

CONTROLLABILITY OF NONLINEAR IMPLICIT FRACTIONAL INTEGRODIFFERENTIAL SYSTEMS

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In this paper, we study the controllability of nonlinear fractional integrodifferential systems with implicit fractional derivative. Sufficient conditions for controllability results are obtained through the notion of the measure of noncompactness of a set and Darbo's fixed point theorem. Examples are included to verify the result.

Keywords: controllability, fractional derivative, integrodifferential equations, fixed point theorem.

1. Introduction

Integrodifferential equations arise in many fields of science and engineering such as fluid dynamics, biological models, and chemical kinetics. A detailed investigation of integrodifferential equations and their solution via the Laplace transform method can be found in the work of Burton (1983). Recently, fractional integrodifferential equations have been used to model various physical phenomena such as heat conduction in materials with memory, combined conduction, convection and radiation problems (Caputo, 1967; Olmstead and Handelsman, 1976; Sabatier *et al.*, 2007), and numerical methods for such equations can be found in the works of Mittal and Nigam (2008) as well as Rawashdeh (2011). Models represented by neutral differential equations are encountered in theoretical epidemiology, physiology and population dynamics. It is interesting to introduce a fractional derivative for these models and study their qualitative behaviors.

Controllability is one of the fundamental concepts in control theory and plays a major role in many control problems such as stabilization of unstable systems by feedback or optimal control (Klamka, 1993). This problem can be studied by using different techniques, among which the fixed-point technique is the most powerful method for establishing the controllability results of nonlinear dynamical systems (see Balachandran and Dauer, 1987; Klamka, 1975a; 1975b; 1975c; 1999; 2001; 2008). Dacka (1980) introduced a method based on the measure of non compactness of a set and Darbo's

fixed-point theorem for studying the controllability of nonlinear systems with an implicit derivative. This method was extended to a larger class of dynamical systems by Balachandran (1988).

Anichini *et al.* (1986) addressed the controllability problem for nonlinear systems through the notion of the measure of noncompactness, the condensing operator and the Sadovskii fixed point theorem (Sadovskii, 1972), whereas Balachandran and Balasubramaniam (1992; 1994) considered the same problem for nonlinear Volterra integrodifferential systems with an implicit derivative. Klamka (2010) discussed the minimum energy control problem of infinite-dimensional fractional-discrete time linear systems and established necessary and sufficient conditions for exact controllability of such systems. Recently, Balachandran *et al.* (2012a; 2013a; 2013b; 2012b; 2012c; 2012d) studied the controllability problem for various types of nonlinear fractional dynamical systems by using fixed point theorems.

However, no work has been reported on the controllability of nonlinear implicit fractional integrodifferential systems in the literature. Therefore, in this paper we study the controllability of nonlinear implicit fractional integrodifferential systems and neutral fractional integrodifferential systems by using the measure of non compactness of a set and the Darbo fixed-point theorem.

2. Preliminaries

In this section we give some basic definitions and properties of fractional operators, the special function and the solution representation of fractional integrodifferential equations (Kexue and Jigen, 2011; Kilbas *et al.*, 2006; Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999; Samko *et al.*, 1993; Kaczorek, 2011).

For $\alpha, \beta > 0$, with $n - 1 < \alpha < n, n - 1 < \beta < n$, and $n \in \mathbb{N}$, D is the usual differential operator and suppose that $f \in L_1(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$.

The Riemann–Liouville fractional integral operator is defined as

$$(I_{0+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

$$(D_{0+}^\alpha f)(t) = D^n (I_{0+}^{n-\alpha} f)(t),$$

and the Caputo fractional derivative is taken as

$${}^C D_{0+}^\alpha f(t) = (I_{0+}^{n-\alpha} D^n f)(t), \quad 0 < \alpha < 1,$$

and, in particular, $I_{0+}^\alpha {}^C D_{0+}^\alpha f(t) = f(t) - f(0)$.

The following is a well-known relation for the Riemann–Liouville and the Caputo derivative:

$${}^C D_{0+}^\alpha f(t) = D_{0+}^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+),$$

$$n = \Re(\alpha) + 1.$$

An interesting class of functions introduced by Mittag-Leffler is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{for } \alpha, \beta > 0, \quad z \in \mathbb{C}$$

and, in particular, for $\beta = 1$,

$$E_{\alpha,1}(az^\alpha) = E_\alpha(az^\alpha) = \sum_{k=0}^{\infty} \frac{a^k z^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad a, z \in \mathbb{C}.$$

Further, the Laplace transform of the Caputo fractional derivative and the Mittag-Leffler function are

$$(i) \mathcal{L}\{{}^C D_{0+}^\alpha f(t)\}(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-k-1},$$

$$(ii) \mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm at^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}$$

for $\Re(s) > |a|^{\frac{1}{\alpha}}$ and $\Re(\beta) > 0$,

$$(iii) \mathcal{L}\{E_\alpha(\pm at^\alpha)\}(s) = \frac{s^{\alpha-1}}{s^\alpha \mp a} \text{ for } \beta = 1.$$

For brevity, let us take I_{0+}^α as I^α and ${}^C D_{0+}^\alpha$ as ${}^C D^\alpha$.

