.0	1.33	1.74	2.15	2.57	2.89	
	1.0	1.0	1.40	1.88	2.36	2.84	3.21	

b) Additional emittance ratio due to "self-produced" W

Energy	E	20	18	16	14	12	10	GeV
N ($\beta = 60$)	8	0.0	0.03	0.13	0.33	0.76	1.67	
	10	0.0	0.02	0.08	0.21	0.48	1.07	
	12	0.0	0.01	0.06	0.15	0.34	0.74	
	14	0.0	0.01	0.04	0.11	0.25	0.55	
	16	0.0	0.01	0.03	0.08	0.19	0.42	

PART 2 REVERSE POLARIZATION

2.1 *Introduction*

At low energies almost all of the polarization is produced in the wiggler, the contribution of the lattice bending being very small. It is thus natural to consider the effect of reversing the polarity of the wiggler magnets. It is not immediately obvious that the wiggler is capable of achieving a complete reversal of the polarization but it will be shown that this is in fact possible over a large range, the limiting factor being the total rf power available. This feature is of great interest when taken in context with the proposal for rotating the polarization vector into the longitudinal plane in the intersection region.[3] If the polarization vector is reversed in the wiggler, then the longitudinal polarization will be reversed in the intersection region, a feature that otherwise would require a second rotation channel. The major problem with this proposal is the greatly increased synchrotron radiation dissipation in the wiggler requiring additional cooling.

In the following sections a prime will denote values appertaining to the reversed polarization case.

2.2 *Field required for constant polarization rate (reversed) as a function of energy*

The formula for the reversed polarization rate is given by reversing the signs in Eq. (7) giving

$$\frac{\alpha'_{\text{pol}}}{\bar{\alpha}_{\text{pol}}} = \xi^5\left[1 - \frac{L_w}{4L_0}\frac{|b'_1|^3}{\xi^3}\right] \tag{32}$$

with the sign convention that α'_{pol} is negative for reverse polarization. If we require α'_{pol} to be numerically equal to $\bar{\alpha}_{\text{pol}}$ then

$$|b'_1|^3 = \frac{4L_0}{L_w}\frac{1+\xi^5}{\xi^2}. \tag{33}$$

The maximum value of b'_1 is B'_1 required at the lowest energy ξ_1 where

$$\frac{|B'_1|}{B_1} = \left(\frac{1+\xi_1^5}{1-\xi_1^5}\right)^{1/3}. \tag{34}$$

This ratio is in general very close to unity and while it has been assumed that B_1 is the maximum allowable value, the difference between B_1 and B'_1 is usually sufficiently small to be acceptable.

2.3 *Damping rates as a function of energy for reversed polarization*

The damping rates are affected by the reversal of the wiggler magnets only because of the increased synchrotron radiation loss produced. Thus from Eqs. (11) and (33) we have

$$\frac{\alpha'_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \xi^3\left[1 + \left(\frac{2L_w}{L_0}\right)^{1/3}\left(\frac{1+\xi^5}{\xi^5}\right)^{2/3}\right]. \tag{35}$$

2.4 *Synchrotron radiation loss for reversed polarization*

The synchrotron radiation loss per turn in the wiggler is given by Eqs. (15) and (33)

$$\frac{U'_w}{U_0} = \left(\frac{2L_w}{L_0}\right)^{1/3}(\xi+\xi^6)^{2/3} \tag{36}$$

which, in the range of interest, is a monotonically increasing function of energy. It may be assumed that the upper limit of validity of Eq. (33) will be when the total synchrotron radiation loss $U'_w + U$ is equal to U_0, where U is the radiation loss per turn at energy ξ in the lattice bends and is given by Eq. (10) as

$$\frac{U}{U_w} = \xi^4.$$

The maximum permissible energy denoted by ξ_3 is then given by the equation

$$\left(\frac{2L_w}{L_0}\right)^{1/3}(\xi_3+\xi_3^6)^{2/3} = 1 - \xi_3^4. \tag{37}$$

In Figure 1, ξ_3 is plotted as a function of L_w/L_0. It can be seen that ξ_3 is a very insensitive function of L_w but with a shorter wiggler magnet the maximum energy increases. The maximum energy loss in the wiggler is therefore given by Eqs. (36) and (37)

$$\left(\frac{U'_w}{U_0}\right)_{\max} = 1 - \xi_3^4. \tag{38}$$

The maximum synchrotron radiation power loss per unit length of the reverse (stronger) bend is then given by

$$(P'_1)_{\max} = \left(\frac{2U'_wI}{3L_1}\right)_{\max} = \frac{2(1-\xi_3^4)}{L_w}U_0I. \tag{39}$$

2.5 *Reverse polarization stability limit*

If the total rf power supplied is kept constant above ξ_3, then b'_1 must be reduced accordingly and α'_{pol} will be reduced. The stability limit is given when $\alpha'_{\text{pol}} = 0$. The significance of this number is as follows. If it proves possible to accelerate electrons in the storage ring without depolarizing them (a big if, since depolarizing effects are still not well understood) then they can be maintained in the reversed polarization state only up to the stability limit ξ_4 given by setting $\alpha'_{\text{pol}} = 0$ in Eq. (32). By an identical process to that carried out to obtain Eq. (37) we obtain an expression for ξ_4, the stability limit.

$$\xi_4 = \left[1 + \left(\frac{2L_w}{L_0}\right)^{1/3}\right]^{-1/4}. \tag{40}$$

This expression is also plotted in Figure 1 and is always greater than ξ_3. For the sake of completeness the formulae for the various parameters are given here for the zero polarization rate case, the derivation is completely straightforward, the values are denoted by a double prime

$$|b_1''|^3 = \frac{4L_0}{L_w}\xi_4^3 \tag{41}$$

$$\frac{\alpha''_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \frac{1}{\xi_4} \tag{42}$$

$$\frac{U''_w}{U_0} = 1 - \xi_4^4. \tag{43}$$

2.6 *Beam size control for reversed polarization*

This is simply obtained by substituting Eq. (33) into Eq. (20) giving

$$\frac{\varepsilon'_x}{\bar{\varepsilon}_x} = \xi^2 \frac{1 + \frac{5}{3}\frac{1+\xi^5}{\xi^5}\left\{\frac{\langle W_w\rangle}{\langle W_0\rangle} + \frac{\pi^2}{27}\frac{\beta_x}{N\langle W_0\rangle}\left(\frac{2L_w}{L_0}\right)^{4/3}\left(\frac{1+\xi^5}{\xi^5}\right)^{2/3}\right\}}{1 + \left(\frac{2L_w}{L_0}\right)^{1/3}\left(\frac{1+\xi^5}{\xi^5}\right)^{2/3}} \tag{44}$$

