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CONTROLLABILITY FOR SEMILINEAR FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the regularity properties for a class of semilinear integrodifferential functional differential equations. It is shown the relation between the reachable set of the semilinear system and that of its corresponding linear system. We also show that the Lipschitz continuity and the uniform boundedness of the nonlinear term can be considerably weakened. Finally, a simple example to which our main result can be applied is given.

1. Introduction

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. Identifying the antidual of H with H we may consider $V \subset H \subset V^*$.

In this paper we deal with the approximate controllability for the semilinear equation in H as follows.

$$\text{(SE)} \qquad \begin{cases} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \\ x(0) &= x_0. \end{cases}$$

Here, the nonlinear part is given by

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds,$$

where k belongs to $L^2(0,T)$ and $g : [0,T] \times V \times U \to H$ is a nonlinear mapping satisfying Lipschitz continuous, that is, there exist positive constants L_1, L_2 such that

$$|g(t,x,u) - g(t,\hat{x},\hat{u})| \le L_1||x-\hat{x}|| + L_2||u-\hat{u}||_U.$$

In (SE), the principal operator A generates an analytic semigroup $S(t)$ on H and B is a bounded linear operator from some Hilbert space U to H. Let $x(T; f, u)$ be a solution of (SE) associated with the nonlinear term f and the

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control u at the time T. We say that the system (SE) is approximate controllable on $[0,T]$ if for every desired final state x_1 and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that $||x(T;f,u) - x_1||_H < \epsilon$. Dauer and Mahmudov [2] dealt with the approximate controllability of a semilinear control system as a particular case of sufficient conditions for approximate solvability of semilinear equations by assuming $S(t)$ is compact operator for each $t > 0$, f is continuous and uniformly bounded and

(1) the corresponding linear system (SE) when $f \equiv 0$ is approximately controllable.

As for the some considerations on the trajectory set of (SE) and that of its corresponding linear system (in case $f \equiv 0$) as matters connected with (1), we refer to [8, 9] and references therein.

Sukavanam and Nutan Kumar Tomar [5] studied the approximate controllability for the following general retarded initial value problem:

$$\begin{cases} \frac{dx(t)}{dt} & = Ax(t) + Bx(t) + f(t, x_t, u(t)), \quad 0 < t \leq T, \\ x_0(0) & = \phi(\theta), \quad -h \leq \theta \leq 0 \end{cases}$$

in $C([-h,0];V)$ by assuming (1) and

(2) there exists a constant $\beta > 0$ such that $||Bv|| \geq \beta||v||$ for all $v \in L^2(0,T;U)$ and $L_1 < \beta$, where L_1 is the Lipschitz constant of f.

In this paper, we will deal with the control problems of (SE) on

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

So, we no longer require the compact property of semigroup, and the uniform boundedness and the inequality condition for Lipschitz continuity of f, but instead we need the regularity and a variation of solutions of the given equations in $L^2(0,T;V)$. In Section 2, we will study wellposedness and regularity properties for a class of the semilinear control system. In Section 3, it is shown the relation between the reachable set of the semilinear system and that of its corresponding linear system. We also show that the Lipschitz continuity and the uniform boundedness of the nonlinear term assumed by [5, 8] can be considerably weakened. In last section, a simple example to which our main result can be applied is given.

2. Semilinear functional equations

Let H and V be complex Hilbert spaces such that $V \subset H \subset V^*$ by identifying the antidual of H with H. Therefore, for the brevity, we may regard that $||u||_* \leq |u| \leq ||u||$ for all $u \in V$, where the notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^*, respectively as usual. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u,\ v \in V.$$

Then A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H, \tag{1}$$

where $(V, V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between V and V^* (see [7]). Moreover, for each $T > 0$, by using interpolation theory we have

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

From the following inequalities

$$\begin{aligned} c_0||u||^2 &\le \operatorname{Re} a(u, u) + c_1|u|^2 \le C|Au|\,|u| + c_1|u|^2 \\ &\le (C|Au| + c_1|u|)|u| \le C||u||_{D(A)}|u|, \end{aligned}$$

it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0||u||_{D(A)}^{1/2}|u|^{1/2}. \tag{2}$$

It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \ge 0\}$ is contained in the resolvent set of A.

The following lemma is from Lemma 3.6.2 of [6].

