# Control on TOL and ETOL array systems 

## NALINAKSHI NTRMAL

Department of Mathematics, Madras Christian College, Madras 600059 , India.

AND
KAMALA KRITHIVASAN
Computer Centre. Indian Institute of Technology. Madras 600036 , India.
Roceived on January 1. 1981


#### Abstract

Regular, comext-free and context-sensitive controls are imposed on the tables of TOLAS and the hierarchy established. The effect of control with appearance checking and minimal table interpretations are investigated over the tables of Part ETOLAS. It is interesting to note that the regular control on the tables of Part DETOLAS will not increase the generative capacity of Part DETOLAS.


Key words: Array systems, Chomsky grammars, string languages, table interpretation.

## 1. Introduction

L-systems, introduced by Lindenmayer, originally in connection with some problems in theoretical biology, have later stimulated a substantial amount of rescarch ${ }^{1,2}$. In these systems one doez not distinguish between terminals and non-terminals, productions are applied in parallel on 3 word and the starting string of the system is of lengtin greater than or equal to one.

In trying to extend this concept of parallel rewriting to two-dimensions, we propose OL and TOL array systems where parallel rewriting of every symbol in a rectangular array is considered, each symbol is replaced by an array of the same size or dimension to avoid distortion of rectang.lar arrays and the axiom is a rectangular array ${ }^{8}$. The use of non-terminals is a very well established mechanism in formal language theory. In L-systems, this notion is represented by Extended TOL and Extended OL systens. We have investigated ETOL and EOL array systems in detail4.

In stïing languagus, various contral devices are introduced to increase che generative capacity of given grammars ${ }^{5}$. Once control is intraduced, it is meaningful to consider the idua of appearance checking which is exhaustively studied for string languages ${ }^{5}$.

In Chomsky grammars with regulated rewriting, at any step in a derivation of the grammar, the choice of the praduction to be applied is restricted by a kind of contral mechinism. In L-systems with regulated rewriting, in one step of a derivation, a whale subset of productions is applied where the chaice of this subset is somehaw restricted. Such systems are called Extended partial table OL systems with minimal table interpretation ${ }^{6}$. It is shown that generative capacity of extended partial table OL systems with minimal table interpretation is stronger than that of regular control and appearance checking, whereas it is equal to the combined effect of reguler control and appearance checkinge.

In this p?per, we extend these ideas of minimal table interpretation, regular contral and appearance checking on pertial ETOLAS and we observe that most of the reselts of N elsen ${ }^{6}$ will carry over to Purt ETOLAS. We also consider in this paper the effect of coatext-s:nsitive, context-free and regular control on the tables of TOLAS. We observe that context-free and context-sensitive contral on the tables of TOLAS will generate array languages which are not generable by TOLAS with regular control.

## 2. Control on TOL array systems

First we review some definitions needed for this paper.
Lat $I$ be an alphabet-a finite non-empty set of symbols. A matrix $M_{\text {ron }}$ (or array, over $I$ is an $m \times n$ rectangular array of symbols from $I(m, n \geqslant 1)$ and the dimensians of the matrix $M_{m n}$ is denated by $\left|M_{m n}\right|=(m, n)$. The set of all matrices aver $I$ (including $\lambda$ ) is denoted by $I^{* *}$ and $I^{++}=I^{* *}-\{\lambda\}$.

## Definition 2.1;

A tabled OL array system (TOLAS) is a 3-tuple $G=(\Sigma, \mathscr{P}, \omega)$ where $\Sigma$ is a finite noncmpty set (the alphabet, say $\Sigma=\left\{a_{1}, \ldots, a_{k}\right\}$ ) ; $\omega \in \Sigma^{++}$is the axiom; and $\mathscr{P}_{\text {consists of }}$ a. finite set $\left\{P_{1}, \ldots, P_{t}\right\}$ for $f \geqslant 1$ and $\operatorname{cach} P_{f}$ is a finite subset of $\Sigma \times \Sigma^{* *}$ called a table with the following two conditions.
(i) $(\forall P)_{,}(\forall a)_{\Sigma}(\exists a)_{\Sigma * *}(\langle a, a\rangle \in P)$;
(ii) $(\forall a)_{\Sigma}(\exists\langle a, a\rangle)_{P}, a$ 's are of the same dimension.