Consider the linear fractional integrodifferential equation of the form

$${}^C D^\alpha x(t) = Ax(t) + \int_0^t H(t-s)x(s) ds,$$

$$x(0) = x_0, \tag{1}$$

where $t \in [0, T] := J, 0 < \alpha < 1, x(t) \in \mathbb{R}^n, A$ is an $n \times n$ matrix and H is an $n \times n$ continuous matrix. Taking the Laplace transform on both the sides of the above equation and using the Laplace transform of the Caputo derivative, we get

$$s^\alpha X(s) - s^{\alpha-1}x(0) = AX(s) + H(s)X(s),$$

$$X(s) = s^{\alpha-1}(s^\alpha I - A - H(s))^{-1}x_0.$$

Taking the inverse Laplace transform on both the sides of the above equation, we have

$$\mathcal{L}^{-1}\{X(s)\}(t)$$

$$= \mathcal{L}^{-1}\{s^{\alpha-1}(s^\alpha I - A - H(s))^{-1}\}(t)x_0,$$

$$x(t) = R_\alpha(t)x_0,$$

where $R_\alpha(t)$ is an $n \times n$ matrix satisfying the following conditions:

- (a) $R_\alpha(0) = I$,
- (b) ${}^C D^\alpha R_\alpha(t) = AR_\alpha(t) + \int_0^t H(t-s)R_\alpha(s) ds$,
- (c) $\mathcal{L}\{R_\alpha(t)\}(s) = \int_0^t e^{-st}R_\alpha(t) dt := s^{\alpha-1}(s^\alpha I - A - H(s))^{-1}$.

Consider the linear fractional dynamical system represented by the following fractional integrodifferential equation:

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + \int_0^t H(t-s)x(s) ds \\ \quad + Bu(t), \quad t \in J, \quad 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{2}$$

where A, B are $n \times n, n \times m$ matrices, respectively, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control vectors of the system and H is an $n \times n$ continuous matrix. The solution of the system (2) is given by

$$x(t) = R_\alpha(t) + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s)Bu(s) ds, \tag{3}$$

where

$$R_{\alpha,\alpha}(\theta) = \theta^{1-\alpha} \frac{d}{d\theta} \left(\int_0^t \frac{(\theta-\tau)^{\alpha-1}}{\Gamma(\alpha)} R_\alpha(\tau) d\tau \right).$$

Definition 1. The system (2) is said to be *controllable on J* if, for every $x_0, x_1 \in \mathbb{R}^n$, there exists a control $u(t)$

such that the solution $x(t)$ of the system (2) satisfies the condition $x(0) = x_0$ and $x(T) = x_1$.

Define the controllability Grammian matrix G as

$$G = \int_0^T (T-s)^{\alpha-1} [R_{\alpha,\alpha}(T-s)B][R_{\alpha,\alpha}(T-s)B]^* ds,$$

where $*$ denotes the matrix transpose. It is proved that the linear system (2) is controllable on J if and only if the controllability Grammian matrix G is positive definite for some $T > 0$ (Balachandran and Kokila, 2013a).

3. Fractional integrodifferential systems

In this section we consider the fractional system represented by the fractional integrodifferential equation with an implicit fractional derivative of the form

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + \int_0^t H(t-s)x(s) ds + Bu(t) \\ &\quad + f(t, x(t), {}^C D^\alpha x(t), u(t)), \\ x(0) &= x_0, \end{aligned} \tag{4}$$

where $0 < \alpha < 1$, $t \in J$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B are respectively $n \times n$, $n \times m$ matrices, H is an $n \times n$ continuous matrix and the nonlinear function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous.

In order to study this problem, we need some basic facts about the measure of noncompactness and the related fixed-point theorem due to Darbo.

Definition 2. Let $(X, \|\cdot\|)$ be a Banach space and S be a bounded subset of X . Then the *measure of noncompactness* of a set S is defined by $\mu(S) = \inf\{r > 0; S \text{ can be covered by a finite number of balls whose radii are smaller than } r\}$.

Let us adopt the following settings:

- Let $C_n(J)$ be the space of continuous functions with the norm $\|x\| = \max\{x_i(t) : i = 1, 2, \dots, n, t \in J\}$. Then the measure of noncompactness of a bounded subset E in X is given by

$$\mu(E) = \frac{1}{2} \theta_0(E) = \frac{1}{2} \lim_{h \rightarrow 0^+} \theta(E, h), \tag{5}$$

where

$$\theta(E, h) = \sup_{x \in E} \left\{ \sup |x(t) - x(s)| : |t - s| \leq h \right\}$$

is the common modulus of the continuity of the functions which belong to the set E .

