Control Systems on Three-Dimensional Lie Groups

Equivalence and Controllability

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Outline

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1. Invariant control systems
2. Classification of systems
   - Classification of 3D Lie groups
   - Solvable case
   - Semisimple case
3. Controllability characterizations
4. Conclusion

Invariant control affine systems

Left-invariant control affine system

$$(\Sigma) \qquad \dot{g} = \Xi(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

- state space: G is a connected (matrix) Lie group with Lie algebra $\mathfrak{g}$
- input set: $\mathbb{R}^\ell$
- dynamics: family of left-invariant vector fields $\Xi(\cdot, u)$
- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$
  $A \in \Gamma^0 \longleftrightarrow$ homogeneous, $\qquad A \notin \Gamma^0 \longleftrightarrow$ inhomogeneous.

$$\Sigma \; : \; A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Trajectories, controllability, and full rank

- admissible controls: piecewise continuous curves $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- trajectory: absolutely continuous curve s.t. $\dot{g}(t) = \Xi(g(t), u(t))$
- controllable: exists trajectory from any point to any other
- full rank: $\mathrm{Lie}(\Gamma) = \mathfrak{g}$; necessary condition for controllability.

Characterization of full-rank systems on 3D Lie groups

- 1-input homogeneous: none
- 1-input inhomogeneous: A, B_1, $[A, B_1]$ linearly independent
- 2-input homogeneous: B_1, B_2, $[B_1, B_2]$ linearly independent
- 2-input inhomogeneous: all
- 3-input: all.

Equivalence

Detached feedback equivalence

Σ and Σ' are detached feedback equivalent if

$$\exists \phi : \mathsf{G} \to \mathsf{G}', \ \varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell \text{ such that } T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$$

- specialization of feedback equivalence
- diffeomorphism ϕ preserves left-invariant vector fields

Proposition

Full-rank systems Σ and Σ' equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

Outline

Classification of 3D Lie algebras

Eleven types of real 3D Lie algebras (e.g., [Mub63])

- $3\mathfrak{g}$ — $\mathbb{R}^3$ — Abelian
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ — $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$ — cmpl. solvable
- $\mathfrak{g}_{3.1}$ — Heisenberg $\mathfrak{h}_3$ — nilpotent
- $\mathfrak{g}_{3.2}$ — cmpl. solvable
- $\mathfrak{g}_{3.3}$ — book Lie algebra — cmpl. solvable
- $\mathfrak{g}_{3.4}^0$ — semi-Euclidean $\mathfrak{se}(1,1)$ — cmpl. solvable
- $\mathfrak{g}_{3.4}^a$, $a > 0, a \neq 1$ — cmpl. solvable
- $\mathfrak{g}_{3.5}^0$ — Euclidean $\mathfrak{se}(2)$ — solvable
- $\mathfrak{g}_{3.5}^a$, $a > 0$ — exponential
- $\mathfrak{g}_{3.6}^0$ — pseudo-orthogonal $\mathfrak{so}(2,1)$, $\mathfrak{sl}(2,\mathbb{R})$ — simple
- $\mathfrak{g}_{3.7}^0$ — orthogonal $\mathfrak{so}(3)$, $\mathfrak{su}(2)$ — simple

Classification of 3D Lie groups

3D Lie groups (e.g., [GOV94])

- $3\mathfrak{g}_1$ — 4 — $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}$, $\mathbb{T}^3$
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ — 2 — $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$
- $\mathfrak{g}_{3.1}$ — 2 — H_3, $\mathsf{H}_3^* = \mathsf{H}_3 / Z(\mathsf{H}_3(\mathbb{Z}))$
- $\mathfrak{g}_{3.2}$ — 1 — $\mathsf{G}_{3.2}$
- $\mathfrak{g}_{3.3}$ — 1 — $\mathsf{G}_{3.3}$
- $\mathfrak{g}_{3.4}^0$ — 1 — $\mathsf{SE}(1,1)$
- $\mathfrak{g}_{3.4}^a$ — 1 — $\mathsf{G}_{3.4}^a$
- $\mathfrak{g}_{3.5}^0$ — $\mathbb{N}$ — $\mathsf{SE}(2)$, n-fold cov. $\mathsf{SE}_n(2)$, univ. cov. $\widetilde{\mathsf{SE}}(2)$
- $\mathfrak{g}_{3.5}^a$ — 1 — $\mathsf{G}_{3.5}^a$
- $\mathfrak{g}_{3.6}$ — $\mathbb{N}$ — $\mathsf{SO}(2,1)_0$, n-fold cov. $\mathsf{A}(n)$, univ. cov. $\widetilde{\mathsf{A}}$
- $\mathfrak{g}_{3.7}$ — 2 — $\mathsf{SO}(3)$, $\mathsf{SU}(2)$.