Definition 2.2
Let

where $a_{i j} \in \Sigma$
$M_{u} \in \Sigma^{* *}, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. We write $u \Rightarrow v$ if $a_{i j} \rightarrow M_{i j}$ are in table $P$ in $\mathscr{P}$ and th?t $\left|M_{i j}\right|$ is a constant. $\Rightarrow *$ is the reflrxive, transitive closure of $\Rightarrow$.

## Definition 2.3

Let $G=(\Sigma, \mathscr{P}, \omega)$ be a TOLAS whi re $\mathscr{P}=\left\{P, \ldots, P_{t}\right\} . \quad u \stackrel{l_{1}}{\Rightarrow} v\left(u \Rightarrow v\left(l_{1}\right)\right)$, if $v$ is derived from $u$ using table $P_{1}$ whose label is $l_{1} . u \rightarrow{ }^{*} v(w)$ cxtends $u \Rightarrow v\left(l_{1}\right)$ in the usual manner, where $w \in\left\{I_{1}, l_{2}, \ldots, I_{1}\right\}^{*}$. where $I_{1}$ is the labsl of the: teble $P_{1}$.

## Definition 2.4

A controlled TOLAS is e. 3-tuple (G. C. $L$ ) where $G$ is a TOLAS, $L$ is the sct of labels of the tables of $G$. $C$ is a linguage over $L . \quad L(G, C, L)=\left\{x \in \Sigma^{* *} / \omega \Rightarrow{ }^{*} x\right.$, $a \in C\}$.

## Notation

We denote by $\mathcal{F}$ (TOLAS: $X$ ) the femily of lenguages gencrated by TOLAS with control language from family $X$.

Now let us investigate the effect of regular, context-free and context-sensitive controls on the tables of TOLAS. It is interesting to observe that context-free (cortext-sersitive) control on the tables of TOLAS, will generate array languages which are not generable by regular (context-free) contral or the tables of TOLAS.

Theorem 2.1
$\mathcal{F}$ TOLAL $\underset{+}{\subset} \mathscr{F}($ TOLAS: $R) \underset{+}{\subset} \mathcal{F}($ TOLAS: $C F) \subset_{+} \mathscr{F}($ TOLAS: CS $)$.
Proof: Inclusions follow from definitions. Proper inclusions are cstablishrd by the following examples.
(i) Let $G=\left(\left\{., x_{\}},\left\{P_{1}, P_{2}\right\}, \frac{x}{x}.\right)\right.$ be a TOLAS where

$$
\begin{aligned}
& P_{1}=\{x \rightarrow x ., x \rightarrow x x, \cdot \rightarrow \cdots, \cdot \rightarrow . x\}, P_{2}=\left\{x \rightarrow \frac{\cdot}{x}, x \rightarrow \stackrel{x}{\cdot},\right. \\
& . \rightarrow \dot{x}, \cdot \rightarrow \cdot\} . \text { Let } C=\left(P_{1}^{n} P_{2}^{m} / n, m \geqslant 1\right) \text { be a regular control. }
\end{aligned}
$$

We consider some arrays of $L(G: C)$. Tben

$$
\begin{aligned}
& x . \Rightarrow \begin{array}{c}
x x . x \\
x \ldots x
\end{array} \Rightarrow \begin{array}{c}
x \ldots x \\
\ldots \ldots \\
\ldots x \ldots
\end{array}
\end{aligned}
$$

Here we get a finite number of matrices as the matrix of the lowest otder. Let $G^{\prime}=\left(\Sigma, \mathcal{P}, \omega^{\prime}\right)$ be a TOLAS so that $L\left(G^{\prime}\right)=L(\boldsymbol{G}: C)$ where we have a finite number of matrices as the matrix of the lowsst order. Hence choase one of them as the axiom, say,

$$
\begin{gathered}
x \ldots \\
\omega^{\prime}=\begin{array}{c}
x . . \\
\ldots x x x
\end{array} .
\end{gathered}
$$