2.7 *Critical energy of synchrotron radiation for reversed polarization*

This is also simply obtained by using Eqs. (4), (9), (28) and (33) and the same procedure employed in the derivation of Eq. (29)

$$\frac{\varepsilon'_{c1}}{\bar{\varepsilon}_c} = 2\left(\frac{L_0}{2L_w}\right)^{1/3}(\xi^4 + \xi^9)^{1/3}. \tag{45}$$

The maximum value will be at the maximum permissible energy ξ_3. For the case of zero polarization rate a similar procedure gives

$$\frac{\varepsilon''_{c1}}{\varepsilon_{c0}} = 2\left(\frac{L_0}{2L_w}\right)^{1/3}\xi^3. \tag{46}$$

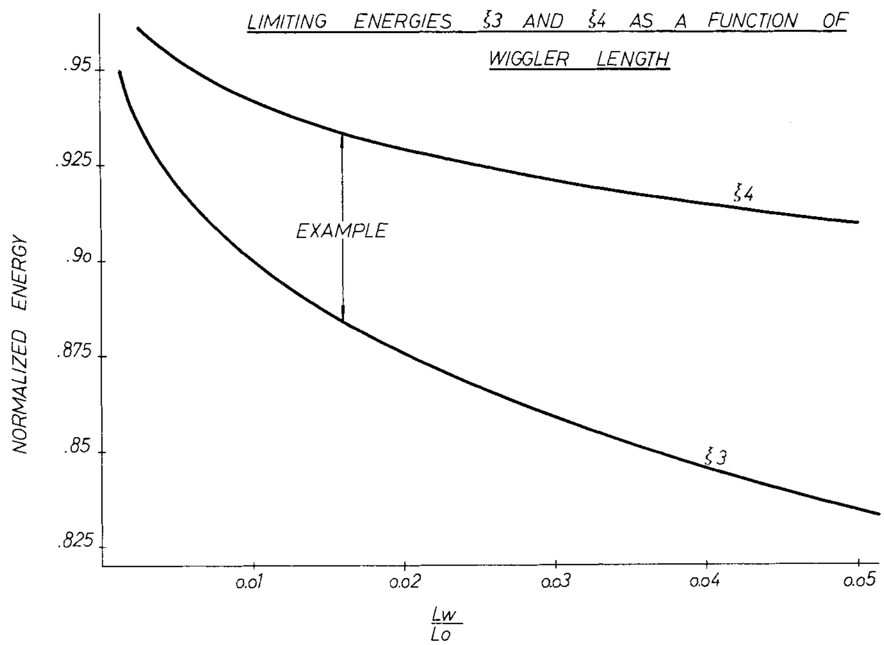

FIGURE 1 Limiting energies as a function of wiggler length. ξ_3 is maximum normalized energy at which it is possible to reverse the polarization. ξ_4 is the stability limit at which there is no net polarization.

TABLE III

Wiggler characteristics as a function of energy for reversed polarization

Energy E	10	12	14	16	17.70	18.68[a]	GeV
Reverse bending field b'_1	−1.33	−1.19	−1.11	−1.06	−1.04	−0.77	tesla
Positive bending field b'_2	0.66	0.60	0.55	0.53	0.52	0.39	tesla
Damp rate ratio $\frac{\alpha'_{damp}}{\alpha_{damp}}$	0.53	0.61	0.74	0.92	1.13	1.07	
Polarization rate ratio $\frac{\alpha'_{pol}}{\alpha_{pol}}$	−1.0	−1.0	−1.0	−1.0	−1.0	0.0	
Radiation loss in wiggler U'_w	5.55	6.46	7.55	8.99	10.63	6.57	MeV/turn
Total radiation loss U'_{tot}	7.27	10.02	14.16	20.26	27.50	27.50	MeV/turn
Power in reverse bend P'_1	55.39	64.41	75.31	89.67	105.99	65.53	kW/m
Power in positive bend P'_2	13.85	16.10	18.33	22.42	26.50	16.38	kW/m
Critical energy rev. bend ε'_{c1}	87.94	113.8	143.6	179.0	215.3	178.7	keV
Critical energy pos. bend ε'_{c2}	43.97	56.9	71.8	89.5	107.7	89.3	keV

[a] For the case of 18.68 GeV the values refer to the zero polarization rate case.

TABLE IV

Emittance ratio as a function of energy for reversed polarization

a) Emittance ratio with $\langle W_w \rangle$ treated as a constant

Energy	E	10	12	14	16	17.70	18.68[a] GeV
$\frac{\langle W_w \rangle}{\langle W_0 \rangle}$	0.0	0.6	0.13	0.23	0.36	0.48	0.70
	0.1	0.38	0.42	0.49	0.60	0.71	0.81
	0.2	0.71	0.72	0.76	0.84	0.94	0.93
	0.3	1.03	1.02	1.02	1.08	1.16	1.05
	0.4	1.36	1.31	1.29	1.32	1.39	1.16
	0.5	1.68	1.61	1.55	1.56	1.62	1.28
	0.6	2.01	1.90	1.82	1.80	1.85	1.40
	0.7	2.33	2.20	2.08	2.04	2.07	1.51
	0.8	2.66	2.49	2.35	2.28	2.30	1.63
	0.9	2.98	2.79	2.61	2.52	2.53	1.75
	1.0	3.31	3.09	2.88	2.76	2.76	1.86

b) Additional emittance ratio due to "self-produced" W

Energy	E	10	12	14	16	17.70	18.68[a] GeV
N ($\beta = 60$)	8	1.80	0.92	0.52	0.33	0.25	0.06
	10	1.15	0.59	0.33	0.21	0.16	0.04
	12	0.80	0.41	0.23	0.15	0.11	0.03
	14	0.59	0.30	0.17	0.11	0.08	0.02
	16	0.45	0.23	0.13	0.08	0.06	0.02

[a] For the case of 18.68 GeV the values refer to the zero polarization rate case.

Example of reversed polarization

The same basic characteristics of the wiggler defined in the example above will be maintained.

