Controller synthesis on non–uniform and uncertain discrete–time domains*

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Abstract. The problem of synthesizing feedback controllers that perform sensing and actuation actions on non–uniform and uncertain discrete time domains is considered. This class of problems is relevant to many application domains. For instance, in engine control a heterogenous and, to some extent, uncertain event–driven time domain is due to the behavior of the 4-stroke internal combustion engine, with which the controller has to synchronize to operate the engine properly. Similar problems arise also in standard discrete–time control systems when considering the behavior of the system with controller implementation and communication effects. The design problem is formalized in a hybrid system framework; synthesis and verification methods, based on robust stability and robust performance results, are presented. The effectiveness of the proposed methods is demonstrated in an engine control application.

1 Introduction

This paper considers the problem of synthesizing feedback controllers that perform sensing and actuation actions on non–uniform and uncertain discrete–time domains. The approach was initially motivated by control problems in the automotive industry, but it is certainly extensible to other application domains.
In engine control applications the existence of non–uniform and, to some extent, uncertain time–domains is a characteristic of the plant behavior itself and the controller implementation. Heterogeneity in time domains arises in engine control from nested control-loops of both:

– *discrete–time domain* control loops with fixed sampling rate, e.g. cruise control (with sampling time of the order of dsec) and throttle valve control (with sampling time of the order of msec);

* This research has been partially supported by the E.C. grant *Control and Computation* IST-2001-33520. The authors are members of the *HyCon* Network of Excellence, E.C. grant IST-511368.

- *event–driven control* actions synchronized with the evolution of the engine cycle[1], such as control of the engine torque (delivered by each cylinder during the power stroke), fuel injection (during the exhaust stroke in multi–point injection engines) and spark ignition (either at the end of the compression stroke or at the beginning of the power stroke).

In particular, event–driven control actions are synchronized with the engine cycle and issued on a non–uniform discrete–time domain, characterized by drifts of the activation times and frequency, which is synchronous with the crankshaft revolution speed.
Moreover, similar problems arise also in standard discrete–time control systems when considering

- the effects of the implementation of control algorithms in embedded systems, which range from uncertain and time–varying delays introduced in the loop (e.g. latency due to scheduling of the algorithms on time–shared CPUs) to the intermittent dropping of some executions of the control algorithm, due to either computation overload of the CPU or communications errors with sensors and actuators;
- sporadic failures on sensors, actuators, embedded controllers or communication.

The Lee-Sangiovanni Vincentelli (LSV) tagged-signal model (TSM) [1] is a formalism for describing aspects of models of computation that very naturally allows the representation of signals defined on non–uniform time–domains. Benveniste *et al.* [2] used the TSM to describe interacting synchronous and asynchronous models of computation and communication. Controller design taking into account implementation constraints was investigated by Bicchi *et al.* [3, 4], who considered input signals quantization, and Palopoli et al. [5], who proposed an optimal trade-off between closed–loop performances and scheduling for a multi–rate control system that is in charge of controlling a number of independent plants.
In this paper we address the problem of synthesizing and verifying control algorithms that are executed at discrete times with phase drifts of the activation event sequence and uncertainties in the activation times. Today, the best practice in industry for dynamic compensators design for this class of control problems is gain scheduling with possibly some on–line adaptation to the varying sampling time. However, the correctness of the controller in terms of stability and closed–loop performance under drifting of the sampling times is not formally guaranteed.
The synthesis and verification problems can be properly formalized and solved using hybrid systems techniques. We show that fundamental results on robust stability and robust performance can be successfully used and reformulated in a hybrid system framework to obtain both:

[1] An interesting topic is the design of efficient interfaces between multi–rate feedback loops characterized by phase and frequency drifts of the activation times. This topic will be the subject of a future paper.

– synthesis procedures that take into account time–domain uncertainties and produce controllers with guaranteed performances, and
– formal verification techniques that guarantee the correctness of controllers designed either abstracting or partially compensating time–domain uncertainties.

We consider the design of dynamic compensators for continuous–time uncertain plants with non–uniform and uncertain activation times. In particular, the final aim is to design linear time–invariant dynamic controllers for sampled–data systems, derived from a non–uniform sampling of the plant model, which guarantee stability and achieve desired rate of convergence despite sampling time variance.

The paper is organized as follows. In Section 2, the motivating automotive application, namely the synthesis of an algorithm for idle speed control, is described. In Section 3, fundamental results on robust stability and robust performance are reviewed. In Section 4, the problem of the design of dynamic compensators under non–uniform and uncertain activation times is formalized. In addition, synthesis and verification methods obtained from the results presented in Section 3 are described. Finally, in Section 5, the proposed techniques are applied to the idle speed control problem showing the degree of robustness of a controller designed without taking into account time–domain uncertainties and a synthesis procedure that, by considering them, produces a controller with improved closed–loop stability. Some concluding remarks are presented in Section 6.

