THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

CONTROL OF TRANSIENT FREE-SURFACE FLOW

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INTRODUCTION

Computational methods for the analysis of unsteady open-channel flow are available and satisfactory verification of the computational procedures have been obtained by comparisons with laboratory and field data. The concept of control of a water surface profile in a desired manner has been referred to in the literature[2,3] but a theory to define the gate motion has not been clearly set forth. The objective of this paper is to provide a theoretical method that will yield a predetermined gate motion to produce a desired water surface profile in a channel. The analytical procedure determines the motion of the control devices in the channel so the unsteady flow conditions are known and controlled during the time that the channel flow is being altered. The method begins with a given initial flow and stage along the channel and prescribes the operation of the gates or valves so the final steady flow is established along the channel in a minimum of time, or so that neither specified stage variations or limitations on the rate of change of the levels are exceeded. Thus, the method falls into the category of a design tool as opposed to analysis. With present technology making on-line computer control of valves and gated structures a reality, a theory to define the manner in which these controls should be operated

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[2] Streeter, V. L., and Wylie, E. B., Hydraulic Transients, McGraw-Hill Book Co., New York, New York, 1967, Chapter 15.

[3] Reynolds, R. R., and Madsen, W. R., "Automation in California State Water Project," Jour. of the Pipeline Division, ASCE, PL3 November 1967, p. 15.

is necessary. Such a theory is available for closed pipes.[4] This paper presents a parallel theory for use in open channels.

To the writer's knowledge the first mention of such a control concept in open channels was made by Streeter.[2] Although the need for a suitable theory has been apparent for years in connection with forebay channels and irrigation works, the major water project of the State of California[3] emphasizes the need for a complete design procedure for gate operation.

The procedure set forth herein permits the solution of such problems as the operation of a gate at the end of a channel in order to establish steady state conditions in a minimum of time, or to change the flow condition from one given condition (steady or unsteady) to another desired condition; the operation of gates at both ends of a channel to accomplish the same objectives; the operation of valves that control forebay channels at a pumping station or hydroelectric plant to best initiate, alter, or stop the flow; the operation of control devices in a canal system to alter the flow from one discharge to another while remaining within specified depth limitations or without exceeding a certain rate of allowable fluctuation of water level; the operation of a valve in a system that involves both conduits flowing full, as at pumping station, and conduits or canals with free surfaces in order to alter flow conditions in a prescribed manner.

The basic theory is first presented and the normal analysis of unsteady open channel flow is discussed. The concepts that are

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4 Streeter, V. L., "Valve Stroking for Complex Piping Systems," <u>Journal of the Hydraulics Division</u>, <u>ASCE</u>, Vol. 93, HY3, May, 1967, p. 81.

necessary to prescribe the motion of the control mechanisms to yield desired results are then presented and the method of numerical solution is set forth. Examples of the application of the theory to two different control problems are included. In these examples the theory specifies the action of the control devices in order to yield the desired channel flow; then, with the initial conditions given and the boundary conditions specified for the duration of the transient, a check can be made on the results of the theory by applying any of the accepted analysis procedures for unsteady flow in open channels. Thus, a verification of the theory is possible by use of accepted analytical techniques, even though laboratory or prototype data are not available.

ANALYSIS

Basic Equations and their Solution

Application of the unsteady momentum and continuity equations to a control volume of small length that extends across the channel yields the following partial differential equations.[2]

$$gy_x + gS - gS_o + VV_x + V_t = 0 \qquad (1)$$

$$\frac{A}{T} V_x + Vy_x + y_t = 0 \qquad (2)$$

The depth of flow in the channel is given by y, average velocity at a section by V, cross-sectional area by A, width of the water surface by T, acceleration due to gravity by g, distance along the channel by x, and time by t. When used as subscripts the independent variables x and t indicate partial differentiation. One-dimensional flow has been assumed in a prismatic channel, the slope of the channel is small and can be defined by S_o, hydrostatic conditions exist along any vertical line in the liquid, and the shear force at the wetted perimeter can be accounted for by use of Manning's equation for steady uniform flow, thus S is the slope of the energy grade line. It is assumed that a finite depth of liquid remains at each section in the channel at all times during the transient and that changes are never so rapid as to create a hydraulic bore.

The basic equations are not amenable to closed form solution and are generally handled by numerical techniques. Two of the most widely accepted current procedures for the solution of the equations involve either the placing of the partial differential equations into finite difference form which leads to a solution of a set of simultaneous equations,(5,6,7) or the conversion of the partial differential equations into four particular ordinary differential equations which are integrated by use of numerical approximations.(2,7,8,9) The advantages and disadvantages of these methods have been discussed by a number of authors and it seems fairly clear that, when properly applied, either method will give the correct analysis of a particular problem. Comparisons between computed and field data show results which are well within the limits of accuracy of the basic input data.

In this study the second procedure is used, namely the method of characteristics. After combining Equations (1) and (2) by use of a linear multiplier and evaluating the particular value of the multiplier, the partial differential equations can be written as four particular total differential equations.(2)

$$\left.\begin{aligned} \sqrt{\frac{gT}{A}}\frac{dy}{dt} + \frac{dV}{dt} + g(S-S_o) &= 0 \qquad (3) \\ \frac{dx}{dt} &= V + \sqrt{\frac{gA}{T}} \qquad (4) \end{aligned}\right\} C^+$$

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5 Desikan, V., "Analysis of Flood Propagation through Channels and Reservoirs: Implicit Finite Difference Method," Zeitschrift für Angewandte Mathematik und Mechanik, Band 46, Heft 6, 1966, p.377.

