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CONTROLLABILITY OF CONVEX PROCESSES

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PREFACE

The purpose of this paper is to provide several characterizations of controllability of differential inclusions whose right-hand sides are convex processes. Convex processes are the set-valued maps whose graphs are convex cones; they are the set-valued analogues of linear operators. Such differential inclusions include linear systems where the controls range over a convex cone (and not only a vector space). The characteristic properties are couched in terms of invariant cones by convex processes, or eigenvalues of convex processes, or a rank condition. We also show that the controllability is equivalent to the observability of the adjoint inclusion.

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Introduction

A convex process A from $\mathbb{R}^n$ to itself is a set-valued map satisfying

$$(0.1) \qquad \forall\, x,y \in \text{Dom } A \,, \quad \lambda,\mu \geqslant 0 \,, \qquad \lambda A(x)+\mu A(y) \subset A(\lambda x+\mu y)$$

or, equivalently, a set-valued map whose graph is a convex cone. Convex processes are the set-valued analogues of linear operators. We shall say that a convex process is closed if its graph is closed and that it is strict if its domain is the whole space.

We associate with a strict closed convex process A the Cauchy problem for the differential inclusion

$$(0.2) \qquad \text{for almost all } \; t \in [0,T] \,, \quad x'(t) \in A(x(t)) \,, \quad x(0) = 0$$

We denote by R_T the reachable set at time T defined by

$$(0.3) \qquad R_T \; := \; \{\, x(T) \,\}_{x(\cdot)} \; \text{ is a solution to (0.2).}$$

We also say that

$$(0.4) \qquad R \; := \bigcup_{T>0} R_T \; \text{ is the reachable set}$$

and that the differential inclusion (0.2) (or the convex process A) is controllable if the reachable set R is equal to the whole space $\mathbb{R}^n$.

Convex processes where introduced and thoroughly studied in Rockafellar [1967], [1970], [1974] and in Aubin-Ekeland [1984], for instance. Derivatives of set-valued maps (see Aubin-Ekeland, [1984], chapter 7) provide examples of closed convex processes. These are used, for instance, in Frankowska [1984], [1985] for deriving local controllability of differential inclusions from the controllability of convex processes which "approximate" in some sense the original differential inclusion around the equilibrium (★) .

We know that for linear problems, the reachable sets are invariant. Hence we have to extend the usual concept of invariant subspace by a linear operator. This can be done in two different ways : let A be a convex process and P be a closed convex cone contained in $\mathrm{Dom}\, A$. We recall that the <u>tangent cone</u> $T_P(x)$ at a point $x \in P$ is defined by

(0;5) $$T_P(x) := \mathrm{cl}\left(\bigcup_{h>0} \frac{1}{h}(P-x)\right) = \mathrm{cl}\,(P + \mathbb{R}x)$$

(★) <u>Theorem</u> (Frankowska). Let F be a set-valued map from $\mathbb{R}^n$ into the compact subsets of $\mathbb{R}^n$, Lipschitzean around zero and $0 \in F(0)$. Denote by $F'(0)$ the derivative of F at zero and by L the closed convex cone spanned by $\mathrm{co}\, F(0)$ (convex hull of $F(0)$). Set

$$A(x) := \overline{F'(0)x + L}$$

Then the differential inclusion

$$x' \in F(x) \quad ; \quad x(0) = 0$$

is locally controllable around zero at time T if the "linearized" inclusion

$$x' \in A(x)$$

is controllable at time T . ■

We shall say that P is <u>invariant by</u> A if

$$(0.6) \qquad \forall\, x \in P \,, \qquad A(x) \subset T_P(x)$$

and that P is a <u>viability domain</u> for A is

$$(0.7) \qquad \forall\, x \in P \,, \qquad A(x) \cap T_P(x) \neq \emptyset \,.$$

When P is a vector space, then $T_P(x) = P$, so that a subspace is invariant by A if $\forall\, x \in P$, $A(x) \subset P$ and is a viability domain for A if $\forall\, x \in P$, $A(x) \cap P \neq \emptyset$.

A first example of invariant cone is provided by the closure of the reachable set.

<u>Theorem 0.1</u>

Let A be a strict closed convex process. Then the closure of the reachable set is the smallest closed convex cone containing $A(0)$ which is invariant by A, the subspace $R - R$ spanned by R is the smallest subspace containing $A(0)$ invariant by A.

Furthermore, if $R - R = \mathbb{R}^n$ and $R \neq \mathbb{R}^n$, there exists $\lambda \in \mathbb{R}$ such that $\mathrm{Im}\,(A - \lambda I) \neq \mathbb{R}^n$. ▲

We could say that a real number λ such that $\mathrm{Im}\,(A - \lambda I) \neq \mathbb{R}^n$ is an <u>eigenvalue of</u> A.

We shall prove this theorem by "duality". Indeed, convex processes can be transposed, as linear operators. Let A be a convex process ; we define its <u>transpose</u> $A^\star$ by

$$(0.8) \qquad p \in A^\star(q) \iff \forall\, (x,y) \in \text{Graph } A \,, \quad \langle p,x\rangle \leqslant \langle q,y\rangle$$

We also replace the orthogonality between subspaces by polarity between cones. If G is a subset of $\mathbb{R}^n$, we denote by G^+ its (positive) <u>polar cone</u> defined by :

(0.9) $$G^{+} := \{p \in \mathbb{R}^n \mid \forall x \in G , \langle p,x\rangle \geq 0\} .$$

We recall that the separation theorem implies that

(0.10) G^{++} is the closed convex cone spanned by G .

Therefore, it is convenient to bear in mind that

(0.11) $$(q,p) \in \text{Graph } (A^{\star}) \Leftrightarrow (-p,q) \in \text{Graph } (A)^{+}$$

so that when A is a closed convex process, then $A = A^{\star\star}$.

<u>Example</u>. Let F be a linear operator from $\mathbb{R}^n$ to itself, L be a closed convex cone of controls and A be the strict closed convex process defined by

(0.12) $$A(x) := Fx + L$$

Then its transpose is equal to

(0.13) $$A^{\star}(q) = \begin{cases} F^{\star}q & \text{if } q \in L^{+} \\ \emptyset & \text{if } q \notin L^{+} \end{cases}$$

When $L = \{0\}$, i.e., when $A = F$, we deduce that $A^{\star} = F^{\star}$, so that transposition of convex processes is a legitimate extension of transposition of linear operators.

When A is a strict closed convex process, we shall prove that $A^{\star}$ is upper semi-continuous with convex compact values, that $A^{\star}(0) = \{0\}$, $\text{Dom } A^{\star} = A(0)^{+}$ is closed and that the restriction of $A^{\star}$ to the vector space $\text{Dom } A^{\star} \cap - \text{Dom } A^{\star}$ is a linear (single-valued) operator.

As expected, we associate with the differential inclusion (0.2) the <u>adjoint inclusion</u> :

$$\text{(0.14)} \qquad \text{for almost all } t \in [0,T] , \quad -q'(t) \in A^{\star}(q(t))$$

We introduce the cones Q_T and Q defined by

$$\text{(0.15)} \qquad \begin{cases} \text{i)} & Q_T := \{\eta \mid q(\cdot) \text{ , a solution to (0.14) satisfying } q(T)=\eta\} \\ \text{ii)} & Q := \bigcap_{T>0} Q_T . \end{cases}$$

To say that $Q = \{0\}$ amounts to saying that the only solution to (0.14) defined on $[0,\infty[$ is $q \equiv 0$, or, in the language of systems theory, that the adjoint system is observable.

The "duality" method lies in the following statement.

Theorem 0.2

Let A be a strict closed convex process. Then

$$\text{(0.16)} \qquad R_T^+ = Q_T \quad \text{and} \quad R^+ = Q$$

Furthermore, a closed convex cone $P \supset A(0)$ is invariant by A if and only if its polar cone $P^+ \subset \text{Dom } A^{\star}$ is a viability domain for $A^{\star}$. ▲

Indeed, it allows to derive theorem 0.1 from

Theorem 0.3

Let A be a strict closed convex process. The cone Q is the largest closed convex cone which is a viability domain for $A^{\star}$ and $Q \cap -Q$ is the largest subspace invariant by (the linear operator) $A^{\star}$.

