J. DIFFERENTIAL GEOMETRY
102 (2016) 501-535

CONTROLLING AREA BLOW-UP IN MINIMAL OR BOUNDED MEAN CURVATURE VARIETIES

Brian White

Abstract

Consider a sequence of minimal varieties M_i in a Riemannian manifold N such that the measures of the boundaries are uniformly bounded on compact sets. Let Z be the set of points at which the areas of the M_i blow up. We prove that Z behaves in some ways like a minimal variety without boundary. In particular, it satisfies the same maximum and barrier principles that a smooth minimal submanifold satisfies. For suitable open subsets W of N, this allows one to show that if the areas of the M_i are uniformly bounded on compact subsets of W, then the areas are in fact uniformly bounded on all compact subsets of N. Similar results are proved for varieties with bounded mean curvature. The results about area blow-up sets are used to show that the Allard Regularity Theorems can be applied in some situations where key hypotheses appear to be missing. In particular, we prove a version of the Allard Boundary Regularity Theorem that does not require any area bounds. For example, we prove that if a sequence of smooth minimal submanifolds converge as sets to a subset of a smooth, connected, properly embedded manifold with nonempty boundary, and if the convergence of the boundaries is smooth, then the convergence is smooth everywhere.

1. Introduction

If M is an embedded, m-dimensional manifold-with-boundary in a Riemannian manifold Ω and if U is a subset of Ω, then $|M|(U)$ and $|\partial M|(U)$ will denote the m-dimensional area of $M \cap U$ and the $(m-1)$-dimensional area of $(\partial M) \cap U$. For most of this article, the reader is not assumed to have any knowledge of varifolds, but for readers who do have such knowledge, if M is an m-dimensional varifold, then $|M|(U)$ denotes the mass of M in U (written $\|M\|(U)$ in [1] and $\mu_M(U)$ in [7]) and $|\partial M|(U)$ denotes the generalized boundary measure of M applied to the set U. (In [1], $|\partial M|(U)$ is written $\|\delta M\|_{\text{sing}}(U)$.)

Let M_i be a sequence of m-dimensional minimal varieties in a Riemannian manifold Ω or, more generally, varieties with mean curvature

Received 4/5/2014.

bounded by some $h < \infty$. Even more generally, the hypothesis that the mean curvature is bounded above by h can be replaced by the hypothesis that not "too much" of M_i has mean curvature $> h$, i.e., that

$$\limsup_{i\to\infty} \int_{M_i \cap K} (|H| - h)^+ \, dA < \infty$$

for every compact subset K of Ω, where H is the mean curvature vector and where t^+ denotes the positive part of t (i.e., $t^+ = \max\{t, 0\}$). We suppose that the boundaries have uniformly bounded measure in compact sets K:

$$\limsup_{i\to\infty} |\partial M_i|(K) < \infty.$$

Let Z be the set of points at which the areas of the M_i blow up:

$$Z = \{x \in \Omega : \limsup_i |M_i|(\mathbf{B}(x, r) = \infty \text{ for every } r > 0\}.$$

Equivalently, Z is the smallest closed subset of Ω such that the areas of the M_i are uniformly bounded as $i \to \infty$ on compact subsets of $\Omega \setminus Z$.

It is useful to have natural conditions that imply that Z is empty, since if Z is empty, then the areas of the M_i are uniformly bounded on all compact subsets of Ω and thus (for example) a subsequence of the M_i will converge as varifolds to a limit varifold of locally bounded first variation. This paper gives some such conditions. It also gives some properties shared by every such area blowup set Z.

First we prove that every such set Z satisfies the following maximum principle:

1.1. Theorem (Maximum Principle, §2.6). *If $f : \Omega \to \mathbf{R}$ is a C^2 function and if $f|Z$ has a local maximum at p, then*

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) \le h\, |\mathrm{D}f(p)|,$$

where $\mathrm{Trace}_m(\mathrm{D}^2 f(p))$ is the sum of the m lowest eigenvalues of the Hessian of f at p.

A closed set Z that satisfies the conclusion of Theorem 1.1 will be called an (m, h) set. The concept of an (m, h) set can be regarded as a generalization of the concept of an m-dimensional, properly embedded submanifold without boundary and with mean curvature bounded by h. In particular, if M is a smooth, properly embedded, m-dimensional submanifold without boundary, then M is an (m, h) set if and only if its mean curvature is bounded by h. (See Corollary 2.8 and Theorem 7.1.)

We also prove that any (m, h) set Z satisfies the same barrier principle that is satisfied by m-dimensional submanifolds of mean curvature bounded by h:

1.2. Theorem (Barrier Principle, §7.1). *Let Ω be a C^1 Riemannian manifold without boundary, and let Z be an (m, h) subset of Ω. Let N*

be a closed region in Ω *with smooth boundary such that* $Z \subset N$*, and let* $p \in Z \cap \partial N$*. Then*

$$\kappa_1 + \cdots + \kappa_m \leq h,$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{n-1}$ *are the principal curvatures of* ∂N *at* p *with respect to the unit normal that points into* N*.*

The converse is also true. (See Theorem 8.2.)

In the case $\dim(N) = m+1$, there is a also a strong barrier principle (Theorem 7.3): in the notation of Theorem 1.2, if $\dim(N) = m+1$, if the mean curvature of ∂N is everywhere greater than or equal to h (with respect to the normal that points into N), and if $Z \subset N$ is an (m, h) set that touches ∂N at a point p, then Z contains the entire connected component of ∂N containing p.

The support of every m-dimensional varifold with mean curvature bounded by h is an (m, h) set. (See Corollary 2.8.) Thus Theorem 1.2 includes as a special case the barrier principle for varifolds proved in [12].

The following theorem allows one to conclude in some circumstances that Z is empty:

1.3. Theorem (Constancy Theorem, §4.1)**.** *Suppose that an* (m, h) *set* Z *is a subset of a connected,* C^1 *properly embedded, m-dimensional submanifold* M *of the ambient space* Ω*. Then* $Z = \emptyset$ *or* $Z = M$*. In other words, the characteristic function of* Z *is constant on* M*.*

1.4. Corollary. *Let* Σ *be a closed, proper subset of a connected,* C^1*-embedded, m-dimensional submanifold* M *of* Ω *. Suppose that* M_i *is a sequence of m-dimensional varieties (or, more generally, varifolds) in* Ω *such that the boundary measures of the* M_i *are uniformly bounded on compact sets and such that the mean curvatures of the* M_i *are uniformly bounded. If the areas of the* M_i *are uniformly bounded on compact subsets of* $\Omega \setminus \Sigma$*, then they are also uniformly bounded on compact subsets of* Ω*.*

In Section 5, we use the results above (specifically, Corollary 1.4) to prove a theorem (§5.4) that extends Allard's Regularity Theorem in the case of integer-multiplicity varifolds. (Allard's Theorem holds more generally for varifolds with densities bounded below by 1, but our theorem is false under that weaker assumption: see §5.6.) For example, for minimal varieties, we have:

1.5. Theorem (§5.1)**.** *Suppose* M_i *is a sequence of proper m-dimensional minimal varieties-without-boundary (or stationary integral varifolds) in a Riemannian manifold* Ω*. Suppose the* M_i *converge as sets (see Remark* 1.6*) to a subset of an m-dimensional, connected,* C^1 *properly embedded submanifold* M *of* Ω*. If the* M_i *converge weakly to* M *with multiplicity 1 anywhere, then they converge smoothly to* M *everywhere.*

The key word here is "anywhere": to invoke Allard's theorem directly, one needs to assume the weak, multiplicity 1 convergence $M_i \to M$ everywhere.

An analogous result (§5.4) holds if the M_i have uniformly bounded mean curvatures.

Allard's Regularity Theorem does not require bounded mean curvature, but rather only mean curvature in $\mathcal{L}^p$ for some p greater than the dimension. Similarly, Theorem 5.4 does not require that the surfaces M_i in question have bounded mean curvature, but rather that they satisfy the weaker hypothesis that

$$\limsup_i \int_{M_i \cap K} (|H| - h)^+)^p \, dA < \infty \tag{1}$$

for some $h < \infty$, for some $p > m$, and for every compact $K \subset \Omega$. In that case, the conclusion is not smooth convergence but rather C^1 convergence with local $C^{1,1-m/p}$ bounds.

1.6. Remark. In Theorem 1.5 (and elsewhere in this paper), we say that the sets $S_i \subset \Omega$ *converge as sets* to $S \subset \Omega$ provided that

$$S = \{x \in \Omega : \limsup_i d(x, S_i) = 0\} = \{x \in \Omega : \liminf_i d(x, S_i) = 0\},$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$. This notion (due to Kuratowski) is in general weaker than convergence with respect to the Hausdorff metric on closed sets. (For compact Ω, the two notions are equivalent.) Such convergence has a nice compactness property: if S_i is a sequence of closed subsets of a Riemannian manifold Ω, then (according to the Arzela–Ascoli theorem) by passing to a subsequence, we can assume that the functions $d(\cdot, S_i)$ converge to a limit function f. It follows that the S_i converge as sets to the zero set of f.

Section 6 gives a version of Allard's Boundary Regularity Theorem that does not assume any area bounds.

Sections 9 and 10 give additional results for the case of codimension 1 varieties, i.e., the case of (m, h) sets in an $(m+1)$-dimensional manifold. For example, as a special case of those results, we have the following:

1.7. Theorem. *Let $m < 7$, and let N be closed, mean convex region in $\mathbf{R}^{m+1}$ with smooth boundary. Suppose that ∂N is not a minimal surface, and that N does not contain any smooth, stable, properly embedded minimal hypersurface. If Z is an $(m, 0)$ set contained in N, then $Z = \emptyset$.*

(We remark that in $\mathbf{R}^3$, the hypothesis that N not contain a smooth, stable properly embedded minimal hypersurface is redundant.)

We also prove (Corollaries 7.4 and 9.2) that the Hoffman–Meeks Halfspace Theorems for proper minimal surfaces in $\mathbf{R}^3$ hold for arbitrary $(2, 0)$ sets in $\mathbf{R}^3$.

Finally, in Section 11, we prove

1.8. Theorem. *If $Z \subset \mathbf{R}^n$ is an (m,h) set, then the set $Z(s)$ of points at distance $\le s$ from Z is also an (m,h) set.*

We also prove an analogous result for (m,h) subsets of Riemannian manifolds.

Readers who are interested in minimal varieties (rather than varieties of bounded curvature) may skip Section 10 and much of Sections 5 and 6. (The portions of 5 and 6 that may be skipped are indicated there.)

Acknowledgments. This research was supported by the National Science Foundation under grants DMS 0406209, DMS 1105330, and DMS 1404282.

2. Area Blow-Up

2.1. Definition. Let Ω be a smooth manifold without boundary and with a C^1 Riemannian metric g. Let Z be a closed subset of Ω. We say that Z is an (m,h) *subset* of (Ω, g) provided it has the following property: if $f : \Omega \to \mathbf{R}$ is a C^2 function such that $f|Z$ has a local maximum at p, then

$$\text{Trace}_m(\mathrm{D}^2 f(p)) \le h\,|\mathrm{D}f(p)|. \tag{2}$$

If there is such a function f for which (2) does not hold, we say that Z fails to be an (m,h) set at the point p.

Here $|\mathrm{D}f|$ is the norm of the derivative of f (or, equivalently, the norm of the gradient of f) with respect to the metric g, and $\text{Trace}_m(\mathrm{D}^2 f)$ is the sum of the lowest m eigenvalues of the Hessian of f with respect to the metric g. In other words, $\text{Trace}_m(\mathrm{D}^2 f(p))$ is the sum of the lowest m eigenvalues of the matrix of second partial derivatives of f in any system of normal coordinates at p. Note this matrix is also the matrix (with respect to the same normal coordinates) for the endomorphism $\mathrm{D}(\nabla f(p))$, where D is the covariant derivative (with respect to the Levi–Civita connection) and ∇f is the gradient of f. Thus $\text{Trace}_m(\mathrm{D}^2 f)$ is equal to $\text{Trace}_m(\mathrm{D}\nabla f)$.