2 Idle speed control problem formulation

The motivating application for the work presented in this paper is the synthesis of an algorithm for idle speed control. The objective is to keep the speed of the crankshaft within a specified range despite the actions of unpredictable but bounded load torques acting on the crankshaft, when the engine is idle.
In Figure 1, a hybrid model describing the behavior of a 4–stroke 4–cylinder spark ignition engine at idle is depicted (more details on the model are given in [6–8]). The hybrid model has a urgent semantic, some nonlinear continuous dynamics and some continuous variables with piece-wise constant evolutions. Engine control inputs are:

– The throttle valve command u_α, used to control the engine air charge[2] m;
– The spark advance angle[3] u_φ, which defines ignition timing.

The command u_α to the throttle valve motor is a discrete–time signal produced with a constant sampling period τ_A. The timer τ, the piece–wise constant variable V, and the self-loop transition with guard condition $\tau = \tau_A$ model the

[2] Fuel injection is set according to the evolution of the air charge m so to have stoichiometric mixtures, as requested for tailpipe emission control.
[3] It denotes the angle performed by the crankshaft from the time at which the spark is ignited to the time at which the piston reaches the next top dead center. It is negative if the spark is given after the top dead center.

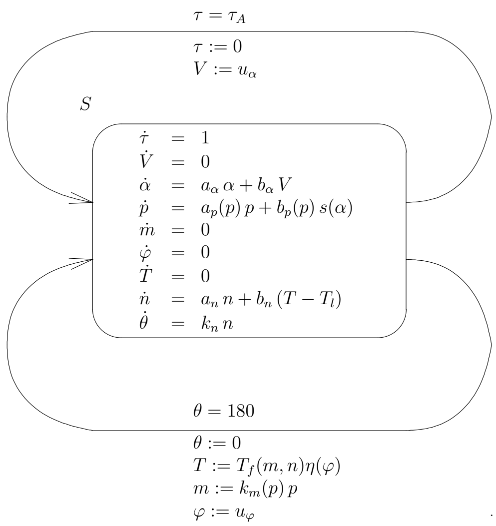

Fig. 1. Hybrid model of the cylinders.

uniform sampling of the discrete–time throttle valve control. To take into account the actuation delay, the desired spark advance u_φ has to be set for each cylinder at the bottom dead center at the end of the intake stroke, so that the ignition subsystem can be programmed to ignite the spark at the proper time. Dead center events are modelled by the self-loop transition with guard condition $\theta = 180$ and reset $\theta := 0$, where θ denotes the crankshaft angle.
The continuous state variables with no trivial dynamics are: the throttle valve angle α; the intake manifold pressure p; the crankshaft revolution speed n and the crankshaft angle θ. The evolution of the intake manifold pressure p depends on the throttle valve angle α, which is controlled by the input u_α. The crankshaft speed n depends on the engine torque T and defines the evolution of the crankshaft angle θ. At each dead–center, i.e. when θ reaches 180, the crankshaft angle θ is reset and the engine torque T is set according to the applied spark advance φ and the air charge m. Moreover, the air charge m and the desired spark advance φ for the next expansion cycle are initialized according to the current value of the intake manifold pressure p and the input u_φ, respectively.
The design of the spark advance control algorithm is particularly challenging since it is defined on a non–uniform discrete–time domain, with sampling period varying according to the evolution of the crankshaft revolution speed.

3 Robust stability of time-varying linear systems

In this section, the problem of designing a controller for non–uniform and uncertain discrete–time domains is formally introduced and some relevant results on robust stability and robust performance are briefly reviewed.

Consider a Linear Time Invariant (LTI) continuous–time system

$$\begin{aligned} \dot{x}(t) &= A^c x(t) + B^c u(t) \\ y(t) &= Cx(t) \ , \end{aligned} \tag{1}$$

with $x(t) \in R^n$ the continuous state, $u(t) \in R^p$ the control signal, $y(t) \in R^o$ the output signal, and $A^c \in R^{n\times n}$, $B^c \in R^{n\times p}$ and $C \in R^{o\times n}$ constant matrices.

The objective is to design a digital controller for system (1), which reads the output y and issues a command u at some sampling times $\{\tau_k\}$ that are not uniformly spaced in time, as usually assumed. This kind of control problems arises in standard discrete–time control when considering controller implementation and communication issues. It also includes the case of hybrid systems with no resets and controller activation times defined by the automaton transitions. In this case, the non–uniformity of the time domain is given by the hybrid behavior of the plant itself[4].