Furthermore, if Q is not reduced to $\{0\}$ and contains no line, there exists a solution $q \neq 0$ and $\lambda \in \mathbb{R}$ to the inclusion $\lambda q \in A^{\star}(q)$. ▲

We could say that such a q is an eignevector of $A^{\star}$.

It will be convenient to introduce the following definition. We say that A satisfies the <u>rank condition</u> if

(0.17) $\left\{ \begin{array}{l} \text{the subspace spanned by the cone } A^m(0) \text{ is the whole space} \\ \mathbb{R}^n \text{ for some integer } m \geq 1 \end{array} \right.$

This is motiveted by the terminology used for linear systems. Indeed, when $A(x) := Fx + L$ where F is a linear operator and L is a convex cone of controls, we observe that $A^m(0) = L + F L + \ldots + F^{m-1} L$.

We shall derive from these results the following characterization of controllability of convex processes.

<u>Theorem 0.4</u>

Let A be a strict closed convex process. The following conditions are equivalent.

a) differential inclusion (0.2) is controllable (i.e., $R = \mathbb{R}^n$)

b) differential inclusion (0.2) is controllable at some time $T > 0$ (i.e., $R_T = \mathbb{R}^n$)

c) the adjoint inclusion (0.14) is observable (i.e., $Q = \{0\}$)

d) the adjoint inclusion (0.14) is observable at some time $T > 0$ (i.e., $Q_T = \{0\}$)

e) $\mathbb{R}^n$ is the smallest closed convex cone containing $A(0)$ which is invariant by A

f) $\{0\}$ is the largest closed convex cone which is a viability domain for $A^\star$

g) A has neither proper invariant subspace nor eigenvalues

h) $A^\star$ has neither proper invariant subspace nor eigenvectors

i) the rank condition holds true and A has no eigenvalues

j) the rank condition is satisfied and $A^\star$ has no eigenvectors.

k) for some $m \geq 1$, $A^m(0) = (-A)^m(0) = \mathbb{R}^n$ ▲

Example. Let F be a linear operator from $\mathbb{R}^n$ to itself and L be a closed convex cone of controls. We consider the differential inclusion

$$(0.18) \qquad x'(t) \in Fx(t) + L \quad , \quad x(0) = 0$$

and its adjoint inclusion

$$(0.19) \qquad -q'(t) = F^{\star}q(t) \quad , \quad \forall\, t \geq 0 \, , \quad q(t) \in L^{+}$$

Corollary 0.5

The following conditions are equivalent.

a) the system (0.18) is controllable

b) the adjoint equation (0.19) is observable (the only solution of $-q' = F^{\star}q$ remaining in L^{+} on $[0,\infty[$ is $q \equiv 0$)

c) $\{0\}$ is the largest closed convex cone contained in L^{+} which is invariant by $F^{\star}$

d) $F^{\star}$ has neither proper invariant subspace contained in L^{+} nor eigenvector in L^{+}

e) the subspace spanned by $L, FL, \ldots, F^{n-1}L$ is equal to $\mathbb{R}^n$ and $F^{\star}$ has no eigenvector in L^{+} (see Brammer [1])

f) for some $m \geq 1$, $L + FL + \ldots + F^mL = L - FL + \ldots + (-1)^mF^mL = \mathbb{R}^n$ (See Korobov [1980]). ▲

This example also illustrates another advantage of duality, because some properties bearing on the adjoint system have a simpler formulation. This explains why some criteria mentionned in Theorem 0.4 disappear in Corollary 0.5.

When L is a vector space, statements c), d) and f) are the same and the mention of eigenvector in statement e) is redondant. This is not the case when L is a proper cone. It is sufficient to consider the example :

$$x' \in -x + \mathbb{R}_+ \quad , \quad x(0) = 0$$

The rank condition is satisfied $(A^2(0) = \mathbb{R})$ and the reachable set is $\mathbb{R}_+$.

We summarize in the first section the results on convex processes and their transpose that we will need later. Section 2 is devoted to the proof of the duality Theorem 0.2, characterizing the positive polar cones of the reachable set. We then derive the characterization of the closure of the reachable set as the smallest invariant cone by A and its dual version in section 3 and the existence of eigenvalues of A and eigenvectors of $A^\star$ in the fourth section. These results are used to prove Theorem 0.4 in the fifth section.

Contents

1. Convex processes and their transposes

Definition 1.1

A set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to be a convex process if its graph is a convex cone. It is closed if its graph is closed. It is called strict if

$$\text{Dom } A := \{x \in \mathbb{R}^n \mid A(x) \neq \emptyset\} \quad \text{is the whole space} .$$

▲

Definition 1.2

Let X be a Hilbert space and $G \subset X$ be a subset. We denote by G^+, the (positive) polar cone of G, the closed convex cone defined by

$$(1.1) \qquad G^+ := \{p \in X^\star \mid \forall x \in G , \ \langle p,x\rangle \geq 0\}$$

▲

The separation theorem implies that the "bipolar" G^{++} is the closed convex cone spanned by G. We shall use the following consequence of this fact.

Lemma 1.3 (Closed image Lemma).

Let X,Y be two Hilbert spaces, ϕ be a continuous linear operator from X to Y and L be a closed convex cone of Y. Assume that

$$(1.2) \qquad \text{Im } \phi - L = Y \qquad \text{(surjectivity condition)}$$

Then

$$(1.3) \qquad \phi^{-1}(L)^+ = \phi^\star(L^+)$$

▲

Proof.

a) We prove first that $\phi^\star(L^+)$ is closed. Let $q_n \in L^+$ be a sequence such that $\phi^\star(q_n)$ converges to some p in $X^\star$ and let us prove that p belongs to $\phi^\star(L^+)$.

We begin by showing that q_n is weakly bounded. Indeed, for any $v \in Y$, there exist $x \in X$ and $y \in L$ such that $v = \phi(x) - y$. Hence :

$$<q_n,v> = <\phi^\star(q_n),x> - <q_n,y> \leqslant <\phi^\star(q_n),x>$$

$$\leqslant \|\phi^\star(q_n)\|\cdot\|x\|$$

Therefore, since X is reflexive, the sequence q_n is in a weakly compact subset and a subsequence $q_{n'}$ converges weakly to some $q \in Y^\star$. Since L^+ is closed and convex, and thus, weakly closed, q belongs to L^+. Since $\phi^\star(q_{n'})$ converges weakly to $\phi^\star(q)$ and strongly to p, we deduce that $p = \phi^\star(q) \in \phi^\star(L^+)$.

b) We observe that $\phi^\star(L^+)^+ = \phi^{-1}(L)$ because $x \in \phi^\star(L^+)^+$ if and only if $<\phi^\star q,x> = <q,\phi(x)> \geqslant 0$ for all $q \in L^+$, i.e., if and only if $\phi(x)$ belongs to $L^{++} = L$. Hence, since $\phi^\star(L^+)$ is closed, we deduce that

$$\phi^\star(L^+) = \phi^\star(L^+)^{++} = \phi^{-1}(L)^+ \qquad \blacksquare$$

We now recall some properties of convex process, some of them already known (see Rockafellar [1967], [1970] § 39, [1974], Aubin-Ekeland, [1984], chapter 3).

Definition 1.4

Let A be a convex process from $\mathbb{R}^n$ to itself. The transpose $A^\star$ of A is the set-valued map from $\mathbb{R}^n$ to itself given by

(1.4) $$p \in A^\star(q) \Leftrightarrow \forall (x,y) \in \text{Graph } (A), \; <p,x> \leqslant <q,y> \qquad \blacktriangle$$

In other words,

(1.5) $$(q,p) \in \text{Graph } (A^\star) \Leftrightarrow (-p,q) \in (\text{Graph } A)^+$$

The transpose of $A^\star$ is obviously a closed convex process and $A = A^{\star\star}$ if and only if the convex process A is closed. When A is a linear operator, its transpose as a linear operator coincides with its transpose as a convex process.