Lemma 2.1. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^* :*

$$|S(t)x| \le M|x|, \tag{3}$$

$$||S(t)x||_* \le M||x||_*, \tag{4}$$

$$|S(t)x| \le Mt^{-1/2}||x||_*, \tag{5}$$

$$||S(t)x|| \le Mt^{-1/2}|x|. \tag{6}$$

The following initial value problem for the abstract linear parabolic equation

$$\text{(LE)} \qquad \begin{cases} \frac{dx(t)}{dt} = Ax(t) + k(t), & 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

By virtue of Theorem 3.3 of [1] (or Theorem 3.1 of [3]), we have the following result on the corresponding linear equation of (LE).

Proposition 2.2. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold*:

1) *Let* $F = (D(A), H)_{1/2,2}$, *where* $(D(A), H)_{1/2,2}$ *is the real interpolation space between* $D(A)$ *and* H (*see* [7, Section 1.3.3]). *For* $x_0 \in F$ *and* $k \in L^2(0, T; H)$, $T > 0$, *there exists a unique solution* x *of* (LE) *belonging to*

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; F)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \leq C_1(||x_0||_F + ||k||_{L^2(0,T;H)}), \tag{7}$$

where C_1 *is a constant depending on* T.

2) *Let* $x_0 \in H$ *and* $k \in L^2(0, T; V^*)$, $T > 0$. *Then there exists a unique solution* x *of* (LE) *belonging to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{8}$$

where C_1 *is a constant depending on* T.

Lemma 2.3. *Suppose that* $k \in L^2(0, T; H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \leq t \leq T$. *Then there exists a constant* C_2 *such that*

$$||x||_{L^2(0,T;D(A))} \leq C_1||k||_{L^2(0,T;H)}, \tag{9}$$

$$||x||_{L^2(0,T;H)} \leq C_2T||k||_{L^2(0,T;H)}, \tag{10}$$

and

$$||x||_{L^2(0,T;V)} \leq C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{11}$$

Proof. The assertion (9) is immediately obtained by (7). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T \left| \int_0^t S(t-s)k(s)ds \right|^2 dt \leq M \int_0^T \left(\int_0^t |k(s)|ds \right)^2 dt \\ &\leq M \int_0^T t \int_0^t |k(s)|^2 dsdt \leq M\frac{T^2}{2} \int_0^T |k(s)|^2 ds, \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \leq T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2), (9), and (10) it holds that

$$||x||_{L^2(0,T;V)} \leq C_0\sqrt{C_1T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\left\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\right\},$$

the proof is complete. □

Consider the following initial value problem for the abstract semilinear parabolic equation

$$\text{(SE)} \qquad \begin{cases} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \\ x(0) &= x_0. \end{cases}$$

Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Let $g : \mathbb{R}^+ \times V \times U \to H$ be a nonlinear mapping satisfying the following:

(F1) For any $x \in V$, $u \in U$ the mapping $g(\cdot, x, u)$ is strongly measurable;

(F2) There exist positive constants L_0, L_1, L_2 such that

(i) $|g(t,x,u) - g(t,\hat{x},\hat{u})| \le L_1||x-\hat{x}|| + L_2||u-\hat{u}||_U$,

(ii) $|g(t,0,0)| \le L_0$

for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $u, \hat{u} \in U$.

For $x \in L^2(0,T;V)$, we set

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds,$$

where k belongs to $L^2(0,T)$

Lemma 2.4. *Let $x \in L^2(0,T;V)$ for any $T > 0$. Then $f(\cdot,x,u) \in L^2(0,T;H)$ and*

$$\begin{aligned} &||f(\cdot,x,u))||_{L^2(0,T;H)} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}). \end{aligned} \tag{12}$$

Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, then

$$\begin{aligned} &||f(\cdot,x,u) - f(\cdot,\hat{x},\hat{u})||_{L^2(0,T;H)} \\ &\le ||k||_{L^2(0,T)}\sqrt{T}(L_1||x-\hat{x}||_{L^2(0,T;V)} + L_2||u-\hat{u}||_{L^2(0,T;U)}). \end{aligned} \tag{13}$$

Proof. From (F1), (F2), and using the Hölder inequality, it is easily seen that

$$\begin{aligned} &||f(\cdot,x,u)||_{L^2(0,T;H)} \\ &\le ||f(\cdot,0,0)|| + ||f(\cdot,x,u) - f(\cdot,0,0)|| \\ &\le \left(\int_0^T \left|\int_0^t k(t-s)g(s,0,0)ds\right|^2 dt\right)^{1/2} \\ &\quad + \left(\int_0^T \left|\int_0^t k(t-s)\{g(s,x(s),u(s)) - g(s,0,0)\}ds\right|^2 dt\right)^{1/2} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}||g(\cdot,x,u) - g(\cdot,0,0)||_{L^2(0,T;H)} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}). \end{aligned}$$