By sampling the continuous–time dynamics (1) on a non–uniform time domain $\{\tau_k\}$, the following Linear Time Variant (LTV) discrete–time system is obtained:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= Cx(k) \ , \end{aligned} \tag{2}$$

where $x(k) = x(\tau_k)$, $u(k) = u(\tau_k) = u(t)\ \forall t \in [\tau_k, \tau_{k+1})$, and $y(k) = y(\tau_k)$ are samples of the corresponding continuous signals, and the system matrices are obtained by integration of (1) over the interval $[\tau_k, \tau_{k+1}]$, i.e.

$$\begin{aligned} A(k) &= e^{A^c(\tau_{k+1}-\tau_k)} \\ B(k) &= \int_0^{\tau_{k+1}-\tau_k} e^{A^c(\tau_{k+1}-\tau_k-\tau)} d\tau \ B^c \ . \end{aligned} \tag{3}$$

Let the time domain $\{\tau_k\}$ be such that the sampling intervals $\tau_{k+1} - \tau_k$ satisfy

$$\tau_{k+1} - \tau_k = \tau^0 + \delta(k) \qquad \text{with } |\delta(k)| \le \Delta \text{ and } \Delta > 0 \ , \tag{4}$$

where τ^0 is a nominal constant sampling period and $\delta(k)$ is a bounded perturbation. Then, in (2),

$$\begin{aligned} A(k) &= \bar{A} + \Delta_A(k) \\ B(k) &= \bar{B} + \Delta_B(k) \ , \end{aligned} \tag{5}$$

where $\bar{A} = e^{A^c\tau^0}$ and $\bar{B} = \int_0^{\tau^0} e^{A^c(\tau^0-\tau)} d\tau B^c$ are the contributions associated to the nominal sampling period τ^0 and $\Delta_A(k) = A(k) - \bar{A}$, $\Delta_B(k) = B(k) - \bar{B}$ take into account sampling time variations.

Perturbations $\Delta_A(k)$ and $\Delta_B(k)$ in (5) are bounded[5]as follows:

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[4] For instance, in the idle speed control problem, activation times are triggered by dead–center events that are produced when the crankshaft angle θ reaches 180.

[5] Unless differently specified, we consider the Euclidean norm of matrices and vectors defined as $\|z\| = \sqrt{\sum_{i=1}^n z_i^2}$ for $z \in R^n$ and $\|M\| = \sigma_{\max}(M) = \max\{\lambda | \lambda^2 \in \lambda(M^T M)\}$, for $M \in R^{n\times n}$, with $\sigma_{\max}(M)$ the maximum singular value of M and $\lambda(M)$ the set of eigenvalues of M.

– If $A^c = 0$, then $\Delta_A(k) = 0$ and

$$\|\Delta_B(k)\| \leq \|B^c\| \, \Delta \; ; \tag{6}$$

– If $A^c \neq 0$ and if the geometric multiplicity of the eigenvalues of A^c is equal to their algebraic multiplicity, then

$$\begin{aligned} \|\Delta_A(k)\| &= \sigma_{max}(\Delta_A(k)) \\ &\leq \left\| e^{-A^c \tau^0} \right\| \left\| e^{A^c \delta(k)} - I \right\| \\ &\leq \left\| \bar{A}^{-1} \right\| \, \|A^c\| \; k\left(A^c\right) \; \frac{e^{\bar{\alpha}(A^c)\Delta} - 1}{\bar{\alpha}(A^c)} \; , \end{aligned} \tag{7}$$

$$\begin{aligned} \|\Delta_B(k)\| &= \sigma_{max}(\Delta_B(k)) \\ &\leq \left\| \bar{A} \right\| \, \|B^c\| \; k\left(A^c\right) \; \frac{e^{\bar{\alpha}(A^c)\Delta} - 1}{\bar{\alpha}(A^c)} \; , \end{aligned} \tag{8}$$

where $k(A) = \|T\| \, \|T^{-1}\|$ denotes the condition number with respect to inversion of the matrix T such that $T^{-1}AT$ is in the Jordan normal form[6]and $\bar{\alpha}(A) = \max\{\alpha(A), \alpha(-A)\}$, with $\alpha(A) = \max\{\mathrm{Re}(\lambda) | \lambda \in \lambda(A)\}$ the spectral abscissa of A. Note that, since $\lim_{\Delta \to 0} \frac{e^{\bar{\alpha}(A^c)\Delta} - 1}{\bar{\alpha}(A^c)} \approx \lim_{\Delta \to 0} \Delta$, then the upper bounds (7–8) converge to zero with Δ.

Upper bounds similar to (7–8) can be obtained when the geometric multiplicity of the eigenvalues of A^c is lower than their algebraic multiplicity, in which case the Jordan normal form has blocks of order greater than 1 (details on the approximation of the norm of the exponential matrix can be found in [9]).