6 Isaacson, E., Stoker, J. J., and Troesch, A., "Numerical Solutions of Flood Prediction and River Regulation Problems," New York Univ., Inst. of Mathematical Sci., Report No. IMN-NYU-235, 1956.

7 Baltzer, R. A., and Lai, C., "Computer Simulation of Unsteady Flows in Rivers and Estuaries," ASCE, Hydraulics Div. Conf., Madison, Wisc., August, 1966.

8 Fletcher, A. G., and Hamilton, W. S., "Flood Routing in an Irregular Channel ASCE, Journal of the Engineering Mechanics Division, EM3, June, 1967, p.45.

9 Liggett, J. A., "Unsteady Open Channel Flow with Lateral Inflow, Tech. Report No. 2, Dept. of Civil Engineering, Stanford Univ., Calif, July, 1959.

$$\left.\begin{aligned} -\sqrt{\frac{gT}{A}}\ \frac{dy}{dt} + \frac{dV}{dt} + g(S-S_o) = 0 \qquad (5) \\ \frac{dx}{dt} = V - \sqrt{\frac{gA}{T}} \qquad (6) \end{aligned}\right\} C^-$$

These four equations describe the flow conditions in the channel just as Equations (1) and (2) since no approximations have been necessary in recasting the equations in this form. However, Equations (3) to (6) have restrictions on the manner in which they can be applied. The previous independent variable x is now dependent upon t, as represented by Equation (4), and Equation (3) can only be applied when Equation (4) is satisfied. These are the so-called C^+ characteristic equations. Equations (5) and (6) form the C^- characteristic equations and are subject to corresponding restrictions.

Equations (4) and (6) can be represented in the xt plane, Figure 1. At the intersection point P, all four equations are valid and a solution for the variables x, t, V, and y is theoretically possible. Thus if conditions are known at the points designated R and S in Figure 1, a numerical integration of Equations (3) to (6) will yield conditions at point P.

It can also be recognized that if conditions are known at some point along the dashed continuation of the C^+ characteristic, say at R' in place of R, an equally valid solution can be obtained for conditions at point P. Similarly conditions may be known at S' instead of S and a solution can again be determined for conditions at point P. As an illustration for further clarification, if conditions are known at point S'' as well as at point R, it is not possible to find conditions at point P (unless the C^- characteristic

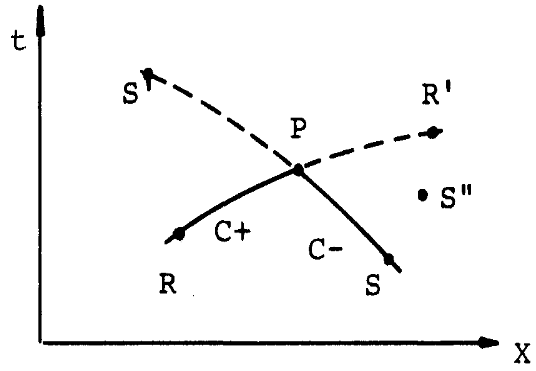

Figure 1. Characteristic Lines in the xt Plane.

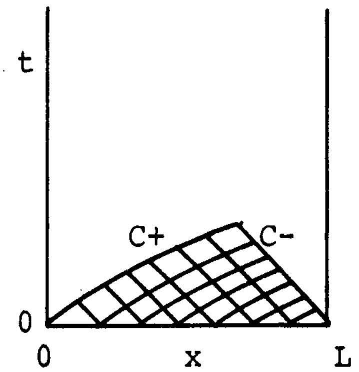

Figure 2. Region of Dependence, Initial Conditions Given.

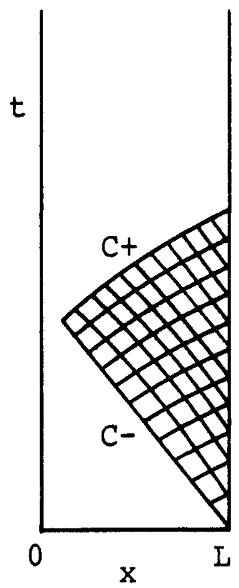

Figure 3. Region of Dependence, Conditions Given at $x = L$.

through S" also happens to pass through the given point P). Additional data would have to be available to obtain the exact solution at the desired point P .

When this interpretation is placed on the characteristics in the xt plane it can be seen that if the initial conditions along a channel are known, a solution can be obtained at any point within the cross-hatched zone of dependence shown in Figure 2. By providing boundary conditions at $x = 0$ and $x = L$, a complete solution can be carried forward in time. This is the standard method of analysis of a transient open-channel flow problem when the characteristics method is used.(2)

Concepts of Control Computation

In order to extend these same concepts of analysis into the realm of design to control transient conditions in a channel it is necessary for computations to be made which will permit the specification of the motion of the control mechanisms (boundary conditions) as a function of time during the period of the transients. This can be accomplished by completely specifying conditions (depth and discharge as a function of time) at the downstream boundary, $x = L$.

By specifying depth and discharge at the downstream boundary for a short duration a complete solution is possible in the cross-hatched portion of Figure 3. This is the domain of dependence since conditions within the hatched region are uniquely determined by the given conditions. By extending the duration of time over which conditions are specified at the downstream boundary, the entire xt plane can be made a domain of dependence except above the upper C^+ and below the lower C^- characteristics, Figure 4.

