

Controller Parameterization for Disturbance Response Decoupling: Application to Vehicle Active Suspension Control

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Abstract—This paper derives a parameterization of the set of all stabilizing controllers for a given plant which leaves some prespecified closed-loop transfer function fixed. This result is motivated by the need to independently shape several different disturbance transmission paths in vehicle active suspension control. The result is studied in the context of quarter-, half-, and full-car vehicle models, to derive appropriate controller structures. A controller design is carried out for the full-car case and simulated with a nonlinear vehicle dynamics model.

Index Terms—Active suspension, controller parameterization, disturbance response decoupling, H_∞ , loop-shaping, vehicle dynamics.

I. INTRODUCTION

THIS paper studies the problem of control system design when there are independent performance goals for several disturbance transmission paths. We are motivated by the problem of vehicle active suspension control where a key feature is the need to insulate the vehicle body from both road irregularities and load disturbances (e.g., inertial loads induced by braking and cornering). It is well known that these are conflicting requirements when passive suspensions are used, but the conflict may be removed when active control is employed with appropriate hardware structure, e.g., choice of sensor location, number, and type. Once a suitable hardware structure is selected, there remains the problem of designing the controller to achieve satisfactory responses for all of the disturbance transmission paths. It is this latter problem that is the main subject of this paper. To this end, we consider the problem of parameterizing the set of all stabilizing controllers for a given plant which leaves the transfer function for a given disturbance transmission path the same as when some nominal stabilizing controller is employed. In this way, the design for each disturbance path can be carried out successively, providing there is sufficient freedom to adjust the responses independently.

A. Vehicle Active Suspension Control

The use of active suspension on road vehicles has been considered for many years [8], [15], [17], [20], [26]. A large

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number of different arrangements from semiactive to fully active schemes has been investigated [2], [16], [25], [27]. There has also been interest in characterizing the degrees of freedom and constraints involved in active suspension design. Constraints on the achievable response have been investigated from “invariant point,” transfer-function, and energy/passivity points of view in [7], [9], [10], [18], [19]. In [18], a complete set of constraints was derived on the road and load disturbance response transfer-functions and results on the choice of sensors needed to achieve these degrees of freedom independently were obtained for the quarter-car model (see [4] for generalization of these results to half- and full-car models). In [19] it was shown that the road and load disturbance responses could not be adjusted independently for any passive suspension applied to a quarter-car model.

The need to design the road and load disturbance responses independently has been considered elsewhere in the active suspension literature. For example, in [15] a hardware and sensing arrangement was devised so that the feedback part of the scheme would not affect the response to road disturbances, which were designed to be suitably soft by means of passive elements in the scheme. In [24], [25] the actuator was placed in series with a spring and damper, which were chosen to give a suitably soft response to road irregularities in the absence of a feedback signal. A controller structure using a filtered combination of the sensor measurements was then selected so that the road disturbance responses were unaffected by the feedback. The present paper represents a continuation of this idea by finding in general the required controller structure to achieve this property for any set of measurements.

In active suspension design for full-car models, it has been found advantageous to decompose the motion into bounce, pitch and roll components for the vehicle body and additionally warp for the wheels in contact with the road [6], [12], [13], [19]. This paper will also exploit such transformations, at least to a partial extent. In the full-car case we will exploit symmetry to decompose into the bounce/pitch and roll/warp half-cars. In the half-car case we will use our results to determine the feedback structure to allow road and load disturbances to be shaped independently and discuss the simplicity assumptions which allow a further decomposition of the half-car into two quarter-cars.

B. Controller Parameterization

The idea of parameterizing all stabilizing controllers in a linear feedback system is a standard one [28], [30]. The extension of this idea to two-degree-of-freedom schemes

(which allows the response to reference commands and the return ratio of the feedback path to be optimized independently) is also standard [21], [29]. The generalization to additional degrees-of-freedom to include some exogenous disturbances has also been considered [14]. The parameterization of all stabilizing controllers which leaves some prespecified closed-loop transfer function fixed, as considered in this paper, represents a continuation of these ideas and techniques.

Our approach makes use of algebraic properties of the ring of stable, proper rational functions [21]. To facilitate the parameterization at the required level of generality we will introduce the idea of left and right normal rank factorizations of a rational matrix (Definition 1).

C. Outline of Paper

Section II sets up in generality the problem of parameterizing all stabilizing controllers which leave some prespecified closed-loop transfer function fixed. Our basic results, which characterize the required structure of the Youla Q -parameter, are given in Theorems 1 and 2. Section III considers the standard quarter-car model employing a ‘‘Sharp’’ actuator with various choices of measured variables. The required controller structures to leave the road disturbance responses the same as in the passive case are derived using the results of Section II. Section IV considers a simple half-car model with acceleration and strut deflection measurements, and again derives the control structure required to keep the road disturbance responses the same as in the passive case. ‘‘Simplicity’’ conditions which allow the design to be carried out for two separate quarter-cars are presented. Section V considers a simple linearized full-car model and shows how this may be separated into two half-car models under a mild symmetry assumption. Section VI presents a design for the full-car model with acceleration and strut deflection measurements. The bounce/pitch half-car is treated according to the theory in Section IV. The roll and warp modes are each treated as quarter-cars with the warp mode being handled in a special way. The controller design is simulated with a nonlinear vehicle model using the multibody simulation package *AutoSim*.

II. CONTROLLER PARAMETRISATION RESULTS

We consider the LFT (linear fractional transformation) model in Fig. 1, where the Laplace transfer function of the generalized plant P is partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and further partitioned conformably with the disturbance signals as

$$\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{y} \end{bmatrix} = \begin{bmatrix} P_{11,11} & P_{11,12} \\ P_{11,21} & P_{11,22} \\ P_{21,1} & P_{21,2} \end{bmatrix} \begin{bmatrix} P_{12,1} \\ P_{12,2} \\ P_{22} \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{u} \end{bmatrix} \quad (1)$$

where $w_1 \in \mathbb{R}^{m_1}$, $w_2 \in \mathbb{R}^{m_2}$, $u \in \mathbb{R}^{m_3}$, $z_1 \in \mathbb{R}^{p_1}$, $z_2 \in \mathbb{R}^{p_2}$, $y \in \mathbb{R}^{p_3}$ at any time instant \hat{u} denotes the Laplace transform of $u(t)$ etc. We consider the problem of parameterizing all stabilizing

controllers which leave $T_{\hat{w}_1 \rightarrow \hat{z}_1}$ (the transfer function from \hat{w}_1 to \hat{z}_1) the same as for some given stabilizing controller K_0 .

Let $P_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be the right and left coprime factorizations of P_{22} over \mathbb{RH}_∞ . Then all stabilizing controllers can be parameterized by

$$K = (Y - MQ)(X - NQ)^{-1} \quad (2)$$

$$= (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}) \quad (3)$$

for $Q \in \mathbb{RH}_\infty^{m_3 \times p_3}$ where $X, Y, \tilde{X}, \tilde{Y}$ are matrices with elements in \mathbb{RH}_∞ which satisfy the Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

For our first result (Theorem 1) it is convenient if the factorizations are chosen so that $Q = 0$ corresponds to the desired stabilizing controller, i.e., $K_0 = YX^{-1} = \tilde{X}^{-1}\tilde{Y}$. (This assumption will be relaxed in the corollary to Theorem 2.)

The results we will establish in this section make use of certain algebraic properties of the set \mathbb{RH}_∞ , namely its ring structure. The reader is referred to [21] for the necessary background on this topic. Here we will be content to recall a few facts. The set \mathbb{RH}_∞ has the property of being a Euclidean domain with degree function defined by the total number of zeros of the element in the closed right half plane and at infinity (counting multiplicities). The invertible elements in \mathbb{RH}_∞ are called *units*, and are the elements with degree equal to zero. A matrix $U \in \mathbb{RH}_\infty^{m \times m}$ is called *unimodular* if it has an inverse whose elements belong to \mathbb{RH}_∞ , or equivalently, if its determinant is a unit in \mathbb{RH}_∞ . The normal rank of a matrix $Q \in \mathbb{RH}_\infty^{m \times n}$, denoted $\text{normal rank}(Q(s))$, is the maximum rank of $Q(s)$ for any $s \in \mathbb{C}$ which is not a pole. Equivalently, the normal rank is equal to the rank for almost all $s \in \mathbb{C}$. We now introduce a type of matrix factorization which will be useful in proving the subsequent results.

Definition 1: Let T be a matrix with elements in \mathbb{RH}_∞ . T is said to have a left normal rank factorization (lnf) if there exist matrices E and $V = (V_1', V_2)'$ over \mathbb{RH}_∞ with E of full column normal rank and V unimodular such that $T = EV_1$. T is said to have a right normal rank factorization (rnf) if there exist matrices F and $U = (U_1, U_2)$ over \mathbb{RH}_∞ with F of full row normal rank and U unimodular such that $T = U_1F$.

Lemma 1: For any $T \in \mathbb{RH}_\infty^{r \times c}$, there exists an lnf and an rnf of T .

Proof: See the Appendix. \square

We return to the problem of parameterizing all stabilizing controllers which leave the transfer function $T_{\hat{w}_1 \rightarrow \hat{z}_1}$ the same as when the controller $K_0 = YX^{-1} = \tilde{X}^{-1}\tilde{Y}$ is applied. From [5], the closed-loop transfer function in Fig. 1 can be expressed as

$$T \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = T_1 - T_2QT_3 \quad (4)$$

where T_1, T_2, T_3 have elements in \mathbb{RH}_∞ and are given by

$$\begin{aligned} T_1 &= P_{11} + P_{12}M\tilde{Y}P_{21} \\ T_2 &= P_{12}M \\ T_3 &= \tilde{M}P_{21}. \end{aligned}$$

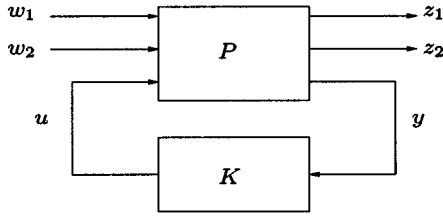


Fig. 1. Generalized model in LFT form.

Thus the problem reduces to parameterizing all stabilizing controllers which leave $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{11}$. These are characterized by all $Q \in \mathbb{RH}_\infty^{m_3 \times p_3}$ such that $(T_2 Q T_3)_{11} = T_{2,1} Q T_{3,1} = 0$, where $T_{2,1} = P_{12,1} M$ and $T_{3,1} = M P_{21,1}$. We now introduce an lnf of $T_{2,1}$ and an rnf of $T_{3,1}$ as follows:

$$T_{2,1} := E V_1 \quad (5)$$

$$T_{3,1} := U_1 F \quad (6)$$

where $E \in \mathbb{RH}_\infty^{p_1 \times r_2}$, $F \in \mathbb{RH}_\infty^{r_3 \times m_1}$, $U_1 \in \mathbb{RH}_\infty^{p_3 \times r_3}$, $V_1 \in \mathbb{RH}_\infty^{r_2 \times m_3}$, r_2 and r_3 are the normal rank of $T_{2,1}$ and $T_{3,1}$ respectively. Note that r_2 and r_3 are also the normal rank of $P_{12,1}$ and $P_{21,1}$, respectively. Furthermore, we have the inequalities $r_2 \leq \min(p_1, m_3)$, $r_3 \leq \min(m_1, p_3)$.