- We may proceed in a similar way in the case where the space considered is the space $C_n^\alpha(J)$ with the norm

$$\|x\|_{C_n^\alpha} = \|x\|_{C_n} + \|{}^C D^\alpha x\|_{C_n}.$$

Then the measure of the noncompactness of a set E is given by

$$\mu(E) = \frac{1}{2} \theta_0({}^C D^\alpha E), \tag{6}$$

where

$${}^C D^\alpha E = \{{}^C D^\alpha x; x \in E\}.$$

- Set the space of Cartesian product $C_{n+m}^\alpha(J) = C_n^\alpha(J) \times C_m(J)$ with the norm

$$\|(x, u)\|_{C_{n+m}^\alpha} = \max\{\|x\|_{C_n^\alpha}, \|u\|_{C_m}\}.$$

Then the measure of noncompactness of any bounded set E in $C_{n+m}^\alpha(J)$ is given by the relation

$$\mu(E) = \max[\mu(E_1), \mu(E_2)], \tag{7}$$

where E_1, E_2 denote the natural projections of the set E on the spaces $C_n^\alpha(J)$ and $C_m(J)$, respectively.

Assume that there exist constants $K > 0, k > 0$ such that

$$\begin{cases} |f(t, x, y, u)| \leq K, \\ |f(t, x, y, u) - f(t, x, \bar{y}, u)| \leq k|y - \bar{y}|, \end{cases} \tag{8}$$

for all $x, y, \bar{y} \in \mathbb{R}^n, u \in \mathbb{R}^m$, and $t \in J$.

The following version of Darbo's fixed point theorem, being a generalization of the Schauder fixed-point theorem, shows the usefulness of the measure of noncompactness.

Theorem 1. (Darbo's theorem (Dacka, 1980)). *If M is a nonempty bounded closed convex subset of X and $P : M \rightarrow M$ is a continuous mapping such that for any set $E \subset M$ we have $\mu(PE) \leq k\mu(E)$ where k is a constant $0 \leq k < 1$, then P has a fixed point.*

For each fixed point $(z, v) \in C_{n+m}^\alpha(J)$, consider the fractional integrodifferential system of the form

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + \int_0^t H(t-s)x(s) ds + Bu(t) \\ &\quad + f(t, z(t), {}^C D^\alpha z(t), v(t)). \end{aligned} \tag{9}$$

The solution of the above system with $x(0) = x_0$ can be written as (Balachandran and Kokila, 2013a)

$$\begin{aligned} x(t) &= R_\alpha(t)x_0 \\ &\quad + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s)Bu(s) ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s) \\ &\quad \times f(s, z(s), {}^C D^\alpha z(s), v(s)) ds. \end{aligned} \tag{10}$$

Now we prove the main result of the paper.

Theorem 2. *If the linear system (2) is controllable on J and the function f satisfies the condition (8), then the nonlinear system (9) is controllable on J .*

Proof. Define the operator $\Psi : C_{n+m}^\alpha(J) \rightarrow C_{n+m}^\alpha(J)$ as in the work of Balachandran and Kokila (2013b) by

$$\Psi(z, v) = (x, u),$$

where

$$\begin{aligned} u(t) = & B^* R_{\alpha, \alpha}^*(T-t)G^{-1} \left[x_1 - R_\alpha(T)x_0 \right. \\ & - \int_0^T (T-s)^{\alpha-1} R_{\alpha, \alpha}(T-s) \\ & \left. \times f(s, z(s), {}^C D^\alpha z(s), v(s)) ds \right], \end{aligned} \quad (11)$$

$$\begin{aligned} x(t) = & R_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} R_{\alpha, \alpha}(t-s)B \\ & \times B^* R_{\alpha, \alpha}^*(T-s)G^{-1} \left[x_1 - R_\alpha(T)x_0 \right. \\ & - \int_0^T T^{\alpha-1} R_{\alpha, \alpha}(T-\tau) \\ & \left. \times f(\tau, z(\tau), {}^C D^\alpha z(\tau), v(\tau)) d\tau \right] ds \\ & + \int_0^t (t-s)^{\alpha-1} R_{\alpha, \alpha}(t-s) \\ & \times f(s, z(s), {}^C D^\alpha z(s), v(s)) ds. \end{aligned} \quad (12)$$

It can be easily verified that $x(T) = x_1$ by substituting $t = T$ in (12). Now we consider the right-hand side of (11) and (12) as a pair of operators $\Psi_2([z, v])(t)$ and $\Psi_1([z, v])(t)$, respectively, and define the operator $\Psi : C_{n+m}^\alpha(J) \rightarrow C_{n+m}^\alpha(J)$ by

$$\Psi([z, v])(t) = [\Psi_1([z, v])(t), \Psi_2([z, v])(t)].$$

Since all the functions involved in the definition of the operator Ψ are continuous, Ψ is continuous. To prove that Ψ maps the space $C_{n+m}^\alpha(J)$ into itself, consider the closed convex set E of $C_{n+m}^\alpha(J)$ defined by

$$E = \{[z, v]; \|v\| \leq N_1, \|z\| \leq N_2, \|{}^C D^\alpha z\| \leq N_3\},$$