Maximum reverse bending field B'_1	1.327 T	(34)
Maximum positive bending field B'_2	0.664 T	
Maximum energy for constant polarization rate ξ_3	17.70 GeV	(37)
Maximum energy for zero polarization rate ξ_4	18.68 GeV	(40)
Maximum energy loss in the wiggler $(U'_w)_{max}$	10.63 MeV/turn	(38)
Maximum linear power loss $(P'_1)_{max}$, constant current	105.96 kW/m	(39)
Maximum critical energy $(\varepsilon'_c)_{max}$	215.4 keV	(45)

The detailed characteristics are shown in Tables III and IV. Notice that the linear power loss is over 50% greater than in the normal polarization case. The problem of dissipating the heat produced by the synchrotron radiation will, however, be helped by the higher critical energy.

The emittance ratio is given by the sum of the two relevant lines in Table IV.

PART 3 SUMMARY OF USEFUL FORMULAE

The equations derived can be put into a unified form for the three cases considered and these equations also have a more general validity.

$$\text{Strong magnetic field ratio} = b_1 = \frac{2\xi k^{1/3}}{l^{1/3}} \qquad \text{(S1)}$$

$$\text{Weak magnetic field ratio} = b_2 = -[2r]\frac{\xi k^{1/3}}{l^{1/3}} \tag{S2}$$

$$\text{Polarization rate ratio} = \frac{\alpha_{\text{pol}}}{\bar{\alpha}_{\text{pol}}} = \xi^5(1 + k[4r(1-r)]) \tag{S3}$$

$$\text{Damping rate ratio} = \frac{\alpha_{\text{damp}}}{\bar{\alpha}_{\text{damp}}} = \xi^3(1 + l^{1/3}k^{2/3}[2r]) \tag{S4}$$

$$\text{Synchrotron radiation loss ratio} = \frac{U_w}{U_0} = l^{1/3}k^{2/3}\xi^4[2r] \tag{S5}$$

$$\text{Power loss per metre ratio} = \frac{P_1}{P_0} = 4\frac{k^{2/3}\xi^4}{l^{2/3}} \quad \text{constant current} \tag{S6}$$

$$\text{Critical energy ratio} = \frac{\varepsilon_{c1}}{\varepsilon_c} = \frac{2|k|^{1/3}\xi^3}{l^{1/3}} \tag{S7}$$

$$\text{Emittance ratio} = \frac{\varepsilon_x}{\bar{\varepsilon}_x}$$

$$= \xi^2 \frac{1 + \left[\frac{12r}{5}\frac{1+r^2}{1+r}\right]\frac{5}{3}|k|\left(\frac{\langle W_w\rangle}{\langle W_0\rangle} + \frac{\pi^2}{27}\left[\frac{3r}{1+r}\right]^2 \frac{\beta_x}{N^2\langle W_0\rangle} l^{4/3}k^{2/3}\right)}{1 + [2r]l^{1/3}k^{2/3}} \tag{S8}$$

where $r = -\frac{b_2}{b_1}$ the magnetic field ratio. (The factors in square brackets are unity for $r = \frac{1}{2}$.)

$\xi = \frac{E}{E_0}$ the beam energy ratio

$l = \frac{2L_w}{L_0}$ twice the wiggler length ratio

$\frac{\langle W_w\rangle}{\langle W_0\rangle}$ the Courant and Snyder "invariant" ratio

β_x the average value of β_x over the wiggler magnets

N the number of wiggler "blocks" in the ring.

In this study the last variable k has been defined via Eq. (S3)

$$k = \left(\frac{1}{\xi^5}\frac{\alpha_{\text{pol}}}{\bar{\alpha}_{\text{pol}}} - 1\right)$$

and imposing the constraint that $\alpha_{\text{pol}} = \bar{\alpha}_{\text{pol}}$, $\alpha_{\text{pol}} = -\bar{\alpha}_{\text{pol}}$ or $\alpha_{\text{pol}} = 0$ respectively for the three cases considered. This is not the only way of defining k, however. If it is defined via Eq. (S1) then the equations give all the relevant parameters for an arbitrary field and energy. It is also possible that once the wiggler has been defined as in the first section, we may require that the emittance be a given function of energy (constant for example) and k will then be defined via Eq. (S8), allowing the polarization rate to vary slightly with energy, which is not unreasonable if the depolarization effects are always small over the whole range.

There will be three constraints on the value of k. Firstly, the field limitation that b_1 should not be greater than the allowable maximum determined by saturation effects. Secondly, the rf limit, that the total synchrotron radiation loss in the ring be less than the rf voltage available for acceleration. Thirdly, the power loss limit which depends on the exact geometry of the wiggler and the allowable surface temperature, the total heat that may be dissipated, the outgassing from the surface, etc.

PART 4 PHYSICAL DISPOSITION OF THE WIGGLER IN THE RING

4.1 *Aperture Requirement*

The equations so far derived are primarily concerned with the total length of wiggler magnets except for Eqs. (27) and (44) where N, the number of wiggler blocks, enter explicitly into the equations for the emittance. A further consideration is the aperture requirement at the centre of the strong magnet. Thus, if all the magnets are of length L_m and separated by a distance L_d then the orbit displacement at the centre of the stronger magnet, a, is given by[8]

$$a \approx \frac{L_m^2}{2\rho_2} + \frac{L_m L_d}{\rho_2} + \left(\frac{L_m}{2}\right)^2 \frac{1}{2\rho_1} \tag{47}$$

and substituting for ρ_1 and ρ_2 from Eq. (4) gives

$$a \approx \frac{L_m^2}{2\rho_0}\frac{b_1}{\xi}\left(\frac{3}{4} + \frac{L_d}{L_m}\right). \tag{48}$$

This function is plotted in Figure 2 for the maximum values of the example calculated above as a function of L_m for values of L_d equal to 0.2 m and 0.4 m. Note that a is inversely proportional to ξ so that if we require a large damping rate at the injection energy, achieved by keeping b_1 at the maximum value, then the orbit displacement is doubled. In practice it is probable that at injection b_1 will be determined by Eq. (48) giving damping times somewhere between those quoted in Eqs. (13) and (14).