To g:t oth r matriecs of th sam" size, we must have c table $\{x \rightarrow ., x \rightarrow x, . \rightarrow ., \rightarrow x\}$. But with th's table we git arrays $1 \mathrm{ke}(\ldots)_{4} \notin L(G: C)$. Hence $L(G: C)$ cennot be gererated by any TOLAS.
(ii) Corsid r the TOLAS $G=\left(\{a, b\},\left\{P_{1}, P_{2}\right\}, a b\right)$, where

$$
P_{1}=\left\{a \rightarrow{ }_{a b}^{a b}, b \rightarrow{ }_{a b}^{a b}\right\}, P_{2}=\left\{a \rightarrow{ }_{a a}^{a a}, b \rightarrow \frac{b b}{b b}\right\} . \text { Let }
$$

$C=\left\{P_{1}{ }^{*} P_{z}^{*} \mid n \geqslant 1\right\}$ be a CFL. It is not diffizult to show thet $L(G: C)$ cannot be generated by any TOLAS with reguler control.
(iii) Consider the TOLAS $G=(\{a\},\{a \rightarrow a a\}, a)$ and the $\operatorname{CScontrol} C=\left\{P_{i}^{n^{\prime}} \mid n \geqslant 1\right\}$. It can be proved that th's lengeag: $L(G: C)$ will rat be gererated by any TOLAS with $C F$ control. Hence $\mathcal{F}$ (TOLAS : $C F) \underset{+}{C} \mathscr{F}$ (TOLAS: $C S$ ).

## 3. Control on partial EDTOLAS

We first define the concepts used in this section. The idias of eppearance checking and minimal table interpritation on the tables of partial ETOL systcms and appcararcc checking on the formal langl:ages heve becr exhat:stively studied ${ }^{\text {b, }}$. Now, we define partial extended table OL array systems with appearence checking, minimal table inter-
pretation and with regular control. Intuitively, at each step of derivation if the elements of an ariey is exactly equal to the left hend sides of the rules of a table in $\mathscr{P}$ of part ETOLAS then we say that the system is a partial ETOI,AS with minimal table interpretation.

## Definition 3.1

Let $t$ be a finite subset of $\Sigma \times \Sigma^{* *}$ then reg ( $t$ )

$$
=\left\{\sigma \in \Sigma \mid \exists_{a} \in \Sigma^{* *}, \sigma \rightarrow a \in I\right\} .
$$

If $X \in \Sigma^{++}$, then $\operatorname{Min} X=\{a / a \in X\}$.
Definition 3.2
A partial ETOLAS (Part ETOLAS) is in ordered 6.tuple $G=\left(V, \mathcal{P}, L, L^{* \bullet}, \omega, \Sigma\right)$, where $V$ is a firite, ron-empt! set (the alphebet of $G$ ), $\Sigma \subset V$, the target alphabet, $\omega \in V^{+}+$ is the axiom. $\mathcal{P}$ is a finite, non-empty collection of finite subsets of $\Sigma \times \Sigma^{* *}$ such that if $t \in P$, th: $\boldsymbol{t} \boldsymbol{t}=\left\{\Sigma \times \Sigma^{* *} \Sigma^{* *}\right.$ are arrays of the same dimension $\}$. The elements of $\mathscr{P}$ are called tebles and th. clements of the tables are called productions. $L$ is the set of labels of $\mathscr{P}$ (there is a one-to-one correspondence between the elements of $\mathscr{P}_{\text {and }} L$ ). $L^{\circ 0}$ is a subset of $L$.

## Definition 3.3

Let $G=\left(V, \mathcal{P}, L, L^{\infty}, \omega, \Sigma\right)$ be a Part ETOLAS and let $X \in V^{++}, Y \in V^{* *}$. $X$ is said to derive $Y$ directly in $G, X_{\boldsymbol{G}}^{\Rightarrow} Y$ if and only if

$$
a_{1} \ldots \ldots a_{n} \quad M_{11} \ldots \ldots M_{1 n}
$$

(i) $X=\ldots \ldots \ldots$ and $Y=\ldots \ldots \ldots \ldots$ where $a_{4} \in V$

