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Controllability of Linear Systems with Input and State Constraints

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Abstract— **This paper presents necessary and sufficient conditions for controllability of linear systems subject to input/state constraints.**

Index Terms— **Linear systems, state constraints, controllability.**

I. Introduction

The notion of controllability has played a central role throughout the history of modern control theory. For linear systems Kalman [5] and Hautus [3] studied this property in the sixties and early seventies and came up with complete characterizations in the well-known algebraic conditions Also in the case that input constraints are present on the linear system the controllability property has been characterized by Brammer [2]. However, in the situation when state constraints are active on the linear system such characterizations are not available in the literature. In this paper we will fill this gap by establishing necessary and sufficient conditions for the controllability in the case of a continuous-time linear system that has constraints on its output variables. The only condition that we impose on the system is right-invertibility of its transfer matrix. In other words, for the class of "right-invertible" linear systems we fully characterize controllability of linear systems involving both state and input constraints or combinations of them. The original results of Kalman, Hautus, and Brammer are recovered as particular cases of these conditions.

II. Notation

The spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and nonnegative integers, respectively. For a matrix $A \in \mathbb{C}^{n\times m}$, we write A^T for its transpose and A^* for its complex conjugate transpose. Moreover, for a matrix $A \in \mathbb{R}^{n\times m}$, its kernel $\ker A$ is defined as $\{x \in \mathbb{R}^m \mid Ax = 0\}$ and its image $\operatorname{im} A$ by $\{Ax \mid x \in \mathbb{R}^m\}$. For two subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathbb{R}^n$, we write $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathbb{R}^n$, when $\mathcal{X}_1 \cap \mathcal{X}_2 = \{0\}$ and the direct sum $\mathcal{X}_1 + \mathcal{X}_2 = \{x_1 + x_2 \mid x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2\} = \mathbb{R}^n$. For a set $\mathcal{Y} \subseteq \mathbb{R}^n$, we define its dual cone $\mathcal{Y}^*$ as $\{w \in R^n \mid w^T y \geqslant 0 \text{ for all } y \in \mathcal{Y}\}$. For two vectors $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $\operatorname{col}(x_1, x_2)$ will denote the vector in $\mathbb{R}^{n_1+n_2}$ obtained by stacking x_1 over x_2. The space of arbitrarily often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^p$ is denoted by $C^\infty(\mathbb{R}, \mathbb{R}^p)$. By $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^p)$ we denote the space of locally Lebesgue integrable functions from $\mathbb{R}$ to $\mathbb{R}^p$. When p is clear from the context, we often write C^∞ or L^1_{loc}, respectively. By $f^{(k)}$ we denote the k-th derivative of f, provided it exists.

III. Problem definition

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state at time $t \in \mathbb{R}$, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, and all matrices are of appropriate sizes. For a given initial state x_0 and input $u \in L^1_{\mathrm{loc}}$, there exists a unique absolutely continuous solution to (1) with $x(0) = x_0$, which is denoted by $x^{x_0,u}$. The corresponding output will be denoted by $y^{x_0,u}$.

Together with (1), we consider the constraints

$$y(t) \in \mathcal{Y} \tag{2}$$

where $\mathcal{Y} \subseteq \mathbb{R}^p$ is a solid closed polyhedral cone, i.e. there exists a matrix $Y \in \mathbb{R}^{q\times p}$ such that $\mathcal{Y} = \{y \in \mathbb{R}^p \mid Yy \geqslant 0\}$. and $\mathcal{Y}$ has a non-empty interior.

We say that a state $x_0 \in \mathbb{R}^n$ is *feasible as initial state* for (1)-(2) if there exists an input $u \in L^1_{\mathrm{loc}}$ such that $y^{x_0,u}(t) \in \mathcal{Y}$ for almost all $t \geqslant 0$. The set of all such initial states is denoted by $\mathcal{X}_0$. Reversely, we say that $x_f \in R^n$ is *feasible as final state*, if x_f is feasible as initial state for the time-reversed system of (1) being

$$\dot{x}(\tau) \;=\; -Ax(\tau) - Bu(\tau) \tag{3a}$$

$$y(\tau) \;=\; Cx(\tau) + Du(\tau). \tag{3b}$$

The set of finally feasible states is denoted by $\mathcal{X}_f$. A closely related set of initial states is $\mathcal{X} = \{\bar{x} \in \mathbb{R}^n \mid \text{there exists } \bar{u} \in \mathbb{R}^m \text{ such that } C\bar{x} + D\bar{u} \in \mathcal{Y}\}$. The relevance of these sets will be illustrated by Example III.1 below. First we will show that $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\mathcal{X}_f \subseteq \mathcal{X}$.