4.2 *Position of wiggler*

It is clear from Tables I and II that the average value of W over the wiggler magnets must be less than the average value of W over the lattice and so it cannot all be placed in the normal unperturbed lattice. Also, unless the individual magnets are long, it cannot all be put in a dispersion-free zone. The most elegant solution appears to be to use a missing magnet dispersion suppressor. As is well known, this configuration has the advantage of leaving the quadrupole strengths unaltered so that the α, β functions are undisturbed. The places left vacant by the removal of magnets from the normal structure, can then be used for siting the wiggler magnets. The various gaps will not have equal values of W_w and by judicious placing of the magnet blocks almost any required value of $\langle W_w \rangle$ may be obtained. The dispersion suppressors next to the interaction regions are clearly unsuited to this purpose because of the high synchrotron radiation background produced by the wiggler. However, the rf system in an electron ring must also be in a dispersion-free section and the dispersion suppressors required for this purpose would be well adapted for wiggler magnets.

4.3 *Synchrotron radiation height*

The synchrotron radiation has a "natural" opening angle given by Ref. 6. In addition there is a component due to the fact that the electrons themselves have an angular distribution given by

$$(\varepsilon_v \gamma)^{1/2} \quad \text{where} \quad \gamma_v = \frac{1 + \alpha_v^2}{\beta_v}$$

where ε_v is the vertical emittance of the beam, α_v and β_v the vertical betatron functions. γ_v is an invariant in drift spaces and pure bending magnets and changes in focussing elements. Therefore, in a

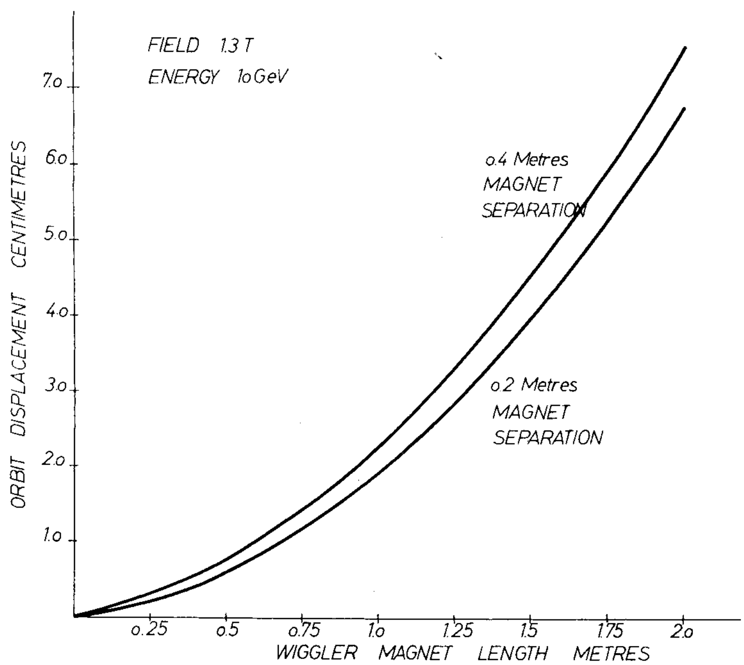

FIGURE 2 Orbit displacement as a function of wiggler length.

separated function lattice the opening angle is independent of the position of the wiggler in the cell. The vertical height of the electron beam is given by

$$(\varepsilon_v \beta_v)^{1/2}.$$

Thus the height is largest close to horizontally defocussing (D) quadrupoles. In view of the large amounts of synchrotron radiation being produced, it is clearly beneficial to have the vertical height of the synchrotron radiation as large as possible reducing the power per unit area incident on the surfaces being irradiated. Therefore the wigglers will be sited next to D quadrupoles.

This implies a small value of β_h and decreases the importance of the "self-induced" W [see Eqs. (27) and (44)].

4.4 *Choice of magnet length*

In the example L_w was defined as 50.16 m, but it is clear that for constructional purposes, the wiggler magnets should be a length that is a round number. Reasonable choices are as follows

N	8	10	12	14	16	
L_m	2.0	1.6	1.4	1.2	1.0	metres
L_w	48.0	48.0	50.4	50.4	48.0	metres

It will be assumed that two missing magnet dispersion suppressors will be used having 4, 5, 6, 7 or 8 blocks disposed in each. They may be placed either side of D quadrupoles in pairs and it would be preferable if the blocks were as close together as possible to reduce the length of special vacuum chamber that will be required with special cooling facilities. The final choice is more or less a trial-and-error process of matching a dispersion suppressor with suitable spaces next to the D quadrupoles, calculating the W values at the spaces and examining the emittance characteristics of the different length magnets placed in these sites from Tables II and IV. To minimize the complexity of the wigglers, N was chosen to be 8 and a low W_w value chosen. The dispersion suppressor was matched into one of the lattices at present under consideration for the LSR e–p option and is shown in Figure 3.

4.5 *Limiting values*

The field limit will be set mainly by saturation in the core of the stronger wiggler magnet. It is assumed that the two weaker magnets in a block are in parallel and the stronger magnet is in series with them. Thus the main excitation of the wiggler requires only one bus-bar. However, some correction windings will be required for the central magnet since the field is not absolutely proportional to the current at higher field strengths. For the present study a value of 1.4 tesla is assumed.

The aperture limit will be important at low energies. In Figure 4 the orbit displacement at 10 GeV, 1.4 T is shown. If two small kickers are placed either side of the wiggler so that a bump can be put into the closed orbit, then half the total aperture is required or conversely, with the same aperture a lower energy can be accepted with the maximum field. This is also shown in Figure 4 indicating an off-centred orbit at injection energy 5 GeV and 1.4 T.

The rf limit has been set at U_0, the value at the nominal maximum energy. Some increase may be obtained at the expense of reduced luminosity but the cavities are usually designed such that this increase is marginal.

The power limit has been arbitrarily set at the value required at the maximum energy at which it is possible to reverse the polarization without exceeding the rf limit (ξ_3) i.e. 110 kW/m. At present this value may be looked upon as a design goal rather than the maximum value shown to be possible. It is assumed that the beam current is constant with energy which may not be the case in reality.

4.6 *Program* "WIGGLE"

The equations in the summary have been incorporated in a program "WIGGLE" written by M. Hanney. The various limits described above are included as are the exact betatron functions from Figure 3. The program finds the fields required to obtain any given value of one of the parameters at a given energy, or the nearest value possible without exceeding one of the limits. The program uses the exact formulation for the average of W over the wiggler magnets developed in the Appendix. This program was used to find the range of variation of the field in the stronger magnet, the polarization rate ratio and the emittance ratio as shown in Figures 5, 6 and 7 respectively.