Theorem 1: Consider any stabilizable P in the configuration of Fig. 1. All stabilizing controllers such that the closed-loop transfer function $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{11}$ are given by K expressed in the form of (2) and (3) with

$$Q = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_1 \tilde{U}_2 \\ Q_2 \end{pmatrix} \quad (7)$$

for $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$ and $Q_2 \in \mathbb{RH}_\infty^{(m_3 - r_2) \times p_3}$, V_1 and U_1 defined from the lnf and rnf factorizations (5) and (6), U_2, V_2 are chosen such that (U_1, U_2) and $(V_1', V_2)'$ are unimodular, and $\tilde{U}_2 \in \mathbb{RH}_\infty^{(p_3 - r_3) \times p_3}$ is a partition of U^{-1}

$$U^{-1} = (U_1, U_2)^{-1} = \begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix}.$$

Proof: See the Appendix. \square

The control structure given in (7) is arrived at by completing the matrix U_1 to a unimodular matrix and then extracting \tilde{U}_2 from the resulting matrix inverse. Since U_1 and the completion are not unique then neither is \tilde{U}_2 . It will be useful to characterize this nonuniqueness in terms of the parameterization of the set $\{Q_1 \tilde{U}_2 : Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}\}$ directly. This is done in the following lemma.

Lemma 2: Given two sets $A = \{Q_1 \tilde{U}_2 : Q_1 \in \mathbb{RH}_\infty^{r_2 \times c}\}$ and $B = \{Q_1^\dagger \tilde{U}_2^\dagger : Q_1^\dagger \in \mathbb{RH}_\infty^{r_2 \times c}\}$ where $\tilde{U}_2, \tilde{U}_2^\dagger \in \mathbb{RH}_\infty^{c \times q}$ are full row normal rank, then

- 1) $B \subseteq A$ if and only if \tilde{U}_2^\dagger is a left multiple of \tilde{U}_2 over \mathbb{RH}_∞ , i.e., there exists a $W \in \mathbb{RH}_\infty^{c \times c}$ such that $\tilde{U}_2^\dagger = W \tilde{U}_2$.
- 2) $B = A$ if and only if there exists a unimodular matrix $W \in \mathbb{RH}_\infty^{c \times c}$ such that $\tilde{U}_2^\dagger = W \tilde{U}_2$.

Proof: See the Appendix. \square

Throughout this paper the vehicle dynamics examples will satisfy some special assumptions on the open-loop plant. This

allows the controller parameterization of Theorem 1 to take a simplified form, and it turns out that a further useful structural simplification can then be made. It will be convenient to summarize these simplifications in the theorem below, which will then be applied directly throughout the paper. (The first two special assumptions on the open-loop plant arise because of some passive elements in the suspension system which ensure that the road disturbance responses are satisfactory without any feedback control. The third assumption is a rather technical one which says that the number of outputs to be left invariant is no smaller than the number of actuators and that this transmission path has full normal rank.)

Theorem 2: Let (i) P be (open-loop) stable, (ii) $K_0 = 0$, (iii) $m_3 = r_2$.

- 1) There exists $\tilde{U}_2 \in \mathbb{RH}_\infty^{(p_3 - r_3) \times p_3}$ such that all stabilizing controllers which give $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{1,1}$ can be parameterized as

$$K = -(I - Q_1 \tilde{U}_2 P_{22})^{-1} Q_1 \tilde{U}_2 \quad (8)$$

for $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$.

- 2) A particular $\tilde{U}_2 \in \mathbb{RH}_\infty^{(p_3 - r_3) \times p_3}$ for which (8) parameterizes all stabilizing controllers such that $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{1,1}$ can be calculated as follows: choose $M = I$, $\tilde{M} = I$, $N = \tilde{N} = P_{22}$, $X = I$, $\tilde{X} = I$, $Y = 0$, $\tilde{Y} = 0$ and $V_1 = I$, define U_1 from the rnf (6), and calculate U_2, \tilde{U}_2 as in Theorem 1.
- 3) Consider any \tilde{U}_2^\dagger such that $K^\dagger = -(I - Q_1 \tilde{U}_2^\dagger P_{22})^{-1} Q_1 \tilde{U}_2^\dagger$ parameterizes all stabilizing controllers which give $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{1,1}$. Then there exist a unimodular matrix W such that $\tilde{U}_2^\dagger = W \tilde{U}_2$, where \tilde{U}_2 is defined in 2).
- 4) Let \tilde{U}_2 be defined in 2) and let K_1 be any stabilizing controller for $\tilde{U}_2 P_{22}$. Then $K_1 \tilde{U}_2$ is a stabilizing controller for P_{22} for which $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{1,1}$.
- 5) Let \tilde{U}_2 be defined in 2) and consider any stabilizing controller K for P_{22} for which $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{1,1}$. Then we can write $K = K_1 \tilde{U}_2$, where K_1 is a stabilizing controller for $\tilde{U}_2 P_{22}$.

Proof: See the Appendix. \square

Corollary 1: Let conditions (i) and (iii) of Theorem 2 hold and suppose that $K_0 = -(I - \bar{Q} P_{22})^{-1} \bar{Q}$ for some $\bar{Q} \in \mathbb{RH}_\infty^{m_3 \times p_3}$. Then all stabilizing controllers which leave $(T)_{\hat{w}_1 \rightarrow \hat{z}_1}$ the same as when K_0 is applied can be parameterized as

$$K = -(I - (\bar{Q} + Q_1 \tilde{U}_2) P_{22})^{-1} (\bar{Q} + Q_1 \tilde{U}_2) \quad (9)$$

for some $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$ and \tilde{U}_2 defined in Theorem 2 (2).

Proof: See the Appendix. \square

The controller structure given in (8), which is a special case of the general parameterization given in Theorem 1, may be represented in the block diagram form shown in Fig. 2(a). Theorem 2(4, 5) shows that the essential feature in this controller structure is the presence of \tilde{U}_2 as the rightmost term in (8). This is illustrated in the block diagram Fig. 2(b) where K_1 may be any stabilizing controller for the transformed plant P^\dagger .

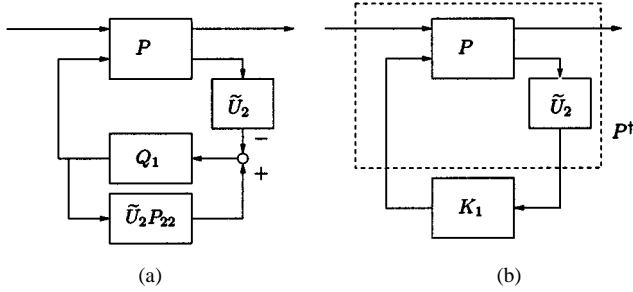


Fig. 2. (a) General controller structure. (b) Equivalent controller.

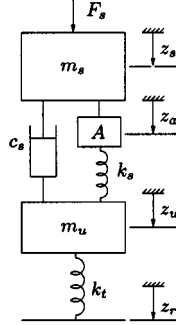


Fig. 3. The quarter-car model.

III. THE QUARTER-CAR MODEL

A. The Quarter-Car With Two Measurements

We begin with the quarter-car model of Fig. 3 where the sprung and unsprung masses are m_s and m_u and the tire is modeled as a linear spring with constant k_t . The suspension consists of a passive damper of constant c_s in parallel with a series combination of an actuator A and a spring of constant k_s (sometimes referred to as a ‘‘Sharp’’ actuator [27]). Following [16] the actuator is modeled so that the relative displacement across the actuator will be a low-pass filtered version of the actuator’s command signal, i.e.,

$$\hat{z}_s - \hat{z}_a = \gamma(s)\hat{u}. \quad (10)$$

As in [16], we use a second-order filter to represent the actuator dynamics

$$\gamma(s) = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}. \quad (11)$$

The external disturbances are taken to be a load F_s and a road displacement z_r , and the measurements are taken to be \ddot{z}_s and $z_s - z_u$. The dynamic equations of the model are given by

$$m_s \ddot{z}_s = F_s - u_p \quad (12)$$

$$m_u \ddot{z}_u = u_p + F_r \quad (13)$$

where

$$u_p = c_s(\dot{z}_s - \dot{z}_u) + k_s(z_a - z_u) \quad (14)$$

$$F_r = k_t(z_r - z_u). \quad (15)$$

We wish to parameterize all controllers which leave the transmission path from the road disturbance to z_s and z_u the same as in the open-loop, i.e., with $u = 0$. This assumes that k_s and c_s are chosen to give satisfactory responses for this transmission

path. In effect, this gives the choice of $K_0 = 0$. We now write the system in the form of Fig. 1 with $z_1 = [z_s, z_u]'$, $w_1 = z_r$, $w_2 = F_s$, $y = [\ddot{z}_s, z_s - z_u]'$, u equals the actuator command signal as in (10) and z_2 omitted. The corresponding dimensions are $m_1 = m_2 = m_3 = 1$, $p_1 = 2$, $p_2 = 0$, $p_3 = 2$. Equation (1) then takes the form

$$\begin{bmatrix} \hat{z}_s \\ \hat{z}_u \\ \frac{\hat{z}_s}{s^2} \\ \hat{z}_s - \hat{z}_u \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \hat{z}_r \\ \hat{F}_s \\ \hat{u} \end{bmatrix}$$

with the transfer functions being given by

$$P_{11} = \frac{1}{d(s)} \begin{bmatrix} k_t(c_s s + k_s) & p_1(s) \\ k_t(m_s s^2 + c_s s + k_s) & (c_s s + k_s) \end{bmatrix} \quad (16)$$

$$P_{12} = \frac{\gamma(s)}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t) \\ -k_s m_s s^2 \end{bmatrix} \quad (17)$$

$$P_{21} = \frac{1}{d(s)} \begin{bmatrix} k_t(c_s s + k_s)s^2 & p_1(s)s^2 \\ -k_t(m_s s^2) & (m_u s^2 + k_t) \end{bmatrix}$$

$$P_{22} = \frac{\gamma(s)}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t)s^2 \\ k_s((m_u + m_s)s^2 + k_t) \end{bmatrix}$$

where $p_1(s) = m_u s^2 + c_s s + k_s + k_t$ and

$$d(s) = (m_s m_u) s^4 + c_s(m_s + m_u) s^3 + (m_s(k_s + k_t) + m_u k_s) s^2 + c_s k_t s + k_s k_t. \quad (18)$$

As expected, all roots of $d(s)$ are in left-half plane, which can be confirmed by the Routh–Hurwitz criterion.

We now observe that $P_{12,1} = P_{12}$, which has normal rank equal to one, i.e., $r_2 = m_3$. Since P is open-loop stable and $K_0 = 0$, the conditions of Theorem 2 are satisfied. We can then apply Theorem 2(2) to find the matrix \tilde{U}_2 which defines the required control structure. Following the definitions in Theorem 2(2) we first find

$$T_{3,1} = \tilde{M}P_{21,1} = \frac{k_t s^2}{d(s)} \begin{bmatrix} c_s s + k_s \\ -m_s \end{bmatrix}$$

which has normal rank equal to one, i.e., $r_3 = 1$. Hence we can select

$$F = \frac{k_t s^2 (s + \alpha)}{d(s)}$$

$$U_1 = \frac{1}{(s + \alpha)} \begin{bmatrix} c_s s + k_s \\ -m_s \end{bmatrix}$$

for any $\alpha > 0$, to give a *mf* of $T_{3,1}$. We can choose U_2 to complete a unimodular matrix U as follows:

$$U = (U_1, U_2) = \begin{bmatrix} \frac{c_s s + k_s}{s + \alpha} & 0 \\ -m_s & 1 \end{bmatrix}$$

which gives $U^{-1} = [\tilde{U}'_1, \tilde{U}'_2]'$ and

$$\tilde{U}_2 = \begin{bmatrix} \frac{m_s}{c_s s + k_s} & 1 \end{bmatrix}. \quad (19)$$

Thus, all stabilizing controllers which leave $T_{\hat{z}_r \rightarrow [\hat{z}_s, \hat{z}_u]}'$ the same as in the open-loop can then be expressed as shown in Fig. 4 where

$$K_1 = -(I - Q_1 \tilde{U}_2 P_{22})^{-1} Q_1 \quad (20)$$

for $Q_1 \in \mathbb{RH}_\infty^{1 \times 1}$ or equivalently K_1 is any stabilizing controller for $\tilde{U}_2 P_{22}$ (see Theorem 2).

