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Controlling coherence of noisy and chaotic oscillators by a linear feedback

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ABSTRACT

We analyze the dynamics of a noisy limit cycle oscillator coupled to a general passive linear system. We analytically demonstrate that the phase diffusion constant, which characterizes the coherence of the oscillations, can be efficiently controlled. Theoretical analysis is performed in the framework of linear and Gaussian approximations and is supported by numerical simulations. We also demonstrate numerically the coherence control of a chaotic system.

1. Introduction

Coherence, or stability of the oscillation frequency, is an important characteristic of clocks, lasers, electronic circuits and other self-sustained oscillators. Often, it directly determines the quality of these systems. Another important feature of the coherence is that it determines a predisposition of an oscillatory system to synchronization. Coherence is quantified by the width of the spectral peak of the oscillation or, equivalently, by the coefficient of the phase diffusion. This notion of coherence applies not only to noisy limit cycle oscillators, but also to many chaotic systems which admit phase description.

In this paper we address a question: how does the coherence change if a noisy or a chaotic oscillator is coupled to a passive linear system? This question can be equivalently formulated as a control problem: how can we affect the coherence of oscillations by attaching a passive linear system to a noisy or to a chaotic one? In this formulation the setup can be treated as a feedback control scheme. In a general context, a feedback control is useful not only for engineering aspects of experiments but also has found applications in various fields of physics [1] such as chaos theory and nonlinear dynamics, statistical mechanics and optics. Particularly, a delayed feedback is a commonly employed tool to control different properties of a dynamical system: to make chaotic systems operate periodically (the famous Pyragas control method [2]), to suppress space–time chaos [3–6], to manipulate collective synchrony in ensembles of interacting oscillators [7–10], to stabilize unstable steady states [11], to control noise-induced motion [12], etc. Many examples of feedback regulation can be found in living organisms; e.g., feedback mechanisms play an important role in the regulation of respiratory and cardiac rhythms [13, 14].

For a noisy or chaotic active oscillator, a control of the phase diffusion provides a tool to control stability of its oscillation — an important property for clocks, electronic generators, and other systems. As was recently analytically and numerically shown in Refs. [15–17], such a control can be achieved by means of a simple or multiple delayed feedback. In this paper we

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extend and generalize the results of Refs. [15–17] by considering a general linear feedback control (a delayed one then appears as a particular case). We present a theory for such a control, treating the phase dynamics of noisy limit cycle oscillators in the linear and Gaussian approximations; the theory is supported by numerics for noisy periodic and chaotic oscillators. We believe that our analysis of a general linear feedback control of coherence contributes to the theory and can be useful for other applications.

The paper is organized as follows. In Section 2 we introduce the basic phase model and analyze it in the framework of linear and Gaussian approximations. In Section 3 we consider an important particular case of a noisy autonomous oscillator coupled to a linear damped one. In Section 4 the coherence control of the chaotic Lorenz oscillator is studied. In Section 5 we discuss our results.

2. Phase model and its diffusion property

2.1. Basic phase model

As is well known, the phase of a noisy self-sustained oscillator exhibits a random-walk-like motion, so that one can speak of phase diffusion [18]. We describe an autonomous noisy or chaotic oscillator by the phase dynamics equation

$$\dot{\phi} = \Omega_0 + \xi(t). \tag{1}$$

Here Ω_0 is the mean frequency and the noisy term $\xi(t)$ describes effects of noise and/or chaos on the phase dynamics. The diffusion of the phase is determined, according to the Green–Kubo formula (see, e.g. Ref. [19]), by the spectral component of noise $\xi(t)$ at zero frequency. We assume that the oscillations are nearly harmonic, which is e.g. justified if a self-sustained oscillator operates near the Hopf bifurcation point, so that the process itself can be represented as $x(t) = A\cos\phi(t)$. This process drives a linear passive system, the output signal of which can be generally written in terms of the Green function as $y(t) = \int_{-\infty}^{0} G(t-t')x(t')\mathrm{d}t'$. This signal $y(t)$ now drives the oscillator (1), leading to the phase equation

$$\dot{\phi} = \Omega_0 + a\int_{-\infty}^{t} G(t-t')\cos(\phi(t') - \phi(t))\mathrm{d}t' + \xi(t), \tag{2}$$

where a is the strength of the coupling. In Appendix A we give a derivation of Eq. (2) for a particular case of noisy van der Pol oscillator.

Our main goal is to investigate the diffusion properties of the phase in the framework of Eq. (2). For this purpose it is convenient to split the phase into an average growth and fluctuations, writing $\phi = \Omega t + \psi$. Then, for the fluctuating instantaneous frequency $v(t) = \dot{\psi}$, satisfying $\langle v\rangle = 0$, we obtain from Eq. (2)

$$\begin{aligned} v(t) = \Omega_0 - \Omega + \xi(t) + a\int_0^{\infty} G(\tau)\cos(\Omega\tau)\cos(\psi(t-\tau) - \psi(t))\mathrm{d}\tau \\ + a\int_0^{\infty} G(\tau)\sin(\Omega\tau)\sin(\psi(t-\tau) - \psi(t))\mathrm{d}\tau. \end{aligned} \tag{3}$$

Next, we have to calculate the power spectrum $S_v(\omega)$ of instantaneous frequency $v(t)$; more precisely — the value of the spectral component at zero frequency $S_v(0)$, because the latter defines the diffusion constant of the phase via the Green–Kubo formula $D = 2\pi S_v(0)$.

Let us first consider a noise-free case, $\xi = \psi = v = 0$. Then Eq. (3) reduces to

$$\Omega - a\int_0^{\infty} G(\tau)\cos\Omega\tau\mathrm{d}\tau = \Omega_0. \tag{4}$$

Eq. (4) shows that the linear feedback shifts the oscillation frequency. Furthermore, generally Eq. (4) can have either a unique or multiple solutions for Ω. The latter case is complicated and should be treated separately. In the following we choose the parameters in a way that no multistability occurs.