Suppose, on the contrary, that there exists $x_0 \in \mathcal{X}_0$ such that $x_0 \notin \mathcal{X}$. Let $u \in L^1_{\mathrm{loc}}$ be an input such that $y^{x_0,u}(t) \in \mathcal{Y}$ for almost all $t \geqslant 0$. Note that if u is continuous, then the result is immediate. To prove this in the general case of $u \in L^1_{\mathrm{loc}}$, we observe that $x_0 \notin \mathcal{X}$ is equivalent to $Cx_0 + \operatorname{im} D \cap \mathcal{Y} = \varnothing$. Since both $Cx_0 + \operatorname{im} D$ and $\mathcal{Y}$ are polyhedra, so is $(Cx_0 + \operatorname{im} D) + (-\mathcal{Y})$. Therefore, there should be a strongly separating hyperplane (see e.g. [6, Thm. 2.39]), i.e. there exist $h \in \mathbb{R}^m$ and $g_1 < g_2$ such that $h^T y + g_1 < 0$ for

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all $y \in Cx_0 + \operatorname{im} D$ and $h^T y + g_2 > 0$ for all $y \in \mathcal{Y}$. Since $x^{x_0,u}$ is continuous, there must exist a positive number ϵ such that $h^T y + g_2 < 0$ for all $y \in Cx^{x_0,u}(t) + \operatorname{im} D$ and for all $t \in [0, \epsilon)$. Since $y^{x_0,u}(t) \in Cx^{x_0,u}(t) + \operatorname{im} D$ for all $t \in [0, \epsilon)$, one gets $y^{x_0,u}(t) \notin \mathcal{Y}$ for all $t \in [0, \epsilon)$. Contradiction! A similar reasoning applies to $\mathcal{X}_f$ by considering the time-reversed system.

The converse inclusion does not hold in general as illustrated by the following example.

Example III.1 Consider the double integrator

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = u; \quad y = x_1 \tag{4}$$

together with the "position" constraint $y \geqslant 0$. Clearly, one has $\mathcal{X} = \{\bar{x} \mid \bar{x}_1 \geqslant 0\}$,

$$\mathcal{X}_0 = \{\bar{x} \mid (\bar{x}_1 > 0) \text{ or } [(\bar{x}_1 = 0) \text{ and } (\bar{x}_2 \geqslant 0)]\} \tag{5}$$

$$\mathcal{X}_f = \{\bar{x} \mid (\bar{x}_1 > 0) \text{ or } [(\bar{x}_1 = 0) \text{ and } (\bar{x}_2 \leqslant 0)]\}. \tag{6}$$

We say that a linear system of the form (1) is *controllable under the constraints* (2) if for each pair of states $(x_0, x_f) \in \mathcal{X}_0 \times \mathcal{X}_f$ there exist an input $u \in L^1_{\text{loc}}$ and a positive number T such that $x^{x_0,u}(T) = x_f$ and $y^{x_0,u}(t) \in \mathcal{Y}$ for almost all $t \in [0, T]$.

IV. Classical controllability results

Two particular cases of our framework are among the classical results of systems theory.

A. Linear systems.

Let $\mathcal{Y} = \mathbb{R}^p$. Clearly, one gets $\mathcal{X}_0 = \mathcal{X}_f = \mathcal{X} = \mathbb{R}^n$. In this case, the following is a classical theorem that gives an answer to the controllability problem.

Theorem IV.1 *Consider the linear system* (1) *and the constraints* (2) *with* $\mathcal{Y} = \mathbb{R}^p$. *Then, it is controllable if, and only if,*

$$\lambda \in \mathbb{C}, \quad z \in \mathbb{C}^n, \quad z^* A = \lambda z^*, \quad z^* B = 0 \quad \Rightarrow \quad z = 0. \tag{7}$$

B. Linear systems with input constraints.

Let $C = 0$ and $D = I$. Note that the problem reduces now to establishing controllability for the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with input constraints $u(t) = y(t) \in \mathcal{Y}$ almost everywhere. Clearly, one gets $\mathcal{X}_0 = \mathcal{X}_f = \mathcal{X} = \mathbb{R}^n$. In this case, the answer to the controllability question is given by Brammer [2] as quoted in the following theorem.

Theorem IV.2 *Consider the linear system* (1) *and the constraints* (2) *with* $C = 0$ *and* $D = I$. *Then, it is controllable if, and only if, the following implications hold:*

$$\lambda \in \mathbb{C}, \quad z \in \mathbb{C}^n, \quad z^* A = \lambda z^*, \quad z^* B = 0 \quad \Rightarrow \quad z = 0 \tag{8a}$$

$$\lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \quad B^T z \in \mathcal{Y}^* \quad \Rightarrow \quad z = 0. \tag{8b}$$

Interestingly, under the hypothesis of Theorem IV.2 there exists a (uniform) $T > 0$ such that for all $x_0,\ x_f \in \mathbb{R}^n$ there is an input $u \in L^1_{\text{loc}}$ such that $x^{x_0,u}(T) = x_f$ (see [2]).

The main contribution of the paper is to give necessary and sufficient conditions for controllabilityin the presence of input/state constraints.

V. Linear systems with input/state constraints

We will use the following assumption in the paper.

Assumption V.1 The transfer matrix $D + C(sI - A)^{-1}B$ is right invertible as a rational matrix.

To make it easier to deal with constraints as in (2), we will transform (1) into a canonical form that is based on [1]. We will briefly recall some of the notions from [1] and [7] and refer to Appendix VIII for some more particular facts.

A. Preliminaries in geometric control theory

Consider the linear system (1). We define the *controllable subspace* and *unobservable subspace* as $\langle A \mid \operatorname{im} B \rangle := \operatorname{im} B + A \operatorname{im} B + \cdots + A^{n-1} \operatorname{im} B$ and $\langle \ker C \mid A \rangle := \ker C \cap A^{-1} \ker C \cap \cdots \cap A^{1-n} \ker C$, respectively.

We say that a subspace $\mathcal{V}$ is *output-nulling controlled invariant* if for some matrix K the inclusions

$$(A - BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C - DK) \tag{9}$$

hold. As the set of such subspaces is non-empty and closed under subspace addition, it has a maximal element $\mathcal{V}^*$.