Lemma 1.5

If A is a closed convex process, then

(1.6) $$A(0) = (\mathrm{Dom}\, A^{\star})^{+}$$ ▲

Proof

We observe that y belongs to $A(0)$ if and only if $0 = \langle p,0\rangle \leqslant \langle q,y\rangle$ for all $q \in \mathrm{Dom}\, A^{\star}$ and $p \in A^{\star}(q)$, i.e., if and only if $\langle q,y\rangle \geqslant 0$ for all $q \in \mathrm{Dom}\, A^{\star}$. ■

Definition 1.5

Let B denote the unit ball. When A is a closed convex process, we define its norm by

(1.7) $$\|A\| := \sup_{x \in B \cap \mathrm{Dom}\, A} \; \inf_{y \in A(x)} \|y\| \in [0,+\infty] \quad .$$ ▲

Proposition 1.6

Let A be a strict closed convex process. Then

a) $\forall\, x,y \in \mathbb{R}^{n}$, $A(x) \subset A(y) + \|A\|\, \|x-y\| B$ (i.e., A is Lipschitzean with finite Lipschitz constant equal to $\|A\|$).

b) $\mathrm{Dom}\, A^{\star} = A(0)^{+}$ and $A^{\star}$ is upper semicontinuous with compact convex images, mapping the unit ball into the ball or radius $\|A\|$.

c) the restriction of $A^{\star}$ to the vector space $\mathrm{Dom}\, A^{\star} \cap -\mathrm{Dom}\, A^{\star}$ is single-valued and linear (and thus, $A^{\star}(0) = 0$). ▲

Proof

a) The first statement is a reformulation of Robinson-Ursescu's theorem (see Robinson [1967], Ursescu [1975], and Aubin-Ekeland [1984], Corollary 3.3.3, p. 132).

b) We observe that :

(1.8) $$\forall\, q \in \text{Dom } A^{\star}, \quad \sup_{p \in A^{\star}(q)} \|p\| \leq \|A\|\, \|q\|$$

because, for all $x \in \text{Dom } A = \mathbb{R}^n$, for all $p \in A^{\star}(q)$, we have

$$\|p\| = \sup_{x \in \mathbb{R}^n} \frac{<p,x>}{\|x\|} \leq \sup_{x \in \mathbb{R}^n} \inf_{y \in A(x)} \frac{<q,y>}{\|x\|}$$

$$\leq \sup_{x \in \mathbb{R}^n} \inf_{y \in A(x)} \frac{\|q\|\,\|y\|}{\|x\|} = \|A\|\,\|q\| \;.$$

Then $A^{\star}$ maps bounded sets to bounded sets. Since its graph is a closed convex cone we deduce that $A^{\star}$ is upper semicontinuous with compact convex images. By Lemma 1.5, $\overline{\text{Dom } A^{\star}} = A(0)^{+}$. Therefore it remains to prove that $\text{Dom } A^{\star}$ is closed. Indeed let $q_n \in \text{Dom } A^{\star}$ be a sequence converging to some q and let $p_n \in A^{\star}(q_n)$. The sequence $\{p_n\}$ being bounded contains a subsequence $\{p_{n'}\}$ converging to some p. Thus

$$(q,p) = \lim_{n' \to \infty} (q_{n'}, p_{n'}) \quad , \quad (q_{n'}, p_{n'}) \in \text{graph } A^{\star}$$

The graph of $A^{\star}$ being closed, we proved that $q \in \text{Dom } A^{\star}$.

■

We observe that we always have

$$\sup_{p \in A^{\star}(q_o)} <p,x_o> \leq \inf_{y \in A(x)} <q_o,y> \;.$$

<u>Lemma 1.7</u>

Let A be a closed convex process.

For any $x_o \in \text{Int Dom } A$, and $q_o \in \text{Dom } A^{\star}$,

(1.9) $$\sup_{p \in A^{\star}(q_o)} <p,x_o> = \inf_{y \in A(x_o)} <q_o,y>$$ ▲

(See Rockafellar [1970]).

We now extend to the case of closed convex cones the concepts of invariant subspaces. When K is a subspace and F is a linear operator, we recall that K is invariant by F when $Fx \in K$ for all $x \in K$. When A is a convex process, there are two ways of extending this notion : we shall say that K is invariant by A if, for any $x \in K$, $A(x) \subset K$ and that K is a viability domain for A if, for any $x \in K$, $A(x) \cap K \neq \emptyset$. We also need to extend these notions to the case when K is a closed convex cone. We recall the

Definition 1.8

If K is a closed convex set and x belongs to K, we say that

$$T_K(x) := \mathrm{cl}\left(\bigcup_{h>0} \frac{1}{h}(K-x)\right) \quad \blacktriangle$$

is the tangent cone to K at x.

Lemma 1.9

When K is a vector subspace, then, for all $x \in K$, $T_K(x) = K$ and when K is a closed convex cone, then,

(1.10) $$\forall\, x \in K, \quad T_K(x) = \mathrm{cl}\,(K + \mathbb{R}x) \quad \blacktriangle$$

(See Aubin-Ekeland, [1984], Proposition 4.1.9, p. 171).

Now, we can introduce

Definition 1.10

Let K be a closed convex cone and A be a convex process. We say that K is invariant by A if

(1.11) $$\forall\, x \in K, \quad A(x) \subset T_K(x)$$

and that K is a viability domain for A if

(1.12) $$\forall\, x \in K, \quad A(x) \cap T_K(x) \neq \emptyset \quad \blacktriangle$$

These are dual notions, as the following proposition shows.

Proposition 1.11

Let A be a strict closed convex process and K be a closed convex cone containing $A(0)$. Then K is invariant by A if and only if K^+ is a viability domain for $A^\star$. ▲

Proof

By Proposition 1.6 b) the condition $A(0) \subset K$ implies that $K^+ \subset A(0)^+ = \text{Dom } A^\star$. To say that K is invariant by A amounts to saying that

$$(1.13) \qquad \forall\, x \in K \;,\; \forall\, q \in T_K(x)^+ \;, \quad \inf_{y \in A(x)} \langle q,y\rangle \geq 0$$

Lemma 1.9 states $T_K(x) = \overline{\mathbb{R}x + K}$, $T_{K^+}(q) = \overline{\mathbb{R}q + K^+}$. Therefore

$$(1.14) \qquad q \in T_K(x)^+ \Leftrightarrow \langle q,x\rangle = 0 \quad , \quad q \in K^+ \Leftrightarrow x \in T_{K^+}(q)^+$$

On the other hand, Lemma 1.7 implies that $\inf_{y \in A(x)} \langle q,y\rangle = \sup_{p \in A^\star(q)} \langle p,x\rangle$.
Therefore condition (1.13) is equivalent to the condition :

$$(1.15) \qquad \forall\, q \in K^+ \;,\; \forall\, x \in T_{K^+}(q)^+ \;, \quad \sup_{p \in A^\star(q)} \langle p,x\rangle \geq 0$$

By proposition 1.6 b) for all $q \in K^+$ the set $A^\star(q)$ is compact. The separation theorem implies that $A^\star(q)$ has a nonempty intersection with $T_{K^+}(q)$ if and only if for all $x \in \mathbb{R}^n$, $\sup_{p \in A^\star(q)} \langle p,x\rangle \geq \inf_{z \in T_{K^+}(q)} \langle z,x\rangle$.
Since $T_{K^+}(q)$ is a cone the latter inequality is equivalent to (1.15). This ends the proof. ■

We introduce now the concepts of eigenvalues and eigenvectors of closed convex processes.

Definition 1.12

We shall say that $\lambda \in \mathbb{R}$ is an eigenvalue of a convex process A if $\text{Im}\,(A - \lambda I) \neq \mathbb{R}^n$ and that $x \in \text{Dom}\, A$ is an eigenvector of A if $x \neq 0$ and if there exists $\lambda \in \mathbb{R}$ such that $\lambda x \in A(x)$. ▲

We observe that half-lines spanned by eigenvectors of $A^\star$ are viability domains for $A^\star$.