2.2. Remark. Let $Z \subset \Omega$ be a closed set. It follows immediately from the definition that the set of h for which Z is an (m,h) set either is empty or has the form $[\eta, \infty)$ for some $0 \le \eta < \infty$.

2.3. Remark. Suppose N is a smooth Riemannian n-manifold with boundary and with a C^1 Riemannian metric g. Then N can be embedded into a smooth open n-manifold Ω and the metric g can be extended to be a C^1 Riemannian metric on all of Ω. A closed subset Z of N is called an (m,h) subset of (N,g) if and only if it is an (m,h) subset of (Ω, g). It is straightforward to show that this condition is indepedent of the choice of Ω and of choice of the extension of the metric.

The following lemma implies that in the definition of (m,h) subset, it suffices to consider test functions f with additional properties:

2.4. Lemma. *Suppose $Z \subset \Omega$ is a closed subset that fails to be an (m,h) subset at the point $p \in Z$. Then there is a C^2 function $f : \Omega \to \mathbf{R}$ such that we have the following*

(1) $\mathrm{Trace}_m(\mathrm{D}^2 f(p)) > h\,|\mathrm{D}f(p)|$.

(2) *The restriction of f to Z attains its maximum value of 0 uniquely at the point p:*

$$f(x) < f(p) = 0 \textit{ for all } x \in Z,\ x \neq p.$$

(3) *The set $\{x : f(x) \geq a\}$ is compact for every $a \in \mathbf{R}$. Indeed, if $u : \Omega \to (-\infty, 0]$ is any smooth, proper function, we can choose f so that f coincides with u outside of some compact set.*

Proof. By hypothesis, there is a C^2 function $f : \Omega \to \mathbf{R}$ such that (1) holds and such that $f|Z$ has a local maximum at p,

$$\max f|Z \cap \mathbf{B} = f(p),$$

where $\mathbf{B}$ is some open neighborhood of p. By replacing f by $f - f(p)$, we can assume that $f(p) = 0$.

Let $u : \Omega \to (-\infty, 0]$ be a smooth, proper function. By modifying u on a compact set, we can assume that $u(p) = 0$, that $u(x) < 0$ for all $x \neq p$, and that $\mathrm{D}^2 u(p) = 0$.

Let $\phi : \Omega \to \mathbf{R}$ be a smooth, nonnegative function that is supported in $\mathbf{B}$ and that is equal to 1 in some neighborhood of p. Replacing f by $\phi f + u$ gives a function with all the asserted properties. q.e.d.

The following corollary says that we can choose the function f in Lemma 2.4 to be smooth (not just C^2), provided we are allowed to move the point p slightly:

2.5. Corollary. *Suppose $Z \subset \Omega$ is a closed subset that fails to be an (m,h) subset at the point $q \in Z$. Then there is a point $p \in Z$ (which may be chosen arbitrarily close to q) and a smooth function $f : \Omega \to \mathbf{R}$ having the properties asserted in Lemma 2.4.*

Proof. Let f be a C^2 function having all the properties asserted by Lemma 2.4 with q in place of p. Let $f_i : \Omega \to \mathbf{R}$ be a sequence of smooth functions such that f_i converges to f uniformly and also locally in C^2. It follows that each $f_i|Z$ attains its maximum at some point p_i, and that the p_i converge to q. Furthermore, the local C^2 convergence implies that

$$\mathrm{Trace}_m(\mathrm{D}^2 f_i(p_i)) > h\,|\mathrm{D}f_i(p_i)|$$

for all sufficiently large i. For each such i, we can modify f_i exactly as in the proof of Lemma 2.4 to get a smooth function $\tilde{f}_i$ that has properties (1), (2), and (3) (with $\tilde{f}_i$ and p_i in place of f and p.) q.e.d.

2.6. Theorem. *Let Ω be a smooth, n-dimensional manifold without boundary. Let g_i ($i = 1, 2, 3, \dots$) and g be C^1 Riemannian metrics on Ω such that the g_i converge to g in C^1.*

For each i, let M_i be an m-dimensional varifold in Ω such that the mean curvature of M_i with respect to g_i is bounded by $h < \infty$ and such that the boundaries of the M_i are uniformly bounded on compact subsets of Ω:

$$\limsup_i |\partial M_i|(U) < \infty \textit{ for all } U \subset\subset \Omega. \tag{3}$$

Let

$$Z = \{x \in \Omega : \limsup_i |M_i|(\mathbf{B}(x, r)) = \infty \textit{ for every } r > 0\}. \tag{4}$$

Then Z is an (m, h) subset of Ω.

More generally, the conclusion remains true if the hypothesis that the mean curvatures are bounded by h is replaced by the hypothesis that

$$\limsup_i \int_{M_i \cap K} (|H| - h)^+ \, dA < \infty \tag{5}$$

for every compact $K \subset \Omega$, where H is the mean curvature vector and where $t^+ = \max\{t, 0\}$.

2.7. Remark. Readers who are primarily interested in minimal varieties may wish to read the following proof under that assumption that the M_i are minimal, i.e., that $h = 0$: in that case a number of terms in the proof drop out. Similarly, readers primarily interested in bounded mean curvature varieties may wish to make the assumption that M_i has mean curvature bounded by h (instead of making the more general assumption (5)), since a few terms in the proof then drop out.

Proof. To simplify notation, we give the proof in the case where the M_i are properly embedded manifolds-with-boundary. But (aside from the notation) exactly the same proof works for general varifolds. We prove the result by contradiction. Thus suppose Z fails to be an (m, h) set at a point $p \in Z$.

By Lemma 2.4, there is a C^2 function $f : \Omega \to \mathbf{R}$ such that

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) > h\,|\mathrm{D}f(p)|, \tag{6}$$

$$f(x) < f(p) \text{ for every } x \in Z \setminus \{p\}, \text{ and} \tag{7}$$

$$\{f \ge a\} \text{ is compact for every } a \in \mathbf{R}. \tag{8}$$

Choose $\delta > 0$ so that

$$\left[\mathrm{Trace}_m(\mathrm{D}^2 f)(p) - h\,|\mathrm{D}f(p)|\,\right]_g > \delta,$$

where the subscript g indicates that the expression inside the brackets is with respect to the metric g. Let $\mathbf{B} \subset \Omega$ be a compact ball centered at p such that

$$\min_{\mathbf{B}} \left[\mathrm{Trace}_m(\mathrm{D}^2 f) - h\,|\mathrm{D}f|\,\right]_g > \delta. \tag{9}$$

(Such a set $\mathbf{B}$ exists because the inequality $[\mathrm{Trace}_m(\mathrm{D}^2 f) - h\,|\mathrm{D}f|\,]_g > \delta$ defines an open set containing p.)

By (7) and (8),

$$\max_{Z \setminus \mathrm{interior}(\mathbf{B})} f < f(p).$$

By adding a constant to f, we can assume that

$$\max_{Z \setminus \mathrm{interior}(\mathbf{B})} f < 0 < f(p). \tag{10}$$

Let $N = \{f \geq 0\}$. By (8) and (10), $N \setminus \mathrm{interior}(\mathbf{B})$ is a compact subset of Z^c, so by definition of Z,

$$\limsup_i |M_i|\,(N \setminus \mathbf{B}) < \infty. \tag{11}$$

Let $\mathbf{B}^*$ be a small closed ball centered at p such that $\mathbf{B}^*$ is in the interior of $\mathbf{B} \cap N$. Choose constants Γ, $\gamma > 0$, and $\tau \geq 0$ such that

$$\max_N f\,|\mathrm{D}f|_g < \Gamma, \tag{12}$$

$$\min_{\mathbf{B}^*} f > \gamma, \quad \text{and} \tag{13}$$

$$\min_N [\,f\ \mathrm{Trace}_m(\mathrm{D}^2 f)]_g > -\tau. \tag{14}$$

Note that the left sides of (9), (12), and (14) all depend C^1-continuously on the metric g. Thus for all sufficiently large i, the inequalities hold with g_i in place of g; for the rest of the proof, we restrict ourselves to such i. All metric-dependent quantities below are with respect to g_i.

Let X_i be the gradient of $\frac{1}{2} f^2$ with respect to the metric g_i:

$$X_i = \nabla\left(\frac{1}{2} f^2\right) = f \nabla f.$$

Thus

$$|X_i| = f\,|\mathrm{D}f| \leq \Gamma$$

on N by (12). Also,

$$\mathrm{D}X_i = f\mathrm{D}(\nabla f) + \mathrm{D}f \otimes \nabla f,$$

so

$$\mathrm{Trace}_m(\mathrm{D}X_i) = \mathrm{Trace}_m(f\mathrm{D}(\nabla f) + \mathrm{D}f \otimes \nabla f). \tag{15}$$

The endomorphisms $f\mathrm{D}(\nabla f)$ and $\mathrm{D}f \otimes \nabla f$ of the tangent space (to the ambient space) correspond (using the metric) to the quadratic forms $f\mathrm{D}^2 f$ and $df \otimes df$, so (15) implies that

$$\begin{aligned} \mathrm{Trace}_m(\mathrm{D}X_i) &= \mathrm{Trace}_m(f\mathrm{D}^2 f + df \otimes df) \\ &\geq \mathrm{Trace}_m(f\mathrm{D}^2 f). \end{aligned}$$

This last inequality holds because $df \otimes df$ is positive semidefinite, which implies that the eigenvalues of $f\mathrm{D}^2 f + df \otimes df$ are bounded below by the corresponding eigenvalues of $f\mathrm{D}^2 f$. (See Lemma 12.3.) Thus

$$\mathrm{Trace}_m(\mathrm{D}X_i) \ge \mathrm{Trace}_m(f\mathrm{D}^2 f) = f\,\mathrm{Trace}_m(\mathrm{D}^2 f)$$

on N.

Since $\mathrm{div}_{M_i} X_i \ge \mathrm{Trace}_m(\mathrm{D}X_i)$, this implies that

$$\mathrm{div}_{M_i} X_i \ge \begin{cases} f\,(\,|\mathrm{D}f|\,h + \delta) & \text{on } N \cap \mathbf{B} \\ -\tau & \text{on } N \setminus \mathbf{B} \end{cases} \tag{16}$$

by (9) and (14) (for g_i).

Since $X_i = 0$ on ∂N, we have

$$\begin{aligned} \int_{M_i \cap N} \mathrm{div}_{M_i} X_i \, dA &= \int_{M_i \cap N} -H \cdot X_i \, dA + \int_{\partial M_i \cap N} X_i \cdot \nu \, dS \\ &\le \int_{M_i \cap N} h\,|X_i|\,dA + \int_{M_i \cap N} (|H| - h)^+\,|X_i|\,dA + \int_{\partial M_i \cap N} |X_i|\,dS \\ &\le \int_{M_i \cap N} hf\,|\mathrm{D}f|\,dA + \Gamma \int_{M_i \cap N} (|H| - h)^+\,dA + \Gamma\,|\partial M_i|(N) \\ &\le \int_{M_i \cap N} hf\,|\mathrm{D}f|\,dA + O(1), \end{aligned}$$

where $O(1)$ stands for any quantity that is bounded independent of i. Thus

$$\int_{M_i \cap N \cap \mathbf{B}} (\mathrm{div}_{M_i} X_i - hf\,|\mathrm{D}f|\,)\,dA \le \int_{M_i \cap (N \setminus \mathbf{B})} (hf\;|\mathrm{D}f| - \mathrm{div}_{M_i} X_i)\,dA + O(1).$$

Thus by (16),

$$\begin{aligned} \int_{M_i \cap N \cap \mathbf{B}} \delta f \, dA &\le \int_{M_i \cap (N \setminus \mathbf{B})} (hf\,|\mathrm{D}f| + \tau)\,dA + O(1) \\ &\le (\Gamma h + \tau)\,|M_i|\,(N \setminus \mathbf{B}) + O(1) \\ &\le O(1), \end{aligned}$$

where the last step is by (11). Since $\mathbf{B}^* \subset N \cap \mathbf{B}$ and since $f > \gamma$ on $\mathbf{B}^*$, this implies that

$$\delta\gamma\,|M_i|\,(\mathbf{B}^*) \le O(1). \tag{17}$$

However, the left side of (17) is unbounded since $p \in Z$ and $\mathbf{B}^*$ is a ball centered at p. The contradiction proves the theorem. q.e.d.