The design problem on non–uniform discrete–time domains can be successfully approached by exploiting interesting results on robust stability (see [10–25]) for perturbed systems of type

$$x(k+1) \; = \; [A + \Delta A(k)]x(k) \; . \tag{9}$$

To the best of our knowledge, the work of Bauer *et al.* [10], along with [16–18], gives the tightest stability conditions for parametric uncertainties of type

$$\Delta A(k) \; = \; \sum_{j=0}^{r} a_j(k) A_j \; , \tag{10}$$

with $A_j \in R^{n \times n}$ and $a_j(k) \in [\underline{a}_j, \overline{a}_j]$, for $j = 1, \cdots, r$. Introduce the set $\tilde{\mathcal{A}}$ of extremal matrices

$$\tilde{\mathcal{A}} = \{ \tilde{A} = A + \sum_{j=0}^{r} a_j A_j \mid a_j = \underline{a}_j \, \text{or} \, a_j = \overline{a}_j, \text{ for all } j = 1, \cdots, r \;\; \} \; . \tag{11}$$

The set $\tilde{\mathcal{A}}$ in (10) defines a polytope $\mathcal{P}_{\tilde{\mathcal{A}}}$ whose vertices coincide with extremal matrices. In [10], the following result was presented:

[6] It is well known that the matrix T is not unique; in what follows, less conservative conditions will be obtained for T such that $k(A)$ is minimized.

Proposition 1. *System (9) with time–varying dynamical matrix inside the polytope* $\mathcal{P}_{\tilde{\mathcal{A}}}$ *is asymptotically stable*[7] *in norm-1 (norm–*∞*) if and only if there exists a* $\bar{k} > 0$ *such that, for any sequence of* $\bar{k}$ *matrices* $\tilde{A}_j \in \tilde{\mathcal{A}}$*,*

$$\textbf{[C1]} \qquad \|\prod_{j=1}^{\bar{k}} \tilde{A}_j\|_1 < 1 \qquad (\|\prod_{j=1}^{\bar{k}} \tilde{A}_j\|_\infty < 1, \textit{respectively}) \ . \tag{12}$$

Molchanov *et al.* [16] extended the previous result by proving that condition $[C1]$ can be formulated for any norm. Condition $[C1]$ is strong since robust stability for a dynamic matrix varying inside the polytope $\mathcal{P}_{\tilde{\mathcal{A}}}$ can be tested by checking combinations of the extremal matrices in $\tilde{\mathcal{A}}$ only. However, it could require many computations if the number r of elements in the linear combination (10) is large. Indeed, the cardinality of $\tilde{\mathcal{A}}$ is 2^r and the stability test on k steps requires 2^{rk} matrix multiplications. In [17, 18], simplified stability tests had been proposed for specific classes of systems.

In [26], Blanchini compares stabilizability via gain scheduling (with measurement of time–varying parameters) and robust state feedback for perturbed systems of type (9–10) and shows that the two approaches are equivalent.

Sufficient conditions for robust stability, based on the Lyapunov approach, had been proposed in [11–14]. Given a positive-definite function $V(X)$, system (9) asymptotically converges[8] to the equilibrium with convergence rate $\mu > 0$ if the difference $V(X(k+1)) - \mu V(X(k))$ is negative for any $k \geq 0$. By slightly extending the work in [11], the following sufficient condition can be obtained:

Proposition 2. *System (9) is globally asymptotically stable with rate of convergence* $0 < \mu < 1$ *if*

$$\textbf{[C2]} \qquad \sigma_{max}(\Delta A(k)) < -\sigma_{max}(A) + \sqrt{\sigma^2_{max}(A) + \frac{\sigma_{min}(Q)}{\sigma_{max}(P)}} \ , \tag{13}$$

where $P = P^T > 0$ *is the solution of the discrete-time Lyapunov equation*

$$A^T P A - \mu P = -Q, \qquad \textit{for } Q = Q^T > 0 \ . \tag{14}$$

Finally, if the nominal matrix A in (9) verifies $\|A\|_p < 1$, for some norm $\|\cdot\|_p$, then a further condition that ensures robust stability is given by

$$\textbf{[C3]} \qquad \|\Delta A(k)\|_p < 1 - \|A\|_p \ . \tag{15}$$

A simple proof is obtained by noting that, since

$$\|A + \Delta A(k)\|_p \leq \|A\|_p + \|\Delta A(k)\|_p < 1$$

[7] Given $z \in R^n$, $\|z\|_1 = \sum_{i=1}^n |z_i|$ and $\|z\|_\infty = \max_{i=1}^n |z_i|$. Given $M \in R^{n\times n}$, $\|M\|_1 = \max_{j=1}^n \sum_{i=1}^n |m_{ij}|$ and $\|M\|_\infty = \max_{i=1}^n \sum_{j=1}^n |m_{ij}|$. A system is asymptotically stable in norm-1 (norm–∞) if $\lim_{k\to\infty} \|x(k)\|_1 = 0$ $(\lim_{k\to\infty} \|x(k)\|_\infty = 0)$.