$$
a_{m} \ldots a_{m n} \quad M_{m 1} \ldots \ldots M_{m_{n}}
$$

for $1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n, M_{n} \in \Sigma^{* *}, \quad l \leqslant r \leqslant m, \quad \backslash \leqslant s \leqslant n$ and $M_{n}$ are of the same dimension;
(ii) $\mathscr{F}_{t} \in \mathscr{P}:\left[\left(a_{3}, M_{4}\right) \in t\right.$ for $\left.i=1,2, \ldots, m, i=1,2, \ldots, n\right]$. We also write $X \Rightarrow Y$ where $l$ is the label fiom $L$ associated with the table $t$. $X$ is said to derive $Y$ directly i:nder the minimal table interpretation $S \xlongequal{\boldsymbol{G m t}} Y$ iff (i) and (ii) above are satisfied and (iii) $\operatorname{reg}(t)=\operatorname{Min}(X)$ for $t \in \mathcal{P}$ satisfying (ii). We aliso write $\underset{m t}{\Rightarrow} \underset{m}{\boldsymbol{y}} \underset{\text { ( }}{ }(l)$ where $l$ is the label associated with $t . \Rightarrow^{*}\left(\Rightarrow \begin{array}{c}* \\ m t\end{array}\right)$ is the reflexive transitive closure of $\Rightarrow\left(\Rightarrow_{m t}\right)$.

## Definition 3.4

Let $G=\left(V, \mathscr{P}, L, L^{\infty}, \omega, \Sigma\right)$ be a Part ETOLAS and let $X \in V^{++}, Y \in V^{* *}$ and $Z \in L^{*}$. Then $X$ is said to derive $Y$ with contral word $Z$ (under the minimal: able interpretation)

$$
\underset{G}{\underset{G}{*}} Y(Z)(X \underset{G m t}{*} Y(Z)) \text { if and only if, }
$$

(i) $Z=l_{1} l_{2} \ldots l_{d}$ where $l_{1} \in L$ for $i=1,2, \ldots, d$;
(ii) there exists arrays $M_{\bullet}, M_{1}, \ldots, M_{d}$ from $V^{* *}$ such that $M_{\bullet}=x, M_{d}=Y$ and $M_{t-1} \Rightarrow M_{i}\left(l_{i}\right)\left(M_{i-t} \Rightarrow m t M_{i}\left(l_{i}\right)\right)$ for every $i=1,2, \ldots, d$.
$X$ is said to derive $\boldsymbol{Y}$ with contral ward $\boldsymbol{Z}$ (under minimal table interpretation) with appearance checking. $X \Rightarrow{ }^{* \oplus} Y(Z)(X \stackrel{\bullet}{\Rightarrow} a c, Y(Z))$ if and only if
(i) $Z=I_{1} l_{2} \ldots I_{d}$ where $I_{1} \in L$ for $i=1,2, \ldots, d$;
(ii) there exists arrays $M_{e}, M_{1}, \ldots, M_{d}$, from $V^{* *}$ st!ch that $X=M_{0}, Y=M_{d}$ and for every $i=1,2, \ldots, d$ ( $t_{1}$ denotes the table asso:iated with the label $l_{i}$ ) if Min $\left(M_{i-i}\right) \leq \operatorname{reg}\left(l_{i}\right)\left(M \ln \left(M_{i-k}\right)=\operatorname{reg}\left(l_{i}\right)\right)$ then $M_{i-1} \Rightarrow M_{i}\left(l_{i}\right)\left(M_{i-1} \Rightarrow{ }_{m t} M_{i}\left(l_{i}\right)\right.$ otherwise $I_{4} \in L^{\bullet e}$ and $M_{i-1}=M_{4}$. The language generated by $G$ is defined as $L_{i}^{\prime}(G)=\left\{x \in \Sigma^{* *} \mid \exists Z \in L^{*}: \omega \underset{G}{*}{ }_{i}^{j} x(Z)\right\}$, where $i$ may be the index $m t$ or not and $j$ may be the index $a c$ or not.

## Definition 3.5

A Part ETOLAS with regular control (RC-Part ETOLAS) is a 7-tuple $G=(V, \mathscr{P}, L$, $L^{\infty}, \omega, \Sigma, R$ ) where $G^{\prime}=\left(V, \mathcal{P}, L, L^{\iota e}, \omega, \Sigma\right)$ is a Part ETOLAS and $R$ is a regular contral over $L$. Then $L_{i}^{\prime}(G)=\left\{X \in \Sigma^{* *} / \mathrm{Z} Z \in R, \omega \Rightarrow: X(z)\right.$, wh re $i$ may be the index $m t$ or not and $j$ may be the index $a c$ or not.