where the positive constants N_1, N_2 and N_3 are defined by

$$\begin{aligned} N_1 &= d_2(|x_1| + d_1), \\ N_2 &= d_1 + d_3 N_1, \\ N_3 &= aN_2 + b_1 N_1 + dN_2 T + K, \end{aligned}$$

and

$$\begin{aligned} a &= \|A\|, \quad b_1 = \|B\|, \\ b_2 &= \|G^{-1}\|, \quad d = \sup \|H(t)\|, \\ a_1 &= \sup \|R_\alpha(T)\|, \\ a_2 &= \sup \|R_{\alpha, \alpha}(T)\|, \\ d_1 &= a_1|x_0| + a_2 K T^\alpha \alpha^{-1}, \\ d_2 &= a_2 b_1 b_2, \\ d_3 &= a_2 b_1 T^\alpha \alpha^{-1}. \end{aligned}$$

Using the above notation, we have

$$\begin{aligned} \|u(t)\| &\leq \|B^*\| \|R_{\alpha, \alpha}^*(T-s)\| \|G^{-1}\| \\ &\quad \left[|x_1| + \|R_\alpha(T)\| |x_0| \right. \\ &\quad \left. + \int_0^T (T-s)^{\alpha-1} \|R_{\alpha, \alpha}(T-s)\| \right. \\ &\quad \left. \times \|f(s, z(s), {}^C D^\alpha z(s), v(s))\| ds \right], \\ &\leq a_2 b_1 b_2 [x_1 + a_1 |x_0| + a_2 K T^\alpha \alpha^{-1}], \\ &\leq d_2(|x_1| + d_1) = N_1, \end{aligned}$$

$$\begin{aligned} \|x(t)\| &\leq \|R_\alpha(T)\| |x_0| \\ &\quad + \int_0^t (t-s)^{\alpha-1} \|R_{\alpha, \alpha}(t-s)\| \|B\| \|u(s)\| ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \|R_{\alpha, \alpha}(t-s)\| \\ &\quad \times \|f(s, z(s), {}^C D^\alpha z(s), v(s))\| ds \\ &\leq a_1 |x_0| + a_2 K T^\alpha \alpha^{-1} + a_2 b_1 N_1 T^\alpha \alpha^{-1} \\ &\leq d_1 + d_3 N_1 = N_2 \end{aligned}$$

and

$$\|{}^C D^\alpha x(t)\| \leq aN_2 + b_1 N_1 + dN_2 T + K = N_3.$$

Hence the operator Ψ transforms the set E in $C_{n+m}^\alpha(J)$ into itself. It can be easily seen that, for each pair $[z, v] \in E$, we have

$$\theta(\Psi_2([z, v], h)) \leq \theta(S^*, h)k_1,$$

where

$$S^*(T, s) = B^* R_{\alpha, \alpha}^*(T-s)$$

and

$$\begin{aligned} k_1 = & \sup_{[z, v] \in E} \|G^{-1}\| \left\{ |x_1| + \|R_\alpha(T)\| |x_0| \right. \\ & \left. + \int_0^T (T-s)^{\alpha-1} \|R_{\alpha, \alpha}(T-s)\| \right. \\ & \left. \times \|f(s, z(s), {}^C D^\alpha z(s), v(s))\| ds \right\}. \end{aligned}$$

Since the function S^* does not depend on the choice of the points in E , all the functions $\Psi_2([z, v](t))$ have a uniformly bounded modulus of continuity and hence they are equi-continuous. Also all the functions involved in $\Psi_1([z, v](t))$ are equicontinuous, since they have uniformly bounded derivatives. Next we have to find an estimate for the modulus of continuity of the functions ${}^C D^\alpha \Psi_1([z, v](t))$. For that, we have

$$\begin{aligned} & |{}^C D^\alpha \Psi_1([z, v](t)) - {}^C D^\alpha \Psi_1([z, v](s))| \\ & \leq |A\Psi_1([z, v](t)) - A\Psi_1([z, v](s))| \\ & \quad + |B\Psi_2([z, v](t)) - B\Psi_2([z, v](s))| \\ & \quad + \left| \int_0^t H(t-s)\Psi_1([z, v](\eta)) d\eta \right. \\ & \quad \left. - \int_0^s H(s-\eta)\Psi_1([z, v](\eta)) d\eta \right| \\ & \quad + |f(t, z(t), {}^C D^\alpha z(t), v(t)) \\ & \quad - f(s, z(s), {}^C D^\alpha z(s), v(s))|. \end{aligned}$$

For the first three terms of the right side of the inequality, we give the upper estimate as $\beta_0(|t-s|)$ and $\lim_{h \rightarrow 0} \beta_0(h) = 0$. Also, it may be chosen independent of the choice of (z, v) . For the fourth term, we give the following estimate:

$$\begin{aligned} & |f(t, z(t), {}^C D^\alpha z(t), v(t)) - f(s, z(s), {}^C D^\alpha z(s), v(s))| \\ & \leq |f(t, z(t), {}^C D^\alpha z(t), v(t)) \\ & \quad - f(t, z(t), {}^C D^\alpha z(s), v(t))| \\ & \quad + |f(t, z(t), {}^C D^\alpha z(s), v(t)) \\ & \quad - f(s, z(s), {}^C D^\alpha z(s), v(s))|. \end{aligned}$$