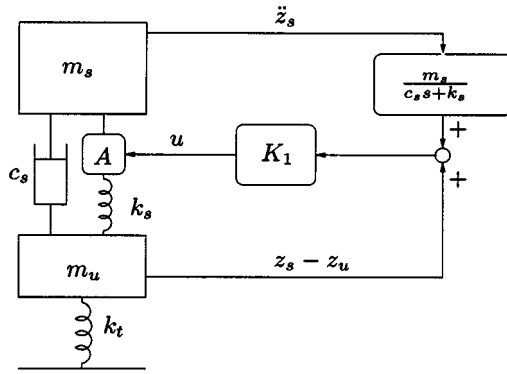


Fig. 4. Controller structure for the quarter-car with two feedbacks.

Example 1: We now apply the above result to a specific case in which we make the suspension stiff to load disturbances but soft to road disturbances. We select the following parameters for the quarter-car model as in [19] which correspond roughly to a small saloon car [3]: $m_s = 250$ kg, $m_u = 35$ kg, $k_t = 150$ kN/m. We choose $\omega_n = 100$ rad/s and $\delta = 0.7071$ as the parameters for the actuator dynamics. We also choose $k_s = 12$ kN/m, $c_s = 4$ kNs/m as the spring-damper coefficients which we consider to give a suitable “soft” response from the road disturbances in the passive implementation. It is now required to design the active controller to achieve desirable responses from the load disturbances.

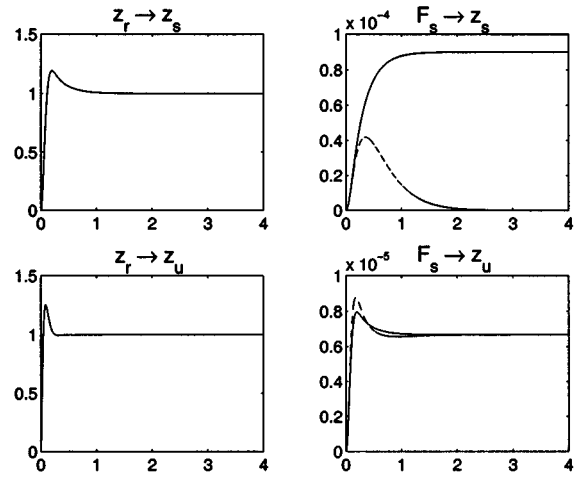
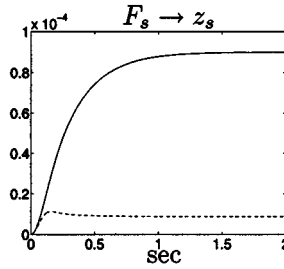
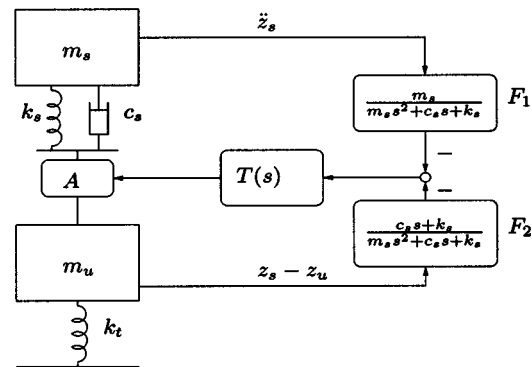
A simple approach is to take Q_1 to be constant in (20) and to minimize the steady-state response from load disturbances to sprung mass position. A straightforward calculation shows that $T_{F_s \rightarrow z_s}(0) = (k_s + k_t - Q_1 k_t)/(k_s k_t)$ which can be made to equal zero when $Q_1 = (k_s + k_t)/k_t = 1.08$. The step responses for the passive and active suspensions are shown in Fig. 5, which clearly illustrates the zero steady-state response to loads achieved by the active controller. The following closed-loop eigenvalues were obtained: -3.00 , -3.95 , $-21.18 \pm 13.09j$, -83.97 , $-70.71 \pm 70.71j$.

As a second approach we can employ the H_∞ loop shaping design procedure [11], [30] to the plant $\tilde{U}_2 P_{22}$. We select a weighting function $W_1 = 10(s + 80)/(s + 8)$, so that the open-loop loop shape, $W_1 \tilde{U}_2 P_{22}$, has a gain crossover frequency at about 44 rad/s, somewhat below the actuator cutoff frequency of 100 rad/s. The use of a lag compensator in W_1 allows the gain to be increased relatively at low frequencies in order to achieve a smaller value of $T_{F_s \rightarrow z_s}(0)$. This choice of weighting function gives a stability margin of 0.3864. The final controller takes the form $K = K_1 \tilde{U}_2$ (see Fig. 4) where

$$K_1 = \frac{-23.87(s + 25.37)(s + 72.55 \pm 75.70j)}{(s + 8.10)(s + 92.22 \pm 108.18j)}$$

which has been reduced to third order by balanced truncation. For this controller the step response from F_s to z_s is shown in Fig. 6, which exhibits an improved transient response in comparison to Fig. 5 but inferior steady-state behavior. The following closed-loop eigenvalues were obtained: -3.95 , $-21.18 \pm 13.09j$, $-35.52 \pm 25.75j$, -83.97 , $-56.52 \pm 64.31j$, $-76.44 \pm 93.06j$. \square

At this point it is instructive to compare the control structure shown in Fig. 4 with a scheme presented by Williams *et*


 Fig. 5. Passive suspension ($Q_1 = 0$, solid) and active suspension ($Q_1 = 1.08$, dashed).

 Fig. 6. Passive suspension (solid) and active suspension using H_∞ loop shaping design (dashed).

 Fig. 7. Controller structure of Williams *et al.*

al. in [25]. The stated aim of their controller is to provide a rapid closed-loop levelling system which does not respond to unwanted road disturbances, and this is achieved “by filtering and summing the sprung mass acceleration and suspension displacement signals to eliminate the effects of the road inputs.” A block diagram of the scheme in [25] is shown in Fig. 7 where $T(s)$ is a phase lead compensator. Although the damper is placed in series with the actuator this is not an essential difference. It may be observed that the ratio between the two filters F_1 and F_2 is equal to $m_s/(c_s s + k_s)$ so the scheme operates in a similar way to that of Fig. 4. In fact it can be shown that, if the denominators in F_1 and F_2 are both replaced by $(s + \alpha)$, and $T(s)$ is any stabilizing controller, then the scheme parameterizes all controllers which leave the

road disturbance responses the same as in the open loop. Thus the scheme lacks full generality only by virtue of the fact that the filters F_1 and F_2 have an extra order of roll-off at high frequency, which is a minor difference since it may be useful to provide some high-frequency roll-off in K_1 in practice.

We point out that our chosen scheme, as well as the approach of [25], assumes that the ride performance is satisfactory with $K_0 = 0$. If this is not the case, a controller $K_0 \neq 0$ may first be design to give any other desired road disturbance responses. Thereafter, Corollary 1 may be utilized to shape the load disturbance responses as well.

Finally, it is useful to comment on the full set of performance requirements that are usually considered in suspension design. In addition to the sprung mass position as a function of road disturbances, which can be analyzed with regard to driver comfort, there are also issues such as tire normal loads (i.e., tire deflection) and rattle space (i.e., strut deflection). It was shown in [7] that if the transfer function from z_r to z_s is determined, then there is no additional freedom left in the road disturbance transmission path, i.e., the transfer functions $T_{\tilde{z}_r \rightarrow \tilde{z}_r - \tilde{z}_u}$ and $T_{\tilde{z}_r \rightarrow \tilde{z}_s - \tilde{z}_u}$ can be deduced directly. A similar fact was shown in [18] for the load disturbance transmission path. Thus, in the above approach to active suspension design, it is assumed that for each disturbance transmission path that is being dealt with, all the relevant factors (e.g., comfort, tire loads, suspension deflection) are taken account of together.

B. The Quarter-Car With Three Measurements

We continue to illustrate our basic theory by considering the quarter-car model with the additional measurement \tilde{z}_u . We now write the system in the form of Fig. 1 with $y = [\tilde{z}_s, z_s - z_u, \tilde{z}_u]^T$ and all other variables the same as in Section III-A. The general plant of (1) then has P_{11}, P_{12} the same as (16), (17) and

$$P_{21} = \frac{1}{d(s)} \begin{bmatrix} k_t(c_s s + k_s)s^2 & p_1(s)s^2 \\ -k_t(m_s s^2) & (m_u s^2 + k_t) \\ k_t(m_s s^2 + c_s s + k_s)s^2 & (c_s s + k_s)s^2 \end{bmatrix}$$

$$P_{22} = \frac{\gamma(s)}{d(s)} \begin{bmatrix} k_s(m_u s^2 + k_t)s^2 \\ k_s((m_u + m_s)s^2 + k_t) \\ -k_s m_s s^4 \end{bmatrix}$$

where $d(s), p_1(s)$ are given in and before (18).

As before we wish to parameterize all controllers which leave the transmission path from the road disturbance to z_s and z_u the same as in the open-loop, i.e., with $u = 0$, which assumes that k_s and c_s are chosen to give satisfactory responses for this transmission path. We can check that the conditions of Theorem 2 again hold, so that we can follow the procedure to obtain the required controller structure.

Following the definitions in Theorem 2(2) we find that

$$T_{3,1} = \tilde{M}P_{21,1} = \frac{k_t s^2}{d(s)} \begin{bmatrix} c_s s + k_s \\ -m_s \\ m_s s^2 + c_s s + k_s \end{bmatrix}$$

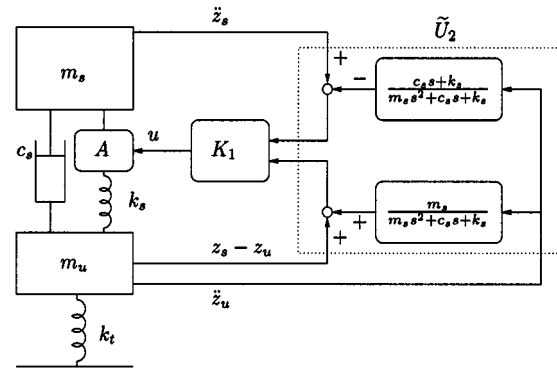


Fig. 8. Controller structure for the quarter-car with three feedbacks.

which gives $r_3 = 1$. Hence we can select

$$F = \frac{k_t s^2 p_2(s)}{d(s)},$$

$$U_1 = \frac{1}{p_2(s)} \begin{bmatrix} c_s s + k_s \\ -m_s \\ m_s s^2 + c_s s + k_s \end{bmatrix}$$

where $p_2(s)$ is any second-order Hurwitz polynomial, to give a mf of $T_{3,1}$. We can choose $U_2 = [I_{2 \times 2}, 0_{2 \times 1}]^T$ to complete a unimodular matrix U to give $U^{-1} = [\tilde{U}'_1, \tilde{U}'_2]^T$ with

$$\tilde{U}'_2 = \begin{bmatrix} 1 & 0 & -\frac{c_s s + k_s}{m_s s^2 + c_s s + k_s} \\ 0 & 1 & \frac{m_s}{m_s s^2 + c_s s + k_s} \end{bmatrix}. \quad (21)$$

Thus, all stabilizing controllers which leave $T_{\tilde{z}_r \rightarrow [\tilde{z}_s, \tilde{z}_u]}$ the same as in the open-loop can then be expressed as shown in Fig. 8 where

$$K_1 = -(1 - Q_1 \tilde{U}'_2 P_{22})^{-1} Q_1 \quad (22)$$

for any $Q_1 \in \mathbb{RH}_\infty^{1 \times 2}$ or equivalently K_1 is any stabilizing controller for $\tilde{U}'_2 P_{22}$ (see Theorem 2).