Lemma 1.13

Let A be a strict convex process. Then $A^\star$ has an eigenvector if and only if $\text{Im}\,(A - \lambda I) \neq \mathbb{R}^n$ for some $\lambda \in \mathbb{R}$. ▲

Proof

a) Let η be an eigenvector of $A^\star$, a solution to $\lambda\eta \in A^\star(\eta)$, $\eta \neq 0$. Thus, for all $y \in A(x)$, $\langle \eta, y - \lambda x \rangle \geq 0$ and thus, $\text{Im}\,(A - \lambda I) \subset \{\eta\}^+ \neq \mathbb{R}^n$.

b) Conversely, assume that for some $\lambda \in \mathbb{R}$, $\text{Im}\,(A - \lambda I) \neq \mathbb{R}^n$. Since it is a convex cone of a finite dimensional space, there exists a non zero $\eta \in \mathbb{R}^n$ such that $\langle \eta, z \rangle \geq 0$ for all $z \in \text{Im}\,(A - \lambda I)$. This implies that for all $x \in \mathbb{R}^n$ and $y \in A(x)$,

$$\lambda \langle \eta, x \rangle \leq \langle \eta, y \rangle$$

By the very definition of $A^\star$, we deduce that $\lambda\eta$ belongs to $A^\star(\eta)$. ∎

Exemple 1.14

Let F be a linear operator from $\mathbb{R}^n$ to itself, L be a closed convex cone of controls and A be the strict closed convex process defined by $A(x) := Fx + L$.

A cone K is invariant by A if

$$\forall\, x \in K, \qquad Fx + L \subset T_K(x)$$

and λ is an eigenvalue of A if

$$\text{Im}\,(F - \lambda I) + L \neq \mathbb{R}^n .$$

The transpose $A^{\star}$ of A is defined by

$$A^{\star}q = \begin{cases} F^{\star}q & \text{if} \quad q \in L^{+} \\ \emptyset & \text{if} \quad q \notin L^{+} \end{cases}$$

A cone $P \subset L^{+} = \text{Dom}\, A^{\star}$ is a viability domain for $A^{\star}$ if and only if

$$\forall\, q \in P , \qquad F^{\star}q \in T_P(q)$$

An element $q \neq 0$ is an eigenvector of $A^{\star}$ if and only if q is an eigenvector of $F^{\star}$ which belongs to the cone L^{+} .

2. The duality theorem.

We devote this section to the duality theorem, which characterizes the polar cones of the reachable sets.

We denote by $W^{1,p}(0,T)$, $p \in [1,\infty]$, the Sobolev space of functions $x \in L^p(0,T;\mathbb{R}^n)$ such that $x'(\cdot)$ belongs to $L^p(0,T;\mathbb{R}^n)$.

Let us consider the Cauchy problem for the differential inclusion

$$(2.1) \qquad \begin{cases} \text{i)} \quad x'(t) \in A(x(t)) & \text{for almost all} \quad t \in [0,T] \\ \text{ii)} \quad x(0) = 0 \end{cases}$$

We recall that the reachable set R_T is defined by

$$(2.2) \qquad R_T := \{x(T) \mid x \in W^{1,1}(0,T) \text{ is a solution to } (2.1)\} \quad .$$

We shall characterize its positive polar cone R_T^+ . For that purpose, we associate with the differential inclusion (2.1) the adjoint inclusion

$$(2.3) \qquad \begin{cases} \text{i)} \quad -q'(t) \in A^\star(q(t)) & \text{for almost all} \quad t \in [0,T] \\ \text{ii)} \quad q(T) = \eta \end{cases}$$

and we denote by $Q_T \subset \text{Dom}\, A^\star$ the set of "final" values η such that the differential inclusion (2.3) has a solution.

$$(2.4) \qquad Q_T := \{\eta \mid \exists q \in W^{1,1}(0,T) \text{ a solution to } (2.3)\}$$

The statement of the duality theorem is the following.

Theorem 2.1

Let A be a strict closed convex process. Then

$$(2.5) \qquad R_T^+ = Q_T \quad .$$

▲

We need the following technical lemma.

<u>Lemma 2.2</u>

Let A be a strict closed convex process. Then the $W^{1,\infty}(0,T)$ solutions to (2.1) are dense in $W^{1,1}(0,T)$ solutions to (2.1) in the metric of uniform convergence on $[0,T]$. ▲

<u>Proof</u>

Indeed let $w \in W^{1,1}(0,T)$ be a solution of (2.1) and $\varepsilon > 0$ be a given number. Denote by $C \geqslant 1$ a Lipschitz constant of A which exists thanks to Proposition 1.6 a). Let $M \subset [0,T]$ be such that w' is bounded on $[0,T] \setminus M$ and

$$(2.6) \qquad 2C(T+1)\, e^{CT} \int_M (\|w(s)\| + \|w'(s)\|)\, ds < \varepsilon$$

Set

$$y'(t) := \begin{cases} 0 & \text{if } t \in M \\ w'(t) & \text{otherwise} \end{cases}$$

and

$$y(t) := \int_0^t y'(s)\, ds$$

Then

$$(2.7) \qquad \|y(t) - w(t)\| \leqslant \int_M \|w'(s)\| ds \leqslant \varepsilon/2$$

and

$$p(t) := \operatorname{dist}(y'(t), A(y(t))) \leqslant \begin{cases} C\|y(t)\| & \text{if } t \in M \\ C\|w(t)-y(t)\| & \text{otherwise} \end{cases}$$

Thus

$$(2.8) \qquad \int_0^T p(t)dt \leqslant C \left(\int_M \|w(t)\| dt + \int_0^T \|w(t)-y(t)\| dt \right)$$

$$\leqslant C \int_M \left(\|w(s)\| + T\|w'(s)\| \right) ds$$

By a Filippov Theorem (see Aubin-Cellina [1984] p. 120) there exists a solution $x(\cdot)$ to (2.1) satisfying, by (2.6) and (2.8),

$$(2.9)\quad \begin{cases} \text{i)} & \|x(t)-y(t)\| \leqslant e^{CT}\int_0^T p(t)\,dt < \varepsilon/2 \\ \text{ii)} & \|x'(t)-y'(t)\| \leqslant C\,e^{CT}\int_0^T p(t)dt + p(t) \qquad \text{a.e.} \end{cases}$$

Since $p(\cdot)$ is a bounded function and $y \in W^{1,\infty}(0,T)$, the solution $x(\cdot)$ belongs to $W^{1,\infty}(0,T)$. Moreover by (2.7), (2.9), for all $t \in [0,T]$,

$$\|x(t)-w(t)\| \leqslant \|x(t)-y(t)\| + \|y(t)-w(t)\| < \varepsilon$$

Since ε is an arbitrary positive number the proof ensues. ■

Proof of Theorem.

a) We denote by S the closed convex cone of solutions to the differential inclusion (2.1) in the Hilbert space

$$(2.10)\qquad X := \{x \in W^{1,2}(0,T) \mid x(0) = 0\}$$

Consider the continuous linear operator

$$\gamma_T : x(\cdot) \in X \rightarrow x(T) \in \mathbb{R}^n$$

The transpose $\gamma_T^\star$ maps $\mathbb{R}^n$ into the dual $X^\star$ of X and for all $\eta \in R_T^+$

$$(2.11)\qquad \forall\, x \in S\,,\quad \langle \gamma_T^\star \eta, x\rangle = \langle \eta, \gamma_T x\rangle \geqslant 0$$

By Lemma 2.2, S is dense in the $W^{1,1}(0,T)$ solutions to (2.1) in the metric of uniform convergence on $[0,T]$. This and (2.11) yield

$$(2.12)\qquad R_T^+ = \{\eta : \gamma_T^\star \eta \in S^+\}$$

Let us set

$$(2.13)\quad \begin{cases} \text{i)} & Y := L^2(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^n) \\ \text{ii)} & L := \{(x,y) \in Y : y(t) \in A(x(t)) \quad \text{a.e.}\} \\ \text{iii)} & D \text{, the differential operator defined on } X \text{ by } Dx = x' \end{cases}$$