2.8. Corollary. *Let M be a proper, m-dimensional submanifold of Ω with no boundary and with mean curvature everywhere $\le h$. Then M is an (m, h) subset of Ω.*

More generally, let M be an m-dimensional varifold (not necessarily rectifiable) of locally bounded first variation with mean curvature everywhere $\le h$ and with no generalized boundary. Then the support of M is an (m, h) subset of Ω.

Proof. If M is a manifold, let M_i (for $i = 1, 2, \dots$) be obtained by multiplying the multiplicity of M everywhere by i. Then the area blow-up set Z is M itself, and so $M = Z$ is an (m, h) set by Theorem 2.6. Similarly, if M is a general m-varifold (i.e., a measure on a certain Grassman bundle), one lets M_i be the result of multiplying M by i. Exactly the same argument shows that the support of M is an (m, h) set. q.e.d.

Conversely, if a smooth m-dimensional manifold is an (m, h) set, then its mean curvature is everywhere $\leq h$. (This follows from the Barrier Principle 7.1.)

3. Limits of (m, h) subsets

3.1. Theorem. *Suppose for $i = 1, 2, 3, \dots$ that Z_i is an (m, h_i) subset of C^1 Riemannian manifold (Ω, g_i). Suppose that the g_i converge in C^1 to a Riemannian metric g, that the Z_i converge as sets (see remark 1.6) to a closed set Z, and that the h_i converge to a limit h.*

Then Z is an (m, h) subset of (Ω, g).

Proof. We prove the theorem by contradiction. Suppose that Z fails to be an (m, h) subset at some point $p \in Z$. By Lemma 2.4, there is a C^2 function $f : \Omega \to \mathbf{R}$ such that

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) > h\,|\mathrm{D}f(p)| \tag{18}$$

$$f(x) < f(p) \text{ for every } x \in Z \setminus \{p\}, \text{ and} \tag{19}$$

$$\{f \geq a\} \text{ is compact for every } a \in \mathbf{R}. \tag{20}$$

Now Z_i is nonempty for all sufficiently large i, so by properness (20), $f|Z_i$ will attain its maximum at a point p_i. Furthermore, p_i converges to p as $i \to \infty$ (by (19) and (20)). By (18), and by the convergence of g_i to g, h_i to h, and p_i to p,

$$[\mathrm{Trace}_m(\mathrm{D}^2 f(p_i)) - h_i\,|\mathrm{D}f(p_i)|]_{g_i} > 0,$$

for all sufficiently large i, contradicting the hypothesis that Z_i is an (m, h_i) subset of (Ω, g_i). q.e.d.

3.2. Corollary. *Suppose Z is an (m, h) subset of (Ω, g), where Ω is an open subset of $\mathbf{R}^n$ containing the origin, g is a C^1 Riemannian metric on Ω, and $g_{ij}(0)$ is the Euclidean metric δ_{ij}. Let λ_i be a sequence positive numbers tending to ∞, and suppose that the dilated sets*

$$\lambda_i Z := \{\lambda_i p : p \in Z\}$$

converge to a limit set Z^*. Then Z^* is an $(m, 0)$ subset of $\mathbf{R}^n$ (with respect to the Euclidean metric).

Proof. Let g_i be the metric on $\lambda_i\Omega$ obtained from g by dilation. (In other words, g_i is the result of pushing forward the metric g by the map $x \mapsto \lambda_i x$, and then multiplying by λ_i^2.)

Then Z_i is an $(m, h/\lambda_i)$ subset of (Ω_i, g_i), so by Theorem 3.1, Z is an $(m, 0)$ subset of $\mathbf{R}^n$. q.e.d.

4. The Constancy Theorem

4.1. Theorem (Constancy Theorem)**.** *Let Ω be an open subset of a manifold with C^1 Riemannian metric g. Let Z be an (m, h) set in (Ω, g). Suppose Z is a subset of a connected, m-dimensional, properly embedded submanifold M of Ω. Then $Z = \emptyset$ or $Z = M$. In other words, the characteristic function of Z is constant on M.*

Proof. The result is essentially local, so we may assume that $\Omega \subset \mathbf{R}^n$. Suppose the result is false, i.e., that Z is a nonempty proper subset of M. Then $M \setminus Z$ contains an open geodesic ball B whose boundary contains a point $p \in Z$. (See Lemma 4.3 below if that is not clear.) By translation, we can assume that $p = 0$. By making a linear change of coordinates, we may assume that the metric g is the Euclidean metric at 0 (i.e., that $g_{ij}(0) = \delta_{ij}$.)

Now let λ_i be a sequence of positive numbers such that $\lambda_i \to \infty$. Note that the sets

$$\lambda_i(M \setminus B) := \{\lambda_i x : x \in M \setminus B\}$$

converge to a closed halfspace H of $\mathrm{Tan}_0 M$ with $0 \in \partial H$. Thus by passing to a subsequence, we can assume that the sets $\lambda_i Z$ converge to a closed subset Z^* of H with $0 \in Z^* \cap \partial H$. By rotating, we can assume that H is the halfplane

$$H = \{x \in \mathbf{R}^n : x_1 \le 0 \text{ and } x_i = 0 \text{ for all } i > m\}.$$

By Corollary 3.2, Z^* is an $(m, 0)$ subset of $\mathbf{R}^n$ (with respect to the Euclidean metric).

Now consider the function

$$f : \mathbf{R}^n \to \mathbf{R},$$
$$f(x) = x_1 + (x_1)^2 + \sum_{i>m} (x_i)^2.$$

Note that $f|H$ has a local maximum at 0, so $f|Z^*$ has a local maximum at 0. But

$$\mathrm{Trace}_m(\mathrm{D}^2 f(0)) = 2 > 1 = |\mathrm{D}f(0)|,$$

contradicting the fact that Z^* is an $(m, 0)$ set. q.e.d.

4.2. Corollary. *Suppose that Z is an (m, h) subset of Ω. Suppose also that Z is contained in M, where M is either*

1) a C^1 submanifold of dimension $\le m - 1$, or

2) a connected, m-dimensional, C^1 manifold-with-boundary such that the boundary is nonempty.

Then $Z = \emptyset$.

Proof. Note that (in either case) M is contained in an m-dimensional, C^1 manifold $\hat{M}$ without boundary. Now apply the Constancy Theorem 4.1 to Z and $\hat{M}$. q.e.d.

4.3. Lemma. *Let M be a connected Riemannian manifold without boundary. Let K be a proper, nonempty, closed subset of M. Then $M \setminus K$ contains an open geodesic ball B whose boundary contains a point in K.*

Proof. Let q be a point in the boundary of K, i.e, in $K \cap \overline{M \setminus K}$. Choose a point $p \in M \setminus K$ sufficiently close to q that the closed geodesic ball of radius $\operatorname{dist}(p, q)$ about p is compact. Then the open geodesic ball of radius $\operatorname{dist}(p, K)$ centered at p has the desired properties. q.e.d.

5. Versions of Allard's Regularity Theorem

We begin with the case of minimal varieties:

5.1. Theorem. *Let Ω be a smooth Riemannian manifold (not necessarily complete). Let $M_i \subset \Omega$ be a sequence of m-dimensional, properly embedded minimal submanifolds without boundary. Suppose that the M_i converge as sets (see Remark* 1.6*) to a subset of an m-dimensional, connected, smoothly embedded submanifold M of Ω. Suppose also that some point in M has a neighborhood $U \subset \Omega$ such that $M_i \cap U$ converges weakly to $M \cap U$ with multiplicity 1, i.e., such that*

$$(*) \qquad \int_{M_i} \phi \, dA \to \int_M \phi \, dA$$

for every continuous, compactly supported function $\phi : U \to \mathbf{R}$. Then M_i converges to M smoothly and with multiplicity 1 everywhere.

The result remains true if each M_i is minimal with respect to a Riemannian metric g_i provided the metrics g_i converge smoothly to a limit Riemannian metric. The result is also true if each M_i is a g_i-stationary integral varifold or, more generally, a g_i-stationary varifold with density ≥ 1 at every point in its support.

Proof. By Theorem 2.6, the area blow-up set Z is an $(m, 0)$ set. By hypothesis, the area blowup set Z is disjoint from U and is therefore a proper subset of M. Hence by the Constancy Theorem 4.1, $Z = \emptyset$. In other words, the areas of the M_i are uniformly bounded on compact subsets of Ω. Thus (after passing to a subsequence) the M_i converge in the varifold sense to a stationary varifold V supported in M.

By the constancy theorem for stationary varifolds ([1, §4.6(3)] or [7, §41]), V is M with some constant multiplicity. By hypothesis, the multiplicity is equal to 1 in U. Therefore it is equal to 1 everywhere.

But then the convergence $M_i \to M$ is smooth by the Allard Regularity Theorem. (More precisely, the convergence is $C^{1,\alpha}$ for some $\alpha > 0$ by Allard's theorem, which then implies by standard elliptic regularity that the convergence is smooth.) q.e.d.

5.2. Remark. In the case where the M_i are smooth minimal submanifolds, the proof actually requires very little geometric measure theory. In particular, the existence of a varifold limit and the constancy theorem follow rather directly from the definition of varifold. And if the M_i's are smooth, the required version of the Allard Regularity Theorem has a very elementary proof: see [10, Theorem 1.1]. (The proof in [10] is for compact M, but with minor modification, the proof works for noncompact M.)

5.3. Remark. The multiplicity 1 hypothesis (*) here (and also in Theorem 5.4) can be weakened to

$$\limsup_i \int_{M_i} \phi \, dA \le \int_M \phi \, dA$$

for every continuous, nonnegative, compactly supported function $f : U \to \mathbf{R}$, provided we assume that the M_i converge as sets to a *nonempty* subset of M. The proof is almost exactly as before.

Readers interested in minimal (rather than bounded mean curvature) varieties may skip the rest of this section.

5.4. Theorem. *Let Ω be a smooth Riemannian manifold (not necessarily complete). Let $M_i \subset \Omega$ be a sequence of m-dimensional submanifolds without boundary such that*

$$\limsup_i \int_{M_i \cap K} ((|H| - h)^+)^p \, dA < \infty \tag{21}$$

for some some $h < \infty$, some $p > m$, and for every compact $K \subset \Omega$. Suppose that the M_i converge as sets (see Remark 1.6*) to a subset of an m-dimensional, connected,* C^1 *embedded submanifold M of Ω. Suppose also that some point in M has a neighborhood $U \subset \Omega$ such that $M_i \cap U$ converges weakly to $M \cap U$ with multiplicity 1. Then M_i converges to M in* C^1*. Furthermore, the M_i are locally uniformly bounded in $C^{1,1-m/p}$:*

$$\limsup_i \left(\sup_{x,y \in M_i \cap K,\, x \ne y} \frac{d(\mathrm{Tan}(M_i, x), \mathrm{Tan}(M_i, y))}{d(x,y)^{1-m/p}} \right) < \infty$$

for every compact $K \subset \Omega$.

The result remains true if the M_i are integer-multiplicity rectifiable varifolds or, more generally, if the M_i are varifolds with the gap α property (see Definition 5.5*, below) for some $\alpha > 1$.*

The multiplicity 1 hypothesis in U can be weakened slightly; see Remark 5.3.