The proof of (13) is similar. □

Theorem 2.5. *Under the assumptions* (F1), *and* (F2) *for the nonlinear mapping f, as given by*

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds,$$

there exists a unique solution x of (SE) *such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3 \left(|x_0| + ||u||_{L^2(0,T;U)}\right). \tag{14}$$

Proof. Let us fix $T_0 > 0$ satisfying

$$C_2 L_1 T_0 ||k||_{L^2(0,T)} < 1 \tag{15}$$

with the constant C_2 in Lemma 2.3. Let y be the solution of

$$y(t) = S(t)\phi^0 + \int_0^t S(t-s)\{f(s,x(s),u(s)) + Bu(s)\}ds.$$

We are going to show that $x \mapsto y$ is strictly contractive from $L^2(0,T_0;V)$ to itself. Let y, $\hat{y}$ belong to V with the same initial condition in $[0,T_0]$. Then, noting that $x(s) - \hat{x}(s) = 0$ for $s \in [0,T_0]$, from assumption (F1), (11) and

$$y(t) - \hat{y}(t) = \int_0^t S(t-s)\{f(s,x(s),u(s)) - f(s,\hat{x}(s),u(s))\}ds$$

we have

$$\begin{aligned} ||y - \hat{y}||_{L^2(0,T_0;V)} &\le C_2\sqrt{T_0}||f(\cdot,x,u) - f(\cdot,\hat{x},u)||_{L^2(0,T_0;H)} \\ &\le C_2 L_1 T_0 ||k||_{L^2(0,T_0)}||x(\cdot) - \hat{x}(\cdot)||_{L^2(0,T_0;V)}. \end{aligned}$$

So by virtue of the condition (15) the contraction mapping principle gives that the solution of (SE) exists uniquely in $[0,T_0]$. Let x be a solution of (SE) and $x_0 \in H$. Then there exists a constant C_1 such that

$$||S(t)x_0||_{L^2(0,T_0;V)} \le C_1|x_0| \tag{16}$$

in view of Proposition 2.2. Let

$$x_1(t) = \int_0^t S(t-s)\{f(s,x(s),u(s)) + Bu(s)\}ds.$$

Then from (11), it follows

$$\begin{aligned} &||x_1||_{L^2(0,T_0;V)} \\ &\le C_2\sqrt{T_0}||f(\cdot,x,u) + Bu||_{L^2(0,T_0;H)} \\ &\le C_2\sqrt{T}_0 \left(L_1\sqrt{T_0}||k||_{L^2(0,T)}||x||_{L^2(0,T_0;V)} + ||f(\cdot,0,u) + Bu||_{L^2(0,T_0;H)}\right). \end{aligned} \tag{17}$$

Thus, combining (16) with (17) we have

$$||x||_{L^2(0,T_0;V))} \leq (1 - C_2 L_1 T_0 ||k||_{L^2(0,T)})^{-1} \Big(C_1|x_0| + C_2\sqrt{T_0}||f(\cdot,0,u) + Bu||_{L^2(0,T_0;H)}\Big).$$

Now from

$$\begin{aligned} |x(T_0)| &= |S(T_0)\phi^0| + \left|\int_0^{T_0} S(T_0 - s)\{f(s,x(s),u(s)) + Bu(s)\}ds\right| \\ &\leq M|x_0| + ML_1\sqrt{T_0}||k||_{L^2(0,T)}||x||_{L^2(0,T_0;V))} \\ &\quad + M\sqrt{T_0}||f(\cdot,0,u) + Bu||_{L^2(0,T_0;H)}, \end{aligned}$$

since the condition (15) is independent of initial values, the solution of (SE) can be extended to the interval $[0, nT_0]$ for every natural number n. An analogous estimate to (14) holds for the solution in $[0, nT_0]$, and hence for the initial value x_{nT_0} in the interval $[nT_0, (n+1)T_0]$. □

3. Approximate controllability of semilinear systems

Let $x(T; f, u)$ be a state value of the system (SE) at time T corresponding to the nonlinear term f and the control u. We define the reachable sets for the system (SE) as follows:

$$\begin{aligned} R_T(f) &= \{x(T; f, u) : u \in L^2(0,T;U)\}, \\ R_T(0) &= \{x(T; 0, u) : u \in L^2(0,T;U)\}. \end{aligned}$$

Definition 3.1. The system (SE) is said to be approximately controllable in the time interval $[0,T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T; f, u)$ of (SE) satisfies $|x(T; f, u) - x_1| < \epsilon$, that is, if $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H, then the system (SE) is called approximately controllable at time T.