## Definition 3.6

A partial ETOLAS $G=\left(V, \mathscr{P}, L, L^{\bullet \bullet}, \omega, \Sigma\right)$ is calledian ETOLAS ift for cvery $t \in \mathscr{P}$ and for every $a \in V$, there is an $M_{n} \in V^{* *}, r$ and $s$ are fixed numbers such that $\left(a, M_{n}\right) \in t$. $G$ is called deterministic (Part EDTOLAS) iff for every $t$ and for every $a \in V$ thr re is at most one $M_{r \boldsymbol{r}} \in V^{* *}, r$ and $s$ are fixed numbers, such that $\left(a, M_{r o}\right) \in t$. $G$ is cilled propagating (Part EPTOLAS) iff for every $t \in \mathscr{P}$ and for every $a \in V,(a, \lambda) \notin t$. The languages generated are denoted by Part EDTOLAL and Part EPTOLAL.

Now let us investigate the relations between the families of DETOLAL, Part DETOLAL $;$, RC-Part ETOLAL' ( $i$ may be ac or nat and $j$ may be $m t$ or not).
rineorem 3.1
$\mathcal{F}$ Part ETOLAL $=e^{\mathcal{F}}$ Part ETOLAL"
$\mathcal{F}$ Part ETOLAL $=\mathscr{F}$ Part ETOLAL $_{m t}^{* 厄}$

Proof: Proof is similar to theorem 1 of Nielsen ${ }^{6}$.
Theorem 3.2
$\mathcal{F E T O L} A L=\mathcal{F P} \cdot$ rt ETOLAL.
Proof: Let $G=\left(V, \mathcal{P}, L, L^{\text {es }}, \omega, \Sigma\right)$ be a Part ETOLAS. Let $F$ be a symbol not in $V$. Define an ETOLAS $G^{\prime}=\left(V \cup\{F\}, \mathcal{D}^{\prime}, L, L^{\iota e}, \omega, \Sigma\right)$ where $\mathscr{P}^{\prime}=\left\{P^{\prime} \mid P \in \mathscr{P}\right\}$. If $l \in L$ is the label of $P \in \mathscr{P}$ then $P^{\prime}=P \cup\left\{\sigma \rightarrow M^{\prime}, \sigma \notin \mathrm{reg} P\right\} \cup\left\{F \rightarrow M^{\prime}\right.$, where $M^{\prime}$ is the rejection array whose dimension is cqual to that of the dimension of the right hand side of the relcs of $P$ ). The label of $P^{\prime}$ in $\mathcal{P}^{\prime}$ is $l$. Hence $L(G)=L\left(G^{\prime}\right)$, i.e. $\mathcal{F}$ Part ETOLAL $\subset \mathcal{F}$ ETOLAL. But by definition $\mathcal{F}$ ETOLAL is contained in $\mathcal{F}$ Part ETOLAL. Hence the theorem.

Now we state the following theorems without proof as the proofs can be found in Nirmal'.

## Theorem 3.3

$$
\mathcal{F} \text { Part ETOLAL }=\mathcal{F} \text { RC-Part ETOLAL. }
$$

Theorem 3.4
$\mathcal{F E D T O L A L}=\mathcal{F}$ Part EDTOLAL! $=\mathscr{F} R C$-Part EDTOLAL!
where $i$ (resp. $j$ ) is the index $m t$ (resp. ac) or missing.

## Remark 3.1

It fallows from theorem 3.4 that regular control on the tables of Part DETOLAS will not increase the generative capacity of Part DETOLAS.

## References

1. Herman, G. T. and Developmental systems and languages, North Holland, 1975. Rozenberg, G.
2. Rozenberg, G. and Salomat, A.
3. Krithivasan, K. and Nirmal, N. L-Systems, Springer lecture notes in Computer Science, 1974. OL and TOL array languages, J. Indian Inst. Sci., 1980, 62 (A), 101-110.
4. Nirmal, N. and Krithivasan, K.
5. Salomaa, A.
6. Nielsen, M.
7. Nirmal, $N$.

EOL and ETOL array languages, TR-Maths-10/78, Madras Christian College, Madras. 1978 (submitted for publication).

Formal languages, Academic Press, 1973.
EOL system with control devices, Acta Informatica, 1975, 373-386
Studies in two-dimensional developmental systems and languages, Ph.D. Thesis, University of Madras, 1979.