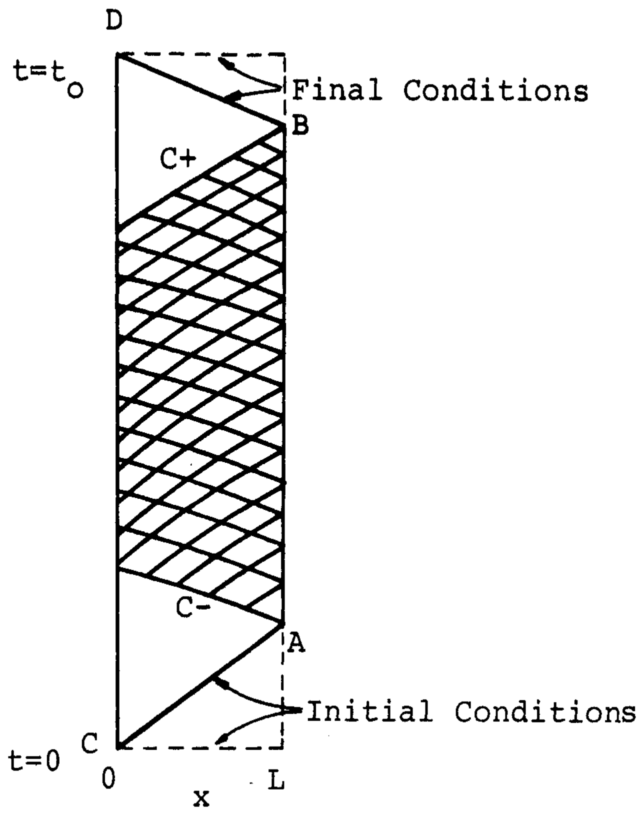

Figure 4. The xt Plane.

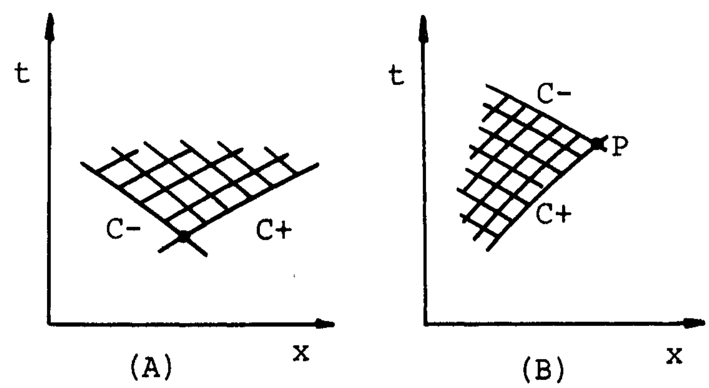

Figure 5. Region of Influence.

Furthermore, if the characteristic lines are made to issue from a point P , Figure 5, the zone above the characteristics in Figure 5a and to the left of the characteristics in Figure 5b is known as the region of influence. If a change is made in conditions at point P conditions within the region are influenced. Thus in Figure 4, conditions at points A and B influence what is happening within both the lower and upper triangles, respectively.

Therefore, in order to obtain the complete gate stroking solution it is necessary to perform computations in the entire region enclosed by the solid lines, Figure 4. Initial conditions exist in the region below the characteristic line AC and final conditions prevail above the upper characteristic BD . It is not necessary to have steady initial conditions in the channel; however, the initial data must be defined. If care is taken not to violate the assumptions placed upon the original differential equations, a given flow condition, steady or unsteady, can be transformed into another desired final flow condition in time t_0 , Figure 4. In a reasonably straight prismatic channel the main assumptions of concern are that some depth must be maintained at all sections along the channel and changes must not be made so rapidly that a hydraulic bore may be created. If a restriction exists on the rate of change of depth or discharge, it is incorporated into the specification of the boundary condition.

The above description outlines the basic concepts that lead to the control of transient free-surface flow. By specifying the depth and flow at the downstream end in a channel with a given initial flow, the transient condition is specified throughout the entire channel and, in particular, the depth and discharge can be computed at the upstream end. With the depth and discharge available as a function of time the necessary gate motion can also be computed. Thus the gate motion as a function of time at each end of the channel is available to change the flow from a certain initial condition to a desired final steady flow condition. It is equally feasible to specify conditions at the upstream end. The analysis then yields the requirements at the downstream end.

Numerical Methods

With the concepts of the control computations established the details of the solution of Equations (3) to (6) are presented. The simplest application of the characteristics method involves the integration of the equations by use of a first order approximation. However in this application a second order procedure[10] is preferred since the accuracy is considerably improved. If a rapid flow change is demanded in a problem the corresponding depth variation produces a significant variation in surge velocity. When the dependent variables change rapidly along a characteristic the second order procedure is needed to produce accurate results.

In Figure 1, it is assumed conditions are known at R and S and it is desired to find x, t, y, and V at P. By substituting $C = \sqrt{gA/T}$ and integrating along the C^+ characteristic utilizing the trapezoidal rule, Equations (3) and (4) become:

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[10] Lister, M., "Numerical Solutions of Hyperbolic Partial Differential Equations by the Method of Characteristics," Mathematical Methods for Digital Computers, Ralston and Wilf, editors, John Wiley and Sons, New York, 1960.

$$\frac{g}{2}\left(\frac{1}{C_R}+\frac{1}{C_P}\right)(y_P-y_R)+V_P-V_R+\frac{g}{2}(S_R+S_P-2S_o)(t_P-t_R)=0 \quad (7)$$

$$x_P-x_R=\left(\frac{V_P+V_R}{2}+\frac{C_R+C_P}{2}\right)(t_P-t_R) \quad (8)$$

where the subscripted letters R and P indicate the point at which the variable is evaluated. A similar integration along the C^- characteristic yields:

$$-\frac{g}{2}\left(\frac{1}{C_S}+\frac{1}{C_P}\right)(y_P-y_S)+V_P-V_S+\frac{g}{2}(S_S+S_P-2S_o)(t_P-t_S)=0 \quad (9)$$

$$x_P-x_S=\left(\frac{V_P+V_S}{2}-\frac{C_S+C_P}{2}\right)(t_P-t_S) \quad (10)$$

In the simultaneous solution of these four equations for the unknown variables at point P , the parameters C_P and S_P must be evaluated using the unknown quantities A_P, T_P and R_P . An iteration procedure that begins with assumed values equal to average conditions from R and S quickly converges to the proper solution, (see Appendix I).