Unlike the minimal case (Theorem 5.1), Theorem 5.4 fails for varifolds if the gap α hypothesis is replaced by the weaker hypothesis that the density is ≥ 1 almost everywhere. See §5.6 for an example of such failure.

5.5. Definition. Let V be an rectifiable m-varifold and $\alpha > 1$. We say that V has the *gap α property* if the density $\Theta(V,x)$ of V at x belongs to

$$\{1\} \cup [\alpha, \infty)$$

for μ_V almost every x.

Proof of Theorem 5.4. Let $h' = h+1$. Then

$$(|H| - h')^+ \leq ((|H|-h)^+)^p.$$

So, by (21),

$$\limsup_i \int_{M_i \cap K} (|H| - h')^+ \, dA < \infty$$

for every compact $K \subset \Omega$. Hence, exactly as in the proof of Theorem 5.1, the areas of the M_i must be uniformly bounded on compact sets. Thus by passing to a subsequence, we can assume that the M_i's converge to a limit varifold V. Also,

$$\begin{aligned} \left(\int_{M_i \cap K} |H|^p \, dA \right)^{1/p} &\leq \left(\int_{M_i \cap K} h^p \, dA \right)^{1/p} + \left(\int_{M_i \cap K} ((|H|-h)^+)^p \, dA \right)^{1/p} \\ &\leq h \operatorname{area}(M_i \cap K)^{1/p} + \left(\int_{M_i \cap K} ((|H|-h)^+)^p \, dA \right)^{1/p}. \end{aligned}$$

Since the area of $M_i \cap K$ is bounded as $i \to \infty$, the hypothesis (21) implies that

$$\limsup_i \left(\int_{M_i \cap K} |H|^p \, dA \right)^{1/p} < \infty.$$

Recall that the density of V at x is

$$\Theta(V,x) = \lim_{r \to 0} \frac{|V|\mathbf{B}(x,r)}{\omega_m r^m},$$

provided the limit exists, where ω_m is the volume of the unit ball in $\mathbf{R}^m$. By the monotonicity formula for the M_i's (which implies the same monotoncity for V), $\Theta(V,x)$ exists everywhere and is upper semicontinuous in x, and it also has the following property:

$$\Theta(V,x) \geq \limsup \Theta(M_i, x_i) \text{ provided } x_i \to x.$$

(See, for example, [7, §17.8] for proof of these upper-semicontinuity properties.) In particular, $\Theta(V,x) \geq 1$ for every point in $\operatorname{spt}(V)$.

Now let W be the set of points where $\Theta(V,x) = 1$. By hypothesis, W is a nonempty subset of M.

We claim that W is a relatively open subset of M. To see that, suppose $x \in W$, i.e., that $\Theta(V,x) = 1$. By the Allard Regularity Theorem, there is a open ball $B \subset \Omega$ around x such that for all sufficiently large $i \geq i_0$, $\text{spt}(M_i) \cap B$ is a C^1 submanifold and such that the $\text{spt}(M_i) \cap U$ converge in C^1 to $M \cap U$. By the upper-semicontinuity of density [7, §17.8], and by replacing B by a smaller open ball around x and by replacing i_0 by a larger number, we can assume that $\Theta(M_i, \cdot) < \alpha$ at almost all points of $\text{spt}(M_i) \cap B$ for $i \geq i_0$. By the gap α property, $\Theta(M_i, \cdot) = 1$ at almost all points of $\text{spt}(M_i) \cap B$ for $i \geq i_0$. Hence the measures μ_{M_i} and $\mathcal{H}^m \llcorner (\text{spt}(M_i))$ coincide in B, which implies (because of the C^1 convergence) that μ_V and $\mathcal{H}^m \llcorner M$ also coincide in B. That in turn implies that $\Theta(V, \cdot) \equiv 1$ in $B \cap M$, so $B \cap M \subset W$. This proves that W is a relatively open subset of M.

Now if $W \neq M$, then there would be an open geodesic ball D in W and a point $x \in \overline{D} \cap W^c$. (Recall Lemma 4.3.) By definition of W,

$$\Theta(V, x) \neq 1.$$

Now the tangent cone C to V at x is a plane with multiplicity $\Theta(V, x)$. (The multiplicity is constant on the plane by the constancy theorem for stationary varifolds ([1, §4.6(3)] or [7, §41]). However, the multiplicity is equal to 1 on a halfplane of that cone–namely, the tangent halfplane to D at x–so $\Theta(V, x) = 1$, contradicting the fact that $x \notin W$. The contradiction proves that $W = M$, i.e, that $\Theta(V, \cdot) \equiv 1$ on M. The conclusion then follows from the Allard Regularity Theorem. q.e.d.

5.6. A counterexample. As mentioned above, Theorem 5.4 fails if the gap α hypothesis is replaced by the hypothesis that the density ≥ 1 almost everywhere. We now give an example of that failure. Let $g : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $g(x) = 0$ if and only if $|x| \geq 1$. Let S_n be the union of the graph of $(1/n)g$ and the x-axis (i.e., the graph of the 0 function). Let $\phi : [1, \infty) \to [1, 2]$ be a smooth function such that $\phi(t) = 2$ for $1 \leq t \leq 2$ and such that $\phi(x) = 1$ for $x \geq 3$.

Now let M_n be the rectifiable varifold whose support is S_n and whose density $\Theta(V, (x, y))$ at $(x, y) \in S_n$ is 1 if $|x| < 1$ and $\phi(|x|)$ for $|x| \geq 1$. Let M be the x-axis. Then M_n, M, and $\Omega = \mathbf{R}^2$ satisfy all the hypotheses except for the gap α hypothesis. Also, $\Theta(M_n, \cdot) \geq 1$ at every point of $\text{spt}(M_n)$, i.e., at every point of S_n. However, we do not have C^1 convergence $\text{spt}(M_i) \to M$. Indeed, none of the M_i are C^1 at the points $(1, 0)$ and $(-1, 0)$.

6. Versions of Allard's Boundary Regularity Theorem

6.1. Theorem. *Let Ω be a smooth Riemannian manifold, and let $M \subset \Omega$ be an m-dimensional smooth, connected, properly embedded manifold-with-boundary such that ∂M is smooth and nonempty. Let M_i be a* *sequence of properly embedded m-dimensional minimal submanifolds-with-boundary of Ω such that the M_i converge as sets to a subset of M, and such that the boundaries ∂M_i converge smoothly to ∂M. Then M_i converges smoothly to M.*

The result remains true if each M_i is minimal with respect to a Riemannian metric g_i provided the metrics g_i converge smoothly to a limit Riemannian metric.

See §6.2 for a generalization to submanifolds M_i of bounded mean curvature or (even more generally) to varifolds with $(|H| - h)^+$ in $\mathcal{L}^p$ for some $p > m$.

Note that we are not assuming any area bounds. To deduce the smooth convergence $M_i \to M$ directly from Allard's Regularity Theorems (boundary and interior), one would need to assume that the M_i converge weakly (in the sense of Radon measures) to M with multiplicity 1. Indeed, we prove Theorem 6.1 by deducing weak, multiplicity 1 convergence from the hypotheses.

Proof. The area blow-up set of the M_i is an $(m, 0)$ set by Theorem 2.6, and it is contained in a connected m-manifold with nonempty boundary, so it is empty by Corollary 4.2. That is, the areas of the M_i are uniformly bounded on compact subsets of Ω. Thus by passing to a subsequence, we can assume that the M_i converge as varifolds to a varifold V supported in M.

Let X be a compactly supported smooth vectorfield on Ω. If we think of M_i as a rectifiable varifold (by assigning it multiplicity 1 everywhere), recall that its first variation operator δM_i is given by

$$\begin{aligned}\delta M_i(X) &= \int_{M_i} \operatorname{div}_{M_i} X \, d\mathcal{H}^m \\ &= -\int_{M_i} H \cdot X \, d\mathcal{H}^m + \int_{\partial M_i} X \cdot \nu_i \, d\mathcal{H}^{m-1}.\end{aligned}$$

Thus

$$\begin{aligned}|\delta M_i(X)| &\le \int_{M_i} |H \cdot X| \, d\mathcal{H}^m + \int_{\partial M_i} |X \cdot \nu_i| \, d\mathcal{H}^{m-1} \\ &\le \int_{\partial M_i} |X| \, d\mathcal{H}^{m-1}.\end{aligned} \tag{22}$$

Taking the limit as $i \to \infty$ gives

$$|\delta V(X)| \le \int_{\partial M} |X| \, d\mathcal{H}^{m-1}. \tag{23}$$

In particular, $\delta V(X) = 0$ for X compactly supported in $\Omega \backslash \partial M$, so by the constancy theorem for stationary varifolds ([1, §4.6(3)] or [7, §41]), V is the rectifiable varifold obtained by assigning some constant multiplicity $a \ge 0$ to M.

(Strictly speaking, the constancy theorem only tells us that V and the varifold M with multiplicity a coincide in $\Omega \setminus \partial M$. However, since V has locally bounded first variation (by 23), $|V|$ is absolutely continuous with respect to $\mathcal{H}^m$ (see [7, §3.2, §40.5]). Thus $|V|(\partial M) = 0$, so in fact the two varifolds coincide throughout Ω.)

Thus by the first variation formula for M,

$$\delta V(X) = a \int_{\partial M} X \cdot \nu \, d\mathcal{H}^{m-1},$$

where ν is the unit normal vectorfield to ∂M that points out of M. Substituting this into (23) gives

$$a \int_{\partial M} X \cdot \nu \, d\mathcal{H}^{m-1} \leq \int_{\partial M} |X| \, d\mathcal{H}^{m-1}. \tag{24}$$

Now let X be a vectorfield whose restriction to ∂M is $f\nu$, where f is a nonnegative function that is strictly positive on some nonempty open set. Then (24) becomes

$$a \int_{\partial M} f \, d\mathcal{H}^{m-1} \leq \int f \, d\mathcal{H}^{m-1},$$

which implies that $a \leq 1$.

We have shown that the M_i converge as varifolds to M with multiplicity a where $a \leq 1$. By Allard's Regularity and Boundary Regularity Theorems (or by the simplified version in [10]), the convergence is smooth on compact subsets of Ω.

(Concerning the simplified versions of Allard's theorems: the proof described in [10] is for interior points, but the method works equally well at the boundary.) q.e.d.

Readers interested in minimal (rather than bounded mean curvature) varieties may skip the rest of this section.

Theorem 6.1 remains true if we replace the hypothesis that the M_i are minimal by the hypothesis that

$$\limsup_{i \to \infty} \int_{K \cap M_i} ((|H| - h)^+)^p \, dA < \infty$$

for every compact $K \subset \Omega$, provided we also replace smooth convergence (in the conclusion) by convergence in C^1 (with uniform local $C^{1,1-m/p}$ bounds). However, the proof of Theorem 6.1 does not work in the more general setting. (As in the minimal case, by passing to a subsequence we can assume that the M_i converge as varifolds to limit varifold V supported in M. However, V need not be stationary in $\Omega \setminus \partial M$, and thus we cannot invoke the constancy theorem for stationary varifolds as we did in the minimal case.) So a different proof is required. In fact, we prove a more general result that also applies to varifolds:

6.2. Theorem. *Let V_i be a sequence of m-dimensional varifolds in a smooth Riemannian manifold Ω such that we have the following:*

(1) *For each i and for each smooth, compactly supported vectorfield X on Ω,*

$$\delta V_i(X) = -\int X \cdot H_i \, d|V_i| + \int_{\Gamma_i} X \cdot \nu_i \, d\mathcal{H}^{m-1},$$

where Γ_i is a smooth, proper, $(m-1)$-dimensional submanifold of Ω, H_i is a Borel vectorfield on Ω, and ν_i is a Borel vectorfield on Γ_i with $|\nu_i(x)| \le 1$ for all x.

(2) $\Theta(V_i, x) \ge 1$ *at each point of* $\operatorname{spt}(V_i) \setminus \Gamma_i$.

