# CONTROL VOLUME METHOD FOR THE THERMAL CONVECTION PROBLEM IN A ROTATING SPHERICAL SHELL: TEST ON THE BENCHMARK SOLUTION

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ABSTRACT

*The development of the control volume method for the thermal convection problem in a rotating spherical shell is presented. In contrast to the spectral methods, commonly used in geodynamo simulations, the control volume method belongs to the class of grid methods (the solution is approximated by a set of discrete values in physical space). In the present paper we concentrate on some problems of convergence and stability of the method. Case 0 of the numerical dynamo benchmark (Christensen et al., 2001, Phys. Earth Planet. Inter., 128, 25-34) was used to check the correctness of our computer code. The results demonstrate good convergence to the suggested standard solution.*

Keywords: Liquid core, dynamo benchmark, finite volume method, SIMPLE algorithm

## 1. INTRODUCTION

In recent decades numerical studies of the MHD process in the liquid core of the Earth have become an important part of the dynamo theory as well as geophysics in general. This problem considers thermal and compositional convection and magnetic field generation by the conductive fluid in the core (see overview of the recent results in *Jones, 2000*). Since only the simplest spatial cases can be checked by analytical solutions, testing of the numerical code in the full non-linear regimes appears to be the issue of the day. For this purpose the benchmark for convection (Case 0) as well as for the full MHD problem was proposed in *Christensen et al. (2001)*. Six groups contributed numerical solutions which showed good agreement. Despite the variety of numerical codes, all are based on similar principles. All the unknowns are expanded in spherical harmonics in angular coordinates and the non-linear terms are evaluated at grid points using the transformation between spectral and grid spaces. Substantial differences are only in the treatment of the radial dependence (expansion into Chebyshev polynomials or finite differences).

The use of regular grids in spectral coordinates is difficult because the Courant condition for the stability is hard to satisfy near the polar axis and the differencing scheme is singular there. *Gilman and Miller (1981)* and *Kageyama et al. (1995)* overcame this problem by imposing regularity conditions and filtering the short wavelengths at the poles. (See also special transformations of the variables used in *Nakajima and Roberts, 1995* and *Hejda and Reshetnyak, 2000*, which helped to overcome numerical instabilities near the axis and the center.)

The other way to overcome the difficulties caused by the singularities of the coefficients is the usage of weighted coefficients. The control volume method, otherwise known as the finite volume method, belongs to this class. The basic strategy of this numerical scheme is to express the differential equations at each point in conservative form, to integrate them over the control volume and convert each such integral into the sum of integrals over the boundary faces by means of Gauss' theorem. As the area of the faces close to the axis of rotation (or to the center of the sphere) is indirectly proportional to the singular coefficients, the resulting grid equations are non-singular. The control volume approach displays very stable numerical behavior even in cases of complex spatial and time behaviour of the simulated fields. No numerical problems appear near the axis and the center.

The application of the control volume method to the solution of dynamo problem was explained in *Hejda and Reshetnyak (2003)*. In the present paper we re-examine the hydrodynamic part of the problem (thermal convection) paying more attention to the problem of stability and convergence of the solution. The correctness of the computer code was checked by Case 0 of the dynamo benchmark (*Christensen et al., 2001*).

## 2. BASIC EQUATIONS AND METHOD OF SOLUTION

The thermal convection of an incompressible fluid ($\nabla\cdot\mathbf{V}=0$) in the Boussinesq approximation in a spherical shell ($r_i < r < r_0$) rotating with angular velocity $\Omega$ is described by the Navier-Stokes equation,

$$R_o\left(\frac{\partial\mathbf{V}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}\right)=-\nabla P+\mathbf{F}+E\nabla^2\mathbf{V} \tag{1}$$

and the heat flux equation,

$$\frac{\partial T}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)T=q\nabla^2 T\ . \tag{2}$$

The equations are scaled with the radius of sphere L as the fundamental length scale, which makes the dimensionless radius $r_O = 1$; the inner core radius $r_i$ is, similarly to that of the Earth, equal to 0.35. Velocity $\mathbf{V}$, pressure $P$ and time $t$ are then measured in units of $\nu/L$, $\rho\nu^2/L^2$ and $L^2/\nu^2$, respectively, where $\nu$ is kinematic viscosity, $\rho$ is density, $R_o = \nu/2\Omega L^2$ is the Rossby number and $E = \nu/2\Omega L^2$ is the Ekman number. Force $\mathbf{F}$ includes the Coriolis and Archimedean forces: $\mathbf{F} = -\mathbf{1}_z \times \mathbf{V} + qR_aTr\mathbf{1}_r$ where ($r$, $\theta$, $\varphi$) is the spherical coordinate system, $\mathbf{1}_z$ is the unit vector along the axis of rotation and $R_a = \alpha g_o \delta TL/2\Omega\kappa$ is the modified Rayleigh number, $\alpha$ is the coefficient of volume expansion, $\delta T$ is the drop of temperature through the shell, $g_o$ is the gravity acceleration at $r = r_0$ and $q = \kappa / \nu$ is the inverse value of the Prandtl number. Eqs (1 – 2) are closed by the non-penetrating and no-slip boundary conditions for the velocity field at the rigid surfaces and constant temperature $T_i = 1$ and $T_0 = 0$ at the inner and outer boundaries of the shell.