Although Equations (7) to (10) are written along the characteristics through R and S which project forward in time, Figure 1, they are equally valid for the C^+ and C^- characteristics through points R' and S , projecting in the negative x direction in Figure 1, or from R and S' , projecting in the positive x direction. Boundary conditions are easily introduced with the appropriate characteristic line to incorporate the influence of any boundaries.[2]

Numerical results obtained by the second order approximations in Equations (7) to (10) yield accurate results if successive changes in x and t are not too large. Specific limits cannot be placed on the size of these changes since the limits are dependent upon the rate of change of the dependent variables.

The two recommended procedures[10] for obtaining an organized numerical solution of Equations (7) to (10) are the grid of characteristics and the method of specified time intervals. The latter method produces a more orderly solution but is subject to some error due to necessary interpolations. For this reason the grid of characteristics is preferred. In this method the grid grows in the xt plane as shown in Figure 6, where depth and discharge have been specified at specific times along the boundary at $x = L$. As can be seen in the figure it is unlikely for a grid intersection to fall exactly on the boundary at $x = o$. It is necessary to assume that the channel extends in the negative x direction to evaluate conditions beyond the channel entrance. Conditions at the exact boundary location are determined by a direct interpolation. For example, y, V, and t are determined at point 3 in Figure 6, by use of a linear interpolation between points 1 and 2.

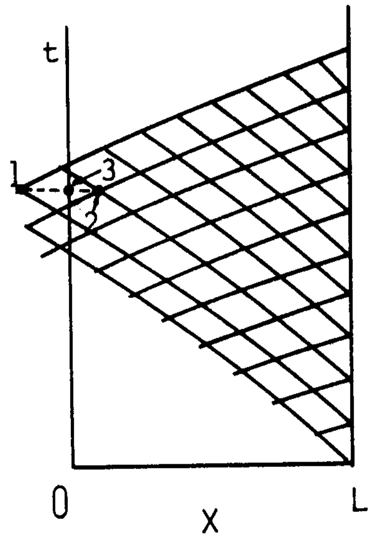

Figure 6. Linear Interpolation at Upstream Boundary.

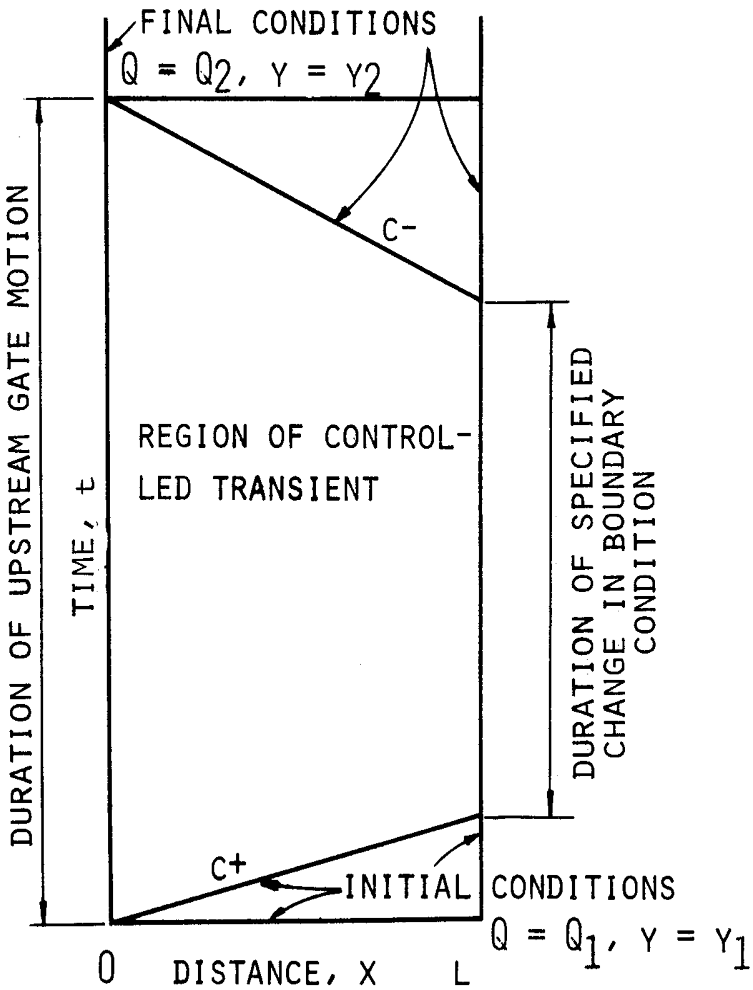

Figure 7. Example 1.

ILLUSTRATIVE EXAMPLES

Example 1

A trapezoidal channel of 4000 foot length delivers water from one reservoir to a second reservoir, both of fixed elevation. Channel properties are $S_o = 0.0002$, $n = 0.015$, $B = 10$, and side slopes 1:1. Gates at each end of the channel are used to control the depth and flow in the channel. Flow is initially steady uniform at a depth, $y_1 = 6$ feet, and discharge $Q_1 = 314.5$ cfs. It is desired to change the flow to steady uniform flow at a depth $y_2 = 4.5$ feet and discharge $Q_2 = 185.2$ cfs.