Now we treat the problem in the linear approximation, assuming that the fluctuations of the phase are weak, i.e. $\psi(t) - \psi(t-\tau) \ll 2\pi$. From Eq. (3) and taking into account Eq. (4) we obtain

$$v(t) = a\int_0^{\infty} G(\tau)\sin\Omega\tau[\psi(t-\tau) - \psi(t)]\mathrm{d}\tau + \xi(t), \tag{5}$$

where Ω is the solution of Eq. (4). Next, we apply the Fourier transform to Eq. (5). Denoting the Fourier transforms of v, ψ, and ξ by F_v, F_ψ, and F_ξ, respectively and using $F_\psi = F_v/\mathrm{i}\omega$, we obtain:

$$F_v(\omega) = \frac{F_\xi(\omega)}{1 - \frac{a}{\mathrm{i}\omega}\int_0^{\infty} G(\tau)\sin\Omega\tau(\mathrm{e}^{-\mathrm{i}\omega\tau} - 1)\mathrm{d}\tau}. \tag{6}$$

Hence, the power spectrum of frequency fluctuations $S_v(\omega)$ is related to the power spectrum of noise $S_\xi(\omega)$ according to

$$S_v(\omega) = \frac{S_\xi(\omega)}{\left|1 - \frac{a}{\mathrm{i}\omega}\int_0^\infty G(\tau)\sin\Omega\tau(\mathrm{e}^{-\mathrm{i}\omega\tau}-1)\mathrm{d}\tau\right|^2}.$$

Considering the limit $\omega \to 0$ we obtain for $S_v(0)$:

$$S_v(0) = \frac{S_\xi(0)}{\left|1 + a\int_0^\infty G(\tau)\tau\sin\Omega\tau\mathrm{d}\tau\right|^2}.$$

As a result, we obtain the expression for the diffusion constant $D = 2\pi S_v(0)$ in the linear approximation as

$$D = \frac{D_0}{\left[1 + a\int_0^\infty G(\tau)\tau\sin\Omega\tau\mathrm{d}\tau\right]^2}, \tag{7}$$

where $D_0 = 2\pi S_\xi(0)$ is the diffusion constant in the absence of the control.

Remarkably, deriving Eq. (4) with respect to Ω we can rewrite Eq. (7) as

$$\frac{D}{D_0} = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega_0}\right)^2. \tag{8}$$

Physically, relation (8)[1] reflects the fact that noise in the basic model (1) is additive. Therefore, the effects of the random perturbation of the phase velocity ξ and of a constant perturbation due to variation $\mathrm{d}\Omega_0$ of the oscillator frequency are described in the framework of linear approximation by the same transform operator (6) at zero frequency. (Indeed, if we consider variation of the oscillator frequency then the noise term in Eq. (5) should be substituted by $\mathrm{d}\Omega_0$ and the transformation $(\mathrm{d}\Omega_0)^2 \to (\mathrm{d}\Omega)^2$ will be described by an equation, similar to (7).)

To perform a statistical analysis beyond the linear approximation analytically, we make an assumption that the phase fluctuations $\psi(t)$ as well as the noisy term $\xi(t)$ are Gaussian. After averaging Eq. (3) over the fluctuations of $v(t) = \dot{\psi}$ (which are also Gaussian distributed), we obtain for the mean frequency Ω:

$$0 = \Omega_0 - \Omega + a\int_0^\infty G(\tau)\cos(\Omega\tau)\langle\cos(\psi(t-\tau)-\psi(t))\rangle\mathrm{d}\tau. \tag{9}$$

The phase difference $\psi(t-\tau)-\psi(t) = \eta(t,\tau)$ is also Gaussian, hence $\langle\cos\eta\rangle = \exp[-\langle\eta^2\rangle/2]$. Thus Eq. (9) takes the form

$$0 = \Omega_0 - \Omega + a\int_0^\infty G(\tau)\cos(\Omega\tau)\exp[-\langle\eta^2\rangle/2]\mathrm{d}\tau. \tag{10}$$

The phase difference η can be represented as an integral of the instantaneous frequency:

$$\eta(t,\tau) = -\int_{t-\tau}^{t} v(s)\mathrm{d}s = -\int_{-\tau}^{0} v(t+z)\mathrm{d}z, \tag{11}$$

where $z = s - t$. For the variance of the phase difference η we obtain:

$$\begin{aligned}\langle\eta^2\rangle &= \left\langle\left[\int_{-\tau}^{0} v(t+z)\mathrm{d}z\right]^2\right\rangle = \left\langle\int_{-\tau}^{0} v(t+t')\mathrm{d}t'\int_{-\tau}^{0} v(t+t'')\mathrm{d}t''\right\rangle \\ &= \int_{-\tau}^{0}\mathrm{d}t'\int_{-\tau}^{0}\mathrm{d}t''K_v(t''-t') = 2\int_{-\tau}^{0}(\tau+t')K_v(t')\mathrm{d}t' \equiv 2R(\tau),\end{aligned} \tag{12}$$

where $K_v(t') = \langle v(t)v(t+t')\rangle$ is the autocorrelation function of the instantaneous frequency. Substituting $\langle\eta^2\rangle = 2R$ into Eq. (10) we obtain:

$$0 = \Omega_0 - \Omega + a\int_0^\infty \mathrm{d}\tau\cos\Omega\tau G(\tau)\mathrm{e}^{-R(\tau)}. \tag{13}$$

Note, that the obtained equation for the frequency of the controlled system is similar to the corresponding Eq. (4), valid in the linear approximation, but contains an additional factor $\mathrm{e}^{-R(\tau)}$. To find quantity R, we introduce the power spectrum $S_v(\omega)$ of the instantaneous frequency $v(t)$ and compute the last integral in Eq. (12):

$$\begin{aligned}R(\tau) &= \int_{-\tau}^{0}(\tau+t')K_v(t')\mathrm{d}t' = \int_{-\tau}^{0}(\tau+t')\left(\int_{-\infty}^{\infty}S_v(\omega)\mathrm{e}^{\mathrm{i}\omega t'}\mathrm{d}\omega\right)\mathrm{d}t' \\ &= \int_{-\infty}^{\infty}\frac{1-\cos\omega\tau}{\omega^2}S_v(\omega)\mathrm{d}\omega.\end{aligned} \tag{14}$$

Here we have used the fact that $S_v(\omega)$ is an even function.

[1] We are grateful to an anonymous reviewer who pointed out this corollary from Eqs. (4) and (7).

The next step is to find the diffusion coefficient that is related to the spectral density of the frequency fluctuations at zero frequency: $D = 2\pi S_v(0)$. Computation of this spectrum in Appendix B yields:

$$S_v(\omega) = \frac{S_\xi(\omega)}{\left|1 - \frac{a}{\mathrm{i}\omega}\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\mathrm{e}^{-R(\tau)}(1-\mathrm{e}^{\mathrm{i}\omega\tau})\right|^2}. \tag{15}$$

Thus, we obtain

$$D = \frac{D_0}{\left[1 + a\int_0^\infty \mathrm{d}\tau\,\tau \sin\Omega\tau G(\tau)\mathrm{e}^{-R(\tau)}\right]^2}, \tag{16}$$

where $D_0 = 2\pi S_\xi(0)$ is the diffusion constant in the absence of the feedback.