[8] The stability of the autonomous system coincide with BIBO stability if $B_C(k)$ is bounded in norm.

then the next–state map is a contraction in the chosen p–norm.
Further robust stability conditions have been obtained using LMI techniques (see [20, 22, 23, 25]) and H_1 and H_∞ formulations (see [19, 21, 24]). Such approaches will be evaluated in future work.

4 Dynamic compensators design under non–uniform and uncertain activation times

Standard design techniques based on frequency domain representations cannot be applied to design control algorithms for system (2), since such system is not time-invariant. However, often linear time–invariant controllers are adopted even for time–varying plants. This is the case for instance when the design is subject to very limiting constraints on the implementation platform. Consider the LTI compensator

$$\begin{aligned} w(k+1) &= Fw(k) + Ge(k) \\ u(k) &= Hw(k) + Le(k) \end{aligned} \tag{16}$$

where $e(k) = r(k) - y(k) \in R^o$ is the error between the controlled output and the reference signal $r(k)$, $w \in R^m$ is the state of the controller, $F \in R^{m\times m}$, $G \in R^{m\times o}$, $H \in R^{p\times m}$ and $L \in R^{p\times o}$ are constant matrices.

By (2) and (16), the closed–loop system is described in the extended state space $X = [x, w]^T$ as follows

$$\begin{bmatrix} x \\ w \end{bmatrix}(k+1) = \begin{pmatrix} A(k) - B(k)LC & B(k)H \\ -GC & F \end{pmatrix} \begin{bmatrix} x \\ w \end{bmatrix}(k) + \begin{pmatrix} B(k)L & I_{n\times n} \\ G & 0_{m\times n} \end{pmatrix} \begin{bmatrix} r \\ d \end{bmatrix}(k) \tag{17}$$

or equivalently, by (5),

$$X(k+1) = [\bar{A}_C + \Delta A_C(k)]\, X(k) \; + \; [\bar{B}_C + \Delta B_C(k)]\, U(k) \;, \tag{18}$$

where $U = [r, d]^T$ and

$$\bar{A}_C = \begin{pmatrix} \bar{A} - \bar{B}LC & \bar{B}H \\ -GC & F \end{pmatrix}, \quad \Delta A_C(k) = \begin{pmatrix} \Delta_A(k) - \Delta_B(k)LC & \Delta_B(k)H \\ 0_{n\times n} & 0_{m\times m} \end{pmatrix},$$

$$\bar{B}_C = \begin{pmatrix} \bar{B}L & I_{n\times n} \\ G & 0_{m\times n} \end{pmatrix}, \qquad \Delta B_C(k) = \begin{pmatrix} \Delta_B(k)L & 0_{m\times n} \\ 0_{m\times 1} & 0_{m\times n} \end{pmatrix}. \tag{19}$$

The closed–loop system (18) is both time–varying, due to the sampling time variations, and parameterized in the controller matrices (16). Upper bounds for the closed–loop perturbation matrices $\Delta A_C(k)$ and $\Delta B_C(k)$ are obtained from (6),(7) and (8), including additional terms p_A and p_B that model parameters uncertainties on $A(k)$ and $B(k)$ in (2). We have:

– If $A^c = 0$, then from (6)

$$\|\Delta A_C(k)\| \le (\|B^c\|\Delta + p_A + p_B)\,\gamma_C \quad \text{with} \quad \gamma_C = \left\| \begin{pmatrix} I & 0 \\ -LC & H \end{pmatrix} \right\|$$

$$\|\Delta B_C(k)\| \le (\|B^c\|\Delta + p_B)\,\|L\|$$

– If $A^c \neq 0$ and if the geometric multiplicity of the eigenvalues of A^c is equal to their algebraic multiplicity, then

$$\|\Delta A_C(k)\| \leq \left[\left(\|\bar{A}^{-1}\| \, \|A^c\| + \|\bar{A}\| \, \|B^c\| \right) k(A^c) \frac{e^{\bar{\alpha}(A^c)\Delta} - 1}{\bar{\alpha}(A^c)} + p_A + p_B \right] \gamma_C$$

$$\|\Delta B_C(k)\| \leq \left[\left\| \bar{A} \right\| \, \|B^c\| \, k(A^c) \frac{e^{\bar{\alpha}(A^c)\Delta} - 1}{\bar{\alpha}(A^c)} + p_B \right] \|L\|$$

Conditions $[C1]$, $[C2]$ and $[C3]$ – as formulated – can be applied for the verification of the correctness of a given dynamic compensator (16), in presence of time–domain and plant parameter uncertainties. On the other hand, they can also be used for controller synthesis if included in an exploration algorithm of the controller parameters space.
Among them, $[C1]$ is the least conservative. However, for synthesis purposes $[C1]$ could be numerically unfeasible, due to the dependency of the extremal matrices in (11) on the controller parameters: $2^{(2m+n)\bar{k}}$ multiplications between extremal matrices are necessary to perform the test on a given set of controller parameters. Then, $[C1]$ is more suitable for verification of a given controller, possibly obtained using either $[C2]$ or $[C3]$.
The Lyapunov approach employed in $[C2]$ allows the designer to set a desired convergence rate in (14) and handle separately in (13) the robustness with respect to time–domain and plant parameter uncertainties.