Only H_3^*, A_n, $n \geq 3$, and $\widetilde{\mathsf{A}}$ are not matrix Lie groups.

Case study: systems on the Heisenberg group H_3

$$\mathsf{H}_3 \;:\; \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathfrak{h}_3 \;:\; \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Theorem

On H_3, any full-rank system is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma^{(1,1)} &: \; E_2 + uE_3 \\
\Sigma^{(2,0)} &: \; u_1 E_2 + u_2 E_3 \\
\Sigma^{(2,1)}_1 &: \; E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma^{(2,1)}_2 &: \; E_3 + u_1 E_1 + u_2 E_2 \\
\Sigma^{(3,0)} &: \; u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

Proof sketch (1/2)

$$d\,\mathrm{Aut}(\mathsf{H}_3) = \mathrm{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} : \begin{array}{c} x, y, z, u, v, w \in \mathbb{R} \\ yw - vz \neq 0 \end{array} \right\}$$

- Single-input system Σ with trace $\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i \right\rangle$.

$$\psi = \begin{bmatrix} a_2 b_3 - a_3 b_2 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{h}_3), \qquad \psi \cdot (E_2 + \langle E_3 \rangle) = \Gamma.$$

  So Σ is equivalent to $\Sigma^{(1,1)}$.

- Two-input homogeneous system with trace $\Gamma = \langle A, B \rangle$; similar argument holds.

Proof sketch (2/2)

- Two-input inhomogeneous system Σ with trace $\Gamma = A + \langle B_1, B_2 \rangle$.
- If $E_1 \in \langle B_1, B_2 \rangle$, then $\Gamma = A + \langle E_1, B_2' \rangle$; like single-input case there exists automorphism ψ such that $\psi \cdot \Gamma = E_3 + \langle E_1, E_2 \rangle$.
- If $E_1 \notin \langle B_1, B_2 \rangle$, construct automorphism ψ such that $\psi \cdot \Gamma = E_1 + \langle E_2, E_3 \rangle$.
- $\Sigma_1^{(2,1)}$ and $\Sigma_2^{(2,1)}$ are distinct as E_1 is eigenvector of every automorphism.
- Three-input system: trivial.

Case study: systems on the orthogonal group $\mathsf{SO}(3)$

$$\mathsf{SO}(3) = \{g \in \mathbb{R}^{3\times 3} \; : \; gg^\top = \mathbf{1}, \; \det g = 1\}$$

$$\mathfrak{so}(3) \; : \; \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Theorem

On $\mathsf{SO}(3)$, any full-rank system is equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma^{(1,1)}_\alpha &: \; \alpha E_1 + uE_2, \quad \alpha > 0 \\
\Sigma^{(2,0)} &: \; u_1 E_1 + u_2 E_2 \\
\Sigma^{(2,1)}_\alpha &: \; \alpha E_1 + u_1 E_2 + u_2 E_3, \quad \alpha > 0 \\
\Sigma^{(3,0)} &: \; u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

Case study: systems on the orthogonal group SO (3)

Proof sketch

$$d\,\mathrm{Aut}(\mathsf{SO}(3)) = \mathrm{Aut}(\mathfrak{so}(3)) = \mathsf{SO}(3)$$