For the first term, we have the upper estimate $k(|{}^C D^\alpha z(t) - {}^C D^\alpha z(s)|)$, whereas for the second term, we may find an estimate $\beta_1(|t-s|)$ with $\lim_{h \rightarrow 0} \beta_1(h) = 0$. Hence

$$\theta({}^C D^\alpha \Psi_1([z, v](t), h) \leq k(\theta({}^C D^\alpha z, h) + \beta(h)),$$

where $\beta = \beta_0 + \beta_1$. Therefore, by (5)–(8), we conclude that, for any set $E \subset C_{n+m}^\alpha(J)$,

$$\theta_0(\Psi_2 E) = 0, \quad \theta_0(\Psi_1 E) \leq k\theta_0({}^C D^\alpha E_2),$$

where E_2 is the normal projection of the set E on the space $C_n^\alpha(J)$. Hence it follows that

$$\mu(\Psi E) \leq k\mu(E).$$

By the Darbo fixed-point theorem, the mapping T has at least one fixed point, therefore there exist functions $u \in C_m(J)$ and $x \in C_n^\alpha(J)$ such that

$$(x, u) = (z, v) = [\Psi_1([z, v](t)), \Psi_2([z, v](t))].$$

This shows that $x(t)$ is the solution of (10) for the control $u(t)$ and these functions are the required solution. Hence the system (9) is controllable on J . ■

4. Neutral fractional integrodifferential systems

Consider the neutral fractional integrodifferential system governed by the neutral fractional integrodifferential equation with an implicit fractional derivative of the form

$$\begin{aligned} & {}^C D^\alpha [x(t) - g(t, x(t))] \\ & = Ax(t) + \int_0^t H(t-s)x(s) ds \\ & \quad + Bu(t) + f(t, x(t), {}^C D^\alpha x(t), u(t)), \end{aligned} \tag{13}$$

$$x(0) = x_0,$$

where A, B, H and f are as in (4) and the function $g : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable. As before, for each fixed point $(z, v) \in C_{n+m}^\alpha(J)$, consider the neutral fractional integrodifferential system

$$\begin{aligned} & {}^C D^\alpha [x(t) - g(t, x(t))] \\ & = Ax(t) + \int_0^t H(t-s)x(s) ds \\ & \quad + Bu(t) + f(t, z(t), {}^C D^\alpha z(t), v(t)), \end{aligned} \tag{14}$$

with $x(0) = x_0$, and the solution of the system (14) is (Balachandran and Kokila, 2013a)

$$\begin{aligned} x(t) & = R_\alpha(t)x_0 \\ & \quad + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s)Bu(s) ds \\ & \quad + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s) \\ & \quad \times f(s, z(s), {}^C D^\alpha z(s), v(s)) ds \\ & \quad + \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1} (s-\tau)^{-\alpha} \\ & \quad \times R_{\alpha,\alpha}(t-s)g'(\tau, x(\tau)) d\tau ds. \end{aligned} \tag{15}$$

Assume the following additional condition.

The function $g : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and there exists a constant $M > 0$ such that

$$\|g'(t, x(t))\| \leq M \quad \text{for all } t \in J \text{ and } x \in \mathbb{R}^n. \tag{16}$$

Theorem 3. *If the linear system (2) is controllable on J and the functions f and g satisfy the condition (8) and (16), then the nonlinear system (14) is controllable on J .*

Proof. Define the operator $\Phi : C_{n+m}^\alpha(J) \rightarrow C_{n+m}^\alpha(J)$, as in the work of Balachandran and Kokila (2013b), by

$$\Phi(z, v) = (x, u),$$

where

$$\begin{aligned}
 u(t) = & B^* R_{\alpha,\alpha}^*(T-t)G^{-1} \left[x_1 - R_\alpha(T)x_0 \right. \\
 & - \int_0^T (T-s)^{\alpha-1} R_{\alpha,\alpha}(T-s) \\
 & \times f(s, z(s), {}^C D^\alpha z(s), v(s)) ds \\
 & - \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s (T-s)^{\alpha-1} (s-\tau)^{-\alpha} \\
 & \left. R_{\alpha,\alpha}(T-s)g'(\tau, x(\tau)) d\tau ds \right], \tag{17}
 \end{aligned}$$

$$\begin{aligned}
 x(t) = & R_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s)B \\
 & \times B^* R_{\alpha,\alpha}^*(T-s)G^{-1} \left[x_1 - R_\alpha(T)x_0 \right. \\
 & - \int_0^T (T-\tau)^{\alpha-1} R_{\alpha,\alpha}(T-\tau) \\
 & \times f(\tau, z(\tau), {}^C D^\alpha z(\tau), v(\tau))d\tau \\
 & - \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s (T-s)^{\alpha-1} (s-\tau)^{-\alpha} \\
 & \times R_{\alpha,\alpha}(t-s)g'(\tau, x(\tau))d\tau \left. \right] ds \\
 & + \int_0^t (t-s)^{\alpha-1} R_{\alpha,\alpha}(t-s) \\
 & \times f(s, z(s), {}^C D^\alpha z(s), v(s)) ds \\
 & + \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1} (s-\tau)^{-\alpha} \\
 & \times R_{\alpha,\alpha}(t-s)g'(\tau, x(\tau)) d\tau ds. \tag{18}
 \end{aligned}$$