This approach can be specialized to the case of the design of dead–beat controllers, obtained when the nominal closed–loop system has all poles in the origin of the complex plane and having finite impulse response.

Proposition 3. *If the nominal closed–loop matrix $\bar{A}_C$ has all eigenvalues in 0, then the closed-loop uncertain time-varying system is asymptotically stable, with convergence rate $\mu \in (0, 1)$, provided that*

$$\textbf{[C4]} \quad \sigma_{max}(\Delta A_C(k)) < -\sigma_{max}(\bar{A}_C) + \sqrt{\sigma^2_{max}(\bar{A}_C) + \mu \frac{\sigma_{min}(Q)}{\sigma_{max}(Q)} \frac{1 - \frac{\sigma^2_{max}(\bar{A}_C)}{\mu}}{1 - \left(\frac{\sigma^2_{max}(\bar{A}_C)}{\mu} \right)^{n+m}}} \quad (20)$$

for some symmetric positive–definite matrix Q.

Proof. The solution to the Lyapunov equation (14) for a given symmetric positive definite matrix Q and convergence rate μ, can be written as

$$P = \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{(\bar{A}_C^T)^k Q \bar{A}_C^k}{\mu^k}.$$

If all eigenvalues of $\bar{A}_C$ are in 0, then $\bar{A}_C$ is nilpotent of order $n + m$ and

$$P = \frac{1}{\mu} \sum_{k=0}^{n+m-1} \frac{(\bar{A}_C^T)^k Q \bar{A}_C^k}{\mu^k}. \quad (21)$$

Then,

$$\sigma_{max}(P) = \|P\| \leq \frac{1}{\mu} \sum_{k=0}^{n+m-1} \frac{\|(\bar{A}_C^T)^k Q \bar{A}_C^k\|}{\mu^k} \leq \frac{\sigma_{max}(Q)}{\mu} \frac{1 - \left(\frac{\sigma_{max}^2(\bar{A}_C)}{\mu}\right)^{n+m}}{1 - \frac{\sigma_{max}^2(\bar{A}_C)}{\mu}} \ . \tag{22}$$

Inequality (22) gives a lower bound for $\frac{\sigma_{max}(Q)}{\sigma_{max}(P)}$, which substituted in (13) gives condition $[C4]$. Q.E.D.

Notice that condition [C4] is much easier to test than [C2], since the Lyapunov equation is explicitly solved. Moreover, it is important to observe that the time-varying closed–loop system does not preserve the dead–beat response due to the time–domain and plant parameters uncertainties.

5 Idle speed control application

In this section, the design methodology proposed in Section 4 is applied to the idle speed control problem described in Section 2. In particular, the design and verification of spark advance control algorithms are illustrated. Spark advance control is activated on the non–uniform discrete–time domain given by the dead-center times $\{\tau_k\}$. Since in idle speed control the engine speed is constrained by specification, then the dead-center times sequence $\{\tau_k\}$ satisfies condition (4) on bounded sampling time variation.

According to the model depicted in Figure 1, the torque generated by the engine during the k–th power stroke depends on: the spark advance command $u_\varphi(\tau_{k-1})$ (set at the beginning of the compression stroke), the mass of loaded air $m(\tau_{k-1})$, and the engine speed at the beginning of the power stroke $n(\tau_k)$. The engine torque, $T(t)$, is modeled as a piece–wise constant signal, with discontinuity points at dead-center times τ_k, i.e.

$$T(t) = T_f(m(\tau_{k-1}), n(\tau_k))\, \eta(u_\varphi(\tau_{k-1})) \qquad \text{for } t \in [\tau_k, \tau_{k+1}) \ . \tag{23}$$

The crankshaft dynamics, discretized on dead–center times $\{\tau_k\}$, is

$$n(k+1) = A(k)\, n(k) + B(k)\, u(k) \ , \tag{24}$$

where the input $u(k)$ comprises both the load disturbance T_l and the engine torque T, i.e.

$$u(k) = T_l(k) + T(k) \ .$$

To control the engine speed n to a given reference value n_r, the engine torque T is modulated, using the spark advance command, so to implement the LTI compensator (16), where $e = n - n_r$ and $u = T$. That is

$$T(k) = c(k) \otimes [n(k) - n_r(k)]$$

with

$$C(z) = H(zI - F)^{-1}G + L = \frac{p_m z^m + p_{m-1} z^{m-1} + \cdots + p_0}{q_m z^m + q_{m-1} z^{m-1} + \cdots + q_0} \ . \qquad (25)$$

The one–step delay between spark advance control and engine torque in (23) is attributed to the controller by fixing $q_0 = 0$.