Let $u \in L^1(0,T;Y)$. Then it is well known that

$$\lim_{h\to 0} h^{-1}\int_0^h ||u(t+s) - u(t)||_Y ds = 0 \tag{18}$$

for almost all point of $t \in (0,T)$.

Definition 3.2. The point t which permits (18) to hold is called the Lebesgue point of u.

First we consider the approximate controllability of the system (SE) in case where the controller B is the identity operator on H under the Lipschitz conditions (F1), (F2) on the nonlinear operator f. So, $H = U$ obviously. Consider

the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) &= Ay(t) + u(t), \\ y(0) &= x_0 \end{cases} \tag{19}$$

and the following semilinear control system

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \\ x(0) &= x_0. \end{cases} \tag{20}$$

Theorem 3.1. *Under the assumptions* (F1) *and* (F2) *we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Therefore, if the linear system (19) *with* $f = 0$ *is approximately controllable, then so is the semilinear system* (20).

Proof. Let $y(t)$ be solution of (19) corresponding to a control u. First, we show that there exists a $v \in L^2(0, T; H)$ such that

$$\begin{cases} v(t) &= u(t) - f(t, y(t), v(t)), \quad 0 < t \leq T, \\ v(0) &= u(0). \end{cases}$$

Let T_0 be a Lebesgue point of u, v so that

$$L_2\sqrt{T_0}||k||_{L^2(0,T_0)} < 1. \tag{21}$$

For a given $u \in L^2(0, T; H)$, we define a mapping

$$W : L^2(0, T; H) \to L^2(0, T; H)$$

by

$$(Wv)(t) = u(t) - f(t, y(t), v(t)), \quad 0 < t \leq T_0.$$

It follows readily from definition of W that

$$\begin{aligned} ||Wv_1 - Wv_2||^2_{L^2(0,T_0;H)} &= ||f(\cdot, y, v_2) - f(\cdot, y, v_1)||^2_{L^2(0,T_0;H)} \\ &\leq L_2^2\, T_0||k||^2_{L^2(0,T_0)}||v_2 - v_1||^2_{L^2(0,T_0;H)} \end{aligned}$$

whence

$$||Wv_1 - Wv_2||_{L^2(0,T_0;H)} \leq L_2\sqrt{T_0}||k||_{L^2(0,T_0)}||v_2 - v_1||_{L^2(0,T_0;H)}.$$

By a well known the contraction mapping principle W has a unique fixed point v in $L^2(0, T_0; H)$ if the condition (21) is satisfied. Let

$$v(t) = u(t) - f(t, y(t), v(t)).$$

Then from (F1), (F2) and Theorem 2.5, it follows

$$
\begin{aligned}
(22)\qquad ||v||_{L^2(0,T_0;H)} &\le ||f(\cdot,y,v)+u||_{L^2(0,T_0;H)} \\
&\le \sqrt{T_0}||k||_{L^2(0,T_0)}\left(L_1||y||_{L^2(0,T_0;V)}+L_2||v||_{L^2(0,T_0;H)}\right) \\
&\quad + ||f(\cdot,0,0)+u||_{L^2(0,T_0;H)} \\
&\le \sqrt{T_0}||k||_{L^2(0,T_0)}\{L_1C_1\left(|x_0|+||u||_{L^2(0,T_0;U)}\right) \\
&\quad + L_2||v||_{L^2(0,T_0;H)}\}+||f(\cdot,0,0)+u||_{L^2(0,T_0;H)}.
\end{aligned}
$$

Thus, from which we have

$$
\begin{aligned}
||v||_{L^2(0,T_0;H))} \le (1-L_2\sqrt{T_0}||k||_{L^2(0,T_0)})^{-1}\{\sqrt{T_0}||k||_{L^2(0,T_0)}L_1C_1(|x_0| \\
+||u||_{L^2(0,T;U)})+||f(\cdot,0,0)+u||_{L^2(0,T_0;H)}\}.
\end{aligned}
$$