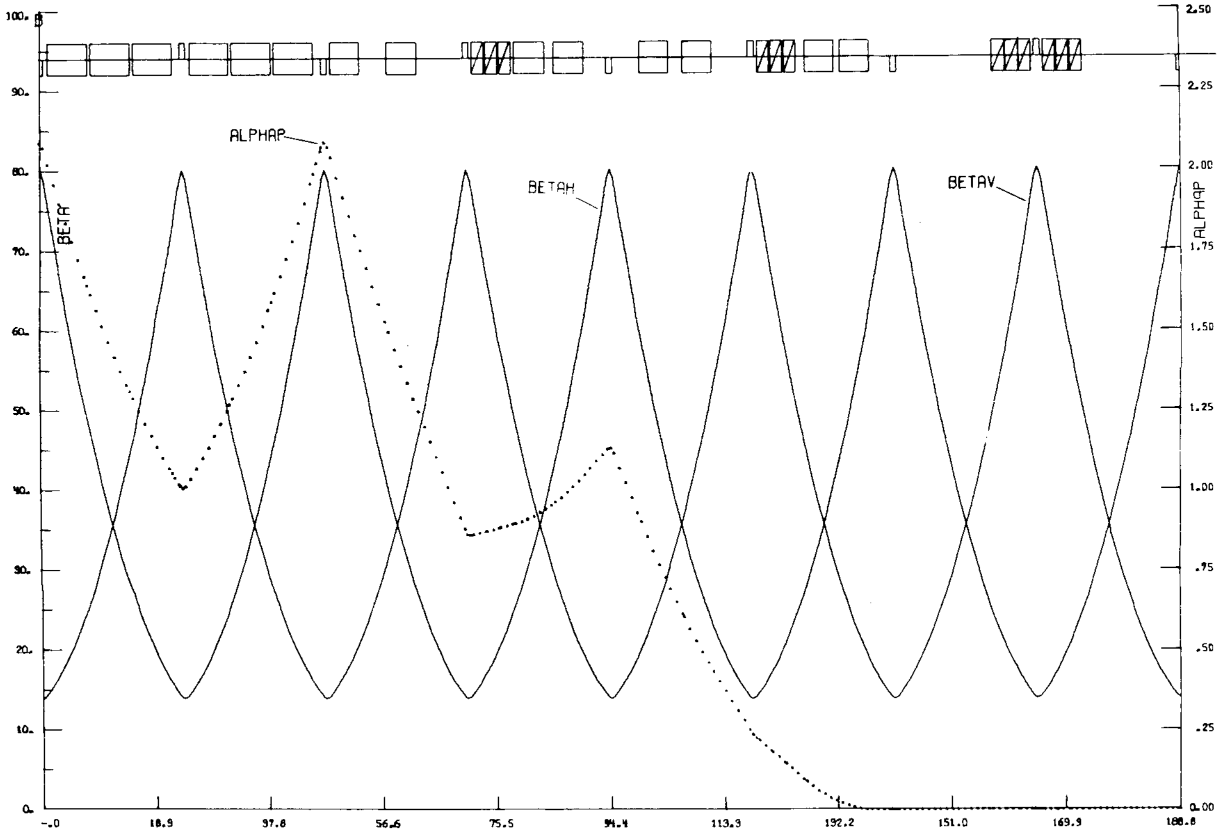

FIGURE 3 Dispersion suppressor incorporating wiggler magnets.

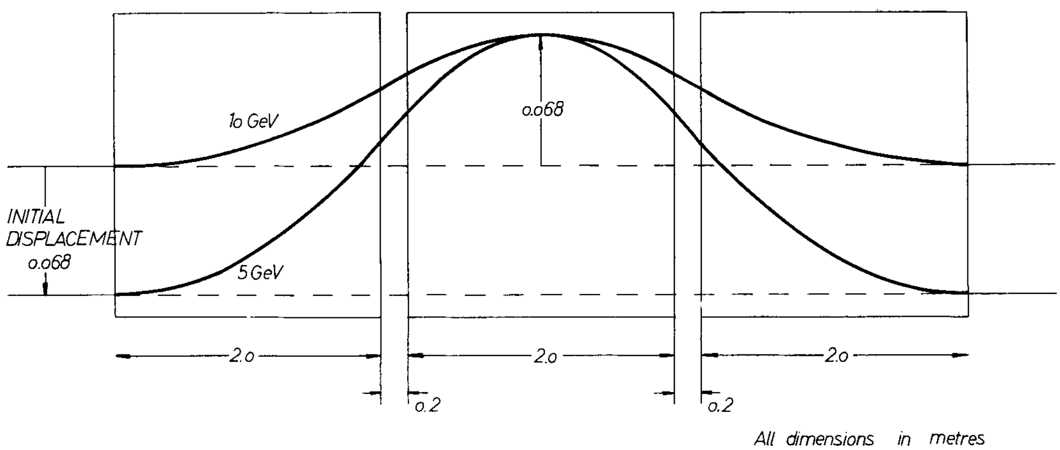

FIGURE 4 Beam trajectory at 10 GeV with centred incoming beam and trajectory at 5 GeV with off-centred incoming beam.