Then $S = (1 \times D)^{-1}(L)$. The closed image Lemma 1.3 applied to the continuous linear operator $\phi = (1 \times D)$ states that

$$(2.14)\qquad S^+ = (1 \times D)^\star(L^+)$$

provided that the "surjectivity assumption"

$$(2.15)\qquad \mathrm{Im}\,(1 \times D) - L = Y$$

is satisfied.

b) It can be written

$$(2.16)\quad \begin{cases} \forall\, (u,v) \in Y \text{ there exists } x \in X \text{ such that} \\ x'(t) \in A(x(t)-u(t)) + v(t) \qquad \text{a.e.} \end{cases}$$

Since the domain of A is the whole space, then A is Lipschitzean

The set-valued map $F(t,x) := A\,(x-u(t)) + v(t)$ is then measurable in t , Lipschitzean with respect to x , has closed images and satisfies the following estimate :

$$d(0,F(t,0)) \leqslant \|A\|\ \|u(t)\| + \|v(t)\|$$

The function $t \to \|A\|\ \|u(t)\| + \|v(t)\|$ being in $L^1(0,T)$ we can apply a Filippov Theorem [1967] (see Clarke [1983]) which states the existence of a solution $x(\cdot)$ to the differential inclusion $x'(t) \in F(t,x(t))$, $x(0) = 0$, satisfying :

$$\|x'(t)\| \leqslant \|A\| e^{\|A\| T} \int_0^T d(0,F(t,0))dt + d(0,F(t,0))$$

Thus $x \in X$ and the surjectivity assumption (2.15) holds true.

c) Therefore, by (2.12) and (2.14), we obtain the formula

$$R_T^+ = \{\eta : \gamma_T^\star \eta \in (1 \times D)^\star (L^+)\} \tag{2.17}$$

Let $\eta \in Q_T$ and q be a solution to the adjoint inclusion (2.3). By Proposition 1.6 b), $q(\cdot) \in W^{1,\infty}(0,T)$ and for all $x \in S$

$$<\eta, x(T)> = <(q',q),(x,x')>_Y$$

This is non negative by the definition of $A^\star$. Thus $Q_T \subset R_T^+$. To prove the opposite, let η belong to R_T^+. By (2.17), there exists $(p,q) \in L^+$ such that

$$<\eta, \gamma_T x> = <p,x>_{L^2} + <q,Dx>_{L^2} \qquad \forall\, x \in X \tag{2.18}$$

By taking x so that $x(T) = 0$ we deduce that $p = Dq$ in the sense of distributions. Since p and q belong to L^2, we infer that q belongs to the Sobolev space $W^{1,2}(0,T)$. Thus $Dq = q'$. Integrating by parts in equation (2.18) and taking into account that $x(0) = 0$, we obtain

$$<\eta, \gamma_T x> = <p-q',x>_{L^2} + <q(T),x(T)> = <q(T),x(T)>$$

The surjectivity of γ_T implies that $\eta = q(T)$. Thus $q(\cdot)$ is a solution to (2.3) and then, η belongs to Q_T. This achieves the proof. ∎

3. Invariant cones and viability domains.

We devote this section to a thorough study of the viability domains for $A^\star$, the transpose of a strict closed convex process. We then derive, thanks to the duality theorem, corresponding properties of the invariant cones.

We consider the Cauchy problem for the differential inclusion

(3.1) $$\text{for almost all } t \in [0,T], \quad x'(t) \in A(x(t)), \quad x(0) = 0,$$

the reachable sets R_T defined by (2.2), the adjoint differential inclusion

(3.2) $$\text{for almost all } t \in [0,T], \quad -q'(t) \in A^\star(q(t))$$

We associate with any $\eta \in \text{Dom } A^\star$ the "solution set" $S_T(\eta)$ of solutions to the differential inclusion (3.2) satisfying $q(T) = \eta$ and we denote by Q_T the domain of the "solution map" S_T :

(3.3) $$Q_T := \{\eta \in \text{Dom } A^\star \mid S_T(\eta) \neq \emptyset\}$$

We shall use the following technical lemma.

Lemma 3.1

Let A be a strict closed convex process. The following properties hold true

a) the graph of the restriction of S_T to any compact subset of $\text{Dom } A^\star$ is compact in $\mathbb{R}^n \times C(0,T;\mathbb{R}^n)$.

b) Any viability domain P for $A^\star$ is contained in Q_T. ▲

Proof

a) Let C be a compact subset of $\text{Dom } A^\star$ and let us consider a sequence (η_n, q_n) where $\eta_n \in C$ and $q_n \in S_T(\eta_n)$. Then a subsequence (again denoted η_n) of η_n converges to some $\eta \in C$ because C is compact.

For almost all $t \in [0,T]$

$$\left|\frac{d}{dt}\|p_n(t)\|^2\right| = 2|\langle p_n(t),p_n'(t)\rangle|$$
$$\leq 2\|p_n(t)\|\ \|p_n'(t)\| \leq 2\ \|A\|\ \|p_n(t)\|^2$$

(by formula (1.8), because $-p_n'(t) \in A^{\star}(p_n(t))$)

Gronwall's Lemma implies that

$$\|p_n(t)\| \leq \|\eta_n\| \exp(\|A\|(t-T)) \tag{3.4}$$

This and formula (1.8) imply that for almost all $t \in [0,T]$,

$$\|p_n'(t)\| \leq \|A\|\ \|\eta_n\| \exp(\|A\|(t-T)) \tag{3.5}$$

Thus, by the Banach-Alaoglu theorem, p_n' lies in a weakly compact subset of $L^{\infty}(0,T;\mathbb{R}^n)$ and by the Ascoli-Arzela theorem, p_n lies in a compact subset of $\mathcal{C}(0,T;\mathbb{R}^n)$. Therefore there exists a subsequence (again denoted) $p_n(\cdot)$ and an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ such that

$$\begin{cases} \text{i)} & p_n \text{ converges uniformly to } p \text{ on } [0,T] \\ \text{ii)} & p_n' \text{ converges weakly to } p' \text{ in } L^1(0,T;\mathbb{R}^n) \end{cases} \tag{3.6}$$

The weak convergence of the pair (p_n,p_n') in $L^1(0,T;\mathbb{R}^n) \times L^1(0,T;\mathbb{R}^n)$ implies the strong convergence of convex combinations of elements of this sequence (Mazur's Lemma). Since $(p_n(t),p_n'(t))$ belongs to Graph $A^{\star}$ for almost all $t \in [0,T]$ and since it is closed and convex, we infer that for almost all $t \in [0,T]$, $(p(t),p'(t)) \in$ Graph $(A^{\star})$. Hence $p(\cdot)$ belongs to $S_T(\eta)$.

b) Let P be a viability domain for $A^{\star}$ and $\eta \in P$. We shall show that there exists a solution $p \in S_T(\eta)$.

The viability theorem (see Haddad [1981]) implies that for all $t_o \leq T$ a solution p of the differential inclusion

$$(3.7) \qquad -p'(t) \in A^{\star}(p(t)) \quad ; \quad p(t) \in P \quad ; \quad p(T) = \eta$$

defined on a time interval $[t_o, T]$, can be extended to a solution of (3.7) defined on a larger time interval $[t_1, T]$, $t_1 < t_o$. Setting $\eta_n = \eta$ in (3.4) and (3.5), we obtain that

$$(3.8) \qquad \begin{cases} \text{i)} \quad \|p(t)\| \leq \|\eta\| & \text{for all } t \in [t_1, T] \\ \text{ii)} \quad \|p'(t)\| \leq \|A\| \, \|\eta\| & \text{for a.e. } t \in [t_1, T] \end{cases}$$

As in the case of ordinary differential equations, one can show that $p(\cdot)$ can be extended to a solution (again denoted) $p(\cdot)$ defined on the time interval $[0, T]$. Thus $p(\cdot)$ belongs to $S_T(\eta)$ and thus, η belongs to Q_T.