The results presented in Section 4 are applied to the closed–loop system given by the plant (24) and the controller (25). The parameters of the compensator (25) are chosen so as to obtain a dead–beat controller for the nominal LTI system. The characteristic polynomial of the closed–loop system is

$$p(\lambda) = \lambda^{m+1} + \frac{q_{m-1} + \bar{b}p_m - \bar{a}q_m}{q_m}\lambda^m + \cdots + \frac{q_0 + \bar{b}p_1 - \bar{a}q_1}{q_m}\lambda + \frac{\bar{b}p_0 - \bar{a}q_0}{q_m} \ .$$

The nominal closed–loop system has all poles in zero, provided that the controller parameters verify

$$\begin{aligned} l_{m+1} &= q_m \\ 0 &= q_{m-1} + \bar{b}p_m - \bar{a}q_m \\ &\vdots \\ 0 &= \bar{b}p_0 - \bar{a}q_0 \ . \end{aligned} \qquad (26)$$

Condition $[C4]$ can be applied to verify the correctness of the proposed spark advance dead–beat controller. Since the open–loop dynamics is scalar, then $\Delta A_C(k)$ has rank one for any realization (16) of (25), hence $\Delta A_C(k)^T \Delta A_C(k)$ has an eigenvalue equal to the trace of $\Delta A(k)^T \Delta A(k)$ and $(n + m - 1)$ zero eigenvalues. In particular, for the canonical reachable–form realization of (25), the maximum singular value of the closed–loop perturbed matrix ΔA_C is upper bounded as follows

$$\sigma_{max}(\Delta A_C) \leq (\|\Delta_A(k)\| + \|\Delta_B(k)\| \left|\frac{p_m}{q_m}\right|)^2 + \|\Delta_B(k)\|^2 \sum_{i=0}^{m-1} (p_i - \frac{q_i p_m}{q_m})^2 \ . \quad (27)$$

In engine control, the drift term $A(k)n(k)$ in (24) is usually compensated by an inner control loop. In this case, (27) simplifies to

$$\sigma_{max}(\Delta A_C) \leq \left[\left(\frac{p_m}{q_m}\right)^2 + \sum_{i=0}^{m-1} \left(p_i - \frac{q_i p_m}{q_m}\right)^2 \right] \|B^c\|^2 \, \delta^2(k) \ . \qquad (28)$$

The robustness of dead-beat controllers with respect to the time–domain uncertainty given by the variability of dead–centers events is evaluated using condition $[C4]$. For given values of desired convergence rates in the continuous–time domain[9]for the closed–loop time–varying system, the controller parameters that maximize the admissible variation Δ of the sampling time and verify condition $[C4]$ are computed. The result is depicted in Figure 2. As expected, the bigger the desired convergence rate, the smaller the admissible variation on the sampling period.

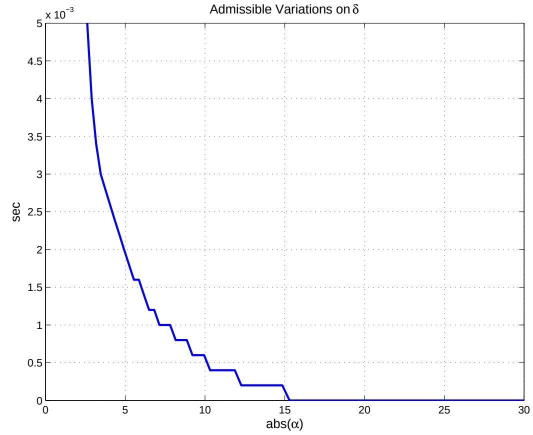

Fig. 2. Maximum variation of the sampling period for given convergence rate.

Condition $[C4]$ can also be used to obtain the largest time–domain uncertainty Δ for which a desired convergence rate is guaranteed for the time-varying closed–loop system, when there are some uncertainties in the plant model. In particular, bounded uncertainties for the drift term of the crankshaft dynamics are considered. The controller parameters that maximize the time–domain uncertainty bound Δ are computed according to condition $[C4]$. Figure 3 reports the result: the higher the desired convergence rate and plant model uncertainties, the smaller the allowed time–domain variation will be.