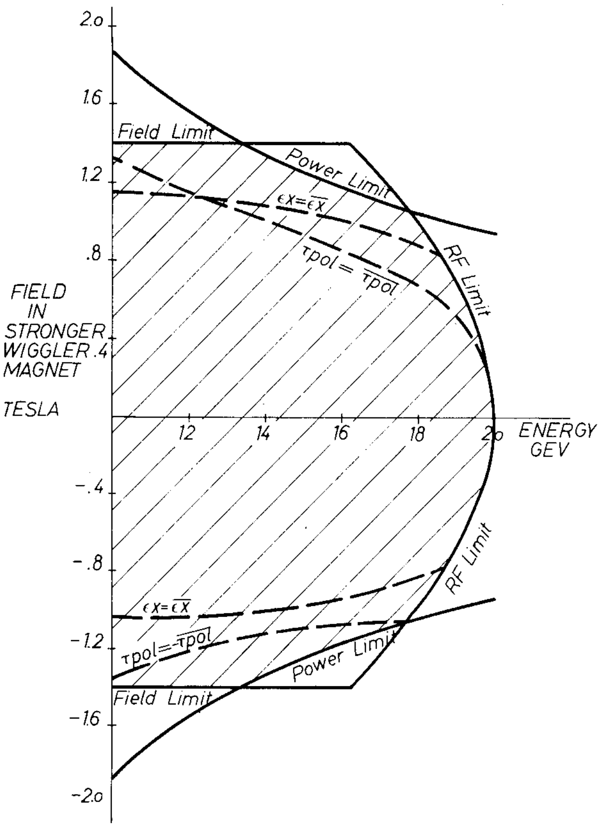

FIGURE 5 Magnetic field in stronger wiggler magnet as a function of energy. Boundary conditions are: field limit ± 1.4 T, rf limit 27.5 MeV and power limit 110 kW/m. Magnet length = 2 m. Also shown are the fields required to keep $\tau_{pol} = \overline{\tau_{pol}}$, $\tau_{pol} = -\overline{\tau_{pol}}$ and $\varepsilon_x = \overline{\varepsilon_x}$.

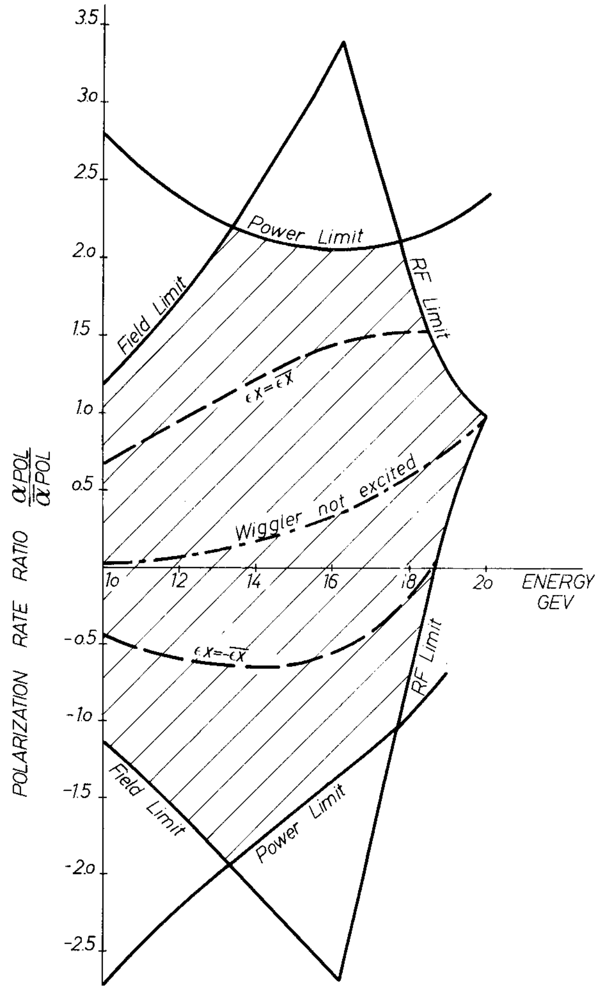

FIGURE 6 Polarization rate ratio as a function of energy. Boundary conditions are: field limit ± 1.4 T, rf limit 27.5 MeV, and power limit 110 kW/m. Magnet length = 2 m. Also shown is the polarization rate obtained when the emittance is kept constant.

Figures 5 and 6 are self-explanatory but Figure 7 requires some comment. At all energies, the emittance first decreases with field in the wiggler magnets and then increases. The lower curve shows the minimum value obtainable. Above about 19 GeV the largest value of emittance is obtained with the wiggler switched off which at first sight appears strange. However, it is a simple outcome of the fact that the emittance obtained is a balance between the quantum excitation produced by the synchrotron radiation and the damping produced by the rf. In this region the damping increases faster than the excitation and therefore the wiggler causes the beam to shrink. Using the accurate formula for the average value of W gives a slight difference between normal and reverse polarization cases. In Figure 7 only the normal polarization case is plotted.

Since the wiggler blocks are either in a zero-dispersion zone or a dispersive zone, it is clear that if the power supplies were separate for these two zones, then a much greater variation of ε_x would be possible. It remains to be seen whether this is a sufficiently desirable feature to warrant the increased complexity and extra bus-bars.

CONCLUSION

The simultaneous control of the polarization time and the emittance at low energies and the additional feature of reverse polarization have been clearly

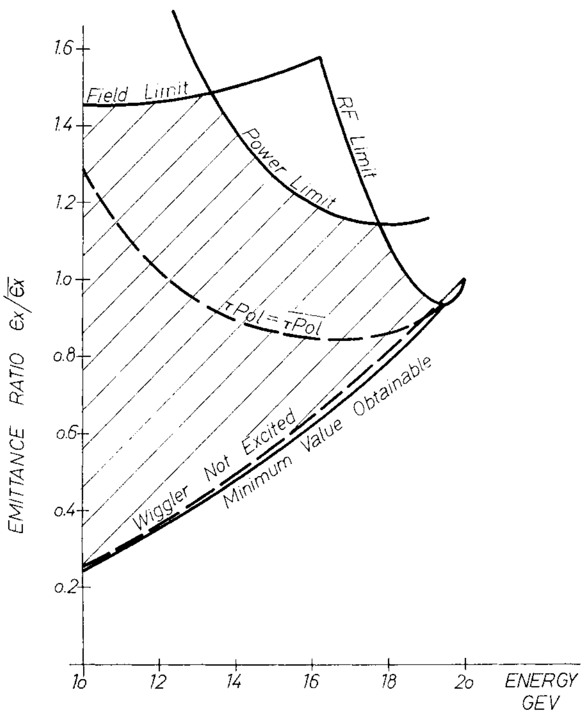

FIGURE 7 Emittance ratio as a function of energy for positive polarization case. Boundary conditions are: field limit 1.4 T, rf limit 27.5 MeV and power limit 110 kW/m. Magnet length = 2 m. Lower bound is the minimum value obtainable which is less than the case of wiggler magnets not excited. Also shown is the emittance obtained when $\tau_{\text{pol}} = \overline{\tau_{\text{pol}}}$.

demonstrated. The length of wiggler magnets is relatively modest and the advantages far outweigh one's natural reluctance to increase the complexity of the ring.