It can be easily verified that $x(T) = x_1$ by inserting $t = T$ in (18). We introduce the right-hand sides of (17) and (18) as a pair of operators $\Phi_2([z, v](t))$ and $\Phi_1([z, v](t))$, respectively, and define the nonlinear operator $\Phi : C_{n+m}^\alpha(J) \rightarrow C_{n+m}^\alpha(J)$ by

$$\Phi([z, v](t)) = [\Phi_1([z, v](t)), \Phi_2([z, v](t))].$$

Obviously, this operator Φ is continuous, since all the functions involved in the operator are continuous. To prove that Φ maps the space $C_{n+m}^\alpha(J)$ into itself, define a closed convex subset \mathcal{H} by

$$\mathcal{H} = \{[z, v]; \|v\| \leq M_1, \|z\| \leq M_2, \|{}^C D^\alpha z\| \leq M_3\},$$

where the positive constants M_1, M_2 and M_3 are defined by

$$\begin{aligned}
 M_1 &= a_2(|x_1| + a_1 + a_4), \\
 M_2 &= a_1 + a_3M_1 + a_4, \\
 M_3 &= aM_2 + \lambda_1M_1 + dM_2T + K + a_5,
 \end{aligned}$$

and

$$\begin{aligned}
 a &= \|A\|, \quad d = \sup \|H(t)\|, \\
 \lambda_1 &= \|B\|, \quad \lambda_2 = \|G^{-1}\|, \\
 \omega_1 &= \sup \|R_\alpha(T)\|, \quad \omega_2 = \sup \|R_{\alpha,\alpha}(T)\|, \\
 a_1 &= \omega_1|x_0| + \omega_2KT^\alpha\alpha^{-1}, \\
 a_2 &= \omega_2\lambda_1\lambda_2, \quad a_3 = \omega_2\lambda_1T^\alpha\alpha^{-1}, \\
 a_4 &= \omega_2M\Gamma(\alpha)T, \quad a_5 = \frac{1}{\Gamma(1-\alpha)}T^{1-\alpha}G^{-1}.
 \end{aligned}$$

Using the above, we have

$$\begin{aligned}
 \|u(t)\| &\leq \omega_2\lambda_1\lambda_2[|x_1| + \omega_1|x_0| + \omega_2KT^\alpha\alpha^{-1} \\
 &\quad + \omega_2M\Gamma(\alpha)T] \\
 &\leq a_2(|x_1| + a_1 + a_4) = M_1,
 \end{aligned}$$

$$\begin{aligned}
 \|x(t)\| &\leq \omega_1|x_0| + \omega_2KT^\alpha\alpha^{-1} + \omega_2\lambda_1M_1T^\alpha\alpha^{-1} \\
 &\quad + \omega_2M\Gamma(\alpha)T \\
 &\leq a_1 + a_3M_1 + a_4 = M_2,
 \end{aligned}$$

and

$$\|{}^C D^\alpha x(t)\| \leq aM_2 + \lambda_1M_1 + dN_2T + K + a_5 = M_3.$$

Hence the operator Φ transforms \mathcal{H} into itself. It can be easily seen that, for each pair $[z, v] \in \mathcal{H}$, we have

$$\theta(\Phi_2([z, v], h)) \leq \theta(P^*, h)k_2,$$

where

$$P^*(T, s) = B^* R_{\alpha,\alpha}^*(T-s)$$

and

$$\begin{aligned}
 k_2 = & \sup_{[z, v] \in \mathcal{H}} \|G^{-1}\| \left\{ |x_1| + R_\alpha(T)|x_0| \right. \\
 & + \int_0^T (T-s)^{\alpha-1} \|R_{\alpha,\alpha}(T-s)\| \|f\| ds \\
 & + \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s (T-s)^{\alpha-1} (s-\tau)^{-\alpha} \\
 & \left. \times \|R_{\alpha,\alpha}(T-s)\| \|g'(\tau, x(\tau))\| d\tau ds \right\}.
 \end{aligned}$$