The stroking solution as it appears in the xt plane is shown in Figure 7. The specified boundary condition at the downstream end of the channel states that the discharge should vary linearly from Q_1 to Q_2 and the depth should vary linearly from y_1 to y_2. The manner in which the variation of the variables is specified is arbitrary; it is not necessary that the changes be made linearly.

Numerical values are presented in Figures 8 and 9 for two possible solutions, differing only in the duration of the transient. In the first, the total operation time of the upstream gate is 26.31 minutes, and the downstream gate is 13.2 minutes beginning 4.4 minutes after the initial motion of the upstream control. The depth and discharge variation at both boundary conditions are shown as a function of time in Figure 8.

In the second case the changes are considerably more rapid but the same final condition is realized, that is, steady uniform flow exists in the channel when the gate motion ceases. The total time of

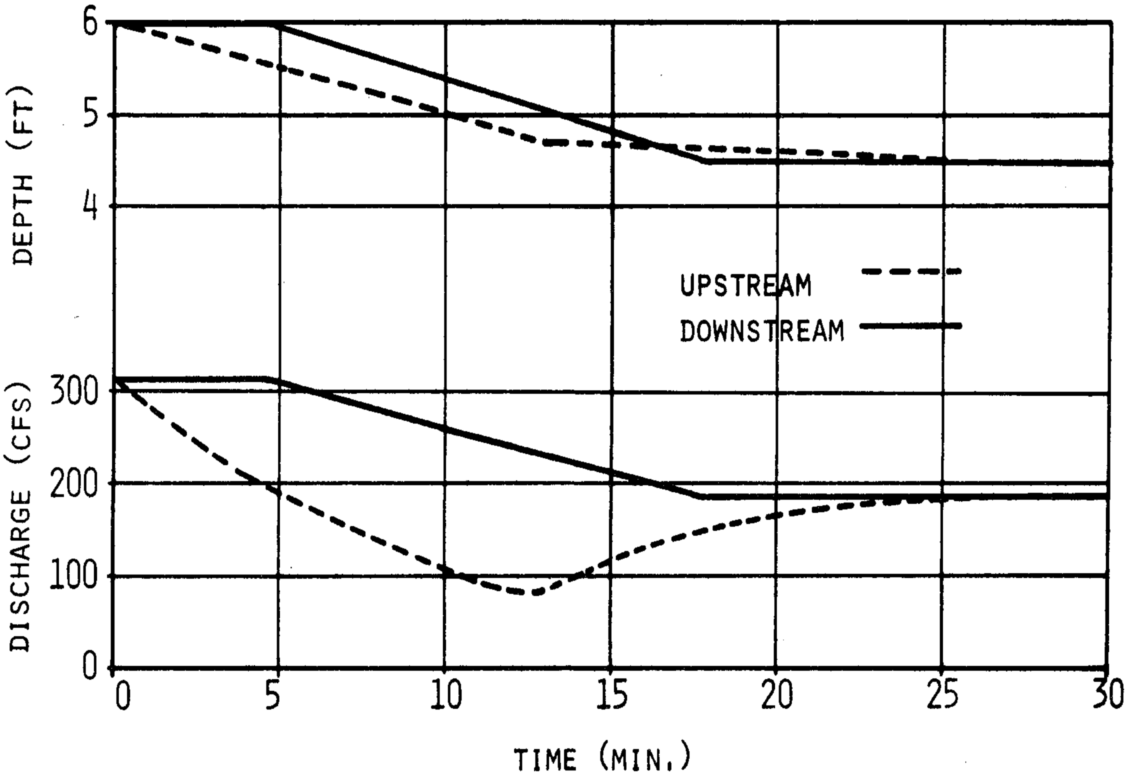

Figure 8. Depth and Discharge at the Boundaries. Example 1, Case 1.

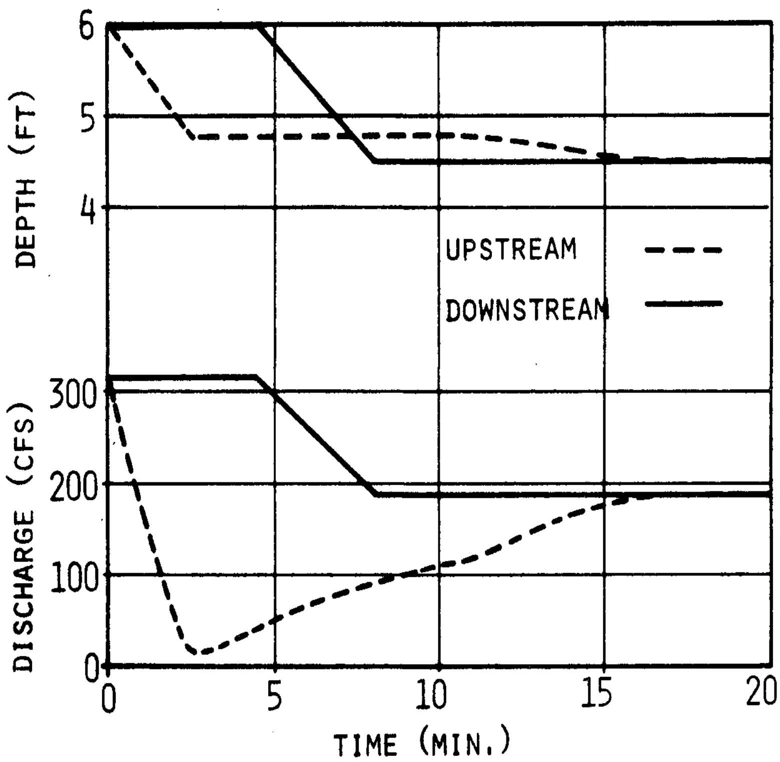

Figure 9. Depth and Discharge at the Boundaries. Example 1, Case 2.

gate operation is 16.6 minutes with the upstream gate in motion over the entire period and the downstream gate commencing 4.4 minutes after the initial upstream motion and continuing for 3.5 minutes. Conditions at the boundaries are shown in Figure 9.