(3) *The* $\operatorname{spt}(V_i)$ *converge as sets (see Remark* 1.6*) to a subset S of a connected, proper,* C^1 *submanifold M with ∂M smooth and nonempty.*

(4) Γ_i *converges smoothy to ∂M.*

(5) *For every compact $K \subset \Omega$,*

$$\limsup_{i \to \infty} \int_K ((|H_i| - h)^+)^p \, d|V_i| < \infty,$$

where p and h are finite constants with $p > m$.

Then, after passing to a subsequence, the V_i converge to a limit V. If z is a point in ∂M, then $\Theta(V, z) = 1/2$ and z has a neighborhood U such that

(i) *for all sufficiently large i, the set* $\operatorname{spt}(V_i) \cap U$ *is a $C^{1,1-m/p}$ manifold-with-boundary in U (the boundary being $\Gamma_i \cap U$), with a $C^{1,1-m/p}$ bound independent of i, and*

(ii) $\operatorname{spt}(V_i) \cap U$ *converges in* C^1 *to $M \cap U$.*

Furthermore, if $\beta > 1$, then U can be chosen so that

(iii) $\sup_{x \in U \setminus \Gamma_i} \Theta(V_i, x) \le \beta$ *for all sufficiently large i.*

The theorem remains true if the V_i satisfy the hypotheses for a sequence g_i of Riemannian metrics on Ω converging smoothly to a limit metric g.

6.3. Corollary. *Suppose that the V_i in Theorem* 6.2 *are integer-multiplicity rectifiable varifolds or, more generally, varifolds with the gap α property (*§5.5*) for some $\alpha > 1$ independent of i. Then $S = M$, and every point (interior or boundary) of M has a neighborhood $U \subset \Omega$ for which* (i) *and* (ii) *hold, and for which $\Theta(V_i, \cdot) \equiv 1$ on* $\operatorname{spt}(V_i) \cap U \setminus \Gamma_i$ *for all sufficiently large i.*

The corollary is false without the gap α assumption: if we let V_i be the portion of M_i from §5.6 in the region $\{(x, y) : |x| \le 5\}$ and if we let $M = [-5, 5] \times \{0\}$, then all the hypotheses of Theorem 6.2 hold, but there are interior points (x, y) of M (namely, the points $(\pm 1, 0)$) such that (x, y) is a singular point of every V_i.

Proof of corollary. Let z be a point in ∂M, and let β be a number such that $1 < \beta < \alpha$. Let U be a neighborhood of z satisfying the conclusions of the theorem. Then by hypothesis (2) and by conclusion (iii),

$$1 \leq \Theta(V_i, x) < \alpha$$

for all $x \in U \cap \operatorname{spt}(V_i) \setminus \Gamma_i$ and $i \geq i_0$, so by the gap α property, $\Theta(V_i, x) \equiv 1$ for such x and i_0.

Now by Theorem 5.4, the multiplicity 1 convergence in U implies such convergence in all of $\Omega \setminus \partial M$. But that implies that S (the limit of the $\operatorname{spt}(V_i)$) is all of M. In particular, S includes all of ∂M, so (by Theorem 6.2) we also get multiplicity 1 convergence everywhere. q.e.d.

Proof of Theorem 6.2. Let $h' = h + 1$. Then, as in the proof of Theorem 5.4,

$$\limsup_{i\to\infty} \int_K (|H_i| - h')^+)\, d|V_i| < \infty,$$

for every compact K. Thus the area blow-up set of the V_i is an (m, h') set by Theorem 2.6 and Definition 2.1, and it is contained in a connected m-manifold with nonempty boundary, so it is empty by Corollary 4.2. In other words, the areas of the V_i are uniformly bounded on compact sets. It follows (using hypothesis (5) and Minkowski's inequality) that

$$\sup_K \int |H_i|^p\, d|V_i| < \infty$$

for compact sets $K \subset \Omega$. By passing to a subsequence, we can assume that the V_i converge to a varifold V. Now let z be a point in $S \cap \partial M$. The remaining conclusions are local, so we can replace Ω by any open set containing z. By isometrically embedding Ω into some $\mathbf{R}^N$ and then enlarging it to get an open subset of $\mathbf{R}^N$, we can assume that Ω is an open subset of $\mathbf{R}^N$ with the Euclidean metric. We may also assume that z is the origin 0. By replacing Ω with an open ball whose closure is in Ω, we can assume that

$$a := \sup_i \left(\int |H_i|^p\, d|V_i| \right)^{1/p} < \infty. \tag{25}$$

From the hypothesis (1) and Holder's inequality, we have

$$|\delta V_i(X)| \leq a \left(\int |X|^q\, d|V_i| \right)^{1/q} + \int_{\Gamma_i} |X|\, d\mathcal{H}^{m-1}$$

for all smooth, compactly supported vectorfields X, where $q = p/(p-1)$. Passing to the limit gives

$$|\delta V(X)| \leq a \left(\int |X|^q\, d|V| \right)^{1/q} + \int_{\partial M} |X|\, d\mathcal{H}^{m-1}. \tag{26}$$

For $r > 0$, let V^r, M^r, and Ω^r be obtained from V, M, and Ω by dilation by $1/r$ about 0.

We claim that

$$\delta V^r(X) \le r^{1-m/p} a \left(\int |X|^q \, d|V^r| \right)^{1/q} + \left(\int_{\partial M^r} |X| \, d\mathcal{H}^{m-1} \right) \tag{27}$$

for every smooth vectorfield X supported in Ω^r. To prove the claim, fix an r and let $\tilde{X}(x) = X(x/r)$. Then

$$\begin{aligned} |\delta V^r(X)| &= r^{1-m} |\delta V(\tilde{X})| \\ &\le r^{1-m} a \left(\int |\tilde{X}|^q \, d|V| \right)^{1/q} + r^{1-m} \left(\int_{\partial M} |\tilde{X}| \, d\mathcal{H}^{m-1} \right) \\ &= r^{1-m/p} a \left(\int |X|^q \, d|V^r| \right)^{1/q} + \left(\int_{\partial M^r} |X| \, d\mathcal{H}^{m-1} \right). \end{aligned}$$

This proves the claim.

Let

(28)

$$\theta := \Theta^*(|V|, 0) := \limsup_{r \to 0} \frac{|V| \, \mathbf{B}(0,r)}{\omega_m r^m} = \limsup_{r \to 0} \frac{|V^r| \, \mathbf{B}(0,1)}{\omega_m} \in [0, \infty].$$

Consider a sequence of r's tending to 0 and let Λ be the set of those r's. Choose Λ so that

$$\lim_{r \in \Lambda \to 0} \frac{|V^r| \, \mathbf{B}(0,1)}{\omega_m} = \theta. \tag{29}$$

By passing to a further subsequence, we can assume that the supports of V^r converge as $r \in \Lambda \to 0$ to a subset of $M' := \operatorname{Tan}(M, 0)$, the tangent halfplane to M at 0. Thus by (27) and Corollary 4.2, the areas of the V^r are uniformly bounded on compact sets, so, by passing to a further subsequence, we can assume that the V^r converge to a limit varifold V' as $r \in \Lambda \to 0$. From (27), we see that

$$\delta V'(X) \le \int_{\partial M'} |X| \, d\mathcal{H}^{m-1} \tag{30}$$

for all smooth, compactly supported X. In particular, V' is stationary in $\mathbf{R}^N \setminus \partial M'$, so, by the constancy theorem for stationary varifolds ([1, §4.6(3)] or [7, §41]), V' is the halfplane M' with some constant multiplicity. By (29), that multiplicity is 2θ. Thus

$$\delta V'(X) = 2\theta \int \operatorname{div}_{M'} X \, d\mathcal{H}^m = 2\theta \int_{\partial M'} X \cdot \nu \, d\mathcal{H}^{m-1},$$

where ν is the unit normal vector to $\partial M'$ that points out from M'. Thus by (30),

$$2\theta \int_{\partial M'} X \cdot \nu \, d\mathcal{H}^{m-1} \le \int_{\partial M'} |X| \, d\mathcal{H}^{m-1},$$

which immediately implies that $\theta \le 1/2$. (Let X be a smooth, compactly supported vecforfield whose restriction to M' is $f\nu$, where f is a nonnegative function that is not identically 0.)

Now $\theta = \Theta^*(|V|, 0) \le 1/2$ implies, for all sufficiently small balls $\mathbf{B}(0, r)$, that $V_i \llcorner \mathbf{B}(0, r)$ satisfies the hypotheses of the Allard Boundary Regularity Theorem [2, p. 429] for all sufficiently large i, which implies the asserted behavior (i) and (ii) in a smaller ball. Also, hypothesis (2) and conclusion (ii) of the theorem imply that $\Theta(V, 0) \ge 1/2$. Therefore, $\Theta(V, 0) = 1/2$.

It remains only to prove (iii). Let U satisfy (i) and (ii). We may assume that (i) and (ii) hold for all i by dropping the first i_0 terms in the sequence. Now suppose that (iii) does not hold for any U. Then, after passing to a subsequence, we can assume that there are points $x_i \in U \setminus \Gamma_i$ such that $x_i \to 0$ and such that

$$\Theta(V_i, x_i) \ge \beta.$$

Let y_i be the point in Γ_i nearest to x_i. Translate V_i, M, and x_i by $-y_i$ and dilate by $1/|x_i - y_i|$ to get $V_i^\dagger$, $M_i^\dagger$, and $x_i^\dagger$. Note that the $M_i^\dagger$ converge to the halfplane $M' = \mathrm{Tan}(M', 0)$. (This follows from the C^1 convergence in (ii).) Now by exactly the same reasoning used for the V^r, we can assume, after passing to a subsequence, that the $V_i^\dagger$ converge to a limit $V^\dagger$ consisting of the halfplane $M' = \mathrm{Tan}(M, 0)$ with some constant multiplicity $c \le 1$. Note that the points $x_i^\dagger$ converge to the point $x^\dagger$ in M' such that $x^\dagger$ is a unit vector in M' perpendicular to $\partial M'$. Now

$$1 \ge c = \Theta(V^\dagger, x^\dagger) \ge \limsup_i \Theta(V_i^\dagger, x_i^\dagger) \ge \beta$$

by the upper semicontinuity of density for varifolds whose mean curvatures satisfy uniform local $\mathcal{L}^p$ bounds [7, §17.8]. However, $\beta > 1$ by hypothesis. The contradiction proves (iii). q.e.d.

6.4. Remark. In Theorem 6.2, the hypothesis that $|\nu(\cdot)| \le 1$ can be relaxed $|\nu(\cdot)| \le \gamma$, where $\gamma > 1$ is a constant (depending on m and on $\dim(\Omega)$) from the Allard Boundary Regularity Theorem. If the V_i have the gap α property, then we can let γ be any number with $1 < \gamma < \alpha$. The proof is almost exactly the same as the proof of Theorem 6.2.

7. The Barrier Principle

The following theorem shows that an (m, h) subset obeys the same barrier form of the maximum principle that is satisfied by smooth m-manifolds with mean curvature bounded by h.

7.1. Theorem (Barrier Principle)**.** *Let Ω be a C^1 Riemannian manifold without boundary, and let Z be an (m, h) subset of Ω. Let N be a closed region in Ω with smooth boundary such that $Z \subset N$, and let $p \in Z \cap \partial N$. Then*

$$\kappa_1 + \cdots + \kappa_m \le h,$$

where $\kappa_1 \le \kappa_2 \le \cdots \le \kappa_{n-1}$ are the principal curvatures of ∂N at p with respect to the unit normal that points into N.