Dually, we say that a subspace $\mathcal{T}$ is *input-containing conditioned invariant* if for some matrix L the inclusions

$$(A - LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \operatorname{im}(B - LD) \subseteq \mathcal{T} \tag{10}$$

hold. As the set of such subspaces is non-empty and closed under subspace intersection, it has a minimal element $\mathcal{T}^*$.

A subspace $\mathcal{R}$ is called an *output-nulling controllability subspace* if for all $x_0, x_1 \in \mathcal{R}$ there exist $T \geqslant 0$ and an integrable function u such that $x^{x_0,u}(0) = x_0$, $x^{x_0,u}(T) = x_1$, and $y(t) = 0$ for all $t \in [0, T]$. The set of all such subspaces admits a maximal element. This maximal element is denoted by $\mathcal{R}^*$. It is known, see e.g. [1], that

$$\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{T}^*. \tag{11}$$

We sometimes write $\mathcal{V}^*(A, B, C, D)$, $\mathcal{T}^*(A, B, C, D)$ and $\mathcal{R}^*(A, B, C, D)$ to make the dependence on (A, B, C, D) explicit.

B. Canonical form

Let $\mathcal{X}_2 := \mathcal{T}^*$ be the smallest input-containing conditioned invariant subspaces of the system (1) and let L be a matrix that satisfies (10) for $\mathcal{T} = \mathcal{T}^*$. Take $\mathcal{X}_1$ a subspace such that $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathbb{R}^n$. Let the dimensions of the subspaces $\mathcal{X}_i$ be n_i. We select now vectors $\{w_1, w_2, \ldots, w_n\}$ to be a basis for $\mathbb{R}^n$ such that the first n_1 vectors form a basis for $\mathcal{X}_1$ and the second n_2 for $\mathcal{X}_2$. As $\operatorname{im}(B - LD) \subseteq \mathcal{T}^*$, one gets

$$B - LD \simeq \begin{bmatrix} 0 \\ \tilde{B}_2 \end{bmatrix}, \tag{12}$$

where $\simeq$ indicates that $B - LD$ is transformed in the coordinates that are adapted to the above basis. Here $\tilde{B}_2$ is a

$n_2 \times m$ matrix. As $(A-LC)\mathcal{T}^* \subseteq \mathcal{T}^*$, the matrix $A-LC$ is of the form $\left[\begin{smallmatrix} A_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{smallmatrix}\right]$ in the new coordinates where the row (column) blocks have n_1 and n_2 rows (columns), respectively. Let the matrices L and C be partitioned according to the basis above as

$$L \simeq \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \text{ and } C \simeq [C_1 \ C_2]$$

where L_k and C_k are $n_k \times m$ and $p \times n_k$ matrices, $k = 1, 2$, respectively. With these partitions, one gets

$$A \simeq \begin{bmatrix} A_{11} + L_1 C_1 & L_1 C_2 \\ A_{21} & A_{22} \end{bmatrix} \tag{13a}$$

$$B \simeq \begin{bmatrix} L_1 D \\ B_2 \end{bmatrix} \tag{13b}$$

where $A_{21} = \tilde{A}_{21} + L_2 C_1$, $A_{22} = \tilde{A}_{22} + L_2 C_2$ and $B_2 = \tilde{B}_2 + L_2 D$. Now, (1) becomes in the new coordinates

$$\dot{x}_1 = A_{11} x_1 + L_1 y \tag{14a}$$

$$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u \tag{14b}$$

$$y = C_1 x_1 + C_2 x_2 + Du. \tag{14c}$$

Note that (14a) indicates that the controllability of the x_1-dynamics can only take place via the "control variable" y, which is constrained to be in $\mathcal{Y}$. Hence, this indicates that at least some input-constrained controllability conditions should hold for the x_1-dynamics as in Theorem IV.2 to guarantee controllability for (1) under the constraints (2).

C. Characterizations of the sets $\mathcal{X}_0$ and $\mathcal{X}_f$

The applied transformation enables the characterizations of the sets $\mathcal{X}_0$ and $\mathcal{X}_f$. To do so, we introduce the notion of lexicographic inequalities. A (finite or infinite) sequence of real numbers $(x^1, x^2, \ldots)$ is said to be *lexicographically nonnegative* if either all entries are zero or the first nonzero entry is positive. If it is lexicographically nonnegative, we write $(x^1, x^2, \ldots) \succcurlyeq 0$. Lexicographical nonpositiveness is defined similarly. A (finite or infinite) sequence of real vectors $(x^1, x^2, \ldots)$ is said to be *lexicographically nonnegative* if the real number sequences $(x_i^1, x_i^2, \ldots)$ of the ith components are lexicographically nonnegative for all i.

The following theorem characterizes the sets $\mathcal{X}_0$ and $\mathcal{X}_f$. The proof is omitted for the sake of shortness.