In either of these cases the actual gate position versus time can be computed if the gate calibration is known since the elevations on each side of each gate is specified as well as the discharge. As a check on the accuracy of the results shown in Figures 8 and 9, the computed gate motions, as specified in the above analysis, were placed as boundary conditions on a standard characteristics method analysis. The same initial conditions were used. The objective was to see if the flow actually returned to steady uniform at the new discharge immediately after the gate motion ceased. This analysis confirmed the gate stroking computations within 0.5 percent in both depth and discharge.

A few simple modifications in this example could make it applicable to a number of more meaningful problems. For example, the downstream reservoir could be a pumping station forebay with the flow and elevation controlled by valves in the pumping station. The stroking analysis could then yield the necessary valve operation and upstream gate operation to change from two pumps operating to one or no pumps operating, and vice versa. Or, the stroking analysis could yield the optimum gate motion if this were a power canal leading to a hydroelectric plant.

Consideration of the mechanics of the flow in a channel in order to reduce the discharge quickly leads to some important modifications to the above concepts of control. Some of the forward momentum must be destroyed in the channel if the flow rate is to be decreased. The stroking can therefore be carried out more effectively by holding the depth constant, or even increasing the depth at the downstream end, either temporarily or permanently. If the flow is to be increased the reverse would be advised. This idea leads to the next example.

Example 2

A long trapezoidal channel has control gates at intervals along its length. One reach is 40000 feet in length between control gates. Channel physical characteristics are $S_o = 0.00004$, $n = 0.0162$, $B = 85$, and side slopes are two horizontal to one vertical. The initial flow in the channel is steady nonuniform at 11000 cfs. It is desired to reduce the flow to steady nonuniform flow at 6000 cfs.

The "controlled volume concept" suggested by Reynolds and Madsen(3) is also incorporated into this solution. This concept says that when a flow change is made in a channel the total volume of water within the reach should remain approximately constant. In this case a depth of 24.5 feet is maintained at the mid-point of the channel.

The specified information and the stroking solution are shown in the xt diagram of Figure 10. A direct way to consider the stroking solution in this example is to view the analysis as two separate problems, one in the upstream half of the channel with conditions specified at the right end, and the other in the downstream half with identical conditions specified at the left end. The specified values at the common mid-point are: the depth remains constant at 24.5 feet, and the flow is reduced linearly from 11000 cfs to 6000 cfs.

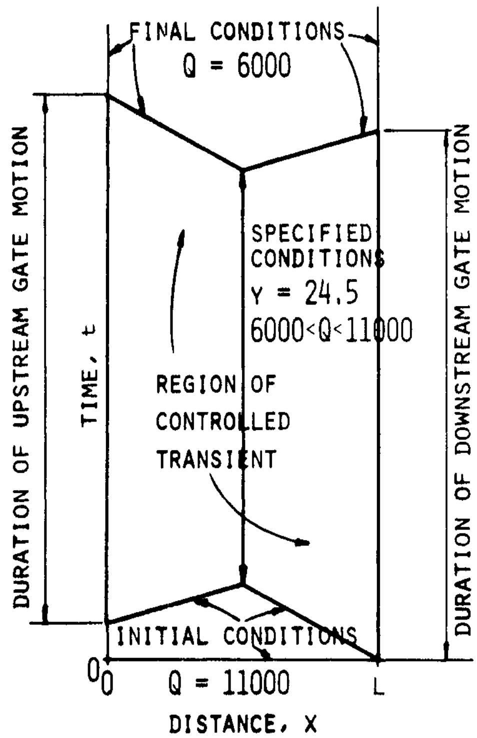

Figure 10. Example 2.