1) *Alternative Controller Structures:* As pointed out in Lemma 2 the exact form of \tilde{U}'_2 in (21) is not unique. Let us suppose that we prefer a structure with

$$\tilde{U}'_2 = \begin{bmatrix} f_1 & 1 & 0 \\ 0 & 1 & f_2 \end{bmatrix}$$

where $f_1, f_2 \in \mathbb{RH}_\infty$ are strictly proper, motivated by a preference to use low-pass filters for the acceleration signals \tilde{z}_s and \tilde{z}_u while keeping the strut deflection unfiltered. For \tilde{U}'_2 given by (21) the identity $\tilde{U}'_2 = W^\dagger \tilde{U}'_2$ implies that

$$W^\dagger = \begin{bmatrix} \frac{m_s}{c_s s + k_s} & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$\tilde{U}'_2 = \begin{bmatrix} \frac{m_s}{c_s s + k_s} & 1 & 0 \\ 0 & 1 & \frac{m_s}{m_s s^2 + c_s s + k_s} \end{bmatrix}.$$

We can see that W^\dagger fails to be unimodular as required by Theorem 2(3). Moreover, it is straightforward to show that the set $\{Q_1^\dagger \tilde{U}'_2 : Q_1^\dagger \in \mathbb{RH}_\infty^{1 \times 2}\}$ is equal to the set $\{Q_1 \tilde{U}'_2 : Q_1 \in \mathbb{RH}_\infty^{1 \times 2}, Q_{1,1}(\infty) = 0\}$, i.e., the full parameterization but with the first element in Q_1 being strictly proper.

TABLE I
THE \tilde{U}_2 STRUCTURES AND THE TRANSFORMED PLANTS OF THE
QUARTER-CAR MODEL WITH VARIOUS MEASUREMENTS

measurements	\tilde{U}_2	$\tilde{U}_2 P_{22}$
$\begin{bmatrix} \ddot{z}_s \\ z_s - z_u \end{bmatrix}$	$\begin{bmatrix} \frac{m_s}{c_s s + k_s} & 1 \end{bmatrix}$	$\frac{k_s \gamma(s)}{c_s s + k_s}$
$\begin{bmatrix} \ddot{z}_s \\ z_s - z_u \\ \ddot{z}_u \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -\frac{c_s s + k_s}{m_s s^2 + c_s s + k_s} \\ 0 & 1 & \frac{m_s}{m_s s^2 + c_s s + k_s} \end{bmatrix}$	$\frac{k_s \gamma(s)}{m_s s^2 + c_s s + k_s} \begin{bmatrix} s^2 \\ 1 \end{bmatrix}$
$\begin{bmatrix} \ddot{z}_s \\ z_s - z_a \\ z_a - z_u \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ \frac{m_s}{c_s s + k_s} & 0 & 1 \end{bmatrix}$	$\gamma(s) \begin{bmatrix} 1 \\ -\frac{c_s s}{c_s s + k_s} \end{bmatrix}$

Thus a controller parameterization with \tilde{U}_2 replaced by \tilde{U}_2^\dagger does not give all possible stabilizing controllers which leave the road disturbance responses the same as in the open-loop (see Lemma 2 and Theorem 2). However, the restriction of the Q -parameter amounts only to an increased roll-off requirement of the controller at high frequency.

Referring to Fig. 8, let us consider another possible controller structure

$$\tilde{U}_2^\dagger = \begin{bmatrix} f_1 & 1 & 0 \\ 1 & 0 & f_2 \end{bmatrix}$$

where $f_1, f_2 \in \mathbb{RH}_\infty$ are strictly proper. The identity $\tilde{U}_2^\dagger = W^\dagger \tilde{U}_2$ now implies that

$$W^\dagger = \begin{bmatrix} \frac{m_s}{c_s s + k_s} & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\tilde{U}_2^\dagger = \begin{bmatrix} \frac{m_s}{c_s s + k_s} & 1 & 0 \\ 1 & 0 & \frac{-(c_s s + k_s)}{m_s s^2 + c_s s + k_s} \end{bmatrix}.$$

Since W^\dagger is unimodular, then \tilde{U}_2 can be replaced by \tilde{U}_2^\dagger in Fig. 8 to give a parameterization of all stabilizing controllers [Theorem 2(3)].

C. Quarter-Car Control Structures and Design

For direct controller design using Theorem 2, it is instructive to compute $\tilde{U}_2 P_{22}$ for the given choice of measurements. Table I shows $\tilde{U}_2 P_{22}$ for three different cases. It is interesting that $\tilde{U}_2 P_{22}$ takes a particularly simple form, which is independent of the sprung and unsprung masses, in the case when the feedback signals \ddot{z}_s and $z_s - z_u$ are used, which means that the controller design would be rather simple in this case.

IV. THE HALF-CAR MODEL

In this section, we shall apply the controller parameterization method to the half-car model shown in Fig. 9. As in the quarter-car model, the actuators A_1 and A_2 are modeled so that the relative displacement across each is equal to a low-pass filtered version of the actuator's command signal, i.e.,

$$\dot{z}_s + l_1 \dot{z}_\psi - \dot{z}_{a1} = \gamma(s) \hat{u}_1 \quad (23)$$

$$\dot{z}_s - l_2 \dot{z}_\psi - \dot{z}_{a2} = \gamma(s) \hat{u}_2 \quad (24)$$

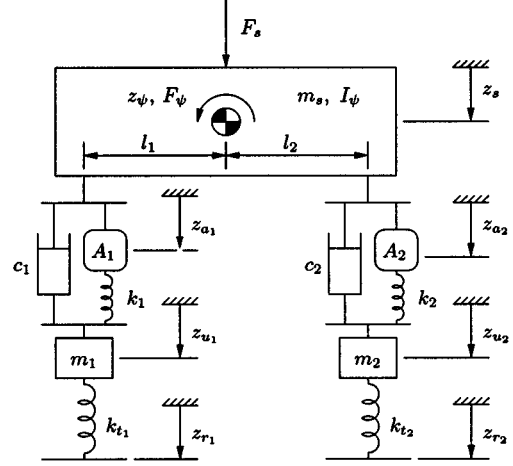


Fig. 9. The half-car model.

where $\gamma(s)$ is defined as in (11). The linearized dynamic equations can be expressed as follows:

$$m_s \ddot{z}_s = F_s - u_{p1} - u_{p2} \quad (25)$$

$$I_\psi \ddot{z}_\psi = F_\psi - u_{p1} l_1 + u_{p2} l_2 \quad (26)$$

$$m_1 \ddot{z}_{u1} = u_{p1} + F_{r1} \quad (27)$$

$$m_2 \ddot{z}_{u2} = u_{p2} + F_{r2} \quad (28)$$

where the passive suspension forces u_{p1}, u_{p2} , and the tire forces F_{r1}, F_{r2} are given by

$$u_{p1} = c_1 (\dot{z}_s + l_1 \dot{z}_\psi - \dot{z}_{u1}) + k_1 (z_{a1} - z_{u1})$$

$$u_{p2} = c_2 (\dot{z}_s - l_2 \dot{z}_\psi - \dot{z}_{u2}) + k_2 (z_{a2} - z_{u2})$$

$$F_{r1} = k_{t1} (z_{r1} - z_{u1})$$

$$F_{r2} = k_{t2} (z_{r2} - z_{u2}).$$

We now write the system in the form of Fig. 1 with $z_1 = [z_s, z_\psi, z_{u1}, z_{u2}]'$, $w_1 = [z_{r1}, z_{r2}]'$, $w_2 = [F_s, F_\psi]'$, $y = [\ddot{z}_s, \ddot{z}_\psi, D_1, D_2]'$ where $D_1 = z_s + l_1 z_\psi - z_{u1}$, $D_2 = z_s - l_2 z_\psi - z_{u2}$ are strut deflections, $u = [u_1, u_2]'$ as in (23), (24) and z_2 omitted. As before we wish to parameterize all controllers which leave the transmission path from the road disturbances $[z_{r1}, z_{r2}]'$ to $[z_s, z_\psi, z_{u1}, z_{u2}]'$ the same as in the open-loop, i.e., with $u_1 = u_2 = 0$, which assumes that k_1, k_2, c_1 and c_2 are chosen to give satisfactory responses for these transmission paths. We can check that the conditions of Theorem 2 again hold, so that we can follow the procedure to obtain the required controller structure.

Following the definitions in Theorem 2(2) we can choose a $U_1 = T_{3,1} \times p_3(s)/s^2 = P_{21,1} \times p_3(s)/s^2 \in \mathbb{RH}_\infty^{4 \times 2}$ and $F = s^2/p_3(s) \times I_{2 \times 2} \in \mathbb{RH}_\infty^{2 \times 2}$, where $p_3(s)$ is any third-order Hurwitz polynomial, to give a nmf of $T_{3,1}$, and complete this U_1 to a unimodular matrix with $U_2 = [0_{2 \times 2}, I_{2 \times 2}]'$ to give

$$\tilde{U}_2 = \begin{bmatrix} \frac{m_s l_2}{(c_1 s + k_1)(l_1 + l_2)} & \frac{I_\psi}{(c_1 s + k_1)(l_1 + l_2)} & 1 & 0 \\ \frac{m_s l_1}{(c_2 s + k_2)(l_1 + l_2)} & \frac{-I_\psi}{(c_2 s + k_2)(l_1 + l_2)} & 0 & 1 \end{bmatrix}. \quad (29)$$

We observe that \tilde{U}_2 “constructs” two combinations of measurements, each of which is a suspension deflection plus a low-pass filtered version of a sum of the two acceleration signals. A block diagram of this control structure is shown in Fig. 10.

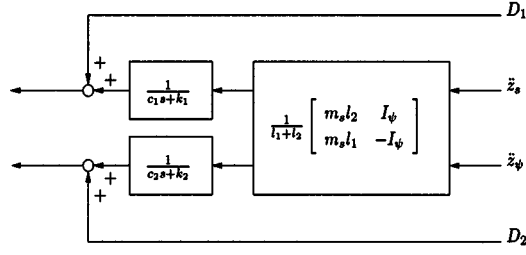


Fig. 10. Controller structure (\tilde{U}_2) for the half-car model.

For direct controller design using the transformed plant P^\dagger in Fig. 2(b), it is interesting to note that $\tilde{U}_2 P_{22}$ takes a particularly simple form in the half-car case when \tilde{U}_2 is determined by (29), namely

$$\tilde{U}_2 P_{22} = \gamma(s) \begin{bmatrix} \frac{k_1}{c_1 s + k_1} & 0 \\ 0 & \frac{k_2}{c_2 s + k_2} \end{bmatrix}. \quad (30)$$

This fact will be exploited in the design example for the full-car model in Section VI.

A. Decoupling by Simplicity

Under certain conditions, the half-car model can be structurally decoupled into two quarter-cars, and in such cases it is useful to exploit the simplified structure. In [19] and [22], assumptions such as *kl-simplicity* were used, in a mechanical network setting, to perform energy-preserving transformations of the external disturbance variables to achieve decoupling. In our setting we will need to use a similar transformation on all of the system variables (but will not necessarily be able to respect the energy-preserving property). For the half-car model shown in Fig. 9, we define it as *simple* if the following equation holds:

$$\frac{m_1}{m_2} = \frac{k_1}{k_2} = \frac{c_1}{c_2} = \frac{k_{t1}}{k_{t2}} = \frac{l_2}{l_1}. \quad (31)$$

Note that half-car roll models are typically symmetric (i.e., $m_1 = m_2$, etc.) so that (31) holds automatically in this case. Condition (31) may sometimes be satisfied also for half-car pitch models. We introduce a transformation matrix L_h as follows:

$$L_h = \frac{1}{l_1 + l_2} \begin{bmatrix} l_2 & l_1 \\ 1 & -1 \end{bmatrix} \quad (32)$$

and define

$$\begin{bmatrix} (x)_b \\ (x)_\rho \end{bmatrix} = L_h \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where x may represent any of the following variables: z_u , z_r , z_a , or u , and the subscripts b and ρ represent the bounce and rotation modes, respectively.