Proof of the Barrier Principle (Theorem 7.1*).* Since the result is local, we may assume that Ω is an open subset of $\mathbf{R}^n$. It suffices to construct a smooth function $f : \Omega \to \mathbf{R}$ such that

$$\max_N f = f(p), \tag{31}$$

$$\mathrm{D}f(p) \neq 0, \tag{32}$$

and such that

$$\frac{\mathrm{Trace}_m(\mathrm{D}^2 f)(p)}{|\mathrm{D}f(p)|} = \mu := \sum_{i=1}^m \kappa_i.$$

Case 1: g is the Euclidean metric. Let $u : \Omega \to \mathbf{R}$ be the signed distance to ∂N:

$$u(x) = \begin{cases} \mathrm{dist}(x, \partial N) & \text{if } x \notin N, \\ -\mathrm{dist}(x, \partial N) & \text{if } x \in N. \end{cases}$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ be unit vectors in $\mathrm{Tan}_p\, \partial N$ in the principal directions of ∂N. These vectors together with $\nabla u(p)$ form an orthonormal basis for $\mathbf{R}^n$, and a standard and straightforward computation shows that these are eigenvectors of $\mathrm{D}\nabla u(p)$ with eigenvalues $\kappa_1, \dots, \kappa_{n-1}$, and 0.

Let $f(x) = e^{\alpha u(x)}$, where α is a positive number to specified later. Then

$$\mathrm{D}f = \alpha e^{\alpha u} \mathrm{D}u.$$

If we work in normal coordinates at p and (by a slight abuse of notation) use $\mathrm{D}^2 f$ to denote the matrix of second partial derivatives of f with respect to those coordinates, then we have (at the point p)

$$\mathrm{D}^2 f = \alpha^2 e^{\alpha u} \mathrm{D}u^T \mathrm{D}u + \alpha e^{\alpha u} \mathrm{D}^2 u.$$

From this we see that the eigenvectors of $\mathrm{D}^2 u(p)$ are also eigenvectors of $\mathrm{D}^2 f(p)$, and that the eigenvalues of $\mathrm{D}^2 f(p)$ are

$$\lambda_i = \alpha \kappa_i \quad (i = 1, \dots, n-1) \tag{33}$$

together with $\lambda_n := \alpha^2$. Choosing α so that

$$\alpha > \max_i |\kappa_i|$$

guarantees that λ_n is the largest eigenvalue and thus by (33) that

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) = \sum_{i=1}^m \alpha \kappa_i = \alpha \mu,$$

so

$$\frac{\mathrm{Trace}_m(\mathrm{D}^2 f(p))}{|\mathrm{D}f(p)|} = \frac{\alpha \mu}{\alpha} = \mu.$$

This completes the proof in Case 1.

Case 2: g is a general C^1 metric. As before, we can assume that $\Omega \subset \mathbf{R}^n$. By a diffeomorphic change of coordinates, we may assume that

$$g_{ij}(p) = \delta_{ij} \tag{34}$$

and that

$$\mathrm{D}g_{ij}(p) = 0. \tag{35}$$

Now by (34) and (35), at the point p, the principal curvatures of ∂M with respect to the Euclidean metric δ are equal to the principal curvatures with respect to the metric g. Thus, by Case 1, there is a smooth function $f : \Omega \to \mathbf{R}$ such that $\mathrm{D}f(p) \neq 0$,

$$\max_N f = f(p),$$

and such that

$$\left[\frac{\mathrm{Trace}_m(\mathrm{D}^2 f(p))}{|\mathrm{D}f(p)|} \right]_\delta = \mu. \tag{36}$$

But by (34) and (35), the left side of (36) does not change if we replace δ by g. This completes the proof in Case 2. q.e.d.

7.2. Corollary. *Suppose Z is an $(m, 0)$ subset of smooth Riemannian $(m+1)$-manifold. If $t \in [0, T] \mapsto M(t)$ is a mean curvature flow of compact hypersurfaces and if $M(0)$ is disjoint from Z, then $M(t)$ is disjoint from Z for all $t \in [0, T]$.*

Here the mean curvature flow can be a classical flow, a Brakke flow of varifolds, or a level-set flow. See [11, Proposition 7.7] for the proof. (There, Z is stated to be the support of a stationary m-varifold, but in fact the proof only uses the Barrier Principle 7.1 and hence establishes Corollary 7.2 for any $(m, 0)$ set Z.)

In the codimension 1 case, we also have a strong barrier principle:

7.3. Theorem (Strong Barrier Principle)**.** *Let Z be an (m, h) subset of a smooth, $(m+1)$-dimensional, Riemannian manifold Ω without boundary.*

Let N be a closed region in Ω with smooth, connected boundary such that $Z \subset N$ and such that

$$H_{\partial N} \cdot \nu \geq h$$

at every point of ∂N, where $H_{\partial N}(x)$ is the mean curvature vector of ∂N at x and $\nu(x)$ is the unit normal at x to ∂N that points into N.

If Z contains any points of ∂N, then it contains all of ∂N.

Proof. See [8] for a proof. Specifically, [8, step 1, page 687] shows that any set Z that violates the conclusion of the strong barrier principle 7.3 also violates the conclusion of the barrier principle 7.1. (The proof there

is written for the case $h = 0$, but the same proof works for arbitrary h.)
q.e.d.

7.4. Corollary (The Halfspace Theorem for $(2,0)$ sets)**.** *Suppose $Z \subset \mathbf{R}^3$ is a nonempty $(2,0)$ set that lies in a halfspace of $\mathbf{R}^3$. Then Z contains a plane. Indeed, if $L : \mathbf{R}^3 \to \mathbf{R}$ is a nonconstant linear function and if*

$$s := \sup_Z L < \infty,$$

then Z contains the plane $L = s$.

Proof. Hoffman and Meeks [4, Theorem 1] proved this in the case where Z is a properly immersed minimal submanifold of $\mathbf{R}^3$, but their proof only uses the strong barrier principle and hence also works for arbitrary $(2,0)$ sets Z. q.e.d.

8. Converse to the Barrier Principle

8.1. Lemma. *Suppose Z is a closed subset of a Riemannian manifold Ω. If Z is not an (m,h) set, then there is smooth function $f : \Omega \to \mathbf{R}$ such that $f|Z$ has a local maximum at a point p where*

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) > h\,|\mathrm{D}f(p)| \tag{37}$$

and where

$$\mathrm{D}f(p) \neq 0. \tag{38}$$

Proof. Since the result is local, we may assume that Ω is diffeomorphic to a ball or, equivalently, to $\mathbf{R}^n$. Thus we may in fact assume that Ω is $\mathbf{R}^n$ with a Riemannian metric. By hypothesis, there is a C^2 function $f : \Omega \to \mathbf{R}$ and a point p such that $f|Z$ has a local maximum at p and such that (37) holds. By Corollary 2.5, there is such an f that is smooth. By translation, we may assume that $p = 0$.

We assume that $\mathrm{D}f(0) = 0$, as otherwise there is nothing to prove. By replacing f by

$$x \mapsto f(x) - \epsilon\,|x|^2$$

for a sufficiently small $\epsilon > 0$, we may assume that $f|Z$ has a strict local maximum at 0 and that $\mathrm{D}f$ has an isolated zero at 0, i.e., that

$$\mathrm{D}f(x) \neq 0 \text{ if } 0 < |x| < r \tag{39}$$

for some $r > 0$.

Since $\mathrm{Trace}_m(\mathrm{D}^2 f(0)) > 0$, the function f does not have a local maximum at 0. Thus 0 is not in the interior of Z. Let p_i be a sequence of points in $\Omega \setminus Z$ converging to 0. Let

$$\begin{aligned} &f_i : \Omega \to \mathbf{R}, \\ &f_i(x) = f(x - p_i). \end{aligned}$$

Since $f|Z$ has a strict local maximum at p, it follows that (for sufficiently large i) $f_i|Z$ has a local maximum at some point q_i with $\lim_i q_i = 0$. By the smooth convergence $f_i \to f$ and by (37),

$$\mathrm{Trace}_m(\mathrm{D}^2 f_i(q_i)) - h\,|\mathrm{D}f_i(q_i)| > 0$$

for all sufficiently large i.

Now $|q_i - p_i| > 0$ since $q_i \in Z$ and $p_i \notin Z$. Also, $|q_i - p_i| \to 0$ since p_i and q_i tend to 0. Thus $|Df_i(q_i)| \neq 0$ by (39).

Thus (for all sufficiently large i) the function f_i and the point q_i have the desired properties. q.e.d.

8.2. Theorem. *Let Z be a closed subset of a Riemannian manifold Ω, and let $m < \dim(\Omega)$. Suppose that Z is not an (m, h) set. Then there is a closed region $N \subset \Omega$ containing Z and a point $p \in Z \cap \partial N$ such that ∂N is smooth and such that*

$$H_m(\partial N, p) > h,$$

where $H_m(\partial N, p)$ is the sum of the smallest m principal curvatures of ∂N at p with respect to the unit normal that points into N.

Proof. By hypothesis, there is a point $p \in Z$ and a smooth function $f : \Omega \to \mathbf{R}$ such that $f|Z$ has a local maximum at p and such that

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) > h\,|\mathrm{D}f(p)|.$$

By Lemma 8.1, we may assume that $\mathrm{D}f(p) \neq 0$. We may also assume that

$$|\mathrm{D}f(p)| = 1.$$

(Otherwise replace f by $f/|\mathrm{D}f(p)|$.) Thus

$$\mathrm{Trace}_m(\mathrm{D}^2 f(p)) > h.$$

By modifying f outside of a compact neighborhood of p, we may assume that $f|Z$ attains its global maximum at p, and that $\mathrm{D}f$ never vanishes on the level set $f = f(p)$. Hence the set $N := \{x : f(x) \leq f(p)\}$ is a closed region with smooth boundary, $Z \subset N$, and $p \in Z \cap \partial N$.

Let

$$\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{n-1}$$

be the principal curvatures of ∂N at p with respect to the unit normal that points into N. We may suppose that we have chosen normal coordinates at p such that the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}$ are the corresponding principal directions of ∂N at p. Let

$$\nu = \frac{\nabla f}{|\nabla f|}$$

and $s = |\nabla f|$, so that $\nabla f = s\nu$. Now

$$\begin{aligned} \mathrm{D}_{\mathbf{e}_i}(\nabla f(p)) &= \mathrm{D}_{\mathbf{e}_i} s\nu \\ &= s\mathrm{D}_{\mathbf{e}_i}\nu + \nu \mathrm{D}_{\mathbf{e}_i} s \\ &= \kappa_i \mathbf{e}_i + \nu \mathrm{D}_{\mathbf{e}_i} s, \end{aligned}$$

so

$$\mathbf{e}_i \cdot \mathrm{D}_{\mathbf{e}_i} \nabla f(p) = \kappa_i.$$

In other words, κ_i is the ii entry of the matrix for $\mathrm{D}\nabla f(p)$ with respect to the orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. Thus

$$h < \mathrm{Trace}_m(\mathrm{D}^2 f(p)) \le \sum_{i=1}^m \kappa_i.$$

(For the last step we are using the following fact from linear algebra: if Q is a symmetric $n \times n$ matrix, then the sum of any m of the diagonal entries of Q is greater than or equal to the sum of the smallest m eigenvalues of Q.) q.e.d.

9. Minimal Hypersurfaces

Here we prove some results in the special case of $(m, 0)$ sets in $(m+1)$-dimensional manifolds. In the next section, we extend the results to (m, h) sets with $h > 0$.

We suppose throughout this section that N is a smooth, $(m+1)$-dimensional Riemannian manifold with smooth, connected boundary. We also suppose that one of the following hypotheses holds:

1) N is complete with Ricci curvature bounded below, or
2) N has an exhaustion by nested, compact, mean convex regions, or
3) N is a subset of a larger $(m+1)$-manifold and $\overline{N}$ is compact and mean convex.

(Each of these hypotheses guarantees that a compact hypersurface in N moving by mean curvature flow cannot escape to infinity in finite time. For hypotheses (2) and (3), this follows immediately from the maximum principle. For hypothesis (1), see [6, 6.2 or 6.4]. In hypotheses (2) and (3), the mean convex regions referred to need not have smooth boundary.)