We show that the operators are equicontinuous. Since the function P^* does not depend on the choice of the points in \mathcal{H} , all the functions $\Phi_2([z, v](t))$ have a uniformly bounded modulus of continuity, and hence they are equicontinuous. Also, all the functions used in $\Phi_1([z, v](t))$ are equicontinuous, since they have uniformly bounded derivatives. Next we have to find an

estimate for the modulus of continuity of the functions ${}^C D^\alpha \Phi_1([z, v](t))$. For that, we have

$$\begin{aligned} & |{}^C D^\alpha \Phi_1([z, v](t)) - {}^C D^\alpha \Phi_1([z, v](s))| \\ & \leq |A\Phi_1([z, v](t)) - A\Phi_1([z, v](s))| \\ & \quad + |B\Phi_2([z, v](t)) - B\Phi_2([z, v](s))| \\ & \quad + \left| \int_0^t H(t-s)\Phi_1([z, v](\eta)) \, d\eta \right. \\ & \quad \left. - \int_0^s H(s-\eta)\Phi_1([z, v](\eta)) \, d\eta \right| \\ & \quad + |f(t, z(t), {}^C D^\alpha z(t), v(t)) \\ & \quad - f(s, z(s), {}^C D^\alpha z(s), v(s))| \\ & \quad + |{}^C D^\alpha g(t, x(t)) - {}^C D^\alpha g(s, x(s))| \\ & \leq |A\Phi_1([z, v](t)) - A\Phi_1([z, v](s))| \\ & \quad + \left| \int_0^t H(t-s)\Phi_1([z, v](\eta)) \, d\eta \right. \\ & \quad \left. - \int_0^s H(s-\eta)\Phi_1([z, v](\eta)) \, d\eta \right| \\ & \quad + |B\Phi_2([z, v](t)) - B\Phi_2([z, v](s))| \\ & \quad + |{}^C D^\alpha g(t, x(t)) - {}^C D^\alpha g(s, x(s))| \\ & \quad + |f(t, z(t), {}^C D^\alpha z(t), v(t)) \\ & \quad - f(s, z(s), {}^C D^\alpha z(s), v(s))|. \end{aligned}$$

For the first four terms of the right-hand side of the inequality, we give the upper estimate as $\beta_0(|t-s|)$ and the last term by $k(|{}^C D^\alpha z(t) - {}^C D^\alpha z(s)|) + \beta_1(|t-s|)$, with $\lim_{h \rightarrow 0} \beta_i(h) = 0$. Hence $\theta({}^C D^\alpha \Phi_1([z, v](t), h) \leq k(\theta({}^C D^\alpha z, h) + \beta(h))$, where $\beta = \beta_0 + \beta_1$. Therefore, by (16) and (5)–(8), we conclude that, for any set $\mathcal{H} \subset C_{n+m}^\alpha(J)$,

$$\theta_0(\Phi_2 \mathcal{H}) = 0, \quad \theta_0(\Phi_1 \mathcal{H}) \leq k\theta({}^C D^\alpha \mathcal{H}_2),$$

where \mathcal{H}_2 is the normal projection of the set \mathcal{H} on the space $C_n^\alpha(J)$. Hence

$$\mu(\Phi \mathcal{H}) \leq k\mu(\mathcal{H}).$$

By the Darbo fixed-point theorem, the mapping Φ has at least one fixed point. Therefore, there exist functions $u \in C_m(J)$ and $x \in C_n^\alpha(J)$ such that $\Phi(x, u) = (x, u)$, that is,

$$u(t) = \Phi_2(x, u)(t), \quad x(t) = \Phi_1(x, u)(t).$$

These functions give the required solution and satisfy $x(T) = x_1$. Hence the system (14) is controllable. ■

5. Examples

In this section we give two examples to illustrate the theory developed in the previous sections.

Example 1. Consider the fractional integrodifferential system with an implicit fractional derivative of the form

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + \int_0^t H(t-s)x(s) \, ds + Bu(t) \\ &\quad + f(t, x(t), {}^C D^\alpha x(t), u(t)), \quad t \in J, \\ x(0) &= x_0, \end{aligned} \tag{19}$$

where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$H(t-s) = \begin{pmatrix} \frac{3(t-s)^{-1/2}}{\Gamma(1/2)} & 0 \\ 0 & \frac{3(t-s)^{-1/2}}{\Gamma(1/2)} \end{pmatrix},$$

$\alpha = 1/2$ and the nonlinear term f is given by

$$\begin{aligned} & f(t, x(t), {}^C D^{1/2} x(t), u(t)) \\ &= \begin{pmatrix} 0 \\ t \cos x(t) + \sin \frac{1}{2}({}^C D^{1/2} x(t)) \end{pmatrix}. \end{aligned}$$

Here

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

with

$$x_1(t) = x(t), \quad {}^C D^{1/4} x_1(t) = x_2(t).$$

First, we consider the homogeneous part of the above system,

$$\begin{aligned} & {}^C D^{1/2} x(t) \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} x(t) \\ &\quad + \int_0^t \begin{pmatrix} \frac{3(t-s)^{-1/2}}{\Gamma(1/2)} & 0 \\ 0 & \frac{3(t-s)^{-1/2}}{\Gamma(1/2)} \end{pmatrix} x(s) \, ds. \end{aligned} \tag{20}$$