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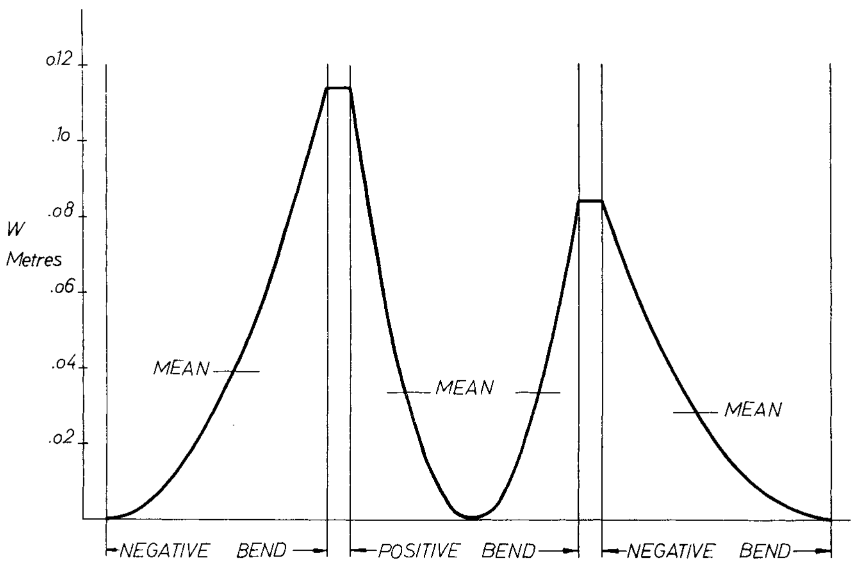
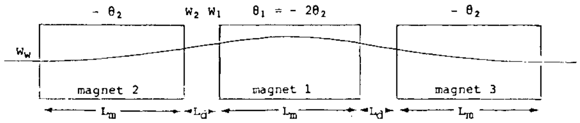


FIGURE 8 Variation of W over 2 m wiggler magnets. Energy 10 GeV. Field 1.3 T. Wiggler is placed in a dispersion free zone and the average value of W in the lattice is 0.062 m.

Appendix

Average value of W over wiggler magnets—exact formulation



A wiggler block is made up of three magnets of length L_m separated by a distance L_d. The central magnet produces a total bend θ_1 where

$$\theta_1 = \frac{L_m}{\rho_1} = \frac{L_w}{3N}\frac{2\pi}{L_0}\frac{b_1}{\xi} = \frac{\pi}{3N}\frac{2L_w}{L_0}\frac{b_1}{\xi},$$

where N is the number of wiggler blocks in the ring. The two outer magnets produce a total bend of $-\theta_2$ where

$$-\theta_2 = \frac{\pi}{3N}\frac{2L_w}{L_0}\frac{b_1}{2\xi}.$$

We require the average W over the central magnet and over the two outer magnets in terms of the betatron functions at the entrance of the first magnet. For convenience we define the following functions

$$\gamma = \frac{1+\alpha^2}{\beta}$$

$$I = \alpha\alpha_p + \beta\alpha'_p$$

$$K = \gamma\alpha_p + \alpha\alpha'_p$$

$$W = \frac{\alpha_p^2 + I^2}{\beta} = \frac{\alpha_p'^2 + K^2}{\gamma} = \gamma\alpha_p^2 + 2\alpha\alpha_p\alpha'_p + \beta\alpha_p'^2$$

and $\alpha_0, \beta_0, \gamma_0, I_0, K_0$ and W_w denote the values at the entrance to the first magnet. W_w is also the value of W over the whole wiggler block when the wiggler magnets are not excited. The changes in these quantities in a drift space and a long magnet have been given[5] and are rewritten in the present notation.

Drift space

$$W_{out} = W_{in}$$
$$I_{out} = I_{in} - L_d K_{in}$$
$$K_{out} = K_{in}$$
$$\beta_{out} = \beta_{in} - 2L_d\alpha_{in} + L_d^2\gamma_{in}$$
$$\alpha_{out} = \alpha_{in} - L_d\gamma_{in}$$
$$\gamma_{out} = \gamma_{in}$$

Bending magnet

$$W_{out} = W_{in} + \theta(2I_{in} - L_m K_{in}) + \theta^2\left(\beta_{in} - L_m\alpha_{in} + \frac{L_m^2}{4}\gamma_{in}\right)$$

$$I_{out} = I_{in} - L_m K_{in} + \theta\left(\beta_0 - \frac{3L_m}{2}\alpha_{in} + \frac{L_m^2}{2}\gamma_{in}\right)$$

$$K_{out} = K_{in} + \theta\left(\alpha_{in} - \frac{L_m}{2}\gamma_{in}\right)$$

$$\beta_{out} = \beta_{in} - 2L_m\alpha_{in} + L_m^2\gamma_{in}$$

$$\alpha_{out} = \alpha_{in} - L_m\gamma_{in}$$

$$\gamma_{out} = \gamma_{in}$$

Average W over a bending magnet

Denoting by W^* the value of W after a distance m in the magnet and defining ψ the bending angle per unit length

$$\psi = \frac{\theta}{L_m}$$

we obtain

$$W^* = W_{in} + m\psi(2I_{in} - mK_{in}) + m^2\psi^2\left(\beta_{in} - m\alpha_{in} + \frac{m^2}{4}\gamma_{in}\right)$$

or

$$W^* = W_{in} + 2I_{in}\psi m + (\beta_{in}\psi^2 - K_{in}\psi)m^2 - \alpha_{in}\psi^2 m^3 + \frac{\gamma_{in}}{4}\psi^2 m^4$$

The average value of W is then given by

$$\langle W\rangle = \frac{1}{L_m}\int_0^{L_m} W^*\,dm$$

and substituting we obtain a simple expression

$$\langle W\rangle = W_{in} + \theta\left(I_{in} - \frac{K_{in}L_m}{3}\right) + \theta^2\left(\frac{\beta_{in}}{3} - \frac{\alpha_{in}L_m}{4} + \frac{\gamma_{in}L_m^2}{20}\right).$$

Average W over the wiggler magnets

We now require the average value of W over each of the three magnets in the wiggler block as a function of the parameters at the entrance to the block. This requires defining W_{in}, I_{in}, K_{in}, β_{in}, α_{in} and γ_{in} in terms of W_w, I_0, K_0, β_0, α_0 and γ_0 for each magnet using the relationships for drift spaces and magnets quoted above. This conceptually simple but exceedingly tedious series of substitutions was carried out using SCHOONSCHIP,[9] the computer program which performs algebraic manipulations of this type. The answers are as follows

