BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 2, April 1989

CONTROLLED TOPOLOGY IN GEOMETRY

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The purpose of the present note is to announce some finiteness theorems for classes of Riemannian manifolds (cf. A, B and D below).

Let $\mathscr{M}_{k,d,v}^{K,D,V}(n)$ denote the class of closed Riemannian n-manifolds with sectional curvatures between k and K, diameter between d and D, and volume between v and V. Here $k \leq K$ are arbitrary, $0 \leq d \leq D$, and $0 \leq v \leq V$.

THEOREM A. *For $n \neq 3, 4$ the class $\mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$ contains at most finitely many diffeomorphism types.*

This unifies and generalizes the following two theorems in high dimensions.

THEOREM (J. CHEEGER [C, P]). *The Class $\mathscr{M}_{k,0,v}^{K,D,\infty}(n)$ contains at most finitely many diffeomorphism types.*

THEOREM (K. GROVE, P. PETERSEN [GP]). *The class $\mathscr{M}_{k,0,v}^{\infty,D,\infty}$ contains at most finitely many homotopy types.*

For $k > 0$ and $n = 3$, the conclusion in Theorem A follows by Hamilton's theorem in [H]. For $k > 0$ and $n = 4$ the fundamental group is either trivial or $\mathbf{Z}_2$ by Synge's theorem. Using Freedman's classification of simply connected topological 4-manifolds together with the above theorem and standard surgery theory then yields (cf. also [HK]).

COROLLARY B. *For $k > 0$ the class $\mathscr{M}_{k,0,v}^{\infty,\infty,\infty}(n)$ contains at most finitely many diffeomorphism (resp. homeomorphism) types when $n \neq 4$ (resp. $n = 4$).*

The basic construction in [GP] exhibits for each $M \in \mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$ a suitable strong deformation retraction of an *a* priori neighborhood of the diagonal in $M \times M$ onto the diagonal. This enables one to find $R, C > 0$ so that for all $p \in M$ the metric r-ball $B(p, r)$ is contractible inside $B(p, C \cdot r)$ whenever $r \leq R$. This latter property carries over to any compact space $X = \lim M_k$ in the Gromov-Hausdorff closure of $\mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$, moreover $\dim X \leq n$, cf. [PV]. Using the local contractibility properties, rather than the deformations as in [GP], one gets homotopy equivalences

$$M_k \underset{g_k}{\overset{f_k}{\rightleftarrows}} X$$

Received by the editors December 21, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C20, 57N99.
The first two authors were supported in part by the National Science Foundation.

for large k, where $\operatorname{diam} f_k = \sup\{\operatorname{diam} f_k^{-1}(x) | x \in X\} \to 0$ and $\operatorname{diam} g_k \to 0$ as $k \to \infty$, cf. [**PV**]. In particular, X is n-dimensional.

It is a fairly easy consequence of results due to Begle [**B**] that $X = \lim M_k$ must be a homology manifold.

Combining all these properties of X allows us to apply a result of F. Quinn [**Q**] to conclude that X admits a resolution for $n \geq 4$. If in addition $n \geq 5$, and X satisfies the disjoint disc property (DDP), it must be a topological manifold according to a theorem of R. D. Edwards, cf. [**E, D**]. To see that $X = \lim M_k$ indeed satisfies the DDP, one uses the deformations associated with M_k together with the homotopy equivalences f_k, g_k. Hence

THEOREM C. *For $n \geq 5$ any compact metric space in the Gromov-Hausdorff closure of $\mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$ is a topological n-manifold.*

Having shown that X is a topological manifold of dimension ≥ 5 a result of T. A. Chapman and S. Ferry [**CF, F**] implies that for k sufficiently large g_k can be deformed to a homeomorphism. Since the closure of $\mathscr{M}_{k,0,v}^{\infty,D,\infty}(n)$ is compact, cf. [**G**], we conclude that this class contains at most finitely many homeomorphism types. By a general result of R. Kirby and L. Siebenmann [**KS**], Theorem A follows.

The same argument as outlined above yields finiteness for diffeomorphism types rather than homotopy types in a finiteness theorem by T. Yamaguchi, cf. [**Y**]. In particular,

THEOREM D. *For $n \geq 5$ the class of closed n-manifolds with injectivity radius bounded from below and volume from above, contains at most finitely many diffeomorphism types.*

ACKNOWLEDGEMENTS. We would like to thank several people for their interest and suggestions concerning this work, in particular K. Corlette, S. Ferry, I. Hambleton, M. Rothenberg, D. Ruberman, and F. Quinn.

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