9.1. Theorem. *Let $m < 7$, and let N be a smooth, mean convex, $(m+1)$-dimensional Riemannian manifold with smooth, nonempty, connected boundary satisfying one of the hypotheses (1)–(3) above.*

Suppose that N contains a nonempty $(m, 0)$ subset Z and that Z does not contain all of ∂N. Then N contains a nonempty, smooth, embedded, stable minimal hypersurface S that weakly separates Z from ∂N in the following sense: if $C \subset N$ is a connected, compact set that contains points of Z and of ∂N, then C intersects S.

The theorem remains true for $m \geq 7$, except that the surface S is allowed to have a singular set of Hausdorff dimension $\leq m - 7$.

(We remark that S has a one-sided minimizing property considerably stronger than stability. See [9, §11] for details. In particular, if any connected component of S is one-sided (i.e., has a nonorientable normal bundle), then its two-sided double cover is also stable.)

Proof. By the Strong Barrier Principle 7.3, the set Z must lie in the interior of N. If ∂N is a stable minimal hypersurface, then we let $S = \partial N$. Thus we may assume that ∂N is not a minimal hypersurface or that it is an unstable minimal hypersurface. We divide the proof into four cases according to whether ∂N is or is not minimal and whether it is or is not compact.

Case 1: ∂N is compact and ∂N is not a minimal surface. Let

$$t \in [0, \infty) \mapsto K(t)$$

be the flow such that $K(0) = N$ and such that $\partial K(t)$ flows by mean curvature flow. Each of the hypotheses (1), (2), and (3) imply that $\partial K(t)$ remains in N (as a compact set) for all time.

Also, since Z is an $(m, 0)$ set, $\partial K(t)$ can never bump into Z (Corollary 7.2.) That is, Z is contained in the interior of $K(t)$ for all t. Thus $Z \subset K_\infty \subset \text{interior}(N)$ where $K_\infty = \cap_t K(t)$. Furthermore, by [9, §11], $S := \partial K_\infty$ is a minimal surface with the indicated regularity properties. This completes the proof in Case 1.

Case 2: ∂N is a compact, unstable minimal hypersurface. The instability means that we can push ∂N slightly into N to get a surface whose mean curvature is everywhere nonzero and points away from ∂N. (For example, we can push Σ into N by the lowest eigenfunction of the Jacobi operator; see [5, Proposition A3] for a proof.) Replacing N by the portion of N on one side of that surface reduces Case 2 to Case 1.

Case 3: ∂N is noncompact and nonminimal. In this case, let p be a point in ∂N where the mean curvature of ∂N is nonzero. Let $f : \partial N \to \mathbf{R}$ be a proper Morse function such that $f(p) = \min f < 0$ and such that 0 is a regular value of f. Let

$$t \in [0, \infty) \mapsto K(t) \subset N$$

be the flow such that

$$K(0) = N,$$
$$K(t) \cap (\partial N) = \{q \in \partial N : f(q) \geq t\},$$

and such that the surfaces

$$M(t) := \partial K(t)$$

move by mean curvature flow.

(Note that $M(t)$ is a (possibly singular) m-dimensional surface with boundary; the boundary of $M(t)$ is $\{q \in \partial N : f(q) = t\}$.)

The rest of the proof is essentially identical to the proof in Case 1.

Case 4: ∂N is a noncompact, unstable minimal hypersurface. Let $f : \partial N \to \mathbf{R}$ be a smooth, proper Morse function that is bounded below. Since ∂N is unstable, it follows that for all sufficiently large t, the surface $(\partial N) \cap \{f < t\}$ will be unstable. In particular, there is a regular value τ of t for which the surface $\Sigma := (\partial N) \cap \{f < \tau\}$ is unstable. By adding a constant to f, we may suppose that $\tau = 0$. The instability implies that we can push the interior of Σ slightly into the interior of N to get a surface Σ' with $\partial\Sigma' = \partial\Sigma$ such that the mean curvature of Σ' is everywhere nonzero and points away from ∂N. For example, we can push Σ into N by the lowest eigenfunction of the Jacobi operator as in Case (2). We make the perturbation small enough that the closed region bounded by $\Sigma \cup \Sigma'$ does not contain any points of Z.

Now let

$$t \in [0, \infty) \mapsto M(t)$$

be the mean curvature flow (constructed by elliptic regularization) such that $M(0) = \Sigma'$ and such that $\partial M(t) = \{x \in \partial N : f(x) = t\}$ for all $t \geq 0$.

The rest of the proof is identical to the proof in Case 3. q.e.d.

9.2. Corollary (Strong Halfspace Theorem for $(2, 0)$ sets)**.** *Let Σ be a connected, properly embedded, separating minimal surface in a complete 3-manifold Ω of nonnegative Ricci curvature. Suppose Z is a nonempty $(2, 0)$ set that lies in the closure N of one of the connected components of $\Omega \setminus \Sigma$, and suppose that $\Sigma \setminus Z$ is nonempty. Then N contains a properly embedded, totally geodesic surface M with Ricci flat normal bundle.*

In particular, if Ω is the flat $\mathbf{R}^3$, then Σ is a plane and Z contains a plane parallel to Σ.

Hoffman and Meeks [4, Theorem 2] proved this in the case where Z is a properly immersed minimal surface.

Proof. The corollary follows from the Theorem 9.1 because by [3, page 210, paragraph 1], every complete, stable, two-sided minimal surface M in Ω is totally geodesic and has Ricci flat normal bundle.

The last assertion ("Z contains a plane parallel to Σ") is Corollary 7.4. q.e.d.

10. Bounded Mean Curvature Hypersurfaces

Here we extend Theorem 9.1 from $(m, 0)$ sets to (m, h) sets.

10.1. Definition. Let N be a smooth Riemannian manifold with smooth boundary, and let $h \geq 0$. We say that N is *h-mean convex* provided

$$H_{\partial N} \cdot \nu \geq h \tag{40}$$

at all points of ∂N, where $H_{\partial N}$ is the mean curvature vector and ν is unit normal to ∂N that points into N.

It is also convenient to allow N with piecewise smooth boundary. In particular, suppose $N = \cap_i N_i$ is the intersection of finitely many smooth Riemannian manifolds with smooth boundary and that the ∂N_i are transverse. (The transversality means that if x belongs to several of the ∂N_i, then the unit normals to those ∂N_i at x are linearly independent.) In that case, we say that N is *h-mean convex* provided (40) holds at all the regular boundary points of N.

In this section, we suppose that $h > 0$ and that N is a smooth, $(m+1)$-dimensional Riemannian manifold that satisfies one of the following hypotheses:

(i) N is complete with Ricci curvature bounded below.
(ii) N has an exhaustion by nested, compact, h-mean convex regions.
(iii) N is a subset of a larger $(m+1)$-manifold and $\overline{N}$ is compact and h-mean convex.

(The exhausting regions in (ii) and the region $\overline{N}$ in (iii) are allowed to have piecewise smooth boundary.)

10.2. Theorem. *Let $h > 0$, let $m < 7$, and let N be a smooth, h-mean convex, $(m+1)$-dimensional Riemannian manifold with smooth, nonempty, connected boundary. Suppose that one of the hypotheses (i), (ii), or (iii) holds, and that N contains a nonempty (m, h) subset Z.*

Then Z is contained in a region K whose boundary is smooth and has constant mean curvature h with respect to the inward unit normal. Furthermore, if ∂N is not contained in Z, then ∂K is stable for the functional $(\text{area}) - h(\text{enclosed volume})$.

The theorem remains true for $m \geq 7$, except that the surface S is allowed to have a singular set of Hausdorff dimension $\leq m-7$.

Proof. The proof is exactly the same as the proof of Theorem 9.1, except that in that proof we let the sets $K(t)$ evolve so that $\partial K(t)$ moves not with velocity H but rather with velocity $H - h\nu$, where H is the mean curvature and $\nu(x)$ is the inward unit normal.

Suitable varifold solutions to the flow can be constructed by elliptic regularization just as in the $h = 0$ case. Furthermore, h-mean convexity is preserved by the flow just as in the $h = 0$ case. Indeed, all the results in [9] for mean convex mean curvature flow continue to hold for arbitrary h, with only very minor modifications in the proofs. In fact, for $h > 0$, the behavior of $\partial K(t)$ as $t \to \infty$ is slightly simpler: in the case $h = 0$, it is possible for $\partial K(t)$ to converge smoothly to a double cover of the limit surface S, whereas for $h > 0$, that is clearly impossible. q.e.d.

11. The Distance to an (m, h) Set

Here we show that (m, h) sets behave well with respect to the distance function. The theorem and its proof are particularly simple when the ambient space is Euclidean, so we consider that case first:

11.1. Theorem. *Suppose Z is an (m, h) subset of $\mathbf{R}^n$. Then for $s > 0$, the set $Z(s)$ of points in $\mathbf{R}^n$ at distance $\leq s$ from Z is also an (m, h) set.*

Proof. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function such that $f|Z(s)$ has a local maximum at $p \in Z(s)$. Let q be a point in Z that minimizes $\operatorname{dist}(q, p)$. Let

$$g(x) = f(x + p - q).$$

Then $g|Z$ has a local maximum at q, so

$$\operatorname{Trace}_m(\mathrm{D}^2 g(q)) - h\,|\mathrm{D}g(q)| \leq 0.$$

Since $\mathrm{D}f(p) = \mathrm{D}g(q)$ and $\mathrm{D}^2 f(p) = \mathrm{D}^2 g(q)$, this implies that

$$\operatorname{Trace}_m(\mathrm{D}^2 f(p)) - h\,|\mathrm{D}f(p)| \leq 0.$$

q.e.d.

11.2. Theorem. *Suppose Z is an (m, h) subset of a connected, Riemannian manifold Ω. For $s > 0$, let $Z(s)$ be the set of points at geodesic distance $\leq s$ from Z.*

(i) *If the sectional curvatures of Ω are bounded below by K, then $Z(s)$ is an $(m, h - mKs)$ set.*
(ii) *If $\dim(\Omega) = m+1$ and if the Ricci curvature of Ω is bounded below by ρ, then $Z(s)$ is an $(m, h - \rho s)$ set.*

Proof. If $\dim(\Omega) = m$, then by the constancy Theorem 4.1, Z is either all of Ω or the empty set, in either of which cases the theorem is trivially true. Thus we suppose that $\dim(\Omega) > m$.

Let N be a closed region in Ω with smooth boundary such that $Z(s) \subset N$ and such that $p \in Z(s) \cap \partial N$. By Theorem 8.2, it suffices to show that

$$H_m(\partial N, p) \leq h - mKs$$

in case (i) or

$$H_m(\partial N, p) \leq h - \rho s$$

in case (ii). Let q be a point in Z such that $\operatorname{dist}(q, p) = s$. Let Γ be the geodesic joining p to q. Note that the signed distance function $\operatorname{dist}(\cdot, \partial N)$ will be smooth on an open set containing $\Gamma \setminus \{q\}$, but that it may not be smooth at q.

We get around that lack of smoothness as follows. Note that for each $\epsilon > 0$, we can find a closed region $N' \subset \Omega$ with smooth boundary such that

(1) $N \subset N'$,
(2) $N \cap \partial N' = \{p\}$,
(3) the principal directions of ∂N at p are also principal directions of $\partial N'$ at p,
(4) each principal curvature of $\partial N'$ at p is strictly less than the corresponding principal curvature of ∂N at p, and
(5) each principal curvature of $\partial N'$ at p is within ϵ of the the corresponding principal curvature of N at p.

By (5),

$$H_m(\partial N', p) \geq H_m(\partial N, p) - m\epsilon. \tag{41}$$

By (4), the function $f(\cdot) := \operatorname{dist}(\cdot, \partial N')$ is smooth on an open subset of N' containing Γ. In particular, if

$$N^* = \{x \in N' : \operatorname{dist}(x, \partial N') \geq s\},$$

then $q \in \partial N^*$ and ∂N^* is smooth near q.