First magnet

$$\langle W'_2\rangle = W_w + \theta_1\left\{-\frac{I_0}{2} + K_0\frac{L_m}{6}\right\} + \theta_1^2\left\{\frac{\beta_0}{12} - \alpha_0\frac{L_m}{16} + \gamma_0\frac{L_m^2}{80}\right\}$$

Third magnet

$$\langle W_3\rangle = W_w + \theta_1\left\{\frac{I_0}{2} - K_0\left(\frac{4L_m}{3} + L_d\right)\right\} + \theta_1^2\left\{\frac{\beta_0}{12} - \alpha_0\left(\frac{7L_m}{16} + \frac{L_d}{3}\right) + \gamma_0\left(\frac{23L_m^2}{40} + \frac{7L_mL_d}{8} + \frac{L_d^2}{3}\right)\right\}$$

Average of weak magnets

$$\langle W_2\rangle = W + \theta_1\left\{-K_0\left(\frac{7L_m}{12} + \frac{L_d}{2}\right)\right\} + \theta_1^2\left\{\frac{\beta_0}{12} - \alpha_0\left(\frac{L_m}{4} + \frac{L_d}{6}\right) + \gamma_0\left(\frac{47L_m^2}{160} + \frac{7L_mL_d}{16} + \frac{L_d^2}{6}\right)\right\}$$

Strong magnet

$$\langle W_1\rangle = W_w + \theta_1\left\{-K_0\left(\frac{5L_m}{6} + L_d\right)\right\} + \theta_1^2\left\{\frac{\beta_0}{12} - \alpha_0\left(\frac{L_m}{4} + \frac{L_d}{6}\right) + \gamma_0\left(\frac{29L_m^2}{80} + \frac{2L_mL_d}{3} + \frac{L_d^2}{3}\right)\right\}$$

And finally the weighted sum required in the emittance calculation

$$4\langle W_1\rangle + \langle W_2\rangle = 5W_w + \theta_1\left\{-K_0\left(\frac{47L_m}{12} + \frac{9L_d}{2}\right)\right\} + \theta_1^2\left\{\frac{5\beta_0}{12} - \alpha_0\left(\frac{5L_m}{4} + \frac{5L_d}{6}\right) + \gamma_0\left(\frac{261L_m^2}{160} + \frac{149L_mL_d}{48} + \frac{3L_d^2}{2}\right)\right\}.$$

If the magnet length and drift spaces are small, we have approximately that

$$\langle W_1\rangle \approx \langle W_2\rangle \approx W_w + \frac{\beta_0\theta_1^2}{12}$$

which was the expression derived in the text by simpler means.

It should be noted that W_{out} for the third magnet is exactly equal to W_w. In general, however, there will be a slight difference due to edge focussing of the wiggler magnets and this will necessitate some compensation.

In Figure 8, the variation of W through the wiggler magnets is shown for one of the wiggler blocks used in the example. The curve was calculated numerically while the mean values are those obtained from the formula above. The agreement is excellent.

Average W over four equal field wiggler magnets

For the sake of completeness the same process is carried out for the case of equal field wiggler magnets of bending angle $-\theta_1, \theta_1, \theta_1, -\theta_1$ respectively.

First magnet

$$W_1 = W_w + \theta_1\left\{-I_0 + K_0\frac{L_m}{3}\right\} + \theta_1^2\left\{\frac{\beta_0}{3} - \alpha_0\frac{L_m}{4} + \gamma_0\frac{L_m^2}{20}\right\}$$

Second magnet

$$\langle w_2\rangle = W_w + \theta_1\left\{-I_0 + K_0\left(-\frac{L_m}{3} - L_d\right)\right\} + \theta_1^2\left\{\frac{\beta_0}{3} - \alpha_0\left(\frac{L_m}{12} - \frac{L_d}{3}\right) + \gamma_0\left(\frac{13L_m^2}{60} + \frac{5L_mL_d}{12} + \frac{L_d^2}{3}\right)\right\}$$

Third magnet

$$\langle w_3\rangle = W_w + \theta_1\left\{I_0 + K_0\left(-\frac{13L_m}{3} - 4L_d\right)\right\}$$
$$+ \theta_1^2\left\{\frac{\beta_0}{3} - \alpha_0\left(\frac{13L_m}{12} + \frac{7L_d}{3}\right)\right.$$
$$\left. + \gamma_0\left(\frac{313L_m^2}{60} + \frac{19L_mL_d}{2} + \frac{13L_d^2}{3}\right)\right\}$$

Fourth magnet

$$\langle W_4\rangle = W_w + \theta_1\left\{I_0 + K_0\left(-\frac{11L_m}{3} - 3L_d\right)\right\}$$
$$+ \theta_1^2\left\{\frac{\beta_0}{3} - \alpha_0\left(\frac{29L_m}{12} - 2L_d\right)\right.$$
$$\left. + \gamma_0\left(\frac{263L_m^2}{60} + \frac{29L_mL_d}{4} + 3L_d^2\right)\right\}$$

And the average value over all the wiggler magnets is

$$\langle W\rangle = W_w - 2K_0\theta_1(L_m + L_d)$$
$$+ \theta_1^2\left\{\frac{\beta_0}{3} - \alpha_0\left(\frac{4L_m}{3} + L_d\right)\right.$$
$$\left. + \gamma_0\left(\frac{37L_m^2}{15} + \frac{103L_mL_d}{24} + \frac{23L_d^2}{12}\right)\right\}$$

If the magnet length and drift spaces are small, we have approximately that

$$W \approx W_w + \frac{\beta\theta_1^2}{3}$$

Note added in proof:

A. N. Skrinsky (private communication, August 1976) has indicated that the presence of reverse bending in the ring affects the asymptotic limit of the degree of polarization which is given by the expression

$$\frac{8}{5\sqrt{3}}\,\frac{\oint H^3\,\mathrm{d}\theta}{\oint |H^3|\,\mathrm{d}\theta}.$$

For the present study this would imply a steady reduction in the degree of polarization from 92% at 20 GeV (wiggler not excited) to 72% at 10 GeV for the normal polarization case and an approximately constant value of 72% over the entire accessible energy range for the reversed, polarization case.