Eq. (16) is still implicit because function $R(t)$ depends on $S_v(\omega)$ in a non-trivial way (14). To proceed we make an approximation of this relation, assuming that the spectrum of velocity fluctuations $S_v(\omega)$ is wide. This means that we can approximate the last integral in (14) setting $S_v(\omega) \approx$ const as

$$R \approx \int_{-\infty}^{\infty} \frac{1-\cos\omega\tau}{\omega^2} S_v(0)\mathrm{d}\omega = \frac{\tau D}{2}. \tag{17}$$

Substituting this expression in Eq. (16) we obtain a closed equation for determination of the diffusion constant in the Gaussian approximation:

$$0 = \Omega_0 - \Omega + a\int_0^\infty \mathrm{d}\tau \cos\Omega\tau G(\tau)\mathrm{e}^{-D\tau/2},$$

$$D = \frac{D_0}{\left[1 + a\int_0^\infty \mathrm{d}\tau\,\tau \sin\Omega\tau G(\tau)\mathrm{e}^{-D\tau/2}\right]^2}. \tag{18}$$

This is an implicit system of two nonlinear equations for unknown mean oscillation frequency Ω and diffusion constant D, but it contains no unknown functions and can be easily solved numerically. Noteworthy is the fact that in the limit of small diffusion constant when one can set $\mathrm{e}^{-D\tau/2} \approx 1$, this system coincides with Eqs. (4) and (7) of the linear approximation. Note, that because the factor $\mathrm{e}^{-D\tau/2}$ implicitly depends on Ω, a relation similar to Eq. (8) cannot be written for this case. This means, that the Gaussian approximation is effectively nonlinear.

3. Linear damped oscillator as a coherence controller

In this section we apply the general theory developed above to an important typical case, where a noisy/chaotic active oscillator is coupled to a passive linear oscillator (which equivalently can be considered as a band pass filter). Such a scheme was used in Ref. [10] for control of collective synchrony in an ensemble of globally coupled oscillators. Here we analyze how this scheme controls the coherence of the noisy van der Pol oscillator:

$$\ddot{x} - \mu(1-x^2)\dot{x} + \Omega_0 x = \varepsilon\dot{u} + \zeta(t), \qquad \langle\zeta(t)\zeta(t')\rangle = 2d^2\delta(t-t'), \tag{19}$$

$$\ddot{u} + \alpha\dot{u} + \tilde{\omega}_0^2 u = x. \tag{20}$$

Here Eq. (19) describes a noisy van der Pol oscillator driven by the output $\dot{u}$ of a passive linear oscillator (20). The Green function of linear damped oscillator is given by:

$$G(\tau) = \mathrm{e}^{-\alpha\tau/2}\left(\cos\omega_0 t - \frac{\alpha}{2\omega_0}\sin\omega_0\tau\right), \tag{21}$$

where $\omega_0 = \sqrt{-\alpha^2/4 + \tilde{\omega}_0^2}$ and the parameter of coupling is $a = -\varepsilon/2\Omega_0$. Substituting this in Eq. (4) we obtain the equation for the frequency Ω

$$\Omega_0 = \Omega - a\int_0^\infty \mathrm{e}^{-\alpha\tau/2}\left(\cos\omega_0\tau - \frac{\alpha}{2\omega_0}\sin\omega_0\tau\right)\cos\Omega\tau\mathrm{d}\tau. \tag{22}$$

Similarly, substituting these expressions in the Eq. (7) we obtain the diffusion constant in the linear approximation as

$$D = \frac{D_0}{\left[1 + a\int_0^\infty \mathrm{e}^{-\alpha\tau/2}\left(\cos\omega_0\tau - \frac{\alpha}{2\omega_0}\sin\omega_0\tau\right)\tau\sin\Omega\tau\mathrm{d}\tau\right]^2}. \tag{23}$$