■

We observe now that the sequence of the closed domains Q_T decreases :

$$(3.9) \qquad \text{if } T_1 \geq T_2 \, , \qquad \text{then } Q_{T_1} \subset Q_{T_2}$$

We introduce the intersection Q of these cones

$$(3.10) \qquad Q := \bigcap_{T > 0} Q_T$$

Since the compact subsets $S^{n-1} \cap Q_T$ form a decreasing sequence, we observe that $Q \neq \{0\}$ if and only if all the cones Q_T are different from 0. We shall say that Q is the largest viability domain, thanks to the following theorem.

Theorem 3.2

Let A be a strict closed convex process.

Then the closed convex cone Q is the largest closed convex cone which is a viability domain for $A^{\star}$.

▲

Proof

Lemma 3.1 b) implies that Q is a closed convex cone which contains any viability domain P . It remains to prove that Q is a viability domain i.e. that

(3.11) $$\forall q \in Q , \quad A^{\star}(q) \cap T_Q(q) \neq \emptyset$$

Assume that $Q \neq \{0\}$.

Thanks to the necessary condition of the viability theorem (see Haddad [1981]), it is sufficient to prove that for some $T > 0$,

(3.12) $$\forall \eta \in Q , \exists p(\cdot) \in S_T(\eta) \quad \text{which is viable on } Q .$$

Since η belongs to Q_{nT} for all $n \geqslant 2$, there exists a solution $p_n(\cdot) \in S_{nT}(\eta)$. By the very definition of Q_t , we know that $p(t) \in Q_t$ for all $t \leqslant nT$.

Therefore, the translated function $\hat{p}_n(\cdot)$ defined on $[0,T]$ by

(3.13) $$\hat{p}_n(t) := p_n(t+(n-1)T)$$

belongs to $S_T(\eta)$ and satisfy for all $t \in [0,T]$, $k \leqslant n-1$

(3.14) $$\hat{p}_n(t) = p_n(t+(n-1)T) \in Q_{t+(n-1)T} \subset Q_{(n-1)T} \subset Q_{kT}$$

By Lemma 3.1 a), $S_T(\eta)$ is compact in $C(0,T;\mathbb{R}^n)$. Thus there exists a subsequence of $\hat{p}_n(\cdot)$ converging to some $\hat{p}(\cdot) \in S_T(\eta)$ uniformly on $[0,T]$. By (3.14) for all $t \in [0,T]$, $k \geqslant 1$, $\hat{p}(t) \subset Q_{kT}$. Therefore

$$\hat{p}(t) \subset \bigcap_{k \geqslant 1} Q_{kT} = Q$$

■

We translate now this result in terms of reachable sets R_T .

Since $0 \in A(0)$ the reachable cones $R(T)$ do form an increasing sequence. We define the reachable set of the inclusion (3.1) to be

(3.15) $$R := \bigcup_{T > 0} R(T)$$

It is a convex cone, which is equal to the whole space if and only if for some $T > 0$, $R(T) = \mathbb{R}^n$.

We say that the closure $\overline{R}$ of R is the <u>smallest invariant cone</u> by A. This definition is motivated by the following consequence of both Theorem 2.1 and Theorem 3.2.

<u>Theorem 3.3</u>

Let A be a strict closed convex process.

Then the closed convex cone $\overline{R}$ is the smallest closed convex cone containing $A(0)$ and invariant by A.

▲

<u>Proof</u>

Indeed Theorem 2.1 and the definition of $\overline{R}$ and Q imply that $\overline{R}^+ = Q$. By Theorem 3.2 and Proposition 1.11, $\overline{R}$ is the smallest closed convex cone containing $A(0) = (\mathrm{Dom}\, A^\star)^+$ which is invariant by A. ■

We consider now the largest subspace

(3.16) $$Q \cap -Q \subset \mathrm{Dom}\, A^\star \cap - \mathrm{Dom}\, A^\star$$

of Q.

<u>Proposition 3.4</u>

Let A be a strict closed convex process. The subspace $Q \cap -Q$ is the largest subspace invariant by $A^\star$ and its orthogonal space $R-R$ is invariant by A in the sense that :

(3.17) $$\forall x \in R-R, \qquad A(x) \subset R-R \quad .$$ ▲

Proof

By Proposition 1.6 c) the restriction of $A^\star$ to $Q \cap -Q$ is a linear (single-valued) operator. We have to check that $A^\star(Q \cap -Q) \subset Q \cap -Q$. Let q belong to $Q \cap -Q$. Then by Theorem 3.2, since $\mathbb{R}q \subset Q \cap - Q$

$$A^\star q \in T_Q(q) = \overline{(\mathbb{R}q + Q)} \subset Q + Q \cap -Q \subset Q$$

Since $-q \in Q \cap -Q$, we also have

$$A^\star q = -A^\star(-q) \subset -Q$$

Thus

$$A^\star q \in Q \cap -Q \ .$$

Since $Q = \overline{R^+}$, the orthogonal space to $Q \cap -Q$ is the (closed) vector space spanned by R . Since we are in finite dimensional space, we infer that

$$(3.18) \qquad (Q \cap -Q)^\perp = R - R$$

Proposition 1.11 implies that the vector space $R - R$ is invariant by A , because we have proved that $Q \cap -Q$ is a viability domain for $A^\star$. ∎

We consider now the cones $A(0)$, $A^2(0) := A(A(0)),\ldots,A^k(0) = A(A^{k-1}(0))$, etc... Since 0 belongs to $A(0)$, these convex cones form an increasing sequence. We introduce the cone

$$(3.19) \qquad N := \mathrm{cl}\left(\bigcup_{k \geq 1} A^k(0)\right)$$

and the vector subspace

(3.20) $\qquad M$ spanned by N

Theorem 3.5

Let A be a strict closed convex process. Then

a) $A(N) \subset N$

b) $\overline{R} \subset N \subset M \subset R - R$

c) $Q \cap -Q \subset \bigcap_{k \geq 1} A^k(0)^{\perp} \subset \bigcap_{k \geq 1} A^k(0)^{+} \subset Q$ ▲

Proof

a) It is clear that $A\left(\bigcup_{k \geq 1} A^k(0)\right) \subset N$.

Let $x \in N$, $y \in A(x)$ and $x_n \in \bigcup_{j \geq 1} A^j(0)$ be a sequence converging to x. Since A is Lipschitzean, there exists a sequence $y_n \in A(x_n) \subset N$ converging to y, which belongs to N because it is closed.

b) Since N is a closed invariant cone containing $A(0)$, Theorem 3.2 implies that N contains the reachable set $\overline{R}$. On the other hand, 0 belongs to $R - R$ and this vector space is invariant by A, thanks to Proposition 3.4. Therefore the cones $A^k(0) = A(A^{k-1}(0))$ are contained in $R - R$ and so does M.

c) We deduce the other inclusions by polarity, noticing that $N^+ = \bigcap_{k \geq 1} A^k(0)^+$ and

$$M^{\perp} = \bigcap_{k \geq 1} A^k(0)^{\perp} .$$ ■

Remark

When the reachable set R is a vector space, the subsets R, N, M and $R - R$ coincide. This happens when, for instance, A is symmetric (in the sense that $A(-x) = -A(x)$), i.e., when the graph of A is a vector subspace.

4. Eigenvectors and eigenvalues of convex processes.

When $Q \cap -Q = \{0\}$ (or $R - R = \mathbb{R}^n$), there is no proper subspace invariant by $A^\star$ (or there is no proper subspace invariant by A). Moreover, when $Q \neq \{0\}$ (or $R \neq \mathbb{R}^n$), we can still prove the existence of an eigenvalue of A (see Definition 1.12 and Lemma 1.13), or eigenvectors of $A^\star$.

Actually, eigenvectors η of $A^\star$, non zero solutions of the inclusion $\lambda\eta \in A^\star(\eta)$, do belong to the largest viability domain Q , because for all $T > 0$ the function $p(t) := \eta \exp (\lambda(T-t))$ belongs to $S_T(\eta)$.

Theorem 4.1

Let A be a strict closed convex process.