Theorem V.2 *Consider the system* (1) *with the constraint* (2). *Suppose that Assumption V.1 holds. Then, the set of initially feasible states can be given by*

$$\begin{aligned}\mathcal{X}_0 = \{x_0 \in \mathbb{R}^n \mid & \text{ there exists } (u^0, u^1, \ldots, u^{n_2-1}) \text{ such} \\ & \text{that } Y(Cx_0 + Du^0,\ CAx_0 + CBu^0 + Du^1, \\ & CA^2x_0 + CABu^0 + CBu^1 + Du^2, \ldots, \\ & CA^{n_2-1}x_0 + CA^{n_2-2}Bu^0 + CA^{n_2-3}Bu^1 + \cdots + \\ & + CBu^{n_2-2} + Du^{n_2-1}) \succcurlyeq 0\}\end{aligned}$$

and the set of finally feasible states can be given by

$$\begin{aligned}\mathcal{X}_f = \{x_f \in \mathbb{R}^n \mid & \text{ there exists } (u^0, u^1, \ldots, u^{n_2-1}) \text{ such} \\ & \text{that } Y(Cx_f + Du^0, -CAx_f - CBu^0 - Du^1, \\ & CA^2x_f + CABu^0 + CBu^1 + Du^2, \ldots, \\ & (-1)^{n_2-1}CA^{n_2-1}x_f + (-1)^{n_2-1}CA^{n_2-2}Bu^0 + \\ & + (-1)^{n_2-1}CA^{n_2-3}Bu^1 + \cdots + (-1)^{n_2-1}CBu^{n_2-1} + \\ & + (-1)^{n_2-1}Du^{n_2-1}) \succcurlyeq 0\}.\end{aligned}$$

Note that X_f is obtained by replacing A by $-A$ and B by $-B$ (i.e. considering the time-reversed system of (A, B, C, D)) in $\mathcal{X}_0$ and replacing u^k by $(-1)^k u^k$, $k = 0, 1, \ldots, n_2 - 1$. Observe that we only have to check the lexicographical inequality up to n_2 on y and its derivatives to establish whether x_0 (x_f) lies in $\mathcal{X}_0$ ($\mathcal{X}_f$) or not. The maximal output-nulling controlled invariant subspace $\mathcal{V}^*(A, B, C, D)$ lies in both $\mathcal{X}_0$ and $\mathcal{X}_f$.

D. Main results

The following theorem is the main result of the paper.

Theorem V.3 *Consider the linear system* (1). *Suppose that Assumption V.1 holds. Then, the system is controllable under the constraints* (2) *if, and only if, the following implications hold*

$$\lambda \in \mathbb{C},\ z \in \mathbb{C}^n,\ z^*A = \lambda z^*,\ z^*B = 0 \ \Rightarrow\ z = 0 \tag{15a}$$

$$\lambda \in \mathbb{R},\ z \in \mathbb{R}^n,\ w \in \mathcal{Y}^*,\ \begin{bmatrix} z^T & w^T \end{bmatrix} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \Rightarrow \Rightarrow z = 0. \tag{15b}$$

Proof: To show the 'only if' part, suppose that the system (1) is controllable under the constraints (2). We first show necessity of (15a). Let $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ be such that $z^*A = \lambda z^*$ and $z^*B = 0$. Let σ and ω be, respectively, the real and imaginary parts of λ. Also let z_1 and z_2 be, respectively, the real and imaginary parts of z. One can write $z^*A = \lambda z^*$ and $z^*B = 0$ in terms of σ, ω, z_1, and z_2 as

$$\begin{bmatrix} z_1^T \\ z_2^T \end{bmatrix} A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} z_1^T \\ z_2^T \end{bmatrix} \tag{16a}$$

$$z_j^T B = 0,\ j = 1, 2. \tag{16b}$$

This implies that any trajectory of (1) satisfies

$$z_1^T \dot{x} = \sigma z_1^T x + \omega z_2^T x,\ z_2^T \dot{x} = -\omega z_1^T x + \sigma x_2^T x.$$

Hence, any trajectory starting with an initial state in $\ker \begin{pmatrix} z_1^T \\ z_2^T \end{pmatrix}$ remains inside $\ker \begin{pmatrix} z_1^T \\ z_2^T \end{pmatrix}$ irrespective of the input. This can happen only if $z = z_1 = z_2 = 0$, as the system (1) is controllable under the constraints (2) and $0 \in \mathcal{X}_0 \cap \ker \begin{pmatrix} z_1^T \\ z_2^T \end{pmatrix}$ and $\mathcal{X}_f \not\subseteq \ker \begin{pmatrix} z_1^T \\ z_2^T \end{pmatrix}$. To show necessity of the second condition, let $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, and $w \in \mathbb{R}^m$ be such that

$$\begin{bmatrix} z^T & w^T \end{bmatrix} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \quad \text{and} \quad w \in \mathcal{Y}^*. \tag{17}$$

By left-multiplying the first equation of (1) by z^T and using (17), we get

$$z^T \dot{x} = \lambda z^T x - w^T y. \quad (18)$$

Since $w \in \mathcal{Y}^*$ and $y \in \mathcal{Y}$, the term $w^T y$ is always nonnegative. Using the characterizations of $\mathcal{X}_0$ and $\mathcal{X}_f$ as in Theorem V.2 it is clear that there exists an $\bar{x} \in \mathcal{X}_0 \cap \mathcal{X}_f$ with $z^T \bar{x} \neq 0$ when $z \neq 0$. Note that also $0 \in \mathcal{X}_0 \cap \mathcal{X}_f$. If $\lambda \geqslant 0$, then it follows from (18) that for any $x_0 \in \mathcal{X}_0$ with $z^T x_0 \leqslant 0$ it holds that $z^T x^{x_0,u}(t) \leqslant z^T x_0$ for all $t \geqslant 0$ and all u. For $\lambda < 0$ a similar reasoning applies for any $x_0 \in \mathcal{X}_0$ with $z^T x_0 \geqslant 0$. This would destroy controllability under constraints unless $z = 0$.