Finally, a dead–beat idle speed controller developed by Magneti Marelli Powertrain is considered. Some experimental results obtained with the controlled engine are reported in Figure 4. Table 1 reports the admissible uncertainties in terms of upper bounds on $\|\Delta_A\|$ and $\|\Delta_B\|$, for which closed–loop stability is guaranteed, according to conditions $[C1]$, $[C2]$, and $[C4]$. It is worthwhile to note that checking the condition $[C1]$ is feasible in this case since controller parameters are fixed (the stability test converged at the 6–th step). Condition $[C3]$ could not be used since the proposed controller does not satisfies the assumption of having nominal matrices with either norm-1, norm-2 or norm-∞ smaller than 1. From the upper bounds on $\|\Delta_A\|$ and $\|\Delta_B\|$, the corresponding maximum time–domain perturbations Δ are obtained. Since the non–uniformity of the dead–centers time domain depends on the crankshaft speed, then the bounds on sampling period Δ are converted into corresponding intervals for the crankshaft speed. Since $[C1]$ is more accurate then $[C2]$ and $[C4]$, then it gives much larger bounds. The range $450 - 1050$ rpm covers typical operating intervals of crankshaft speeds for idle engines, considering nominal idle speed equal to 750 rpm. According to the analysis, the proposed dead-beat controller has no guaranteed stability outside the range $450 - 1050$ rpm. The results given by

[9] The continuous–time convergence rate is computed as $1/\tau^0 \ln(\mu)$.

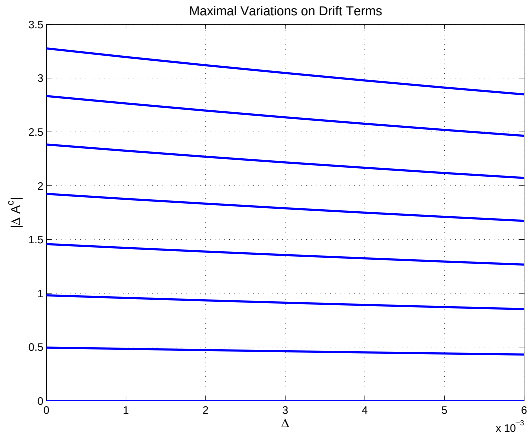

Fig. 3. Admissible uncertainties in the drift term p_A for given values of time–domain uncertainty Δ. The family of curves denote different desired convergence rate for the time-varying closed–loop system: the higher is the desired convergence rate, the smaller is the admissible uncertainties in the drift term.

bounds	$[C1]$	$[C2]$	$[C4]$
$\|\Delta_A\|/\|A\|$	34%	8%	1%
$\|\Delta_B\|/\|B\|$	42%	11%	1.3%
Δ [sec]	0.03	0.008	0.001
n [rpm])	$450 - 1050$	$670 - 830$	$740 - 760$

Table 1. Maximum time–domain variations for idle speed dead–beat controller tuned by Magneti Marelli Powertrain.

$[C2]$ and $[C4]$ are quite conservative and the corresponding ranges of crankshaft speed are not satisfactory.

6 Conclusions

This paper addressed the problem of robustly controlling non–periodically sampled dynamics. The motivating application is the design and verification of an idle speed controller for automotive applications. By sampling the continuous–time crankshaft dynamics at dead–center times, an event–based nonlinear time–variant model is obtained. Robust techniques developed in the last two decades for linear–time variant discrete systems are revised with the aim of proposing a design methodology that takes into account time–domain uncertainties. The proposed methodology has been applied to the design and verification of a dead–beat algorithm for idle speed control of an automotive engine. The largest

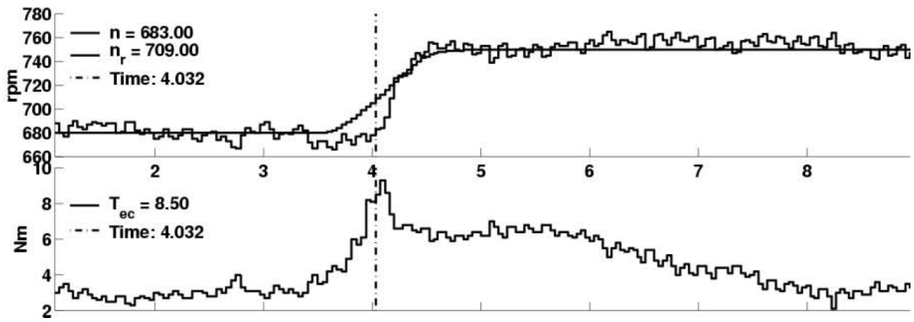

Fig. 4. Experimental results on idle speed control provided by Magneti Marelli Powertrain, for a reference idle speed n_r.

acceptable perturbations due to non-uniform sampling and plant uncertainties for which a desired rate of convergence is guaranteed were evaluated.

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