If the largest viability domain Q for $A^\star$ is different from $\{0\}$ and contains no line, then $A^\star$ has at least an eigenvector. ▲

By Lemma 1.13 and duality theorem 2.1, the following dual version of this theorem holds true.

Theorem 4.2

Let A be a strict closed convex process. Assume that the reachable set R is different from $\mathbb{R}^n$ and spans the whole space. Then A has at least one eigenvalue. ▲

First we recall the following property

Lemma 4.3

Let Q be a closed convex cone of $\mathbb{R}^n$. The following properties are equivalent :

i) $Q \cap -Q = \{0\}$

ii) Q is spanned by a compact convex subset which does not contain zero

iii) The interior of Q^+ is non-empty .

If one of these properties hold true, then for all $x_o \in \text{Int } Q^+$, the compact convex subset

(4.1) $$M := \{q \in Q : \langle q,x_o\rangle = 1\}$$

spans Q . ▲

Proof

We provide the proof for the convenience of the reader.

Condition i) means that zero is the extremal point of Q , which is equivalent to the assertion $0 \notin \text{co } (Q \cap S^{n-1})$. Since the compact convex set $\text{co } (Q \cap S^{n-1})$ spans the cone Q we proved the equivalence of i) and ii). Condition iii) means that $Q^{++} = Q$ contains no line, which is precisely the statement i).

If $x_o \in \text{Int } Q^+$ and $q,q_i \in M$, $i = 1,2\ldots$ are such that $\langle q_i,x_o\rangle = 1$, $\lim_{i\to\infty} q_i/\|q_i\| = q \in Q \cap S^{n-1}$. Then

$$0 < \langle q,x_o\rangle = \lim_{i\to\infty} \langle q_i,x_o\rangle/\|q_i\| = \lim_{i\to\infty} \|q_i\|^{-1}$$

It implies that the norms $\|q_i\|$ are bounded and, therefore, M is bounded. Obviously it is also convex and closed. ■

Proof of Theorem 4.1

Let $x_o \in \text{Int } Q^+$ and let M be defined by (4.1). Then for all $p \in M$

(4.2) $$T_M(p) := \{v \in T_Q(x_o) : \langle v,x_o\rangle = 0\}$$

We introduce the following projectors

(4.3) $$\forall\, p \in M \qquad \pi(p)q = q - \langle q,x_o\rangle\, p$$

For all $p \in M$ and $q \in Q$, $\langle\pi(p)p,x_o\rangle = 0 = \langle\pi(p)q,x_o\rangle$. Hence the projector $\pi(p)$ maps the set $\mathbb{R}p + Q$ into $T_M(p)$. Since $T_Q(p) = \overline{\mathbb{R}p + Q}$ and $\pi(p)$ is a continuous linear operator, we obtain :

$$(4.4) \qquad \forall\, p \in M \,, \qquad \pi(p) \text{ maps } T_Q(p) \text{ into } T_M(p)$$

Consider the set-valued map $p \in M \rightarrow \pi(p)A^{\star}(p)$. It is upper semicontinuous with nonempty compact convex images. By assumptions of Theorem 4.1, for all $p \in M \subset Q$, $A^{\star}(p) \cap T_Q(p) \neq \emptyset$. Thus by (4.4)

$$(4.5) \qquad \forall\, p \in M \,, \qquad \pi(p)A^{\star}(p) \cap T_M(p) \neq \emptyset$$

The assumptions of Theorem 6.4.11 p. 341 of Aubin-Ekeland [1984] are satisfied. Therefore, for some $\bar{p} \in M$, $0 \in \pi(\bar{p})A^{\star}(\bar{p})$. Hence there exists $\bar{q} \in A^{\star}(\bar{p})$ such that $<\bar{q},x_o>\bar{p} = \bar{q} \in A^{\star}(\bar{p})$. In other words $\bar{p}$ is an eigenvector of $A^{\star}$ associated to the eigenvalue $<\bar{q},x_o>$.

∎

5. Characterization of controllable convex processes.

We shall deduce from the preceding results several characterizations of the controllability of differential inclusions

(5.1) $\qquad$ for almost all $t \in [0,T]$, $\quad x'(t) \in A(x(t))$, $x(0) = 0$

or, equivalently, of the observability of the adjoint inclusion

(5.2) $\qquad$ for almost all $t \in [0,T]$, $\quad -q'(t) \in A^{\star}(q(t))$.

Definition 5.1

We shall say that (5.1) is controllable at time T (respectively, controllable) if $R_T = \mathbb{R}^n$ (respectively, $R = \mathbb{R}^n$). We shall say that the adjoint inclusion (5.2) is observable at time T (respectively, observable) if $Q_T = \{0\}$ (respectively, $Q = \{0\}$). ▲

We also observe the following property.

Lemma 5.2

Let A be a strict closed convex process. The three following properties are equivalent.

$$(5.3) \qquad \begin{cases} \text{a)} & \exists\, m \geq 1 \text{ such that } A^m(0) - A^m(0) = \mathbb{R}^n \\ \text{b)} & \exists\, m \geq 1 \text{ such that } A^m(0)^{\perp} = \{0\} \\ \text{c)} & \exists\, m \geq 1 \text{ such that } \operatorname{Int} A^m(0) \neq \emptyset \end{cases}$$

▲

It is convenient to introduce the

Rank condition 5.3.

We say that a convex process A satisfies the rank condition if one of the equivalent properties (5.3) holds true.

▲

Lemma 5.4

Consider the strict closed convex process $A(x) = Fx + L$, where $F \in \mathbb{R}^{n \times n}$

is a matrix and L is a vector subspace of $\mathbb{R}^n$. Then A satisfies the rank condition if and only if $A^n(0) - A^n(0) = \mathbb{R}^n$. ▲

Proof

The rank condition is satisfied if and only if for some $m \geq 1$ the cone $L + AL + \ldots + A^{m-1}L$ spans the whole space. The Cayley-Hamilton Theorem ends the proof. ■

We begin by stating characteristic properties of observability of the adjoint system (5.2) and then, use the duality results to infer the equivalent characteristic properties of system (5.1).

Theorem 5.5

Let A be a strict closed convex process. The following properties are equivalent

$a^\star$) The adjoint inclusion (5.2) is observable

$b^\star$) The adjoint inclusion (5.2) is observable at time $T > 0$ for some T

$c^\star$) $\{0\}$ is the Largest closed convex cone which is a viability domain for $A^\star$

$d^\star$) $A^\star$ has neither proper invariant subspace nor eigenvectors

$e^\star$) the rank condition is satisfied and $A^\star$ has no eigenvectors. ▲

Proof

α) Since the intersections $Q_T \cap S^{n-1}$ of the cones Q_T and the unit sphere S^{n-1} form a decreasing sequence of compact subsets, we deduce that $Q \cap S^{n-1}$ is empty if and only if $Q_T \cap S^{n-1}$ is empty for some T, i.e., that $Q = \{0\}$ if and only if $Q_T = \{0\}$ for some $T > 0$. Thus $a^\star) \Leftrightarrow b^\star)$.

β) Property $c^\star)$ is equivalent to $Q = \{0\}$ by Theorem 3.2, i.e. $a^\star) \Leftrightarrow c^\star)$.

γ) When $Q = \{0\}$, then $Q \cap -Q = \{0\}$ (there is no proper invariant subspace) and there is no eigenvector (because an eigenvector is contained in Q).

When $Q \neq \{0\}$, then either $Q \cap -Q \neq \{0\}$ and by Proposition 3.4 there is a proper invariant subspace or $Q \cap -Q = \{0\}$ and, by Theorem 4.1, there exists at least an eigenvector of A^* . This proves the equivalence of $d^*)$ with $Q = \{0\}$, i.e. $a^*) \Leftrightarrow d^*)$.