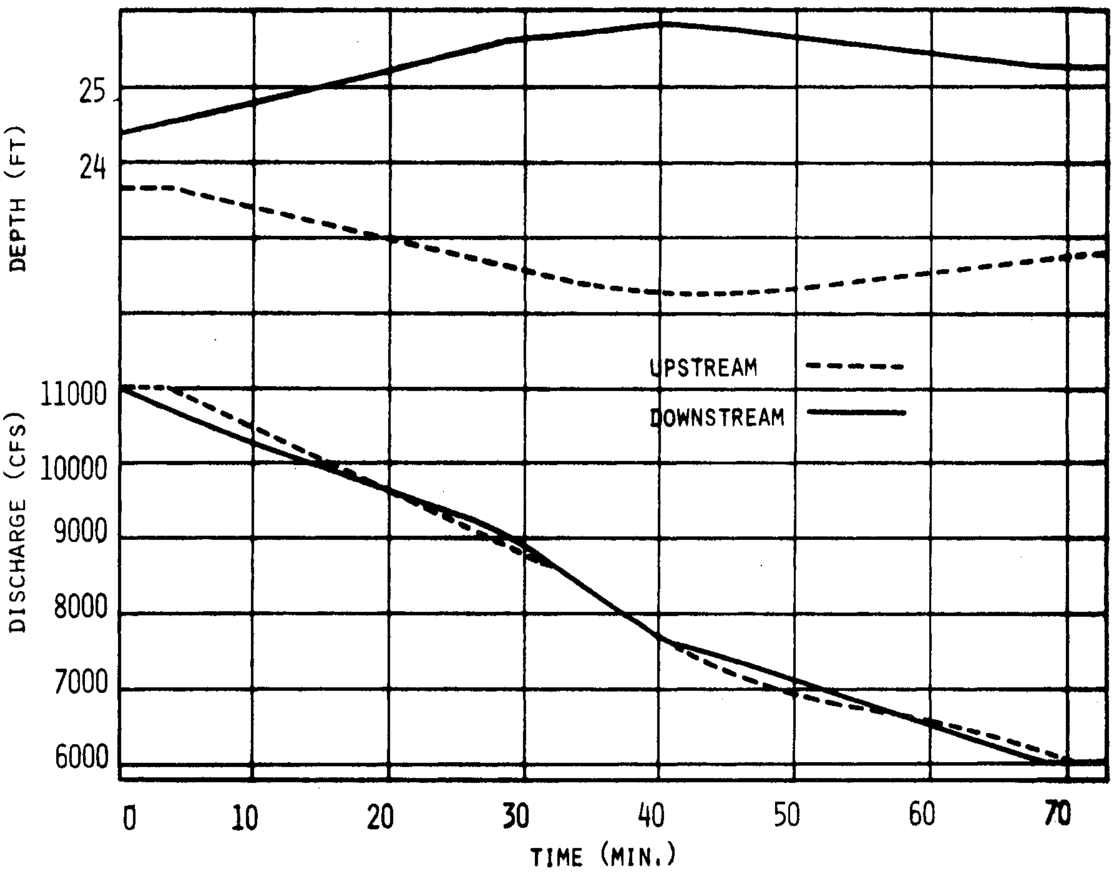

Figure 11. Depth and Discharge at the Channel Ends. Example 2, Case 1.

Some of the numerical results of two possible solutions are presented in Figures 11 and 12. The only difference between the two solutions is the duration of the transient and the extent of variation of the water surface profile while being altered. For both cases the initial and final water surface profiles are shown in Figure 13. It can be seen that the volume of water within the reach remains approximately constant for the initial and final conditions. If is were desirable to have both gates commence operating at the same instant, examination of Figure 10 shows that a shift of the specified conditions line to the right of the channel mid-point would bring this about.

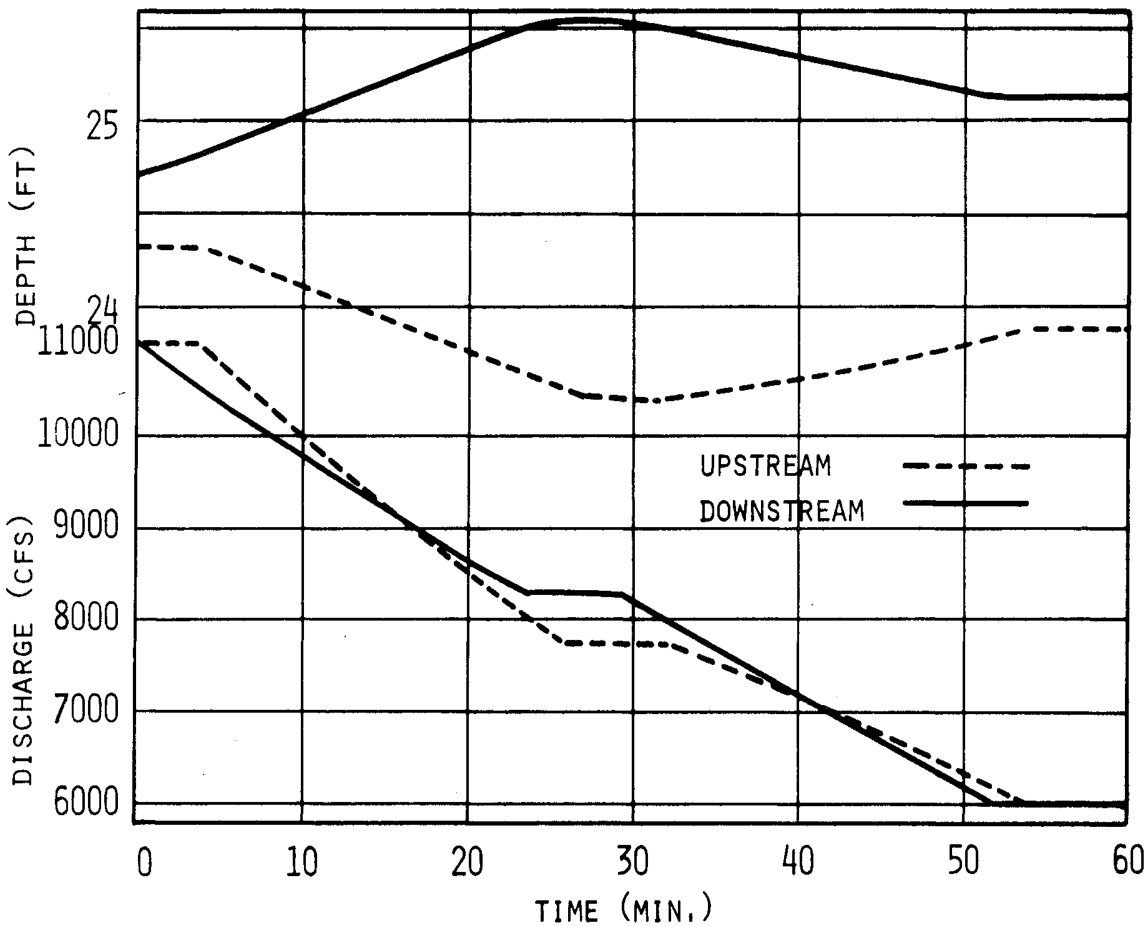

Figure 12. Depth and Discharge at the Channel Ends, Example 2, Case 1.