To show the 'if' part, let an initial state $x_0 \in \mathcal{X}_0$ and a final state $x_f \in \mathcal{X}_f$ be given. Let $x_0 = \operatorname{col}(x_{10}, x_{20})$ and $x_f = \operatorname{col}(x_{1f}, x_{2f})$ in the coordinates related to $\mathcal{X}_1 \oplus \mathcal{X}_2$ as introduced before. We will follow the following steps in constructing an input u that steers the state from x_0 to x_f:

1) We first show that the conditions (15a)-(15b) imply that the system

$$\dot{x}_1 = A_{11} x_1 + L_1 y \quad (19)$$

is controllable where y is treated as input with the constraint (2).

2) Then we show that the "input" y for system (19) that steers x_{10} to x_{1f} in (uniform) time T can be chosen inside C^∞ and it satisfies certain specific boundary conditions on $y(0), y^{(1)}(0), \ldots, y^{(\rho)}(0)$ and $y(T), y^{(1)}(T), \ldots, y^{(\rho)}(T)$ for any $\rho \in \mathbb{N}$.

3) The boundary conditions on y (and its derivatives) will be selected in such a manner that they are related to x_{20} and x_{2f}. Then, we find a $y \in C^\infty$ that generates x_1 as the solution to (19) with initial condition $x_1(0) = x_{10}$ and $x_1(T) = x_{1f}$ and satisfies the boundary conditions.

4) Finally, we construct an input u such that system (14b)-(14c) (with x_1 as a given function) produces the selected function y. Because of the boundary conditions on y and its derivatives, we will conclude (with a minor modification to u) that also the x_2-states of (14b) are steered from x_{20} at time 0 to x_{2f} at time T.

The following lemma achieves the first two steps.

Lemma V.4 *The conditions* (15a)-(15b) *imply that the system* (19) *is controllable under the input constraints* (2). *Moreover, there exists a $T > 0$ such that for any x_{10}, $x_{1f} \in \mathbb{R}^{n_1}$ the function y that steers the initial state x_{10} at time 0 to the final state state x_{1f} at time T for the system* (19) *satisfying* (2) *can be chosen inside C^∞. Moreover, for any $\rho \in \mathbb{N}$ the initial and final values of y and its derivatives $(y(0), y^{(1)}(0), \ldots, y^{(\rho)}(0))$ and $(y(T), y^{(1)}(T), \ldots, y^{(\rho)}(T))$ can be selected arbitrarily as long as they satisfy $Y[y(0), y^{(1)}(0), y^{(2)}(0), \ldots, y^{(\rho)}(0)] \succcurlyeq 0$ and $Y[y(T), -y^{(1)}(T), y^2(T), \ldots, (-1)^\rho y^{(\rho)}(T)] \succcurlyeq 0$.*

Proof: Note that the conditions (15a)-(15b) are invariant under coordinate transformation. Therefore, we can assume without loss of generality that the system (1) is of the form (14). To show the mentioned implication, we will use Theorem IV.2. Hence, let $\lambda \in \mathbb{C}$ and $z_1 \in \mathbb{C}^n$ be such that $z_1^* A_{11} = \lambda z_1^*$ and $z_1^* L_1 = 0$. The condition (15a) for the system (14) (by considering $z = [z_1^T \; 0]^T$) implies $z_1 = 0$. This means that the condition (8a) is satisfied for the system (19). To see that the condition (8b) is also satisfied, let $\lambda \in \mathbb{R}$ and $z_1 \in \mathbb{R}^{n_1}$ be such that $z_1^T A_{11} = \lambda z_1^T$ and $L_1^T z_1 \in \mathcal{Y}^*$. Then, $z = \operatorname{col}(-z_1, 0)$ and $w = L_1^T z_1$ would satisfy the left hand side of (15b) for the system (14). Hence, $z_1 = 0$. Since both conditions (8a) and (8b) are satisfied, Theorem IV.2 implies that the system (19) is controllable with the input constraints (2) and suppose that T is a uniform time in which each initial state can be steered to any final state. In the remainder of the proof we consider all functions and function classes on the interval $[0, T]$ only.

To show that the function y that steers an initial state to a final state for the system (19) can be chosen arbitrarily smooth with restrictions on initial and final values, we will prove that the set $C^\infty_{\text{bound},\mathcal{Y}}$ of C^∞ functions taking values in $\mathcal{Y}$ and satisfying the boundary conditions is dense in $L^1_{\text{loc},\mathcal{Y}}$, being the set of L^1_{loc} functions that take values in $\mathcal{Y}$ almost everywhere. We use density here in terms of the L^1_{loc}-topology. If we can establish this fact, then it is immediate that the set of all states that are reachable in time T from the origin with the constraint $y(t) \in \mathcal{Y}$, i.e. the set

$$\mathcal{X}^r(C^\infty_{\text{bound},\mathcal{Y}}) := \{\bar{x} \in \mathbb{R}^{n_1} \mid \exists y \in C^\infty_{\text{bound},\mathcal{Y}} \text{ with } x_1^{0,y}(T) = \bar{x}\},$$

is dense in the set of states that are reachable from zero with the constraint $y(t) \in \mathcal{Y}$ almost everywhere, i.e. the set

$$\mathcal{X}^r(L^1_{\text{loc},\mathcal{Y}}) := \{\bar{x} \in \mathbb{R}^{n_1} \mid \exists y \in L^1_{\text{loc},\mathcal{Y}} \text{ with } x_1^{0,y}(T) = \bar{x}\}$$