Using the Laplace transform, we find the solution of the system (20) as

$$x(t) = R_{1/2}(t)x(0), \tag{21}$$

where

$$R_{1/2}(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & Q(t) \end{pmatrix}.$$

Here

$$Q(t) = \frac{1}{4}E_{1/2}(t^{1/2}) + \frac{3}{4}E_{1/2}(-3t^{1/2})$$

and $R_{1/2}(t)$ is the resolvent matrix which satisfies the following properties:

(i) $R_{1/2}(0) = I,$

(ii) ${}^C D^{1/2}R_{1/2}(t) = AR_{1/2}(t) + \int_0^t H(t-s)R_{1/2}(s) ds,$

(iii) $\mathcal{L}\{R_{1/2}(t)\}(s) = \int_0^t e^{-st}R_{1/2}(t) dt := s^{-1/2}(s^{1/2}I - A - \mathcal{L}(H))^{-1}.$

Now, taking Laplace transform on (19) and using the property (iii) as well as a simple partial fraction method, we obtain the solution of (19) as (Balachandran and Kokila, 2013a)

$$x(t) = R_{1/2}(t) + \int_0^t (t-s)^{-1/2}R_{1/2,1/2}(t-s)Bu(s) ds, \tag{22}$$

where

$$R_{1/2,1/2}(t) = \begin{pmatrix} L(t) & 0 \\ 0 & L(t) \end{pmatrix},$$

$$L(t) = \frac{1}{4}E_{1/2,1/2}(t^{1/2}) + \frac{3}{4}E_{1/2,1/2}(-3t^{1/2}).$$

By simple matrix calculation, one can see that the controllability matrix

$$G = \int_0^T (T-s)^{\alpha-1} [R_{\alpha,\alpha}(T-s)B] \times [BR_{\alpha,\alpha}(T-s)]^* ds = \int_0^T (T-s)^{-1/2} \begin{pmatrix} L^2 & L^2 \\ L^2 & 2L^2 \end{pmatrix} ds$$

is positive definite for any $T > 0$. Furthermore,

$$|f(t, x, y, u) - f(t, x, \bar{y}, u)| = \left\| \sin \frac{y}{2} - \sin \frac{\bar{y}}{2} \right\| \leq \frac{1}{2}|y - \bar{y}|$$

and there exists $K > 0$ such that $|f(t, x, y, u)| \leq K$, so the hypotheses of Theorem 2 are satisfied. Observe that

the control defined by

$$u(t) = B^*R_{\alpha,\alpha}^*(T-t)G^{-1} \left[x_1 - R_{\alpha}(T)x_0 - \int_0^T (T-s)^{\alpha-1}R_{\alpha,\alpha}(T-s) \times f(s, x(s), {}^C D^\alpha x(s), u(s)) ds \right]$$

steers the system (19) from x_0 and x_1 and hence the fractional system (19) is controllable on $[0, T]$. ♦

Remark 1. It should be noted that, for $\alpha = 1$, the fractional system (19) is reduced to integer order Volterra integrodifferential systems with implicit derivative which was studied by Balachandran and Balasubramaniam (1992).

Example 2. Consider the following neutral fractional integrodifferential system:

$${}^C D^\alpha [x(t) - g(t, x(t))] = Ax(t) + \int_0^t H(t-s)x(s) ds + Bu(t) + f(t, x(t), {}^C D^\alpha x(t), u(t)), \quad t \in J, \tag{23}$$

$$x(0) = x_0,$$

where A, B, H and f are as above, $\alpha = 1/2$ and g is taken as

$$g(t, x(t)) = \begin{pmatrix} \frac{x_1(t)}{1+x_2^2(t)} \\ \frac{x_2(t)}{1+x_1^2(t)} \end{pmatrix}.$$

Note that g is differentiable and the derivative is uniformly bounded. Further, the linear system is controllable and $f(t, x(t), {}^C D^{1/2}x(t), u(t))$ satisfies the hypotheses of Theorem 3, then the non-linear system (23) is controllable on $[0, T]$. ♦

Example 3. Consider the following nonlinear fractional integrodifferential system represented by the matrix fractional integrodifferential equation:

$${}^C D^\alpha x(t) = Ax(t) + \int_0^t H(t-s)x(s) ds + Bu(t) + f(t, x(t), {}^C D^\alpha x(t), u(t)), \quad t \in J, \tag{24}$$

$$x(0) = x_0,$$

where A, B and H are as above, $\alpha = 1/2$ and f is taken as

$$f(t, x(t), {}^C D^{1/2}x(t), u(t)) = \begin{pmatrix} 0 \\ \sin x(t) \cos u(t) \end{pmatrix},$$

since the linear system is controllable and the nonlinear function $f(t, x(t), {}^C D^{1/2}x(t), u(t))$ does not satisfy the condition stated in Theorem 2. However, by Theorem 3.1 of Balachandran *et al.* (2012c), the nonlinear system (24) is controllable on $[0, T]$. ♦

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