And we obtain

$$
\begin{aligned}
(23)\qquad & |v(T_0)| \\
&= |f(T_0,y(T_0),v(T_0))-u(T_0)| \\
&\le \left|\int_0^{T_0} k(T_0-s)\{g(s,y(s),v(s)-g(s,0,0)\}ds\right| \\
&\quad + \left|\int_0^{T_0} k(T_0-s)g(s,0,0)ds+u(T_0)\right| \\
&\le ||k||_{L^2(0,T_0)}||g(\cdot,y,v)-g(\cdot,0,0)||_{L^2(0,T_0;H)} \\
&\quad + L_0||k||_{L^2(0,T_0)}\sqrt{T_0}+|u(T_0)| \\
&\le ||k||_{L^2(0,T_0)}(L_1||y||_{L^2(0,T_0;V))}+L_2||v||_{L^2(0,T_0;H)}+L_0\sqrt{T_0})+|u(T_0)|.
\end{aligned}
$$

If $2T_0$ is a Lebesgue point of u, v, then we can solve the equation in $[T_0, 2T_0]$ with the initial value $v(T_0)$ and obtain an analogous estimate to (22) and (23). If not, we can choose $T_1 \in [T_0, 2T_0]$ to be a Lebesgue point of u, v. Since the condition (21) is independent of initial values, the solution can be extended to the interval $[T_1, T_1+T_0]$, and so we have showed that there exists a $v \in L^2(0,T;H)$ such that $v(t) = u(t) - f(t, y(t), v(t))$. Let $v(t) = u(t) - f(t, y(t), v(t))$ and let y be a solution of (19) corresponding to a control u. Consider the following semilinear system

$$
\begin{cases}
\frac{d}{dt}x(t) &= Ax(t)+f(t,x(t),v(t))+u(t)-f(t,y(t),v(t)),\ 0<t\le T \\
x(0) &= x_0.
\end{cases}
$$

The solution of (19) and (24), respectively, can be written as

$$
y(t) = S(t)x_0 + \int_0^t S(t-s)u(s)ds,
$$

and

$$x(t) = S(t)x_0 + \int_0^t S(t-s)u(s)ds + \int_0^t S(t-s)\{f(s,x(s),v(s)) - f(s,y(s),v(s))\}ds.$$

Then from Theorem 2.5 it is easily seen that $x(\cdot) \in C([0,T];H)$, that is, $x(s) \to x(t)$ as $s \to t$ in H. Let $\epsilon > 0$ be given. For $t \geq \epsilon$, set

$$x^\epsilon(t) = S(t)x_0 + \int_0^{t-\epsilon} S(t-s)u(s)ds + \int_0^{t-\epsilon} S(t-s)\{f(s,x^\epsilon(s),v(s)) - f(s,y(s),v(s))\}ds.$$

Then we have

$$\begin{aligned} x(t) - x^\epsilon(t) = & \int_{t-\epsilon}^t S(t-s)u(s)ds - \int_{t-\epsilon}^t S(t-s)f(s,y(s),v(s))ds \\ & + \int_{t-\epsilon}^t S(t-s)f(s,x(s),v(s))ds \\ & + \int_0^{t-\epsilon} S(t-s)\{f(s,x(s),v(s)) - f(s,x^\epsilon(s),v(s))\}ds. \end{aligned}$$

So, as seen in the proof of Theorem 2.5, for some constant $T_0 > 0$ satisfying $C_2L_1T_0||k||_{L^2(0,T)} < 1$, we see easily that

$$\begin{aligned} ||x - x^\epsilon||_{L^2(0,T_0;V)} \leq & C_2\sqrt{\epsilon}||u||_{L^2(0,T_0;H)} + C_2L_1T_0||k||_{L^2(0,T)}||x - x^\epsilon||_{L^2(0,T_0;V)} \\ & + C_2\sqrt{\epsilon}L_1\sqrt{T}_0||k||_{L^2(0,T)}||x - y||_{L^2(0,T_0;V)}. \end{aligned}$$