Under the assumption of simplicity, we can then rewrite (23)–(28) as follows:

$$\hat{z}_s - (\hat{z}_a)_b = \gamma(s)(\hat{u})_b \quad (33)$$

$$\hat{z}_\psi - (\hat{z}_a)_\rho = \gamma(s)(\hat{u})_\rho \quad (34)$$

$$m_s \ddot{z}_s = F_s - (c_1 + c_2)(\dot{z}_s - (\dot{z}_u)_b) - (k_1 + k_2)((z_a)_b - (z_u)_b) \quad (35)$$

$$I_\psi \ddot{z}_\psi = F_\psi - (c_1 l_1^2 + c_2 l_2^2)(\dot{z}_\psi - (\dot{z}_u)_\rho) + (k_1 l_1^2 + k_2 l_2^2)((z_a)_\rho - (z_u)_\rho)$$

TABLE II
DECOUPLED HALF-CAR BY SIMPLICITY

Quarter-car	Half-car (bounce)	Half-car (rotation)
m_s	m_s	I_ψ
m_u	$m_1 + m_2$	$m_1 l_1^2 + m_2 l_2^2$
k_s	$k_1 + k_2$	$k_1 l_1^2 + k_2 l_2^2$
c_s	$c_1 + c_2$	$c_1 l_1^2 + c_2 l_2^2$
k_t	$k_{t1} + k_{t2}$	$k_{t1} l_1^2 + k_{t2} l_2^2$
F_s	F_s	F_ψ
z_s	z_s	z_ψ
z_u, z_a, z_r	$(z_u)_b, (z_a)_b, (z_r)_b$	$(z_u)_\rho, (z_a)_\rho, (z_r)_\rho$
u	$(u)_b$	$(u)_\rho$

(36)

$$(m_1 + m_2)(\ddot{z}_u)_b = (c_1 + c_2)(\dot{z}_s - (\dot{z}_u)_b) - (k_1 + k_2)((z_a)_b - (z_u)_b) + (k_{t1} + k_{t2})((z_r)_b - (z_u)_b) \quad (37)$$

$$(m_1 l_1^2 + m_2 l_2^2)(\ddot{z}_u)_\rho = (c_1 l_1^2 + c_2 l_2^2)(\dot{z}_\psi - (\dot{z}_u)_\rho) + (k_1 l_1^2 + k_2 l_2^2)((z_a)_\rho - (z_u)_\rho) + (k_{t1} l_1^2 + k_{t2} l_2^2)((z_r)_\rho - (z_u)_\rho). \quad (38)$$

Comparing with (10), (12) and (13), we find that (33), (35) and (37) represent a *bounce* quarter-car, and (34), (36) and (38) represent a *rotation* quarter-car, which are decoupled from each other. The relevant correspondences between variables is summarized in Table II. Furthermore, we can show that, in order to arrive at a decoupled form for (35) and (36) (e.g., z_ψ being absent from (35) etc), then we need both (31) to hold and for L_h to be defined by (32) up to scalar multiplication of each row.

V. THE FULL-CAR MODEL

In this section we shall introduce a standard full-car model with a similar suspension strut arrangement at each wheel-station to the quarter- and half-car cases in Figs. 3 and 9. We assume a *left-right symmetry* which allows a decoupling of the full-car model into two half-cars, namely the *bounce/pitch* and *roll/warp* half-cars. In preparation for a controller design in Section VI we will highlight the special form of the warp quarter-car, which has no “sprung mass dynamics.”

A. The Dynamic Equations

Referring to Fig. 11, the actuators A_1, \dots, A_4 are again modeled so that the relative displacement of each is equal to a low-pass filtered version of the actuator’s command signals, i.e.,

$$\hat{z}_s - l_f \hat{z}_\theta + t_f \hat{z}_\phi - \hat{z}_{a1} = \gamma(s) \hat{u}_1$$

$$\hat{z}_s - l_f \hat{z}_\theta - t_f \hat{z}_\phi - \hat{z}_{a2} = \gamma(s) \hat{u}_2$$

$$\hat{z}_s + l_r \hat{z}_\theta + t_r \hat{z}_\phi - \hat{z}_{a3} = \gamma(s) \hat{u}_3$$

$$\hat{z}_s + l_r \hat{z}_\theta - t_r \hat{z}_\phi - \hat{z}_{a4} = \gamma(s) \hat{u}_4$$

where $\gamma(s)$ is defined as in (11). The linearized dynamics of the full-car can be expressed as

$$m_s \ddot{z}_s = F_s - u_{p1} - u_{p2} - u_{p3} - u_{p4} \quad (39)$$

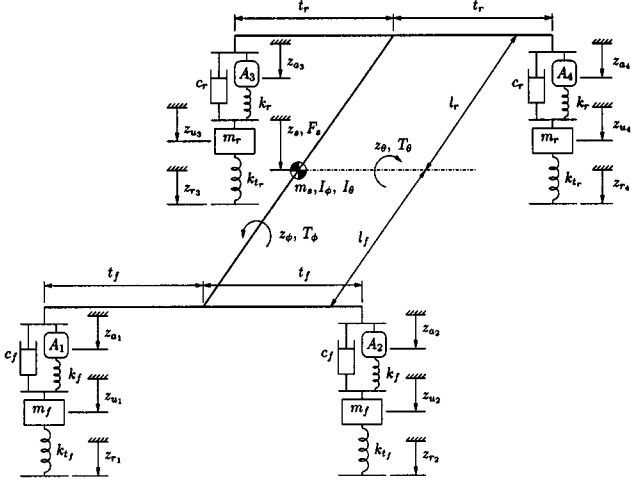


Fig. 11. The full-car model.

$$I_\theta \ddot{\theta} = T_\theta + u_{p1} l_f + u_{p2} l_f - u_{p3} l_r - u_{p4} l_r \quad (40)$$

$$I_\phi \ddot{\phi} = T_\phi - u_{p1} t_f + u_{p2} t_f - u_{p3} t_r + u_{p4} t_r \quad (41)$$

$$m_f \ddot{z}_{u1} = u_{p1} + F_{r1} \quad (42)$$

$$m_f \ddot{z}_{u2} = u_{p2} + F_{r2} \quad (43)$$

$$m_r \ddot{z}_{u3} = u_{p3} + F_{r3} \quad (44)$$

$$m_r \ddot{z}_{u4} = u_{p4} + F_{r4} \quad (45)$$

where the passive suspension forces u_{p1}, \dots, u_{p4} and the tire forces F_{r1}, \dots, F_{r4} are given by

$$\begin{aligned} u_{p_i} &= c_f \dot{D}_i + k_f (z_{a_i} - z_{u_i}) \\ u_{p_j} &= c_r \dot{D}_j + k_r (z_{a_j} - z_{u_j}) \\ F_{r_i} &= k_{t_f} (z_{r_i} - z_{u_i}) \\ F_{r_j} &= k_{t_r} (z_{r_j} - z_{u_j}) \end{aligned} \quad (46)$$

$$F_{r_j} = k_{t_r} (z_{r_j} - z_{u_j}) \quad (47)$$

for $i = 1, 2, j = 3, 4$, and the strut deflections D_1, \dots, D_4 are

$$D_1 = z_s - l_f z_\theta + t_f z_\phi - z_{u1}$$

$$D_2 = z_s - l_f z_\theta - t_f z_\phi - z_{u2}$$

$$D_3 = z_s + l_r z_\theta + t_r z_\phi - z_{u3}$$

$$D_4 = z_s + l_r z_\theta - t_r z_\phi - z_{u4}.$$

B. Symmetric Transformation

Since the full-car model is symmetric, we can decouple it into two half-car models. First, we introduce a transformation matrix $L_{f,1}$:

$$L_{f,1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (48)$$

such that

$$\begin{bmatrix} (x)_{b_f} & (x)_{b_r} & (x)_{\rho_f} & (x)_{\rho_r} \end{bmatrix}' = L_{f,1} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}' \quad (49)$$

where x may represent any of the following variables: z_u, z_r, z_a , strut deflection D or actuator command signal u , while the

TABLE III
DECOUPLED (SYMMETRIC) FULL-CAR

Half-car	Full-car (bounce/pitch)	Full-car (roll/warp)
m_s	m_s	—
I_ψ	I_θ	I_ϕ
m_1	$2m_f$	$2m_f$
m_2	$2m_r$	$2m_r$
k_1	$2k_f$	$2k_f$
k_2	$2k_r$	$2k_r$
c_1	$2c_f$	$2c_f$
c_2	$2c_r$	$2c_r$
k_{t1}	$2k_{t_f}$	$2k_{t_f}$
k_{t2}	$2k_{t_r}$	$2k_{t_r}$
l_1	l_f	t_f
l_2	l_r	t_r
F_s	F_s	—
F_ψ	$-T_\theta$	T_ϕ
z_s	z_s	—
z_ψ	$-z_\theta$	z_ϕ
z_{u1}, z_{u2}	$(z_u)_{b_f}, (z_u)_{b_r}$	$(z_u)_{\rho_f}, -(z_u)_{\rho_r}$
z_{r1}, z_{r2}	$(z_r)_{b_f}, (z_r)_{b_r}$	$(z_r)_{\rho_f}, -(z_r)_{\rho_r}$
z_{a1}, z_{a2}	$(z_a)_{b_f}, (z_a)_{b_r}$	$(z_a)_{\rho_f}, -(z_a)_{\rho_r}$
u_1, u_2	$(u)_{b_f}, (u)_{b_r}$	$(u)_{\rho_f}, -(u)_{\rho_r}$

subscripts b_f, b_r represent the front and rear bounce components ρ_f, ρ_r represent the front and rear roll components.

1) *Bounce/Pitch Half-Car*: After applying the $L_{f,1}$ transformation, (39), (40), and (42)–(45) can be rearranged as follows:

$$\begin{aligned} m_s \ddot{z}_s - F_s &= -2c_f (\dot{z}_s - l_f \dot{z}_\theta - (\dot{z}_u)_{b_f}) \\ &\quad - 2k_f ((z_a)_{b_f} - (z_u)_{b_f}) \\ &\quad - 2c_r (\dot{z}_s + l_r \dot{z}_\theta - (\dot{z}_u)_{b_r}) \\ &\quad - 2k_r ((z_a)_{b_r} - (z_u)_{b_r}), \\ I_\theta \ddot{\theta} - T_\theta &= \{2c_f (\dot{z}_s - l_f \dot{z}_\theta - (\dot{z}_u)_{b_f}) \\ &\quad + 2k_f ((z_a)_{b_f} - (z_u)_{b_f})\} l_f \\ &\quad - \{2c_r (\dot{z}_s + l_r \dot{z}_\theta - (\dot{z}_u)_{b_r}) \\ &\quad + 2k_r ((z_a)_{b_r} - (z_u)_{b_r})\} l_r \\ 2m_f (\ddot{z}_u)_{b_f} &= 2c_f (\dot{z}_s - l_f \dot{z}_\theta - (\dot{z}_u)_{b_f}) \\ &\quad + 2k_f ((z_a)_{b_f} - (z_u)_{b_f}) \\ &\quad + 2k_{t_f} ((z_r)_{b_f} - (z_u)_{b_f}) \\ 2m_r (\ddot{z}_u)_{b_r} &= 2c_r (\dot{z}_s + l_r \dot{z}_\theta - (\dot{z}_u)_{b_r}) \\ &\quad + 2k_r ((z_a)_{b_r} - (z_u)_{b_r}) \\ &\quad + 2k_{t_r} ((z_r)_{b_r} - (z_u)_{b_r}). \end{aligned}$$

It can be observed that the above equations take the same form as (25)–(28) under the transformations listed in the first column of Table III. To decouple into two quarter-cars requires a simplicity assumption

$$\frac{k_f}{k_r} = \frac{c_f}{c_r} = \frac{k_{t_f}}{k_{t_r}} = \frac{m_f}{m_r} = \frac{l_r}{l_f}. \quad (50)$$