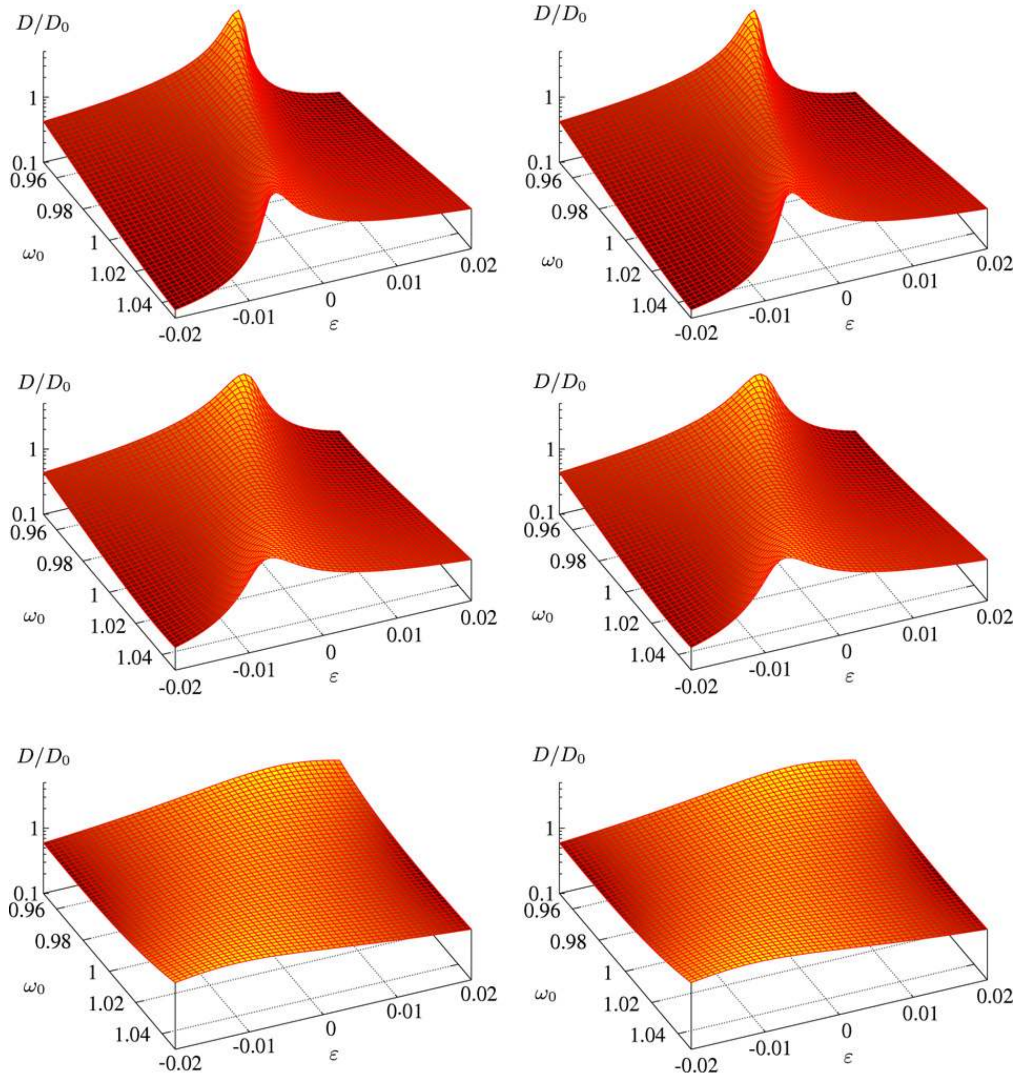

Fig. 1. Theoretical results for the phase diffusion constant D of the controlled noise driven van der Pol oscillator as the function of oscillator frequency $\tilde{\omega}_0$ and feedback strength ε in linear (left row) and Gaussian (right row) approximations for $D_0 = 0.0024$ and for $\alpha = 0.08\tilde{\omega}_0$ (top panels), $\alpha = 0.1\tilde{\omega}_0$ (middle panels), and $\alpha = 0.2\tilde{\omega}_0$ (bottom panels).

In the same way we obtain equations for the frequency and the diffusion constant in the Gaussian approximation:

$$\Omega - a\int_0^\infty \mathrm{e}^{-(\alpha+D)\tau/2}\left(\cos\omega_0\tau - \frac{\alpha}{2\omega_0}\sin\omega_0\tau\right)\cos\Omega\tau\,\mathrm{d}\tau = \Omega_0, \tag{24}$$

$$D = \frac{D_0}{\left[1 + a\int_0^\infty \mathrm{e}^{-(\alpha+D)\tau/2}\left(\cos\omega_0\tau - \frac{\alpha}{2\omega_0}\sin\omega_0\tau\right)\tau\sin\Omega\tau\,\mathrm{d}\tau\right]^2}. \tag{25}$$

Computing the integrals we obtain the main results of our analysis — closed systems of two equations for Ω, D in both approximations. These lengthy expressions are given in Appendix C.

These systems for two variables D and Ω have been solved numerically and the results are presented in Fig. 1. First we analyze the dependence of the diffusion constant D on the the oscillator damping factor α. One can see that the feedback control essentially changes the diffusion constant for small values of α. With an increase of the band pass of the filter the control effect almost vanishes (see bottom panels in Fig. 1).

Next, we analyze the impact of the oscillator frequency $\tilde{\omega}_0$. From Fig. 1 one can see that there is only a slight difference between the two approximations. This difference is more noticeable in Fig. 2, where we show the values of diffusion constant D resulting from linear (a) and Gaussian (b) approximations for different values of oscillator frequency. Below we compare numerical solutions of the analytically obtained equations with direct numerical simulations.

Finally, we note that from Figs. 1 and 2 one can see that dependence $D = D(\varepsilon)$ is not monotonic. To explain this, let us recall that the frequency Ω of the van der Pol oscillator (19) depends on the feedback factor ε according to (22) and (24). On the other hand, the effect of control via linear oscillator (20) depends on the relation between Ω and ω_0, demonstrating

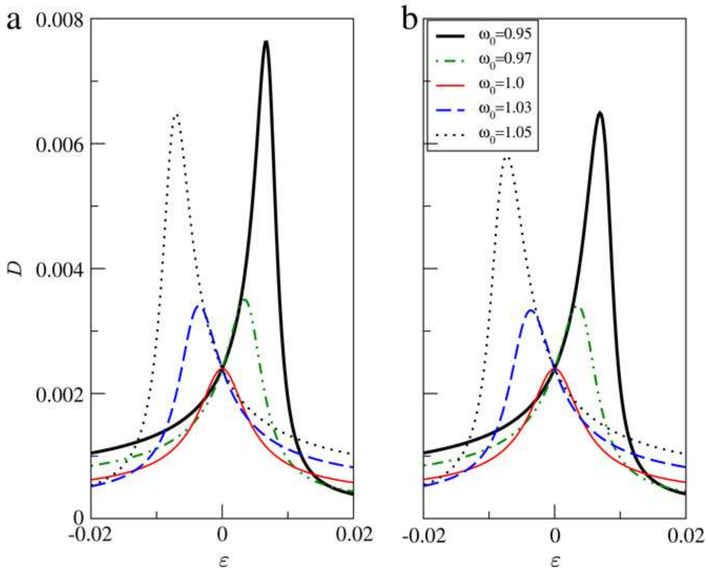

Fig. 2. Theoretical results for linear (a) and Gaussian (b) approximations for diffusion constant D for different values of oscillator frequency and for $\alpha = 0.1\tilde{\omega}_0$.

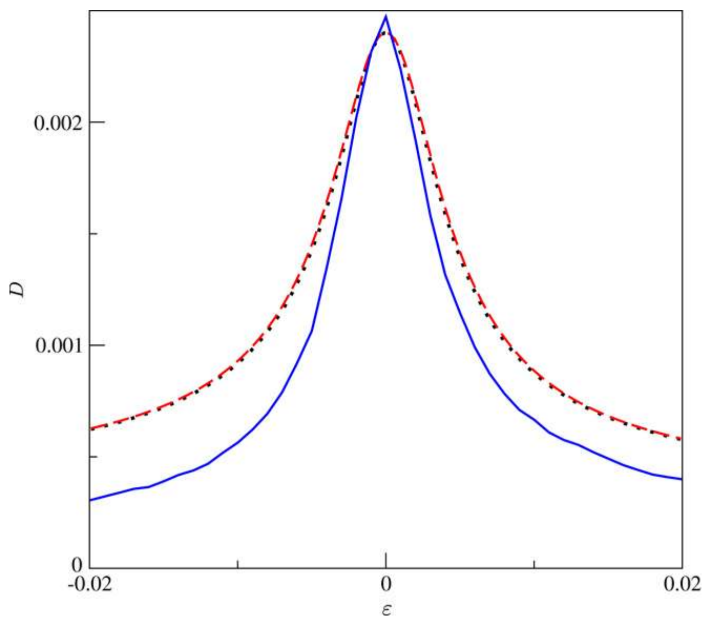

Fig. 3. The dependence of the phase diffusion constant D on the feedback strength ε for the controlled van der Pol model. The control is implemented by linear passive oscillator, Eq. (20). The solid line represents results of numerical simulation (Eqs. (19) and (20)); dotted and dashed lines represent theoretical results of linear equation (C.1) and Gaussian equation (C.3) approximations, respectively.

a resonance-like maximum. These two facts explain that the resonant value of ω_0 shifts with ε, as one can see in Fig. 1. As a result, the dependence $D(\varepsilon)$ for fixed ω_0 also exhibits a pronounced resonance-like behavior (Fig. 2).