in the Euclidean topology. We used here the notation $x_1^{0,y}$ to denote the solution trajectory to (19) with "input" y and initial condition $x_1(0) = 0$. Since the former is a convex set and $\mathcal{X}^r(L^1_{\text{loc},\mathcal{Y}}) = \mathbb{R}^{n_1}$ due to constrained controllability, we can conclude that the former must be equal to $\mathbb{R}^n$ as well. This can be seen most easily by assuming the opposite (suppose there is an $\bar{x} \in \mathbb{R}^{n_1}$ with $\bar{x} \notin \mathcal{X}^r(C^\infty_{\text{bound},\mathcal{Y}})$) and then showing that there must exist a separating hyperplane between the convex sets $\{\bar{x}\}$ and $\mathcal{X}^r(C^\infty_{\text{bound},\mathcal{Y}})$ (as in Section III), which cannot be true since $\mathcal{X}^r(C^\infty_{\text{bound},\mathcal{Y}})$ is dense in $\mathcal{X}^r(L^1_{\text{loc},\mathcal{Y}}) = \mathbb{R}^n$. This proves controllability of (19) when using $C^\infty_{\text{bound},\mathcal{Y}}$ functions.

Hence, if we can prove that the closure (in L^1_{loc}-sense) of $C^\infty_{\text{bound},\mathcal{Y}}$ is equal to $L^1_{\text{loc},\mathcal{Y}}$, then the proof is complete. To do so, we start by observing that $\mathcal{Y} = \{y \in \mathbb{R}^p \mid Yy \geqslant 0\}$ can also be written in an "image representation" $\mathcal{Y} = \{Mw \mid w \geqslant 0\}$ for some matrix M of which the columns form the generators for $\mathcal{Y}$. This shows that it suffices to show that the set of nonnegative C^∞ functions with boundary conditions is dense in the set of nonnegative L^1_{loc} functions. As it is well known that the collection of nonnegative C^∞ functions is dense in the set of nonnegative L^1_{loc}-functions, it only remains to be proven that the nonnegative C^∞ functions *with boundary conditions* are dense in the nonnegative C^∞ functions using the L^1_{loc}-topology. We do this in two steps:

1) We first show that the set of C^∞ functions with $0 = y(0) = y^{(1)}(0) = y^{(2)}(0), \ldots$ and $0 = y(T) = y^{(1)}(T) = y^{(2)}(T), \ldots$ is dense in C^∞.
2) Then we show that this also holds if we replace the zero boundary conditions by arbitrary values for $(y(0), y^{(1)}(0), \ldots, y^{(\rho)}(0))$ and $(y(T), y^{(1)}(T), \ldots, y^{(\rho)}(T))$ satisfying $(y(0), y^{(1)}(0), \ldots, y^{(\rho)}(0)) \succcurlyeq 0$ and $(y(T), -y^{(1)}(T), \ldots, (-1)^\rho y^{(\rho)}(T)) \succcurlyeq 0$.

Therefore, we will use the existence of functions $w_\varepsilon \in C^\infty$ for $\varepsilon > 0$ that satisfy (see e.g. [4])

- $w_\varepsilon^{(l)}(0) = w_\varepsilon^{(l)}(T) = 0$ for all $l \in \mathbb{N}$;
- $w_\varepsilon(t) = 1$ for $t \in [\varepsilon, T - \varepsilon]$;
- $w_\varepsilon(t) \in [0, 1]$ for $t \in [0, T]$

Suppose that $g \in C^\infty$ with $g(t) \geqslant 0$ almost everywhere. Then using the above properties it follows that the products $gw_\varepsilon \in C^\infty$ converge to g when $\varepsilon \downarrow 0$. Observe that $(gw_\varepsilon)^{(l)} = \sum_{j=0}^{l} \binom{l}{k} g^{(l-j)} w_\varepsilon^{(j)}$ where $\binom{l}{k} = \frac{l!}{j!(l-j)!}$ the binomial coefficient with $l!$ denoting the factorial of l, which is equal to the product $l(l-1)(l-2)\ldots 1$. This shows that gw_ε has zero values and derivatives at 0 and T. This completes the proof of the first part.

To prove the second step, note that the lexicographical conditions on the boundary conditions imply that there exists a nonnegative C^∞ function $\bar{y}$ with $(\bar{y}(0), \bar{y}^{(1)}(0), \ldots, \bar{y}^{(\rho)}(0)) = (y(0), y^{(1)}(0), \ldots, y^{(\rho)}(0))$ and $(\bar{y}(T), \bar{y}^{(1)}(T), \ldots, \bar{y}^{(\rho)}(T)) = (y(T), y^{(1)}(T), \ldots, y^{(\rho)}(T))$. Let now again an arbitrary $g \in C^\infty$ with $g(t) \geqslant 0$ be given. Then the C^∞ functions $\tilde{w}_\varepsilon := (1 - w_\varepsilon)\bar{y} + gw_\varepsilon$ converge to g in the L^1_{loc}-topology, when $\varepsilon \downarrow 0$. Moreover, since $\tilde{w}_\varepsilon$ has the same values and derivatives of $\bar{y}$ in 0 and T and $\tilde{w}_\varepsilon$ is nonnegative, this proof is complete.