By the step by step method, we know that $x^\epsilon \to x$ as $\epsilon \to 0$ in $L^2(0,T;V)$ $(T > 0)$ for $\epsilon < t < T$. From (6) it follows that

$$\begin{aligned} & ||x^\epsilon - y||^2_{L^2(0,T;V)} \\ = & \int_0^T \left| \int_0^{t+s-\epsilon} S(t+s-\tau)\{f(\tau,x^\epsilon(\tau),v(\tau)) - f(\tau,y(\tau),v(\tau))\}d\tau \right|^2 ds \\ \leq & (ML_1||k||_{L^2(0,\tau)})^2 \int_0^T \left(\int_0^{t+s-\epsilon} (t+s-\tau)^{-1/2}||x^\epsilon - y||_{L^2(0,\tau;V)}d\tau \right)^2 ds \\ \leq & (ML_1||k||_{L^2(0,\tau)})^2 \int_0^T \log\left(\frac{t}{\epsilon}\right) ||x^\epsilon - y||^2_{L^2(0,\tau;V)}d\tau ds \\ \leq & T(ML_1||k||_{L^2(0,\tau)})^2 \log\left(\frac{t}{\epsilon}\right) \int_0^T ||x^\epsilon - y||^2_{L^2(0,\tau;V)}d\tau. \end{aligned}$$

By using Gronwall's inequality, independently of ϵ, we get $x^\epsilon = y$ in $L^2(0,T;V)$ for almost all $\epsilon \leq t \leq T$, and $x^\epsilon(t) = y(t)$ in H. Therefore, noting that

$x(\cdot)$, $y(\cdot) \in C([0,T;H])$, every solution of the linear system with control u is also a solution of the semilinear system with control w, that is, we have that $R_T(0) \subset \overline{R_T(f)}$. □

From now on, we consider the initial value problem for the semilinear parabolic equation (SE). Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Theorem 3.2. *Let us assume that there exists a constant $\beta > 0$ such that $||Bu|| \geq \beta||u||$ for all $u \in L^2(0,T;U)$, and the assumptions* (F1), (F2), *and $R(f) \subset R(B)$ be satisfied. Then we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Proof. Consider the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) & = Ay(t) + u(t), \\ y(0) & = x_0 \end{cases} \tag{24}$$

and the following semilinear control system

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t, x(t), v(t)) + Bv(t), \\ x(0) & = x_0. \end{cases} \tag{25}$$

Let y be a solution of (25) corresponding to a control u. Set $v(t) = u(t) - B^{-1}f(t, y(t), v(t))$. Then as seen in Theorem 3.1, we know that $v \in L^2(0,T;U)$. Consider the following semilinear system

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t, x(t), v(t)) + Bu(t) - f(t, y(t), v(t)), \ 0 < t \leq T \\ x(0) & = x_0. \end{cases}$$

If we define x^ϵ, y as in proof of Theorem 3.1, then we get

$$x^\epsilon(t) - y(t) = \int_0^{t-\epsilon} S(t-s)\{f(s, x^\epsilon, v(s)) - f(s, y, v(s))\}ds.$$

So using Gronwall's inequality, as in Theorem 3.1, we obtain that $R_T(0) \subset \overline{R_T(f)}$. □

Example. We consider the semilinear heat equation dealt with by Zhou [9], and Naito [4]. Let

$$H = L^2(0,\pi), \ V = H_0^1(0,\pi), \ V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$\text{(SE1)} \qquad \begin{cases} \frac{d}{dt}y(x,t) & = Ay(x,t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \\ y(t,0) & = y(t,\pi) = 0, \quad t > 0, \\ y(0,x) & = \phi^0(x), \quad y(x,s) = \phi^1(x,s), \quad -h \le s < 0, \end{cases}$$

where k belongs to $L^2(0,T)$. The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Let

$$U = \left\{ \sum_{n=2}^{\infty} u_n \phi_n : \sum_{n=2}^{\infty} u_n^2 < \infty \right\},$$

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n \phi_n \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n \in U.$$

It is easily seen that the operator B is one to one and $\mathrm{R}(B)$ is closed. It follows that the operator B satisfies hypothesis as in Theorem 3.2. For example, consider the nonlinear term f given by

$$g(t,y,u) = \alpha(t)\left(||D_x y||\phi_1(x) + ||u||\phi_2(x)\right), \quad \alpha(t) \in C([0,T]).$$

Then f is not uniformly bounded and $\mathrm{R}(g) \subset \mathrm{R}(B)$ and from Theorem 3.2 it follows that the system of (SE1) is approximately controllable. In case where $B = I$ we obtain the approximate controllability of (SE1) without restrictions such as the uniform boundedness and inequality constraints for Lipschitz constant of f or compactness of $S(t)$.

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