3.1. Proportional derivative control: Numerical results

In this section we verify the theory above by direct numerical simulations of a noisy van der Pol oscillator coupled to a linear passive system ((19) and (20)). In the presence of control, the diffusion can be suppressed or enhanced, depending on the feedback strength ε and frequency $\tilde{\omega}_0$, which is confirmed by the numerical results in Figs. 3 and 4. In Fig. 3 we show the numerically obtained dependence of the diffusion constant on the coupling strength ε (solid line) and compare it with theoretical results for linear equation (C.1) and Gaussian equation (C.3) approximations (dotted and dashed lines, respectively). The parameters are: $\Omega_0 = \tilde{\omega}_0 = 1$, $\alpha = 0.1\tilde{\omega}_0$. In numerical simulation we used $d = 0.1$ and $\mu = 0.2$; for theoretical curves we took $D_0 = 0.0024$. The correspondence between the numerics and analytical results is good in the case of small values of the feedback strength ε.

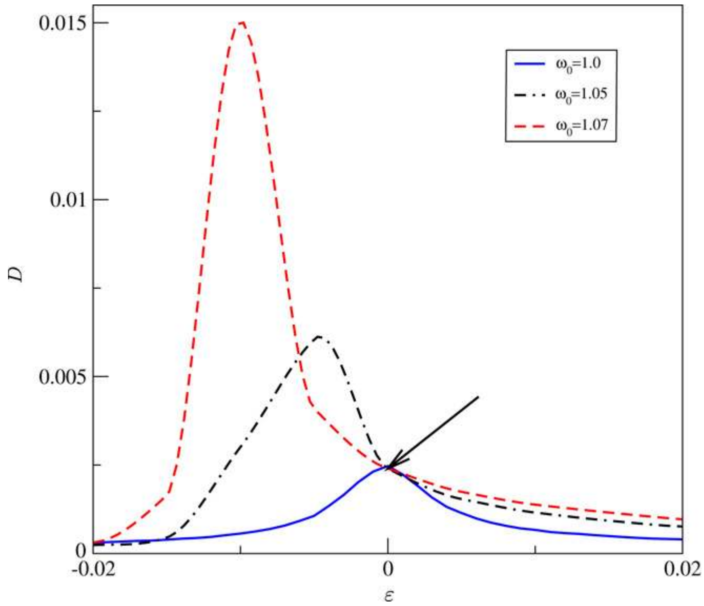

Fig. 4. Results of numerical simulation for the controlled van der Pol model ((19) and (20)) for different values of the oscillator frequency $\tilde{\omega}_0$. The arrow points to the uncontrolled value D_0 of the diffusion coefficient.

In Fig. 4 we plot the diffusion constant D as a function of ε for controlled van der Pol model for different values of the oscillator frequency $\tilde{\omega}_0$. Depending on ω_0 one can see that the curves are shifted with respect to the curve for $\tilde{\omega}_0 = 1.0$, in the same way as in Fig. 2.

4. Coherence control of a chaotic oscillator

Although it is not possible to derive a phase equation for chaotic oscillators explicitly, the phase dynamics of some chaotic oscillators is qualitatively similar to the dynamics of noisy periodic oscillators (see Ref. [20]), thus Eq. (2) can also be used for chaotic oscillators coupled to passive linear oscillators. To prove this numerically, we consider the chaotic Lorenz system, controlled by the passive oscillator (20):

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy + \varepsilon\dot{u}, \\ \ddot{u} + \alpha\dot{u} + \tilde{\omega}_0^2 u &= z, \end{aligned} \tag{26}$$

where $\sigma = 10$, $r = 28$, and $b = 8/3$. The phase of the Lorenz system is well-defined if one uses a projection of the phase space on the plane $(u = \sqrt{x^2 + y^2}, z)$ (see Ref. [20] and Fig. 7):

$$\phi = \arctan \frac{z(t) - z_0}{u(t) - u_0},$$

where the point $\{u_0 = 2b(r-1), z_0 = r - 1\}$ corresponds to the non-trivial fixed points of the Lorenz system. Notice that there is no noise term in Eq. (26). However, due to chaos, the phase of the autonomous system grows non-uniformly, with a non-zero diffusion constant. Diffusion constant D as the function of the feedback strength ε is shown in Fig. 5. One can see that the diffusion can be both significantly enhanced (up to 4 times) and suppressed (up to 2 times) by the feedback. The diffusion constant strongly depends on the frequency ω_0 of the linear damped oscillator. This dependence for the Lorenz system is demonstrated in Fig. 6.

We note that there is no unique way to introduce phase for a chaotic system. Besides the way we use here one can obtain phase by linear interpolation between the times when a trajectory intersects a Poncaré section, or by means of Hilbert or wavelet transform. However, the phases determined in these different ways differ only on a time scale of oscillation period, whereas all the definitions are equivalent on a long (with respect to oscillation period) time scale (see a discussion in Ref. [20]) and thus yield the same values of the diffusion constant.

Noteworthy is the fact that the effect of suppression of diffusion is not due to the suppression of chaos. One can see this from Fig. 7, where we show the projections of the phase portrait for the system without feedback and also for maximal suppression and enhancement; in all cases the dynamics is chaotic.

Another way to represent the effect of the attached linear oscillator on the coherence, is to look at the power spectrum. The power spectrum of $z(t)$ has a peak near frequency Ω_0, and the width of the peak is proportional to the diffusion constant

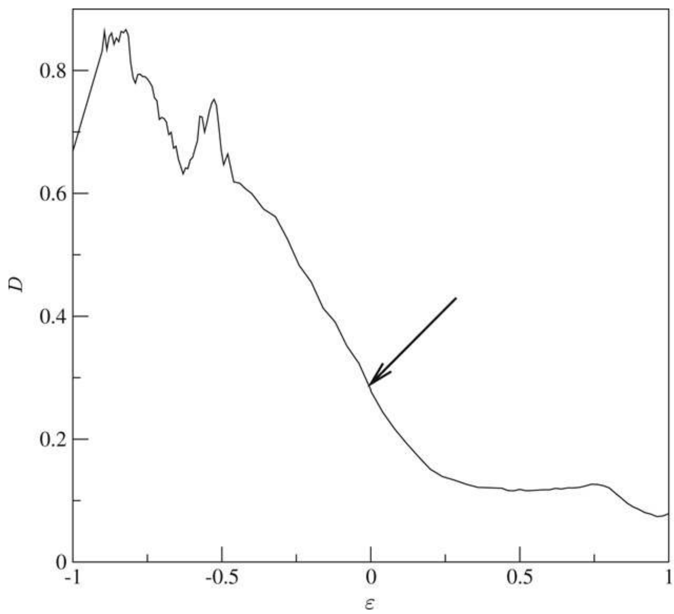

Fig. 5. Diffusion constant D for the phase of the controlled Lorenz system (26) as the function of feedback strength ε, for $\tilde{\omega}_0 = 2\pi/0.76, \alpha = 0.1\tilde{\omega}_0$. The arrow points to the uncontrolled value D_0 of the diffusion coefficient.