■

Construction of an input for the controllability job

Let us return to the given feasible initial state $x_0 \in \mathcal{X}_0$ and the feasible final state $x_f \in \mathcal{X}_f$, which can be written in the new coordinates as $x_0 = \operatorname{col}(x_{10}, x_{20})$ and $x_f = \operatorname{col}(x_{1f}, x_{2f})$. Since $x_0 \in \mathcal{X}_0$, there exists $\{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n_2-1}\}$ such that

$$Y(y(0), y^{(1)}(0), \ldots, y^{(n_2-1)}(0)) \succcurlyeq 0 \tag{20a}$$

where

$$y^{(k)}(0) = CA^k x_0 + CA^{k-1}B\bar{u}_0 + CA^{k-2}B\bar{u}_1 + \cdots + CB\bar{u}_{k-1} + D\bar{u}_k \tag{20b}$$

Moreover, since $x_f \in \mathcal{X}_f$, there exists $\{\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{n_2-1}\}$ such that

$$Y(y(T), -y^{(1)}(T), \ldots, (-1)^{(n_2-1)} y^{(n_2-1)}(T)) \succcurlyeq 0 \tag{21a}$$

where

$$y^{(k)}(T) = CA^k x_f + CA^{k-1}B\tilde{u}_0 + CA^{k-2}B\tilde{u}_1 + \cdots + CB\tilde{u}_{k-1} + D\tilde{u}_k \tag{21b}$$

It follows from Lemma V.4 that one can find a (uniform) time instant T and a function $y \in C^\infty$ satisfying (20) and (21) such that the initial state x_{10} at time 0 is steered to the final state x_{1f} at time T by the application of y to the dynamics (19). Let x_1 be the trajectory generated in this way. Since $y \in C^\infty$, it is clear that $x_1 \in C^\infty$.

Given y and x_1 we now have to construct u (and a corresponding initial state) such that the system (14b)-(14c) produces y as output. For this we will apply Assumption V.1 to show that the transfer matrix related to the system (A_{22}, B_2, C_2, D) is right-invertible. Assumption V.1 implies that $\mathcal{V}^*(A, B, C, D) + \mathcal{T}^*(A, B, C, D) = \mathbb{R}^n$ and that $[C\ D]$ has full row rank for the system (1). As in this case $\mathcal{X}_1$ (see Section V-B) can be taken as a subset of $\mathcal{V}^*(A, B, C, D)$, the fact that $\mathcal{V}^*$ satisfies (9) for some K, implies that C_1 can be taken 0 (possibly after a pre-compensating feedback $u = -Kx + v$, see [1] for more details). Hence, this gives that $\begin{bmatrix} C_2 & D \end{bmatrix}$ is of full row rank. Also note that $\mathcal{T}^*(A_{22}, B_2, C_2, D) = \mathbb{R}^{n_2}$ by construction. It follows from Proposition VIII.1 in the appendix that the transfer matrix $C_2(sI - A_{22})^{-1}B_2 + D$ is right invertible as a rational matrix and a right-inverse $H_2(s)$ can be chosen as a polynomial.

Note that x_1 can be considered as a disturbance in (14b)-(14c). To construct a suitable input u such that the system (14b)-(14c) produces y as output, we define $\tilde{y} \in C^\infty$ as the output generated by the system

$$\dot{x}_2 = A_{22}x_2 + A_{21}x_1;\ \tilde{y} = C_1x_1 + C_2x_2 \tag{22}$$

for initial state $x_2(0) = 0$ and the given trajectory $x_1 \in C^\infty$. We select u as $u(t) = H_2(\frac{d}{dt})[y - \tilde{y}](t)$ and a corresponding initial state $\bar{x}_{20}$ as indicated in Proposition VIII.2 in the appendix. By linearity it follows that the input u for initial state $\bar{x}_{20}$ produces y for the system (14b)-(14c). Let $\bar{x}_{2f}$ be the value of the corresponding x_2-trajectory at time T. Hence, this means that we have constructed an input u that steers x_{10} at time 0 to x_{1f} at time T and produces output y of the system (14b)-(14c), that satisfies the boundary conditions (20)-(21) at times 0 and T. These boundary conditions will be used now to show that a (modified) input function steers x_{20} to x_{2f} as well.

From the boundary conditions (20)-(21) and the fact that the output is created by the system (14b)-(14c) for a specified function x_1, it follows that $x_{20} - \bar{x}_{20} \in \mathcal{V}^*(A_{22}, B_2, C_2, D)$ and $x_{2f} - \bar{x}_{2f} \in \mathcal{V}^*(A_{22}, B_2, C_2, D)$ by applying Proposition VIII.3 in the appendix. Hence, x_{20} and $\bar{x}_{20}$ are equal to each other up to a difference in $\mathcal{V}^*(A_{22}, B_2, C_2, D)$. The same holds for x_{2f} and $\bar{x}_{2f}$. We will compensate for this difference using the following observation. As $\mathcal{T}^*(A_{22}, B_2, C_2, D) = \mathbb{R}^{n_2}$ it follows that both $x_{20} - \bar{x}_{20}$ and $x_{2f} - \bar{x}_{2f}$ are in $\mathcal{V}^*(A_{22}, B_2, C_2, D) \cap \mathcal{T}^*(A_{22}, B_2, C_2, D)$ which is equal to $\mathcal{R}^*(A_{22}, B_2, C_2, D)$ according to (11). Using the definition of $\mathcal{R}^*$, we see that there exists an input $\bar{u}$ that steers $x_{20} - \bar{x}_{20}$ at time 0 to $x_{2f} - \bar{x}_{2f}$ at time T for system

$$\dot{x}_2 = A_{22}x_2 + B_2u;\ y = C_2x_2 + Du$$

with a zero output. Again using linearity, it can be shown that the input function $u + \bar{u}$ steers state x_{20} at time 0 to state x_{2f} at time T for the system (14b)-(14c) and produces y. Since y steers (14a) from x_{10} at time 0 to x_{1f} at time