It follows (Proposition 12.2) that each principal curvature of ∂N^* at q is greater than or equal to Ks plus the corresponding principal curvature of $\partial N'$ at p, and thus (taking the sum of the first m principal curvatures) that

$$H_m(\partial N^*, q) \geq H_m(\partial N', p) + mKs.$$

Since $Z \subset N^*$ and since Z is an (m, h) set, the left side of this inequality is at most h (by the barrier principle 7.1), so

$$\begin{aligned} h &\geq H_m(\partial N', p) + mKs \\ &\geq H_m(\partial N, p) - m\epsilon + mKs \end{aligned}$$

by (41). Since $\epsilon > 0$ can be arbitrarily small, this implies that $h \geq H_m(\partial N, p) + mKs$ or

$$H_m(\partial N, p) \leq h - mKs,$$

from which it follows (Theorem 8.2) that $Z(s)$ is an $(m, h - mKs)$ set.

If $\dim(\Omega) = m + 1$, then (letting N, N', and N^* be as above)

$$H_m(\partial N^*, q) \geq H_m(\partial N', p) + \rho s$$

by Proposition 12.2. Arguing exactly as above with ρ in place of mK, we conclude that

$$H_m(\partial N, p) \leq h - \rho s,$$

from which it follows that $Z(s)$ is an $(m, h - \rho s)$ set. q.e.d.

12. Appendix: Tubular Neighborhoods

For the reader's convenience, we give the basic facts about the second fundamental form of the level sets of the distance function to a smooth hypersurface. (These facts were used in Section 11.)

12.1. Lemma. *Let M be a two-sided, smoothly embedded hypersurface in a smooth, $(n+1)$-dimensional Riemannian manifold N, let $f : N \to \mathbf{R}$ be the signed distance function to M, and let Ω be an open subset of N on which f is smooth with nonvanishing gradient. For $p \in \Omega$, let*

$$M_p := \{x : f(x) = f(p)\}$$

be the level set of f containing p, and let B_p be the second fundamental form of M_p at p with respect to the unit normal $\nu(p) := \nabla f(p)$. Then

$$(\mathrm{D}_\nu B)(\cdot,\cdot) = \mathrm{R}(\cdot,\nu,\cdot,\nu) + \sum_{k=1}^{n} B(\cdot,\mathbf{e}_k)B(\mathbf{e}_k,\cdot),$$

where R is the curvature tensor of N and where $\mathbf{e}_1,\dots,\mathbf{e}_n$ are unit vectors orthogonal to each other and to ν.

Proof. Note that the hypotheses imply that ν is a unit vectorfield and that the integral curves of ν are geodesics:

$$\mathrm{D}_\nu \nu \equiv 0. \tag{42}$$

Let $\mathbf{x}$ and $\mathbf{y}$ be two tangent vectorfields to one of the level sets of f. Extend these vectorfields by parallel transport along the integral curves of ν. Thus

$$\mathrm{D}_\nu \mathbf{x} = \mathrm{D}_\nu \mathbf{y} = 0. \tag{43}$$

Now

$$B(\mathbf{y},\mathbf{x}) = B(\mathbf{x},\mathbf{y}) = (\mathrm{D}_\mathbf{x}\mathbf{y})\cdot\nu = -\mathbf{x}\cdot \mathrm{D}_\mathbf{y}\nu.$$

By (43), $(\mathrm{D}_\nu B)(\mathbf{x},\mathbf{y}) = \nu(B(\mathbf{x},\mathbf{y}))$. Thus by (42) and (43),

$$\begin{aligned}
(\mathrm{D}_\nu B)(\mathbf{x},\mathbf{y}) &= \nu(\mathrm{D}_\mathbf{x}\mathbf{y}\cdot\nu) \\
&= (\mathrm{D}_\nu\mathrm{D}_\mathbf{x}\mathbf{y})\cdot\nu + \mathrm{D}_\mathbf{x}\mathbf{y}\cdot\mathrm{D}_\nu\nu \\
&= (\mathrm{D}_\nu\mathrm{D}_\mathbf{x}\mathbf{y})\cdot\nu + 0 \\
&= \mathrm{R}(\nu,\mathbf{x})\mathbf{y}\cdot\nu + (\mathrm{D}_\mathbf{x}\mathrm{D}_\nu\mathbf{y})\cdot\nu + (\mathrm{D}_{[\nu,\mathbf{x}]}\mathbf{y})\cdot\nu \\
&= \mathrm{R}(\nu,\mathbf{x},\nu,\mathbf{y}) + 0 + (\mathrm{D}_{[\nu,\mathbf{x}]}\mathbf{y})\cdot\nu.
\end{aligned}$$

It remains only to show that

$$(\mathrm{D}_{[\nu,\mathbf{x}]}\mathbf{y})\cdot\nu = \sum_{k=1}^{n} B(\mathbf{x},\mathbf{e}_k)B(\mathbf{e}_k,\mathbf{y}). \tag{44}$$

Now

$$[\nu,\mathbf{x}] = \mathrm{D}_\nu\mathbf{x} - \mathrm{D}_\mathbf{x}\nu = -\mathrm{D}_\mathbf{x}\nu,$$

which is orthogonal to ν, and thus tangent to the level sets of f, so

$$(\mathrm{D}_{[\nu,\mathbf{x}]}\mathbf{y}) \cdot \nu = B([\nu, \mathbf{x}], \mathbf{y}) = B(-\mathrm{D}_{\mathbf{x}}\nu, \mathbf{y}). \tag{45}$$

Now

$$\begin{aligned} -\mathrm{D}_{\mathbf{x}}\nu &= \sum_{k=1}^{n}(-\mathrm{D}_{\mathbf{x}}\nu \cdot \mathbf{e}_k)\mathbf{e}_k \\ &= \sum_{k=1}^{n} B(\mathbf{x}, \mathbf{e}_k)\mathbf{e}_k. \end{aligned}$$

Substituting this into (45) and using the linearity of $B(\cdot, \mathbf{y})$ gives (44).
q.e.d.

12.2. Proposition. *Let Ω, f, ν, B, and M_x (for $x \in \Omega$) be as in Lemma* 12.1. *Let $\kappa_1(x) \le \cdots \le \kappa_n(x)$ be the principal curvatures of M_x at x with respect to the unit normal ν. Let Γ be a geodesic curve perpendicular to the level sets of f (i.e., an integral curve of the vectorfield $\nu := \nabla f$).*

If $p, q \in \Gamma$ and if $f(q) > f(p)$, then

$$\kappa_i(q) \ge \kappa_i(p) + K \operatorname{dist}(p, q), \tag{46}$$

$$\mathrm{Trace}_m B_q \ge \mathrm{Trace}_m B_p + mK \operatorname{dist}(p, q), \tag{47}$$

$$H(q) \ge H(p) + \rho \operatorname{dist}(p, q), \tag{48}$$

where K is a lower bound for the sectional curvature of the ambient space, ρ is a lower bound for the Ricci curvature of the ambient space, and $H(x) = \mathrm{trace}\, B_x$ is the mean curvature of $M(x)$ at x with respect to ν.

Proof. Let V be the space of normal vectorfields $\mathbf{v}$ on Γ such that $\mathrm{D}_\nu \mathbf{v} \equiv 0$. We may regard $B(q)$ and $B(p)$ as both being symmetric bilinear forms on V. (In effect, we are identifying $\mathrm{Tan}_p M_p$ and $\mathrm{Tan}_q M_q$ by parallel transport along Γ.)

Let $\mathbf{v}$ be a vectorfield in V. Then $\mathrm{D}_\nu(B(\mathbf{v}, \mathbf{v})) = (\mathrm{D}_\nu B)(\mathbf{v}, \mathbf{v})$, so by Lemma 12.1,

$$\mathrm{D}_\nu(B(\mathbf{v}, \mathbf{v})) \ge \mathrm{R}(\nu, \mathbf{v}, \nu, \mathbf{v}), \tag{49}$$

and thus

$$\mathrm{D}_\nu(B(\mathbf{v}, \mathbf{v})) \ge K\|\mathbf{v}\|^2.$$

Integrating from p to q gives

$$B_q(\mathbf{v}, \mathbf{v}) \ge B_p(\mathbf{v}, \mathbf{v}) + K \operatorname{dist}(p, q)\|\mathbf{v}\|^2.$$

Now (46) follows from the Rayleigh quotient characterization of the eigenvalues of B_x, i.e., the principal curvatures. (See Lemma 12.3, below.)

Summing from $i = 1$ to m in (46) gives (47).

To prove (48), let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be an orthonormal set of vectorfields in V. Then by (49),

$$D_\nu(B(\mathbf{e}_i, \mathbf{e}_i)) \ge R(\nu, \mathbf{e}_i, \nu, \mathbf{e}_i).$$

Summing from $i = 1$ to n gives

$$D_\nu h \ge \mathrm{Ricci}(\nu, \nu) \ge \rho.$$

Assertion (48) follows by integrating from p to q. q.e.d.

12.3. Lemma. *Let Q and Q' be symmetric bilinear forms on a Euclidean space V. Suppose $Q(\mathbf{v}, \mathbf{v}) \le Q'(\mathbf{v}, \mathbf{v})$ for all unit vectors $\mathbf{v}$. Then each eigenvalue of Q is less than or equal to the corresponding eigenvalue of Q'.*

Proof. This follows immediately from the Rayleigh quotient characterization of the eigenvalues:

$$\lambda_k(Q) = \inf_{W \in G(k,V)} \left(\sup_{w \in W, |w|=1} Q(w, w) \right),$$

where $G(k, V)$ is the set of k-dimensional linear subspaces of V. q.e.d.

References

[1] W.K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491. MR0307015, Zbl 0252.49028.

[2] ———, *On the first variation of a varifold: boundary behavior*, Ann. of Math. (2) **101** (1975), 418–446. MR0397520 (53 #1379), Zbl 0319.49026.

[3] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211, DOI 10.1002/cpa.3160330206. MR562550 (81i:53044), Zbl 0439.53060.

[4] D. Hoffman and W.H. Meeks III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990), no. 2, 373–377, DOI 10.1007/BF01231506. MR1062966 (92e:53010), Zbl 0722.53054.

[5] D. Hoffman and B. White, *Genus-one helicoids from a variational point of view*, Comment. Math. Helv. **83** (2008), no. 4, 767–813. MR2442963 (2010b:53013), Zbl 1161.53009.

[6] T. Ilmanen, *Generalized flow of sets by mean curvature on a manifold*, Indiana Univ. Math. J. **41** (1992), no. 3, 671–705, DOI 10.1512/iumj.1992.41.41036. MR1189906 (93k:58057), Zbl 0759.53035.

[7] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University Centre for Mathematical Analysis, Canberra, 1983. MR756417, Zbl 0546.49019.

[8] B. Solomon and B. White, *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals*, Indiana Univ. Math. J. **38** (1989), no. 3, 683–691. MR1017330, Zbl 0711.49059.

[9] B. White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **13** (2000), no. 3, 665–695 (electronic), DOI 10.1090/S0894-0347-00-00338-6. MR1758759 (2001j:53098), Zbl 0961.53039.

[10] ———, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519, DOI 10.4007/annals.2005.161.1487. MR2180405 (2006i:53100), Zbl 1091.53045.
[11] ———, *Which ambient spaces admit isoperimetric inequalities for submanifolds?*, J. Differential Geom. **83** (2009), no. 1, 213–228. MR2545035, Zbl 1179.53061.
[12] ———, *The maximum principle for minimal varieties of arbitrary codimension*, Comm. Anal. Geom. **18** (2010), no. 3, 421–432. MR2747434 (2012a:53117), Zbl 1226.53061.

Department of Mathematics
Stanford University
Stanford, CA 94305
E-mail address: bcwhite@stanford.edu