TABLE IV
DECOUPLED ROLL AND WARP MODES OF THE FULL-CAR BY *SIMPLICITY*

Quarter-car	Full-car (Warp)	Full-car (Roll)
m_s	—	I_ϕ
m_u	$2(m_f + m_r)$	$2(m_f t_f^2 + m_r t_r^2)$
k_s	$2(k_f + k_r)$	$2(k_f t_f^2 + k_r t_r^2)$
c_s	$2(c_f + c_r)$	$2(c_f t_f^2 + c_r t_r^2)$
k_t	$2(k_{t_f} + k_{t_r})$	$2(k_{t_f} t_f^2 + k_{t_r} t_r^2)$
F_s	—	T_ϕ
z_s	—	z_ϕ
z_u, z_a	$(z_u)_{warp}, (z_a)_{warp}$	$(z_u)_{roll}, (z_a)_{roll}$
z_r, u	$(z_r)_{warp}, (u)_{warp}$	$(z_r)_{roll}, (u)_{roll}$

2) *Roll/Warp Half-Car*: After applying the $L_{f,1}$ transformation, (41)–(45) can be rearranged as follows:

$$\begin{aligned}
 I_\phi \ddot{z}_\phi - T_\phi &= - \{ 2c_f(t_f \dot{z}_\phi - (\dot{z}_u)_{\rho_f}) \\
 &\quad + 2k_f((z_a)_{\rho_f} - (z_u)_{\rho_f}) \} t_f \\
 &\quad - \{ 2c_r(t_r \dot{z}_\phi - (\dot{z}_u)_{\rho_r}) \\
 &\quad + 2k_r((z_a)_{\rho_r} - (z_u)_{\rho_r}) \} t_r \\
 2m_f(\ddot{z}_u)_{\rho_f} &= 2c_f(t_f \dot{z}_\phi - (\dot{z}_u)_{\rho_f}) + 2k_f((z_a)_{\rho_f} - (z_u)_{\rho_f}) \\
 &\quad + 2k_{t_f}((z_r)_{\rho_f} - (z_u)_{\rho_f}), \\
 2m_r(\ddot{z}_u)_{\rho_r} &= 2c_r(t_r \dot{z}_\phi - (\dot{z}_u)_{\rho_r}) + 2k_r((z_a)_{\rho_r} - (z_u)_{\rho_r}) \\
 &\quad + 2k_{t_r}((z_r)_{\rho_r} - (z_u)_{\rho_r}).
 \end{aligned}$$

We observe that one equation is missing in this half-car compared to (25)–(28). This is because the chassis is modeled as being infinitely stiff under torsion, so that there is no dynamic equation corresponding to warp dynamics of the car body. However the above three equations do take a similar form to (26)–(28) under the transformations listed in the second column of Table III.

As in the half-car case, under the assumption of simplicity

$$\frac{k_f}{k_r} = \frac{c_f}{c_r} = \frac{k_{t_f}}{k_{t_r}} = \frac{m_f}{m_r} = \frac{t_r}{t_f} \quad (51)$$

and the $L_{f,2}$ transformation

$$L_{f,2} = \frac{1}{t_f + t_r} \begin{bmatrix} t_r & -t_f \\ 1 & 1 \end{bmatrix} \quad (52)$$

such that

$$\begin{bmatrix} (x)_{warp} \\ (x)_{roll} \end{bmatrix} = L_{f,2} \begin{bmatrix} (x)_{\rho_f} \\ (x)_{\rho_r} \end{bmatrix} \quad (53)$$

where x can be z_u, z_r or z_a , strut deflection D or actuator command signal u , the roll/warp half-car can be further decoupled into roll and warp quarter-cars under the mapping illustrated in Table IV.

VI. A DESIGN EXAMPLE FOR THE FULL-CAR MODEL

In this section, we shall synthesize an active controller for a specific full-car model. As in Section V, the model is chosen to be *left-right symmetric* which allows a decoupling into the *bounce/pitch* and *roll/warp* half-cars. Our design approach for

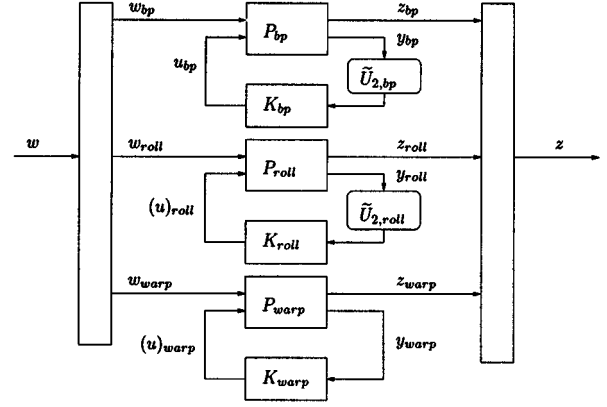


Fig. 12. Control scheme for the full-car model.

the bounce/pitch half-car will make use of the theory outlined in Section IV. The approach for the roll/warp half-car will make use of a *simplicity* assumption which allows it to be decoupled into the two corresponding quarter-cars, namely the *roll* and *warp* quarter-cars. The roll quarter-car will be treated in the same way as the quarter-car of Section III-A. As pointed out in Section V-B2, the warp quarter-car has a different form than the standard quarter-car in that the “sprung mass” is effectively infinite. Furthermore, in warp motion there is good reason to use the active controller to make the road disturbance responses even softer than they would be with the default passive parameter settings. Thus the warp mode will be handled in a different way to the other three modes.

For the controller design the available measurements are assumed to be $\ddot{z}_s, \ddot{z}_\theta, \ddot{z}_\phi, D_1, D_2, D_3$, and D_4 . The control structure will be chosen to have three independent loops, consisting of the roll quarter-car, the warp quarter-car, and bounce/pitch half-car controllers. This scheme is shown in Fig. 12, where the signals are defined as follows:

$$\begin{aligned}
 w &= [F_s, T_\theta, T_\phi, z_{r1}, z_{r2}, z_{r3}, z_{r4}]' \quad (54) \\
 w_{bp} &= [F_s, T_\theta, (z_r)_{b_f}, (z_r)_{b_r}]' \\
 w_{roll} &= [T_\phi, (z_r)_{roll}]' \\
 w_{warp} &= [(z_r)_{warp}]' \\
 z &= [z_s, z_\theta, z_\phi, z_{u1}, z_{u2}, z_{u3}, z_{u4}]' \quad (55) \\
 z_{bp} &= [z_s, z_\theta, (z_u)_{b_f}, (z_u)_{b_r}]' \\
 z_{roll} &= [z_\phi, (z_u)_{roll}]' \\
 z_{warp} &= [(z_u)_{warp}]' \\
 y_{bp} &= [\ddot{z}_s, \ddot{z}_\theta, (D)_{b_f}, (D)_{b_r}]' \\
 y_{roll} &= [\ddot{z}_\phi, (D)_{roll}]' \\
 y_{warp} &= [(D)_{warp}]' \\
 u_{bp} &= [(u)_{b_f}, (u)_{b_r}]'
 \end{aligned}$$

and the subscripts $b_1, b_2, roll$, and $warp$ are defined as in (49) and (53).

The following parameters will be used for the full-car model which are similar to typical parameters for a sports car [3]: $m_s = 1600$ kg, $I_\theta = 1000$ kg m², $I_\phi = 450$ kg m², $t_f = t_r = 0.75$ m, $l_f = 1.15$ m, $l_r = 1.35$ m, $k_f = k_r = 20$ kN/m, $c_f = c_r = 5$ kNs/m, $m_f = m_r = 50$ kg, $k_{t_f} = k_{t_r} = 250$ kN/m. As in

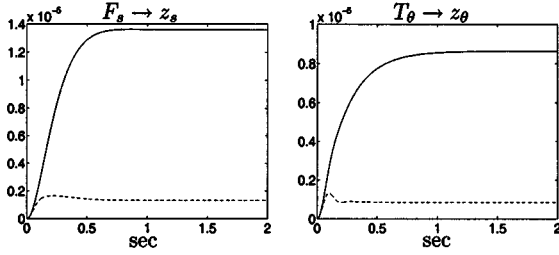


Fig. 13. Step responses of $T_{F_s \rightarrow z_s}$ and $T_{T_\theta \rightarrow z_\theta}$: passive (solid) and active control (dashed).

the previous examples, the actuator dynamics is represented as in (11) with $\omega_n = 100$ rad/s and $\delta = 0.707$.

A. Bounce/Pitch Control

Referring to Table III, the bounce/pitch half-car corresponds to the half-car of Section IV with the following coefficients: $m_s = 1600$ kg, $I_\psi = 1000$ kg m², $m_1 = m_2 = 100$ kg, $k_1 = k_2 = 40$ kN/m, $c_1 = c_2 = 10$ kNs/m, $k_{t_1} = k_{t_2} = 500$ kN/m, $l_1 = 1.15$ m, $l_2 = 1.35$ m, which is not *simple* and cannot be decoupled into two quarter-cars. The H_∞ loop shaping controller design will be applied to this half-car model. The essential controller structure $U_{2,bp}$ is given by (29) as

$$\tilde{U}_{2,bp} = \begin{bmatrix} \frac{m_s l_r}{2(c_f s + k_f)(l_r + l_f)} & \frac{I_\theta}{2(c_f s + k_f)(l_r + l_f)} & 1 & 0 \\ \frac{m_s l_f}{2(c_r s + k_r)(l_r + l_f)} & \frac{-I_\theta}{2(c_r s + k_r)(l_r + l_f)} & 0 & 1 \end{bmatrix}. \quad (56)$$

Setting a weighting function W_{bp} as follows:

$$W_{bp} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \frac{s + 80}{s + 8} \quad (57)$$

means that the weighted plant $W_{bp} \tilde{U}_{2,bp} P_{22}$ has a bandwidth of about 60 rad/s and has an increased low-frequency gain due to the lag compensator terms. By applying the H_∞ loop shaping controller design procedure a sixth-order controller was obtained after balanced truncation as follows:

$$K_{bp} = \begin{bmatrix} K_c & 0 \\ 0 & K_c \end{bmatrix} \quad (58)$$

where

$$K_c = \frac{-24.81(s + 27.94)(s + 72.56 \pm 73.47j)}{(s + 8.06)(s + 101.45 \pm 114.65j)}. \quad (59)$$

It is interesting to note that K_{bp} in (58) is a scalar matrix due to the fact that $W_{bp} \tilde{U}_{2,bp} P_{22}$ is itself scalar (see (30)). This controller gives the dc gains $T_{F_s \rightarrow z_s}(0)$ and $T_{T_\theta \rightarrow z_\theta}(0)$ as 1.32×10^{-6} and 8.39×10^{-7} , respectively, compared with 1.36×10^{-5} and 8.64×10^{-6} using passive control. The step responses using the two controllers are shown in Fig. 13.

B. Roll Control

Referring to Table IV, the roll mode of the full-car corresponds to a quarter-car with the following coefficients: $m_s = 450$ kg, $m_u = 112.5$ kg, $k_s = 45$ kN/m, $c_s = 11.25$ kNs/m, $k_t = 562.5$ kN/m. The required structure of \tilde{U}_2 takes the following form after using Table IV, (19), and (51)

$$\tilde{U}_{2,roll} = \begin{bmatrix} \frac{I_\phi}{2(c_f s + k_f)t_f(l_r + t_f)} & 1 \end{bmatrix}. \quad (60)$$

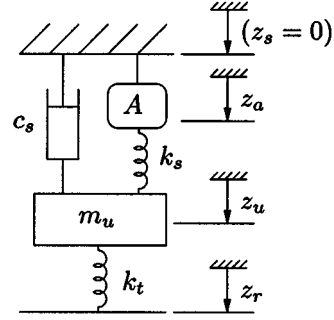


Fig. 14. The warp quarter-car.