T, it can be concluded that input u steers x_0 to x_f for the system (1), thereby satisfying the constraints (2). ■

Note that Kalman's and Brammer's results are recovered as particular cases of Theorem V.3 under Assumption V.1. We consider it to be elegant to remove the right-invertibility assumption, but it does not seem to be straightforward. Removing this assumption would mean that not all "control inputs" y are allowed in (14a). Only the ones that are in the image of the linear system can be applied, which adds additional conditions (next to the boundary and C^∞ conditions) on y. This complicates the decoupling of the controllability proof in two steps as done now: one for (14a) and one for (14b).

VI. Examples

Reconsider Example III.1 with $\mathcal{Y} = [0, \infty)$, i.e.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix};\ C = \begin{pmatrix} 1 & 0 \end{pmatrix};\ D = 0.$$

Note that the transfer function $\frac{1}{s^2}$ for this system is invertible as a rational function. As this system is obviously controllable without any constraints, (15a) is satisfied. To consider (15b) we compute the system matrix

$$\begin{pmatrix} A - \lambda I & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which is invertible for any λ and thus (15b) is satisfied, which implies that the double integrator system is controllable under the *position* constraint $y = x_1 \geqslant 0$.

If we consider the *velocity* constrained double integrator, i.e. $y = x_2$, C becomes $(0\ 1)$ and $\mathcal{Y} = [0, \infty)$, the feasible initial states are $\mathcal{X}_0 = \{x_0 \mid x_{20} \geqslant 0\}$ and the feasible final states are $\mathcal{X}_f = \{x_f \mid x_{2f} \geqslant 0\}$. The transfer function, being $\frac{1}{s}$, is also invertible and the unconstrained system remains, of course, controllable. However, controllability under the output/state constraint $y = x_2 \geqslant 0$ is lost. Indeed,

$$\begin{pmatrix} A - \lambda I & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and $\lambda = 0$ (an invariant zero of the plant, see e.g. [1]), $z^T = (-1\ 0)$ and $w = 1 \in \mathcal{Y}^* = [0, \infty)$ violate condition (15b). This is also intuitively clear as nonnegative velocities x_2 prevent the position x_1 from decreasing and thus the system is not controllable under the velocity constraint $y = x_2 \geqslant 0$.

VII. Conclusions

This paper characterized the controllability of continuous-time linear systems subject to input and/or state constraints under the condition of right-invertibility of the transfer matrix. The characterizations are in terms of algebraic conditions that are of a similar nature as the classical results for unconstrained and input-constrained linear systems [2], [3], [5], which are recovered as special cases of the main result of this paper. Investigating the removal of the right-invertibility condition is future work.

VIII. Appendix: Some facts from geometric control theory

The right invertibility of the transfer matrix is related to the controlled and conditioned invariant subspaces:

Proposition VIII.1 (cf. [1]) *The transfer matrix $D + C(sI - A)^{-1}B$ is right invertible if, and only if, $\mathcal{V}^* + \mathcal{T}^* = \mathbb{R}^n$ and $\begin{bmatrix} C & D \end{bmatrix}$ is of full row rank. Futhermore, this right inverse can be chosen polynomial if, and only if, additionally the condition $\langle A \mid \operatorname{im} B \rangle \subseteq \mathcal{T}^* + \langle \ker C \mid A \rangle$ is satisfied.*

Systems that have transfer functions with a polynomial inverse are of particular interest for our treatment.

Proposition VIII.2 *Consider the linear system* (1). *Suppose that the transfer matrix $D + C(sI - A)^{-1}B$ has a polynomial right inverse. Let $H(s) = H^0 + sH^1 + \cdots + s^h H^h$ be such a right inverse. For a given $C^\infty(p)$-function $\bar{y}$, take*

$$x(0) = \sum_{\ell=0}^{h} \sum_{j=0}^{\ell-1} A^j B H^\ell \bar{y}^{(\ell-1-j)}(0) \qquad (23a)$$

$$u(t) = H(\frac{d}{dt})\bar{y}(t). \qquad (23b)$$

Then, the output y, corresponding to the initial state $x(0)$ and the input u, of system (1) *is identical to $\bar{y}$.*

The proof is omitted for brevity.

The proposition below shows what information about the state at a certain time instant can be obtained from the values of the output and its higher order derivatives at the same time instant.

Proposition VIII.3 *Consider the linear system* (1). *Let the triple (u, x, y) satisfy the equations* (1) *with the pair (u, y) being $(n-1)$-times differentiable. If*

$$y^{(k)}(t) = CA^k\bar{x} + CA^{k-1}B\bar{u}_0 + CA^{k-2}B\bar{u}_1 + \cdots + CB\bar{u}_{k-1} + D\bar{u}_k$$

for $k = 0, 1, \ldots, n-1$, for some t, $\bar{x} \in \mathbb{R}^n$, and $\{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n-1}\}$ then $x(t) - \bar{x} \in \mathcal{V}^*$.

The proof is omitted for brevity.

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