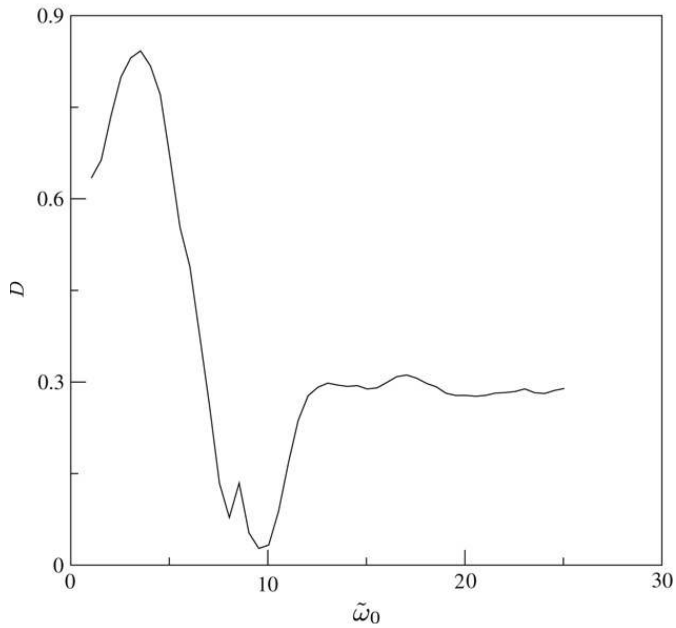

Fig. 6. The dependence of the diffusion constant D for the phase of the controlled Lorenz system (26) on the oscillator frequency $\tilde{\omega}_0$, for $\varepsilon = 0.5, \alpha = 0.1\tilde{\omega}_0$.

D. One can see from Fig. 8 that the feedback control in the case of suppression of the diffusion makes the spectral peak essentially more narrow (a) and vice versa, more wide in the case of enhancement (b).

We conclude this section by comparing our approach to coherence control with the widely used Pyragas chaos control technique [2]. The goal of the latter is to stabilize a periodic orbit within chaos, which in our language means complete suppression of the phase diffusion. We focus on the case where the diffusion can be either enhanced or decreased, but not eliminated, so that the system remains chaotic. (Generally, for a stronger feedback factor one can expect suppression of chaos.) Next, in the Pyragas technique the main parameter of the control loop – time delay – should correspond to the period of the orbit to be stabilized, whereas in our approach the parameters of the feedback can be varied freely in order to achieve the desired result.

5. Summary and discussion

We have demonstrated that a coupling of a chaotic or noisy self-sustained oscillator to a passive linear one significantly changes the coherence of oscillations, which allows one to use this effect as the coherence control by means of a general

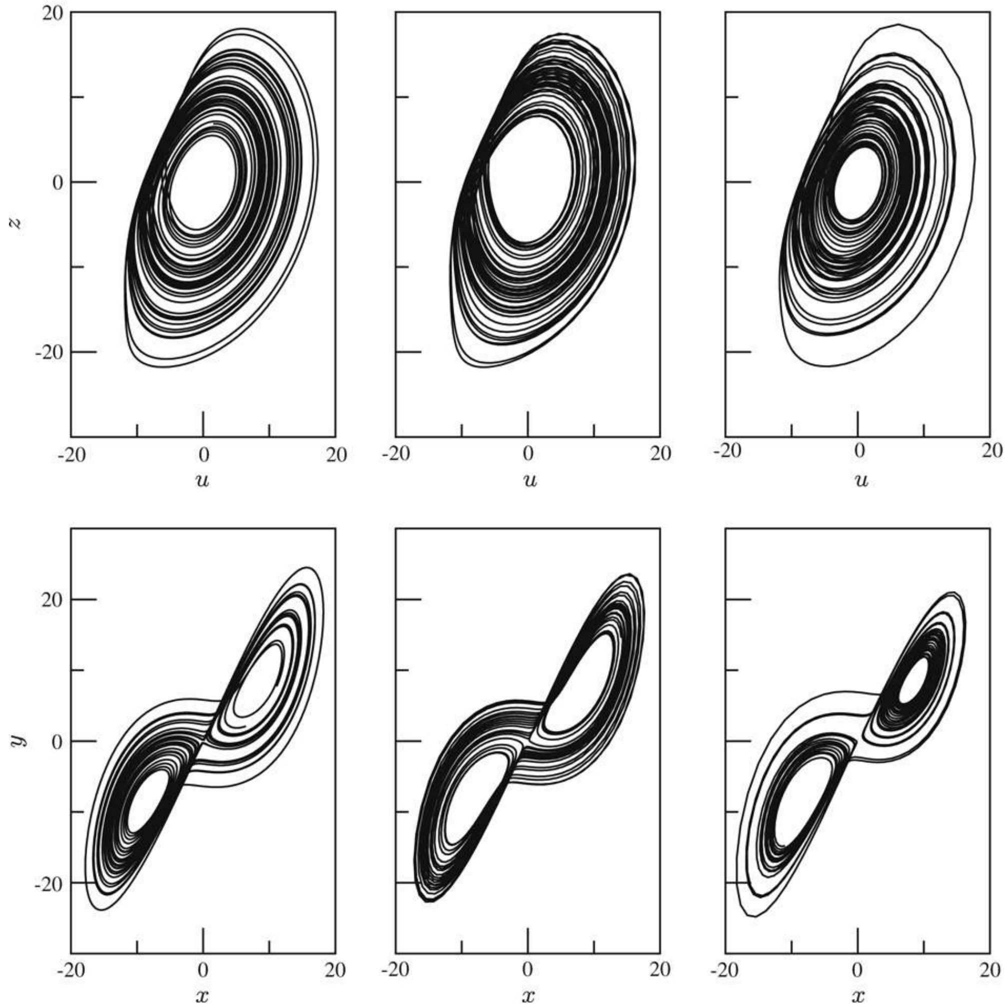

Fig. 7. The projections of the phase portrait for the Lorenz system (26) in the absence of feedback, $D = 0.28$ (left column) and in the presence of feedback for $\varepsilon = 0.96$, $D = 0.08$ (middle column) and for $\varepsilon = -0.88$, $D = 0.85$ (right column). Parameters are: $\tilde{\omega}_0 = 2\pi/0.76$, $\alpha = 0.1\tilde{\omega}_0$.