Given the weighting function

$$W_{roll} = \frac{10(s + 80)}{(s + 8)} \quad (61)$$

such that the weighted plant $W_{roll} \tilde{U}_{2,roll} P_{22}$ has a bandwidth of about 60 rad/s, it is found that the H_∞ loop shaping controller after model reduction is given by

$$K_{roll} = K_c \quad (62)$$

where K_c is given by (59). (This controller is the same as the diagonal terms in K_{bp} since the weighted plant $W_{roll} \tilde{U}_{2,roll} P_{22}$ is the same as the diagonal elements in the scalar matrix $W_{bp} \tilde{U}_{2,bp} P_{22}$, see (30) and Table I.) This controller gives the steady state gain of $T_{T_\phi \rightarrow z_\phi}$ as 2.33×10^{-6} , compared to the passive suspension with dc gain $T_{T_\phi \rightarrow z_\phi}(0) = 2.4 \times 10^{-5}$, a similar result to Fig. 6.

C. Warp Control

For the warp quarter-car, we will take a slightly different approach for the design of the active controller. Since the sprung mass cannot be twisted, i.e., it has no warp motion, there is no corresponding role for the active controller to make the “sprung mass” stiffer to the loads. On the other hand, even though the passive road disturbance responses were designed to be relatively soft, there is no reason why they should not be even softer in the warp mode. We will therefore abandon the goal of keeping the response to the road warp input $(z_r)_{warp}$ invariant under active control. We also note that there is no acceleration measurement associated with warp and so there is only one feedback signal available corresponding to the strut deflections: $(D)_{warp} = -(z_u)_{warp}$. For this reason there is no \tilde{U}_2 block for the warp quarter-car loop in Fig. 12.

Referring to Table IV, the warp quarter-car reduces to the form illustrated in Fig. 14, with $z_s = 0$ and the coefficients $m_u = 200$ kg, $k_s = 80$ kN/m, $c_s = 20$ kNs/m and $k_t = 1000$ kN/m. The dynamic equation then takes the form, using (13)–(15)

$$m_u s^2 \hat{z}_u = -(c_s s + k_s) \hat{z}_u + k_s \hat{z}_a + k_t (\hat{z}_r - \hat{z}_u)$$

which reduces to, using (10) and (11)

$$(m_u s^2 + c_s s + k_s + k_t - k_s \gamma(s) K_{warp}(s)) \hat{z}_u = k_t \hat{z}_r \quad (63)$$

with the correspondences given in Table IV.

We now claim that it is desirable to choose $K_{warp}(s)$ so that the dc gain $T_{z_r \rightarrow z_u}(0) = 1$. We will now give some reasoning

to justify this. If we consider the case where the full-car model is in equilibrium with $F_s = T_\theta = T_\phi = 0$, then the (39)–(41) are equivalent to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ l_f & l_f & -l_r & -l_r \\ -t_f & t_f & -t_r & t_r \end{bmatrix} \begin{bmatrix} u_{p1} \\ u_{p2} \\ u_{p3} \\ u_{p4} \end{bmatrix} = 0. \quad (64)$$

Evidently there is one degree of freedom available in the suspension forces, and indeed we can check that

$$[u_{p1}, u_{p2}, u_{p3}, u_{p4}]' = [t_r, -t_r, -t_f, t_f]' \kappa \quad (65)$$

for some constant κ , completely characterizes that freedom. From the point of view of reducing the amount of “twist” on the vehicle chassis which the suspension forces impose, it would be desirable to achieve a value of $\kappa = 0$ in the steady state. The following result shows that the above mentioned condition achieves this property.

Proposition 1: Suppose the (linearized) full-car model defined in Section V-A is in equilibrium with $F_s = T_\theta = T_\phi = 0$ and z_{r1}, \dots, z_{r4} arbitrary. Then the following equation holds:

$$(z_r)_{\text{warp}} = (z_u)_{\text{warp}} \quad (66)$$

if and only if $u_{p1} = u_{p2} = u_{p3} = u_{p4} = 0$.

Proof: Using (49) and (53), we notice that the warp variable $(x)_{\text{warp}}$ is a combination of variables at the four wheel-stations

$$(x)_{\text{warp}} = \frac{1}{2(t_r + t_f)} [t_r, -t_r, -t_f, t_f] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where x can be z_u or z_r . Using (46) and (47) we see that (66) is equivalent to $[t_r/k_{t_f}, -t_r/k_{t_f}, -t_f/k_{t_r}, t_f/k_{t_r}] \cdot [F_{r1}, F_{r2}, F_{r3}, F_{r4}]' = 0$, which is equivalent to $[t_r/k_{t_f}, -t_r/k_{t_f}, -t_f/k_{t_r}, t_f/k_{t_r}] \cdot [u_{p1}, u_{p2}, u_{p3}, u_{p4}]' = 0$ by (42)–(45), which in turn is equivalent to $\kappa = 0$ by (65). \square

It can be observed that the proposition holds if (66) is replaced by any equation of the form:

$$[\lambda, -\lambda, -\mu, \mu] \cdot [z_{r1} - z_{u1}, z_{r2} - z_{u2}, z_{r3} - z_{u3}, z_{r4} - z_{u4}]' = 0$$

with $\lambda\mu > 0$, i.e., the particular ratios chosen in the definition of the warp variable are not critical to the result.

Now let us return to the warp quarter-car represented in (63). If we choose a simple constant controller and ignore temporarily the actuator dynamics, i.e., set $\gamma(s) = 1$, then the choice of $K_{\text{warp}} = 7.25$ achieves a damping ratio of one, a natural frequency equal to 50 rad/s (which is lower than the bandwidth of the actuator) and a steady-state gain of 2. As shown by Proposition 1 we would like to achieve the condition (66) in the steady-state, which is equivalent to the dc gain $T_{\hat{z}_r \rightarrow \hat{z}_u}(0) = 1$. Setting $K_{\text{warp}}(s) = -0.1036s + 1$ achieves in (63) a damping ratio of one, a natural frequency equal to 70.71 rad/s (which is lower

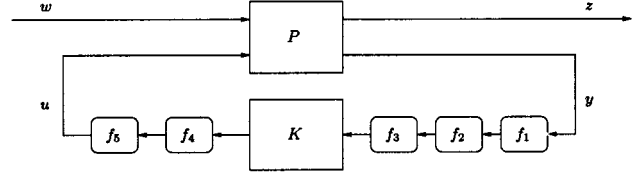


Fig. 15. Final controller structure for the full-car model.

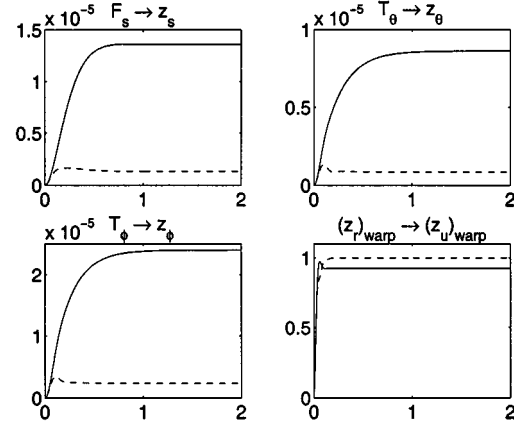


Fig. 16. Step responses using active (dashed) and passive (solid) suspensions.

than the bandwidth of the actuator) and a steady-state gain of one. In order that the controller is proper, we can choose

$$K_{\text{warp}} = \frac{-0.1036s + 1}{(300)^{-1}s + 1}. \quad (67)$$

The response does not change significantly with this modification or when the actuator dynamics are included. The step response with the final controller, with improved warp behavior compared to the passive case, is shown in Fig. 16.

D. The Full-Car Control

As a final step we can redraw Fig. 12 in the form of Fig. 15 where P is the full-car model represented by (39)–(45). In Fig. 15 the measurements y and control signals u are defined as follows:

$$y = [\ddot{z}_s, \ddot{z}_\theta, \ddot{z}_\phi, D_1, D_2, D_3, D_4]' \\ u = [u_1, u_2, u_3, u_4]'$$

where D_i and u_i represent the strut deflection and control command signal at each wheel station. The blocks f_1, \dots, f_5 are defined as follows:

$$f_1 = \text{diag}(1, 1, 1, L_{f,1}) \\ f_2 = \text{diag}(1, -1, 1, 1, 1, L_{f,2}) \\ f_3 = \begin{bmatrix} \frac{m_s l_r}{d_1(s)} & \frac{I_\theta}{d_1(s)} & 0 & 1 & 0 & 0 & 0 \\ \frac{m_s l_f}{d_2(s)} & \frac{-I_\theta}{d_2(s)} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{I_\phi}{d_3(s)} & 0 & 0 & 0 & 1 \end{bmatrix} \\ f_4 = \text{diag}(1, 1, L_{f,2}^{-1}) \\ f_5 = L_{f,1}^{-1}$$

where $L_{f,1}$ is defined in (48), $L_{f,2}$ is defined in (52) and $d_1(s) = 2(c_f s + k_f)(l_r + l_f)$, $d_2(s) = 2(c_r s + k_r)(l_r + l_f)$, $d_3(s) =$

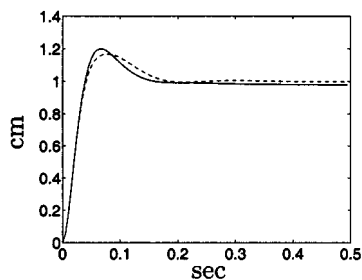


Fig. 17. Response of z_{u1} to a step input of 1 cm at z_{r1} for *AutoSim* model using active (dashed) and passive (solid) suspension.

$2(c_f s + k_f)t_f(t_r + t_f)$. Note that f_3 combines $\tilde{U}_{2,bp}$ defined in (56) and $\tilde{U}_{2,roll}$ defined in (60), and the third row in f_3 reflects the fact that there is only one measurement available for warp control. The controller K is defined as

$$K = \text{diag}(K_{bp}, K_{warp}, K_{roll})$$

where K_{bp} , K_{warp} and K_{roll} are given by (58), (62) and (67), respectively. Compared with the passive suspension, the benefits of using active controllers is shown in Fig. 16. The responses to “bounce,” “pitch,” and “roll” road inputs are not shown since these are the same in the passive and active cases.

E. Vehicle Dynamics Simulations

In this section we present some simulation results for the controller designed in Section VI-D using the multibody simulation package *AutoSim*. A nonlinear dynamical model of the simple full-car shown in Fig. 11 was constructed with the suspension struts constrained to move perpendicularly to the vehicle body. To model a rolling wheel of inertia 1 kg m^2 with tire the *magic formula* [1] was employed to calculate the accelerating and braking forces. The control law given in Section VI-D was implemented together with the actuator structure described in Section V-A.

The model was first tested at zero velocity for various road disturbance inputs and gave similar results, for small displacements, to a *Matlab* simulation of the linearized model. As expected, the bounce, pitch and roll responses were the same in the active and passive cases. Fig. 17 shows the effect of applying a step input to the *AutoSim* model at the right front wheel in both the passive and active cases. The difference in behavior is due to the “warp” mode being treated differently in the active case, as explained in Section VI-C.

The *AutoSim* model was then tested under acceleration and braking. For acceleration, a torque was applied at each front wheel with the opposing reaction torques acting on the vehicle body. A similar approach was taken for braking but with the braking torques applied to the front and rear wheels in a 60:40 ratio. Fig. 18 shows the “squat” and “dive” of the model under acceleration and deceleration, with the forward velocity given in Fig. 18(a) and the pitch angle z_θ given in Fig. 18(c). The simulation shows that the active suspension significantly improves the squat and dive performance.