- Classification procedure similar, though more involved.
- Product $A \bullet B = a_1 b_1 + a_2 b_2 + a_3 b_3$ is preserved by automorphisms.
- *Critical point* $\mathfrak{C}^\bullet(\Gamma)$ at which an inhomogeneous affine subspace is tangent to a sphere $\mathcal{S}_\alpha = \{A \in \mathfrak{so}(3) \, : \, A \bullet A = \alpha\}$ is given by

$$\mathfrak{C}^\bullet(\Gamma) = A - \frac{A \bullet B}{B \bullet B} B$$

$$\mathfrak{C}^\bullet(\Gamma) = A - \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} B_1 \bullet B_1 & B_1 \bullet B_2 \\ B_1 \bullet B_2 & B_2 \bullet B_2 \end{bmatrix}^{-1} \begin{bmatrix} A \bullet B_1 \\ A \bullet B_2 \end{bmatrix}.$$

- $\psi \cdot \mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\psi \cdot \Gamma)$ for any automorphism $\psi \in \mathsf{SO}(3)$.
- Scalar $\alpha^2 = \mathfrak{C}^\bullet(\Gamma) \bullet \mathfrak{C}^\bullet(\Gamma)$ invariant under automorphisms.

Outline

Some controllability criteria for invariant systems

Sufficient conditions for full-rank system to be controllable

- system is homogeneous
- state space is compact
- the direction space Γ^0 generates $\mathfrak{g}$, i.e., $\mathrm{Lie}\,(\Gamma^0) = \mathfrak{g}$
- there exists $C \in \Gamma$ such that $t \mapsto \exp(tC)$ is periodic
- the identity element $\mathbf{1}$ is in the interior of the attainable set $\mathcal{A} = \{g(t_1) \,:\, g(\cdot)$ is a trajectory such that $g(0) = \mathbf{1},\ t_1 \geq 0\}$.

([JS72])

Systems on simply connected completely solvable groups

Condition $\mathrm{Lie}\,(\Gamma^0) = \mathfrak{g}$ is also necessary. ([Sac09])

Characterization for systems on 3D groups

Theorem

1. *On* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, H_3, $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.3}$, $\mathsf{SE}(1,1)$, *and* $\mathsf{G}^a_{3.4}$,
   a full-rank system is controllable if and only if $\mathrm{Lie}(\Gamma^0) = \mathfrak{g}$.
2. *On* $\mathsf{SE}_n(2)$, $\mathsf{SO}(3)$, *and* $\mathsf{SU}(2)$,
   all full-rank systems are controllable.
3. *On* $\mathsf{Aff}(\mathbb{R}) \times \mathbb{T}$, $\mathsf{SL}(2,\mathbb{R})$, *and* $\mathsf{SO}(2,1)_0$,
   a full-rank system is controllable if and only if it is homogeneous or there exists $C \in \Gamma$ *such that* $t \mapsto \exp(tC)$ *is periodic.*
4. *On* $\widetilde{\mathsf{SE}}(2)$ *and* $\mathsf{G}^a_{3.5}$,
   a full-rank system is controllable if and only if $E_3^*(\Gamma^0) \neq \{0\}$.

Characterization for 3D groups

Proof sketch (1/2)

1. Completely solvable simply connected groups; characterization known ([Sac09]).
2. The groups $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ are compact, hence all full-rank systems are controllable.

   $\mathsf{SE}_n(2)$ decomposes as semidirect product of vector space and compact subgroup; hence result follows from [BJKS82].

Characterization for 3D groups

Proof sketch (2/2)

3. Study normal forms of these systems obtained in classification.
   - Full-rank homogeneous systems are controllable.
   - For each full-rank inhomogeneous system we either explicitly find $A \in \Gamma$ such that $t \mapsto \exp(tA)$ is periodic
   - or prove that some states are not attainable by inspection of coordinates of $\dot{g} = \Xi(g, u)$.
   - As properties are invariant under equivalence, characterization holds.
4. Study normal forms of these systems obtained in classification.
   - Condition invariant under equivalence.
   - Similar techniques with extensions ([JS72]); however for one system on $\mathsf{G}_{3.5}^{a}$ we could only prove controllability by showing $\mathbf{1} \in \operatorname{int} \mathcal{A}$.

Outline

Conclusion

Summary

- Characterization of controllability for systems on 3D Lie groups.
- Normal forms for controllable systems on 3D Lie groups.

Outlook

- Cost-extended systems associated to optimal control problems.

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