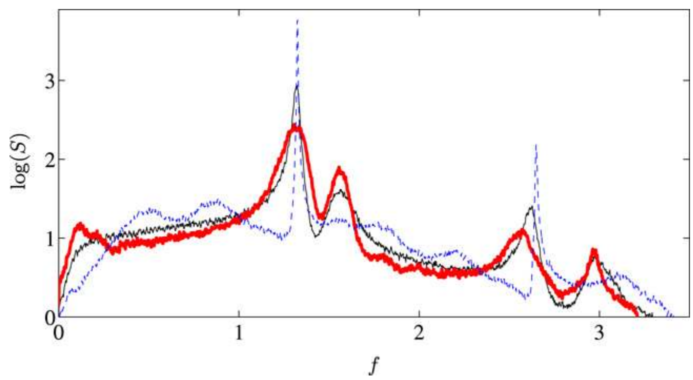

Fig. 8. Spectra $S(f)$ of the z component of the uncontrolled (solid line) and controlled Lorenz system (26) for $\varepsilon = 0.5$ (dashed line) and for $\varepsilon = -0.3$ (bold line). Other parameters are: $\tilde{\omega}_0 = 2\pi/0.76$, $\alpha = 0.1\tilde{\omega}_0$.

linear feedback. For characterization of coherence we have used the phase diffusion constant, which is proportional to the width of the spectral peak of oscillations. We have developed a statistical theory of phase diffusion under the influence of a general linear feedback in the framework of linear and Gaussian approximations and validated it by numerical results. The theory works if the feedback is not very strong, or if the noise is strong enough to suppress multistability in mean frequency. The case of multistability for strong feedback, not considered here, gives a possible direction for a further development of the theory.

It has been shown in Ref. [10] that proportional derivative control implemented by a linear damped oscillator can be used for manipulation of synchrony in ensembles of interacting oscillators. There is good reason to believe that the method suggested here may possibly substitute delayed-feedback schemes in some other applications, e.g., in stabilization of low-dimensional systems [2,21–26,11,1], control of noise-induced oscillations [12], etc.

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Appendix A. Phase model for a noisy van der Pol oscillator

In this appendix we derive the basic phase equation (2) for the noisy van der Pol oscillator:

$$\begin{aligned}\dot{x} &= \Omega_0 y,\\ \dot{y} &= -\Omega_0 x + \mu(1-x^2)y + \frac{\varepsilon}{\Omega_0}\widehat{L}(x) + \frac{1}{\Omega_0}\zeta(t), \qquad \langle \zeta(t)\zeta(t')\rangle = 2d^2\delta(t-t'),\end{aligned} \tag{A.1}$$

where $\widehat{L}$ is a linear operator describing the feedback and ζ is Gaussian noise. The general theory states (see, e.g., Refs. [27,20]) that in the first approximation an external force acting on a limit cycle oscillator affects the phase variable, but not the amplitudes, because the phase is free (i.e., it corresponds to the zero Lyapunov exponent) and can be adjusted by a very weak action. On the contrary, the amplitude variables correspond to negative Lyapunov exponents and are, therefore, stable. For further consideration we rely on this statement and use the phase description, which is valid for small noise and feedback. For small nonlinearity μ and in the absence of noise ($d = 0$) and control ($\varepsilon = 0$), the van der Pol model has a limit cycle solution $x_0 \approx 2\cos\phi$, $\dot{x}_0 \approx -2\Omega_0 \sin\phi$ with a uniformly growing phase $\phi(t) \approx \Omega_0 t + \phi_0$ [28].

The linear operator $\widehat{L}$ can be generally written in terms of the Green function $G(t-t')$ as $\widehat{L}(x) = \int_{-\infty}^{t} G(t-t')x(t')\mathrm{d}t'$. According to Refs. [27,20,15,16] we can apply the standard procedure to derive the phase equation and write

$$\dot{\phi} = \Omega_0 + \frac{\partial\phi}{\partial y_0}\left(\frac{\varepsilon}{\Omega_0}\widehat{L}(x_0) + \frac{1}{\Omega_0}\zeta(t)\right),$$

where $x_0 = 2\cos\phi$, $y_0 = -2\sin\phi$ are the limit cycle solutions and the phase is defined according to $\phi = -\arctan(y_0/x_0)$. Computing $\frac{\partial\phi}{\partial y_0}$ and substituting the variables x_0 and y_0 by $2\cos\phi$ and $2\sin\phi$, we obtain

$$\dot{\phi}(t) = \Omega_0 - \frac{\varepsilon}{2\Omega_0}\cos\phi(t)\widehat{L}(x_0) - \frac{\cos\phi(t)}{2\Omega_0}\zeta(t). \tag{A.2}$$

In terms of the Green function we write

$$\dot{\phi}(t) = \Omega_0 - \frac{\varepsilon}{\Omega_0}\cos\phi(t)\int_{-\infty}^{t} G(t-t')\cos\phi(t')\mathrm{d}t' - \frac{\cos\phi(t)}{2\Omega_0}\zeta(t). \tag{A.3}$$

Since our main goal is to quantify the phase diffusion, we are mostly interested in the long-term dynamics of the phase. Therefore, we average the r.h.s. of (A.3) over the period of oscillations. Using $\langle\cos\phi(t)\cos\phi(t')\rangle = \frac{1}{2}\cos(\phi(t')-\phi(t))$, and the fact that ζ is δ-correlated and independent of ϕ, so that

$$\langle\zeta(t)\zeta(t')\cos\phi(t)\cos\phi(t')\rangle \approx \langle\zeta(t)\zeta(t')\rangle\langle\cos\phi(t)\cos\phi(t')\rangle = d^2\delta(t-t'),$$

we obtain the basic phase equation (2) with $a = -\frac{\varepsilon}{2\Omega_0}$ and $\xi(t)$ being the effective noise, satisfying $\langle\xi(t)\xi(t')\rangle = \frac{d^2}{4\Omega_0^2}\delta(t-t')$.

Note that the derived equations are also valid for the case of more general proportional and proportional derivative feedback, when the control term is designed as a combination of the linear operators from x and $\dot{x}$, i.e.,

$$\begin{aligned}\widehat{L}(x) &= \widehat{L}_0(x) + \widehat{L}_1(\dot{x}) = \int_{-\infty}^{t} G_0(t-t')x(t')\mathrm{d}t' + \int_{-\infty}^{t} G_1(t-t')\dot{x}(t')\mathrm{d}t'\\ &= \int_0^{\tau}\left[G_0(\tau) + 2G_1(0)\delta(\tau) + G_1'(\tau)\right]x(t-\tau)\mathrm{d}\tau = \int_0^{\tau} G(\tau)x(t-\tau)\mathrm{d}\tau,\end{aligned} \tag{A.4}$$

where $G(\tau) = G_0(\tau) + 2G_1(0)\delta(\tau) + G_1'(\tau)$, where $t - t' = \tau$.