δ) Since the sequence of cones $A^k(0)$ is increasing, the sequence of vector spaces $A^k(0)^\perp$ is decreasing, so that

$$\bigcap_{k \geq 1} A^k(0)^\perp = \{0\} \quad \Leftrightarrow \quad \exists\, m \geq 1 \text{ such that } A^m(0)^\perp = \{0\}$$

$$\Leftrightarrow \quad \text{the rank condition is satisfied .}$$

Assume that $Q = \{0\}$. Then, by Theorem 3.5 c), and the above remark, the rank condition is satisfied and there is no eigenvector. Assume now that the rank condition is satisfied. Then $Q \cap -Q = \{0\}$ by Theorem 3.5 c). Then, Theorem 4.1 implies that if A^* does not have an eigenvector, the cone Q is equal to $\{0\}$. Equivalence between $e^*)$ and $Q = \{0\}$ ensues. ∎

Theorem 5.6

Let A be a strict closed convex process. The equivalent properties $a^*)$, $b^*)$, $c^*)$, $d^*)$ and $e^*)$ of Theorem 5.5 are equivalent to the following properties

a) Differential inclusion (5.1) is controllable

b) Differential inclusion (5.1) is controllable at some time $T > 0$

c) $\mathbb{R}^n$ is the smallest closed convex cone containing $A(0)$ which is invariant by A

d) A has neither proper invariant subspace nor eigenvalues

e) The rank condition is satisfied and A has no eigenvalues.

f) for some $m \geq 1$, $A^m(0) = (-A)^m(0) = \mathbb{R}^n$ ▲

Proof

Statements a)-e) follow from the duality results (Proposition 1.11, Lemma 1.13 and Theorem 2.1) and Theorem 5.5. We shall show that a) is also equivalent to f).

Step 1. Consider the closed convex process $A_1(x) = A(-x)$. Then $A_1^\star = -A^\star$. We claim that (5.1) is controllable if and only if the inclusion

$$(5.1)' \qquad x' \in A_1(x) \quad ; \quad x(0) = 0$$

is controllable.

Indeed invariant subspaces and eigenvectors of $A_1^\star$ and $A^\star$ coincide and our claim follows from Theorem 5.5 d).

Step 2. If (5.1) is controllable then by Step 1 and Theorem 3.5 b) $\bigcup_{k \geq 1} A^k(0) = \bigcup_{k \geq 1} A_1^k(0) = \mathbb{R}^n$. Since $\{A^k(0)\}$ and $\{A_1^k(0)\}$ are increasing sequences of convex cones it implies that for some $m \geq 1$, $A^m(0) = A_1^m(0) = \mathbb{R}^n$. Moreover $A_1^m(0) = -(-A)^m(0)$. This implies f).

Step 3. Assume that f) holds true. If (5.1) is not controllable then there exist $\lambda \in \mathbb{R}$, $q \in A(0)^+$, $q \neq 0$ such that $\lambda q \in A^\star(q)$. Then $(-\lambda)q \in A_1^\star(q)$. Therefore,

$$\begin{cases} \lambda^m q \in (A^\star)^m(q) & \text{if } \lambda \geq 0 \\ (-\lambda)^m q \in (A_1^\star)^m(q) & \text{if } \lambda \leq 0 \end{cases}$$

If $\lambda \geq 0$, then for all $y \in A^m(0)$, $0 = \langle \lambda^m q, 0\rangle \leq \langle q, y\rangle$. If $\lambda \leq 0$, then for all $y \in A_1^m(0)$, $0 = \langle (-\lambda)^m q, 0\rangle \leq \langle q, y\rangle$. In both cases we obtain a contradiction with f). The proof is complete. ■

So, the conjunction of Theorems 5.5 and 5.6 imply Theorem 0.4 stated in the introduction.

In the case when the set-valued map A is defined by $A(x) := Fx + L$, we derive known results due to Kalman when L is a vector space of control and to Brammer, Saperstone and Yorke when L is an arbitrary set of controls containing 0 .

Consider the linear control system in $\mathbb{R}^n$

$$(5.4) \qquad \begin{cases} x' = Fx + Gu \quad ; \quad u \in U \\ x(0) = 0 \end{cases}$$

where $F \subset \mathbb{R}^{n \times n}$, $G \subset \mathbb{R}^{n \times m}$ are constant matrices and $U \subset \mathbb{R}^m$ is the given control set. The control system (5.4) is said <u>locally controllable</u> around zero if zero is an interior point of the reachable set of (5.4).

To provide necessary and sufficient conditions for local controllability of (5.4) let us consider convex hull co U of U , and

$$N := \overline{\mathbb{R}_+ \text{ co } U} = \text{cl } \{\lambda u : \lambda \geq 0 \text{ , } u \in \text{co } U\}$$

and the associated control system

$$(5.5) \qquad \begin{cases} x' \in Fx + \overline{GN} \\ x(0) = 0 \end{cases}$$

<u>Lemma 5.7</u>

If $0 \in \overline{\text{co}}\ GU$ then the control system (5.4) is locally controllable around zero if and only if the system (5.5) is controllable. ▲

<u>Proof</u>

The reachable set of system (5.5) is a convex cone equal to

$$(5.6) \qquad \left\{\int_0^t e^{F(t-s)} v(s)\, ds : t \geq 0 \text{ , } v(s) \in \overline{GN}\right\}$$

and containing the reachable set of (5.4). Hence the local controllability of (5.4) implies the controllability of (5.5).

Because $0 \in \overline{co}\ GU = \overline{G\ co\ U}$ by a density argument, it is possible to verify that the cone given by (5.6) is equal to $\mathbb{R}^n$ if and only if

$$(5.7) \qquad 0 \in \text{Int} \left\{ \int_0^t e^{F(t-s)} Gu(s)ds \ : \ t \geq 0 \ , \ u(s) \in co\ U \right\}$$

The sets $\left\{ \int_0^t e^{F(t-s)} Gu(s)ds \ : \ u(s) \in U \right\}$ being convex and dense in $\left\{ \int_0^t e^{F(t-s)} Gu(s)ds \ : \ t \geq 0 \ , \ u(s) \in co\ U \right\}$ (Lee-Markus [1967]) the inclusion (5.7) is equivalent to

$$(5.8) \qquad 0 \in \text{Int} \left\{ \int_0^t e^{F(t-s)} Gu(s)ds \ : \ t \geq 0 \ , \ u(s) \in U \right\} \qquad \blacksquare$$

By Lemma 5.4 the rank condition 5.3 for the closed convex process $Ax = Fx + \overline{GN}$ is equivalent to

$$(5.9) \qquad \begin{cases} N - N = \mathbb{R}^m \\ \text{rank } [G \ , \ FG \ , \dots , \ F^{n-1}G \ ] = n \end{cases}$$

This and Theorem 5.6 f) imply

Theorem (Kalman)

If $U = \mathbb{R}^m$ then the control system (5.4) is controllable if and only if rank $[G \ , \ FG \ , \dots , \ F^{n-1}G \ ] = n$. ▲

Theorem 5.8

Assume that $0 \in \overline{co}\ GU$. Then the system (5.4) is locally controllable around zero if and only if the rank condition (5.9) is satisfied and there is no eigenvector of $F^\star$ in $(GU)^+$. ▲

Proof

Observe that $GU^+ = (\overline{GN})^+$. By Lemma 5.7 it is enough to prove that the system (5.5) is controllable if and only if the rank condition 5.3 is satisfied and $F^{\star}$ has no eigenvector in $(\overline{GN})^{+}$. But this follows from Theorem 5.5 $e^{\star})$ and (5.9). ■

In particular when $m = 1$ we obtain the result from Saperstone-Yorke [1971]. The above theorem is a generalization of Brammer's theorem [1972] (see also Jacobson [1977]). Theorem 5.6 f) and Example 1.14 imply

Theorem 5.9

Let F be an $n \times n$ matrix and L be a closed convex subcone of $\mathbb{R}^n$. The control system

$$x' = Fx + L \quad ; \quad x(0) = 0$$

is controllable if and only if for some $m \geqslant 1$

$$L + FL + \ldots + F^m L = L - FL + \ldots + (-1)^m F^m L = \mathbb{R}^n$$ ▲

The last theorem together with Lemma 5.7 imply a result of Korobov [1980].

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