To solve Eqs (1 – 2) we have used the control volume method[1]. It is assumed that all fields are defined at the nodes which are the centers of grid cells (control volumes). In contrast to the finite element method the faces of the control volumes are perpendicular to the coordinate axes. The basic strategy of the method is to express the differential equations in conservative form, integrate them over the control volumes and convert every such integral into the sum of fluxes over the boundary faces by means of Gauss' theorem. It is advantageous to employ a different grid for each component of vector fields (and an additional grid for the scalar field). Then, if we consider, e.g., the heat flux equation, the velocity components are calculated for the points that lie on the corresponding faces of the control volumes ($v_r$ is calculated at the faces that are normal to the $r$-direction, etc.). The discrete form of the system of linear equations is represented by the band matrix. Note, that it is the flux form of equations which allow us to omit the boundary conditions at the axis (and the center of the sphere if the magnetic field is taken into account) because the flux is zero at the faces with zero area. Nevertheless, extrapolation to the axis is necessary in some situations. It is well known that convection-diffusion problems are prone to instabilities for larger Reynolds numbers. Whereas the simplest remedy for this difficulty is the up-wind scheme, we have used the power-law scheme which is of the second order of accuracy (*Patankar, 1980*). The linear system of equations was solved using tridiagonal solver in $r$-direction and the Gauss-Seidel iterative algorithm with underrelaxation in the tangential directions.

## 3. RESULTS OF THE BENCHMARK SOLUTION

The solution of the dynamo benchmark is quasi-stationary, drifting slowly with frequency $\omega$ in longitude. The solution is symmetric about the equator and has fourfold symmetry in longitude. In accordance with *Christensen et al. (2001)* we present the drift frequency $\omega$, mean kinetic energy over the shell volume as well as the local temperature $T$ and azimuthal velocity $V_\varphi$ at point $P_0 : r = (r_0 + r_i)/2,\ \vartheta = \pi/2$ for which $V_r = 0$ and $\dfrac{\partial V_r}{\partial \varphi} > 0$ .

After renormalization of Eqs (1 – 2) due to the different definition of units the benchmark parameters for the "Case 0" regime correspond to $E = 2.1125 \times 10^{-4}$, $R_a = 76.92$, $R_o = 2.1125 \times 10^{-4}$ and $q = 1$. Starting from the initial condition for the velocity field and temperature distribution recommended in *Christensen et al. (2001)* we integrated our equations in time up to the moment, when the growth rate of kinetic energy was equal to zero with a relative accuracy of less than $10^{-5}$. It was usually reached within

[1] The systematic development of the control volume method (otherwise known as the finite volume method) for the heat flux equation and the Navier-Stokes equations can be found in *Patankar (1980)*. The control volume approach for the full MHD problem in a spherical geometry was explained in more detail by *Hejda and Reshetnyak (2003)*.

1 time unit. The typical space distribution of the velocity field and temperature are presented in Fig. 1. Such figures were obtained for all calculations in which the number of longitudional grid points $N_\varphi$ was higher than 48. For smaller $N_\varphi$ the number of convection columns depended on the number of grid points and was approximately equal to $N_\varphi/2$ thus indicating the non-physical origin of these structures. The number of grid points in radial and latitudional variables does not influence the basic structure of the solution but only its accuracy.

The results which are summed up in Table 1 display good convergence to the suggested standard solution. One can hardly expect that for the benchmark case, where the convective cells are very close to some eigen-functions used in the spherical decomposition, our method could be more effective than the spectral. The lower accuracy of the drift frequency $\omega$ is understandable. As the solution is drifting slowly the difference between subsequent steps is very small and the computed drift depends strongly on the requested accuracy of the iterative process. A few hundred of time steps must be carried until the solution pattern moves to the next grid point. It is thus understandable that the accuracy of $\omega$ is of two orders lower than the accuracy of velocity and temperature. Similar results were obtained by *Matsui and Okuda (2002)* who used the finite element method.