Finally, we note that if the oscillator is anharmonic, Eq. (2) should be generalized to a form containing higher Fourier components of phases ϕ and ϕ'.

Appendix B. Computation of the spectrum of frequency fluctuations

First we rewrite Eq. (3) taking into account Eq. (11), which yields:

$$\begin{aligned} v(t) &= \Omega_0 - \Omega + \xi(t) + a\int_0^\infty \mathrm{d}\tau \cos\Omega\tau G(\tau)\cos\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right] \\ &\quad - a\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\sin\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right]. \end{aligned} \tag{B.1}$$

Next, in order to obtain equations for the autocorrelation function $K_v(t')$, we introduce the autocorrelation function of the noise $K_\xi(t')$ and the cross-correlation function $K_{\xi v}(t')$, i.e.,

$$K_{\xi v}(t') = \langle \xi(t+t')v(t)\rangle, K_\xi(t') = \langle \xi(t+t')\xi(t)\rangle.$$

Multiplying Eq. (B.1) by $\xi(t+t')$, $v(t+t')$ and averaging, we obtain the equations for the correlation functions $K_{\xi v}(t')$ and $K_v(t')$

$$\begin{aligned} K_{\xi v}(t') &= K_\xi(t') + a\int_0^\infty \mathrm{d}\tau \cos\Omega\tau G(\tau)\left\langle \xi(t+t')\cos\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right]\right\rangle \\ &\quad - a\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\left\langle \xi(t+t')\sin\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right]\right\rangle, \end{aligned} \tag{B.2}$$

$$\begin{aligned} K_v(t') &= K_{\xi v}(-t') + a\int_0^\infty \mathrm{d}\tau \cos\Omega\tau G(\tau)\left\langle v(t+t')\cos\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right]\right\rangle \\ &\quad - a\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\left\langle v(t+t')\sin\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right]\right\rangle. \end{aligned} \tag{B.3}$$

For the averaging of Eqs. (B.2) and (B.3) we use the Furutsu–Novikov formula [29,30], valid for zero-mean Gaussian variables x, y:

$$\langle xF(y)\rangle = \langle F'(y)\rangle\langle xy\rangle.$$

Thus, all terms having the form $\langle x\cos y\rangle$ vanish, whereas all terms of type $\langle x\sin y\rangle$ remain:

$$\left\langle \xi(t+t')\sin\left[\int_{-\tau}^0 \mathrm{d}z v(z+t)\right]\right\rangle = \int_\tau^0 \mathrm{d}z K_{\xi v}(t'-z)\mathrm{e}^{-R}.$$

Finally we can rewrite Eqs. (B.2) and (B.3) as:

$$K_{\xi v}(t') = K_\xi(t') - a\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\int_{-\tau}^0 \mathrm{d}z K_{\xi v}(t'-z)\mathrm{e}^{-R}, \tag{B.4}$$

$$K_v(t') = K_{\xi v}(-t') - a\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\int_{-\tau}^0 \mathrm{d}z K_v(t'-z)\mathrm{e}^{-R}. \tag{B.5}$$

Now we introduce the spectrum of the noise S_ξ and the cross-spectrum of instantaneous frequency and noise $S_{\xi v}$. Then Eqs. (B.4) and (B.5) yield

$$S_{\xi v}(\omega) = S_\xi(\omega) - aS_{\xi v}(\omega)\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\mathrm{e}^{-R}\int_{-\tau}^0 \mathrm{d}z\mathrm{e}^{-\mathrm{i}\omega z}, \tag{B.6}$$

$$S_v(\omega) = S_{\xi v}(-\omega) - aS_v(\omega)\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\mathrm{e}^{-R}\int_{-\tau}^0 \mathrm{d}z\mathrm{e}^{-\mathrm{i}\omega z}. \tag{B.7}$$

Excluding $S_{\xi v}(\omega)$ we get

$$S_v(\omega) = \frac{S_\xi(\omega)}{\left|1 - \frac{a}{\mathrm{i}\omega}\int_0^\infty \mathrm{d}\tau \sin\Omega\tau G(\tau)\mathrm{e}^{-R}(1-\mathrm{e}^{\mathrm{i}\omega\tau})\right|^2}. \tag{B.8}$$

Appendix C. Linear oscillator in the feedback loop: Equations for Ω and D

Computing the integrals in Eqs. (22)–(25) we obtain equations for the frequency Ω and diffusion coefficient D in the linear

$$\Omega - \frac{16a\alpha\Omega^2}{\left((\Omega+\omega)^2+\alpha^2/4\right)\left((\Omega-\omega)^2+\alpha^2/4\right)} = \Omega_0,$$
$$D = \frac{D_0}{\left[1+\frac{2a\alpha\Omega\left(\Omega^4-(\omega^2+\alpha^2/4)^2\right)}{\left((\Omega+\omega)^2+\alpha^2/4\right)^2\left((\Omega-\omega)^2+\alpha^2/4\right)^2}\right]^2}, \tag{C.1}$$

and in the Gaussian approximation

$$\Omega + \frac{16a\alpha\Omega^2+2aD\left((\alpha+D)^2+4(\Omega^2+\omega^2)\right)}{\left((\Omega+\omega)^2+\frac{(\alpha+D)^2}{4}\right)\left((\Omega-\omega)^2+\frac{(\alpha+D)^2}{4}\right)} = \Omega_0, \tag{C.2}$$

$$\begin{aligned} D \;=\; D_0 \Big/ \Big[1 &+ (16\Omega a D^5 + 32\Omega a\alpha D^4 - 32\Omega a(\alpha^2+4\omega^2-4\Omega^2)D^3 \\ &+ 128\Omega a\alpha(-2\omega^2-\alpha^3+2\Omega^2)D^2 + 16\Omega a(-7\alpha^4-48\omega^4+8\Omega^2\alpha^2+16\Omega^4 \\ &+ 32\Omega^2\omega^2-40\omega^2\alpha^2)D - 512\Omega a\omega^4\alpha + 512\Omega^5 a\alpha - 256\Omega a\omega_0'^2\alpha^3 - 32\Omega a\alpha^5) \\ &\times \left((\Omega+\omega_0')^2+(\alpha+D)^2/4\right)^{-2}\left((\Omega-\omega_0')^2+(\alpha+D)^2/4\right)^{-2}\Big]^2 . \end{aligned} \tag{C.3}$$

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