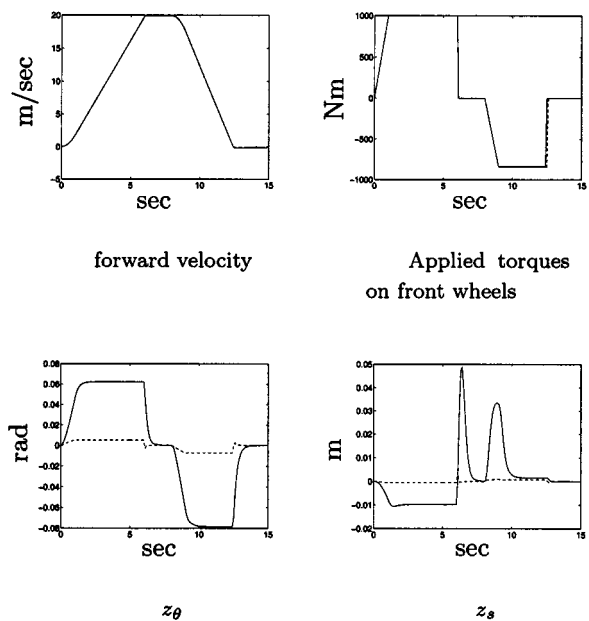


Fig. 18. Antidive and antisquat effect in *AutoSim* model using active suspension (dashed), compared with passive suspension (solid). (a) Forward velocity. (b) Applied torque on front wheels. (c) z_θ . (d) z_s .

VII. CONCLUDING REMARKS

This paper has considered the vehicle active suspension design problem with particular regard for the potentially conflicting performance requirements from two disturbance sources: road irregularities and loads applied to the vehicle body. General theorems (Theorems 1, 2) were derived to parameterize all stabilizing controllers which leave some prespecified closed-loop transfer function fixed. This allowed a feedback controller to be designed taking account of only the load disturbance path objective, given that the controller structure ensured that the road disturbance responses remained satisfactory.

The approach was illustrated for the quarter-car model with various different choices of measurements. The required control structures were derived in parametric form. For a half-car model, a parametric control structure was derived for a typical measurement set: vertical and angular accelerations of the sprung mass and strut deflection measurements. The conditions under which the model structure could be decomposed into two quarter-cars was investigated. For the full-car model, decomposition into two half-cars was exploited under a mild symmetry assumption. This enabled the bounce/pitch half-car design to be carried out with the half-car structure previously derived. For the roll/warp half-car a further decomposition into two quarter-cars was assumed. This allowed the warp quarter-car to be treated in a distinct way, which is necessary since the load disturbance path is absent here and it is also reasonable to change (i.e., soften) the road disturbance response from the passive case. A controller was designed and demonstrated on a nonlinear vehicle dynamics model and showed the effectiveness of the design for reduced dive and squat under acceleration and braking, improved warp response and invariance of other road disturbance responses.

A key step in the method described in this paper is the computation of the matrix \tilde{U}_2 which determines the required controller structure. Throughout the paper it was always possible to calculate \tilde{U}_2 symbolically using *Maple*. For more complicated vehicle models this may not be feasible. In such a case a direct numerical approach may be possible. Let us consider the case $r_3 = m_1$, i.e., the matrix F in (6) is square. (This condition applied throughout this paper and seems quite typical in general.) Then the following procedure can be taken: 1) partition $T_{3,1} = [B', A']'$ where B is square; 2) find a minimal realization of $L := AB^{-1}$; and 3) find a left coprime factorization $L = \tilde{B}^{-1}\tilde{A}$ and set $\tilde{U}_2 = [-\tilde{A}, \tilde{B}]$ (e.g., see [30, Theorem 12.19]). Such an approach was taken in a trailing-arm vehicle model in [23].

APPENDIX

Proof of Lemma 1: T can be decomposed in terms of its Smith form over \mathbb{RH}_∞ [21]: $T = UHV$, where $U \in \mathbb{RH}_\infty^{r \times r}$, $V \in \mathbb{RH}_\infty^{c \times c}$ are unimodular, $H \in \mathbb{RH}_\infty^{r \times c}$. Suppose that H has r_1 nonzero diagonal elements. Then we can write

$$H = \begin{pmatrix} H_1 & \\ 0_{(r-r_1) \times r_1} & \end{pmatrix} (I_{r_1 \times r_1}, 0_{r_1 \times (c-r_1)}) \quad (68)$$

$$= \begin{pmatrix} I_{r_1 \times r_1} & \\ 0_{(r-r_1) \times r_1} & \end{pmatrix} (H_1, 0_{r_1 \times (c-r_1)}) \quad (69)$$

where $H_1 \in \mathbb{RH}_\infty^{r_1 \times r_1}$. We also partition U and V conformably: $U = (U_1, U_2)$, $V = (V_1', V_2')'$ where $U_1 \in \mathbb{RH}_\infty^{r_1 \times r_1}$, $U_2 \in \mathbb{RH}_\infty^{(r-r_1) \times (c-r_1)}$, $V_1 \in \mathbb{RH}_\infty^{r_1 \times c}$, $V_2 \in \mathbb{RH}_\infty^{(c-r_1) \times c}$. We therefore obtain

$$\begin{aligned} T &= (U_1 H_1) V_1 = E V_1 \\ &= U_1 (H_1 V_1) = U_1 F \end{aligned}$$

where $E \in \mathbb{RH}_\infty^{r_1 \times r_1}$, $F \in \mathbb{RH}_\infty^{r_1 \times c}$ are full column normal rank and full row normal rank, respectively. \square

Proof of Theorem 1: A stabilizing controller in the form (3) leaves $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{11}$ if and only if $(T_2 Q T_3)_{11} = E V_1 Q U_1 F = 0$. This is equivalent to $V_1 Q U_1 = 0$ since E (respectively, F) has full column (respectively, row) normal rank. We now show that it requires Q to take the form given in (7). Clearly

$$V_1 Q (U_1, U_2) = (0_{r_2 \times r_3}, Q_1) \quad (70)$$

for some $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$. This gives

$$V_1 Q = (0, Q_1) U^{-1} = Q_1 \tilde{U}_2.$$

Next we see that

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} Q = \begin{pmatrix} Q_1 \tilde{U}_2 \\ Q_2 \end{pmatrix} \quad (71)$$

for some $Q_2 \in \mathbb{RH}_\infty^{(m_3 - r_2) \times p_3}$, which establishes (7).

Conversely, if (7) holds for some $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$ and $Q_2 \in \mathbb{RH}_\infty^{(m_3 - r_2) \times p_3}$, then so does (71), from which follows $V_1 Q = (0, Q_1) U^{-1}$ which again implies $V_1 Q U_1 = 0$. \square

Proof of Lemma 2:

- 1) (\Leftarrow) In this case $B = \{Q_1^\dagger \tilde{U}_2^\dagger\} = \{(Q_1^\dagger W) \tilde{U}_2\}$, which means any element of B is also an element of A . (\Rightarrow) Suppose $B \subseteq A$, then for any $Q_1^\dagger \in \mathbb{RH}_\infty^{r \times c}$ there exist some $Q_1 \in \mathbb{RH}_\infty^{r \times c}$ such that $Q_1^\dagger \tilde{U}_2^\dagger = Q_1 \tilde{U}_2$. Let us now choose the first row of Q_1^\dagger to be $[0, \dots, 1, \dots, 0]$ (with the one in the i th place), and all other rows to be zero, and define q_i to be the first row of the corresponding Q_1 . Then $[0, \dots, 1, \dots, 0] \tilde{U}_2^\dagger = q_i \tilde{U}_2$, from which we conclude that $\tilde{U}_2^\dagger = W \tilde{U}_2$ where $W = [q_1', q_2', \dots, q_c']'$.
- 2) From 1) we know that $B = A$ if and only if $\tilde{U}_2^\dagger = W_1 \tilde{U}_2$ and $\tilde{U}_2 = W_2 \tilde{U}_2^\dagger$ for some $W_1, W_2 \in \mathbb{RH}_\infty^{c \times c}$. Hence we have $\tilde{U}_2^\dagger = W_2 W_1 \tilde{U}_2^\dagger$ which gives $(I - W_2 W_1) \tilde{U}_2^\dagger = 0$. Since \tilde{U}_2^\dagger is full row normal rank, it is equivalent to $W_2 W_1 = I$, which is equivalent to W_1 and W_2 being unimodular and inverses of each other over \mathbb{RH}_∞ . \square

Proof of Theorem 2:

- 1) 2) These conditions allow us to choose $M = I, \tilde{M} = I, N = \tilde{N} = P_{22}, X = I, \tilde{X} = I, Y = 0, \tilde{Y} = 0$ and $V_1 = I$. Then (8) follows directly from (3) and (7).
- 3) Consider \tilde{U}_2 and \tilde{U}_2^\dagger such that all stabilizing controllers can be parameterized as $K = (I - Q_1 \tilde{U}_2 P_{22})^{-1} Q_1 \tilde{U}_2$, $Q_1 \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$ and $K^\dagger = (I - Q_1^\dagger \tilde{U}_2^\dagger P_{22})^{-1} Q_1^\dagger \tilde{U}_2^\dagger$, $Q_1^\dagger \in \mathbb{RH}_\infty^{r_2 \times (p_3 - r_3)}$, respectively. We can check that $K = K^\dagger$ if and only if $Q_1 \tilde{U}_2 = Q_1^\dagger \tilde{U}_2^\dagger$. From Lemma 2, this means there exists a unimodular matrix W such that $\tilde{U}_2^\dagger = W \tilde{U}_2$.
- 4) Since \tilde{U}_2, P_{22} have elements in \mathbb{RH}_∞ , it follows from [30, Corollary 5.5] that K_1 stabilizes $U_2 P_{22}$ if and only if $(I - K_1 \tilde{U}_2 P_{22})^{-1} K_1 \in \mathbb{RH}_\infty^{m_3 \times (p_3 - r_3)}$, which is equivalent to $(I - K_1 \tilde{U}_2 P_{22})^{-1} K_1 \tilde{U}_2 \in \mathbb{RH}_\infty^{m_3 \times p_3}$ since \tilde{U}_2 is right invertible over \mathbb{RH}_∞ , which is the necessary and sufficient condition that $K_1 \tilde{U}_2$ stabilizes P_{22} . To complete the proof, let $Q_1 = -(I - K_1 \tilde{U}_2 P_{22})^{-1} K_1 \in \mathbb{RH}_\infty^{m_3 \times (p_3 - r_3)}$ and note that $K_1 = -(I - Q_1 \tilde{U}_2 P_{22})^{-1} Q_1$. Therefore $K = K_1 \tilde{U}_2$ will take the form of (8), from which the result follows.
- 5) Any stabilizing controller K for which $T_{\hat{w}_1 \rightarrow \hat{z}_1} = (T_1)_{1,1}$ takes the form of (8). We can then define a controller $K_1 = -(I - Q_1 \tilde{U}_2 P_{22})^{-1} Q_1$ for some $Q_1 \in \mathbb{RH}_\infty^{m_3 \times (p_3 - r_3)}$ such that $K = K_1 \tilde{U}_2$. It can also be shown directly that $-(I - K_1 \tilde{U}_2 P_{22})^{-1} K_1 = Q_1 \in \mathbb{RH}_\infty^{m_3 \times (p_3 - r_3)}$, which means that K_1 stabilizes $\tilde{U}_2 P_{22}$. \square

Proof of Corollary 1: Using the parameterization of Theorem 2(2), it follows from (4) that the closed-loop transfer function $(T)_{\hat{w}_1 \rightarrow \hat{z}_1}$ remains the same as when $K_0 = -(I - \tilde{Q} P_{22})^{-1} \tilde{Q}$ is applied if and only if $(T_1 + T_2 Q T_3)_{1,1} = (T_1 + T_2 \tilde{Q} T_3)_{1,1}$, which is equivalent to $(T_2 (Q - \tilde{Q}) T_3)_{1,1} = 0$, which results in (9) by using Theorem 1 where V_2, Q_2 are empty. \square

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