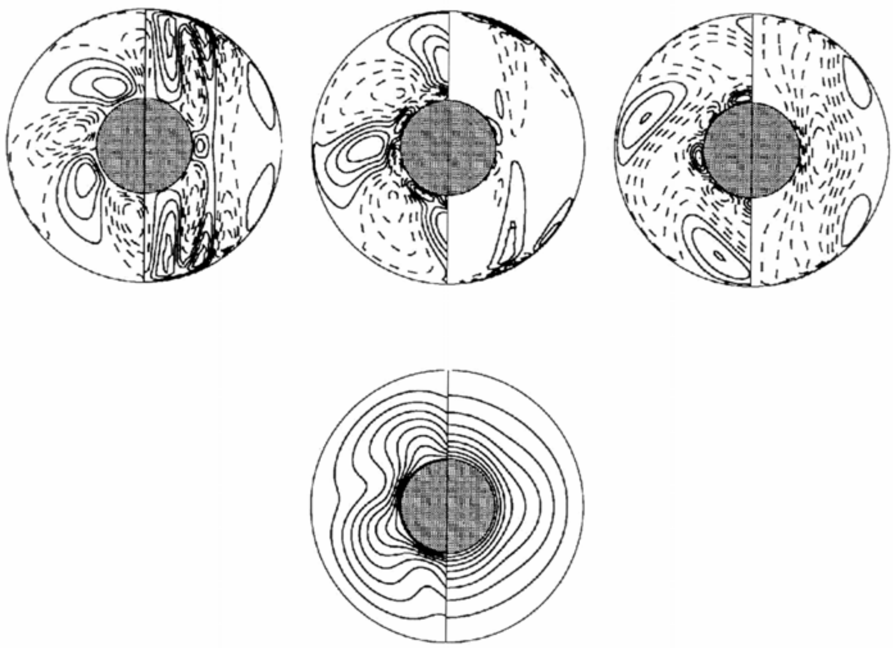

**Fig. 1.** Velocity field components ($V_r$, $V_\theta$, $V_\varphi$) (upper) and temperature $T$ (below); equatorial (left) and axi-symmetrical meridional (right) sections. Minimal and maximal values for velocity components: (-25.1, 20.0), (-4.0, 4.0), (-27.9, 19.5) for meridional and (-0.47, 0.39), (-1.75, 1.75), (-9.80, 2.19) equatorial sections; for the both temperature field projections $T \in (0, 1)$.

**Table 1.** Benchmark Case 0 – non-magnetic convection. $N_r$, $N_\theta$, $N_\varphi$ are numbers of grid points, $E_\kappa$ is the kinetic energy, $T$ is the mean temperature, $V_\varphi$ – velocity and $\omega$ is the angular velocity at point $P_0$. The bottom line corresponds to the suggested standard solution (*Christensen et al., 2001*).

| $N_r$ | $N_\theta$ | $N_\varphi$ | $E_\kappa$ | $T$ | $V_\varphi$ | $\omega$ |
|---|---|---|---|---|---|---|
| 35 | 35 | 64 | 61.986 | 0.4386 | -9.958 | 0.2638 |
| 45 | 45 | 64 | 60.825 | 0.4316 | -9.978 | 0.1123 |
| 55 | 55 | 64 | 60.563 | 0.4313 | -10.029 | 0.0886 |
| 65 | 65 | 64 | 60.450 | 0.4312 | -10.055 | 0.0984 |
| 85 | 85 | 64 | 60.282 | 0.4311 | -10.082 | 0.1338 |
| 105 | 105 | 64 | 60.179 | 0.4310 | -10.095 | 0.1556 |
| 45 | 45 | 96 | 59.994 | 0.4294 | -10.121 | 0.0853 |
| 85 | 85 | 96 | 59.414 | 0.4291 | -10.175 | 0.1187 |
| | | | 58.348 | 0.4281 | -10.157 | 0.1824 |

All test calculations showed very good stability. The time step varied between $10^{-4}$ and $10^{-3}$. An attempt to increase the time step sometimes led to an inadequate increase of the number of iterations and was thus non-productive. The nature of the problem and properties of the computer code allowed our simulations to be carried out on a Pentium-IV PC using double precision accuracy. Sun UltraSPARC III in single processor mode was used for the three largest cases.

As was mentioned above, insufficient resolution in the longitudinal variable generates artificial columnar structures. The boundary between the false and proper solutions for the benchmark case was at about 48 longitudinal grid points. We expected the number of grid points to depend on the Ekman number and that is why we have carried out test calculations for the Ekman number five times higher i.e. $E = 0.001$. The calculations showed that 32 points were sufficient for this case.

## 4. CONCLUSIONS

The dynamo benchmark is a well-established standard solution for verifying any newly developed computer code for (magneto)convection in a rotating spherical shell. Our numerical code based on the control volume method went successfully through the first part of this test – non-magnetic convection. The results are similar to those obtained by finite element method (*Matsui and Okuda, 2002*).

Our experience shows that the single processor code would not be sufficient for a computation of the full dynamo problem, including the dynamo benchmark. That is why we are now developing a parallel version of the computer code. Based on the MPI technology, it will be suitable for both, PC-clusters as well as high performance computers.

It was already mentioned that the control volume method is not expected to do as well on the simple test solution as previous models that used spherical harmonic expansions.

However, control volume method will be more efficient on a parallel computer because only "nearest neighbor" communication between the processors would be needed – instead of the global communication needed for spherical harmonic codes. Therefore, when much higher spatial resolution is desired to simulate strongly turbulent convection, this method may be a better choice than a spherical harmonic method.

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