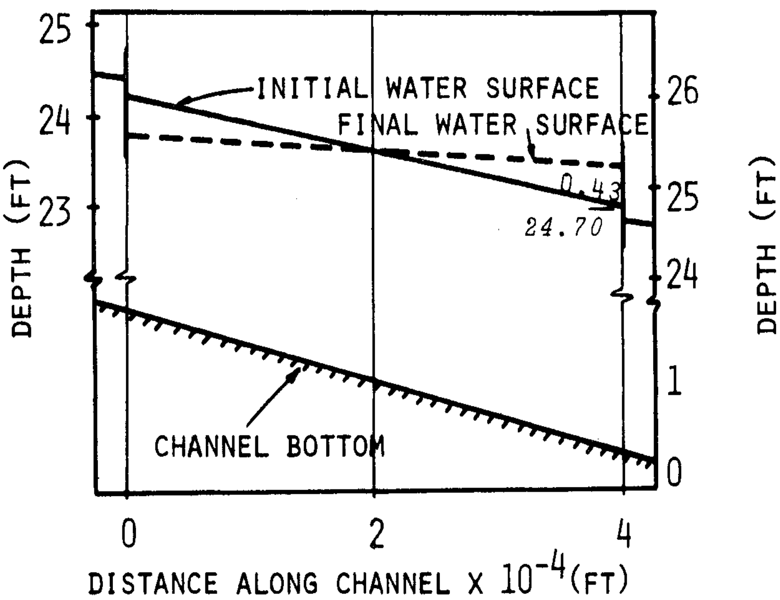

Figure 13. Water Surface in Channel.

CONCLUSIONS

The concept of controlled flow in open channels is introduced. A method of analysis is presented which enables the motion of valves or gates to be specified so that the unsteady flow in the period between the initiation of the first control movement and the final movement is completely determined. In addition, when the final control motion ceases the flow is steady along the entire length of the channel.

The theory is presented and examples are given to show the application of the concepts to specific problems. Restrictions on the method and assumptions in the theory are discussed. The standard characteristics method of analysis is used to provide a check on the gate stroking theory as applied to the specific problems.

Controlled free-surface flow, as introduced in this study, provides design procedures for the theoretical determination of gate or valve movements to control unsteady open-channel flow. If an on-line computer is used in a canal system, input data regarding the instantaneous conditions in the channel together with input or stored data pertaining to normal operational methods and responses to emergency conditions can be analyzed in a few seconds. The computed results are then available for real-time computer control.

ACKNOWLEDGMENT

The research described herein was sponsored by the National Science Foundation, Project GK-721.

APPENDIX I

The variable x_P can be eliminated in a simultaneous solution of Equations (8) and (10)

$$t_P = \frac{2(x_S - x_R) + t_R(V_P + C_P + V_R + C_R) - t_S(V_P - C_P + V_S - C_S)}{V_R + C_R - (V_S - C_S) + 2C_P} \qquad (11)$$

Equation (7) can be written

$$V_P = C_3 - C_4 y_P \qquad (12)$$

where the coefficients C_3 and C_4 are dependent upon known values of velocity, depth and time at R and unknown values of velocity, depth and time at point P.

$$C_4 = \frac{g}{2}\left(\frac{1}{C_R} + \frac{1}{C_P}\right) \qquad (13)$$

$$C_3 = V_R + C_4 y_R - \frac{g}{2}(S_R + S_P - 2S_o)(t_P - t_R) \qquad (14)$$

Similarly Equation (9) can be written

$$V_P = C_1 + C_2 y_P \qquad (15)$$

where C_1 and C_2 are dependent upon known quantities at point S and unknown variables at point P.

$$C_2 = \frac{g}{2}\left(\frac{1}{C_S} + \frac{1}{C_P}\right) \qquad (16)$$

$$C_1 = V_S - C_2 y_S - \frac{g}{2}(S_S + S_P - 2S_o)(t_P - t_S) \qquad (17)$$

The simultaneous solution of Equations (12) and (15) yields

$$y_P = \frac{C_3 - C_1}{C_2 + C_4} \qquad (18)$$

An iteration procedure that will yield a sufficiently accurate result for the unknown variables at point P after the second iteration (unless there are great changes in depth and velocity) is as follows. Begin with assumed values of V_P and y_P equal to the average of the corresponding variables at R and S . After computing a value of C_P based on the assumed y_P , solve for t_P in Equation (11). With this first estimate of t_P and the assumed values of V_P and y_P , the coefficients in Equation (18) can be computed, hence the first estimate of y_P can be evaluated. Either Equation (12) or (15) can be used to obtain the first estimate of V_P . The process is now repeated beginning with the newly computed values of V_P and y_P in place of the originally assumed quantities. This iteration can be continued until the desired degree of accuracy is obtained. In the examples in this study two iterations produced sufficiently accurate results. With the variables V_P, y_P, and t_P known, either Equation (8) or (10) can be used to evaluate x_P .

APPENDIX II

NOTATION

A	area of cross section below water surface
B	width of bottom of trapezoidal section
C	$\sqrt{gA/T}$
C^+, C^-	refers to characteristic lines
g	acceleration of gravity
L	length of the channel
n	Manning's roughness coefficient
P	point in the xt plane where conditions are unknown
Q	discharge
R	hydraulic radius
R	point on the C^+ characteristic where conditions are known
S	point on the C^- characteristic where conditions are known
S	slope of the energy grade line
S_o	slope of the channel bottom
T	width of water surface
t_o	total time of gate motion to control the transients
t	time
V	average velocity at a cross section
x	distance along the channel
y	depth of water at a channel section