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Supporting Online Material for

Controlling Electromagnetic Fields

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Supporting online material for:

Controlling Electromagnetic Fields

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Scaling of ε, μ for a General Coordinate Transformation

The following is taken from reference (*14*) of the main text with corrections for a few typographical errors.

We start from Maxwell's equations in a system of Cartesian coodinates:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mu\mu_0 \, \partial \mathbf{H} / \partial t, \\ \nabla \times \mathbf{H} &= +\varepsilon\varepsilon_0 \, \partial \mathbf{E} / \partial t \end{aligned} \tag{1}$$

where both ε and μ may depend on position. Let us transform to a general system defined by,

$$q_1(x, y, z), \quad q_2(x, y, z), \quad q_3(x, y, z) \tag{2}$$

Lines of constant q_2, q_3 define the generalized q_1 axis, and so on. Thus if we define a set of points by equal increments along the q_1, q_2, q_3 axes, the mesh will appear distorted in the original x, y, z co-ordinate frame: see figure 1.

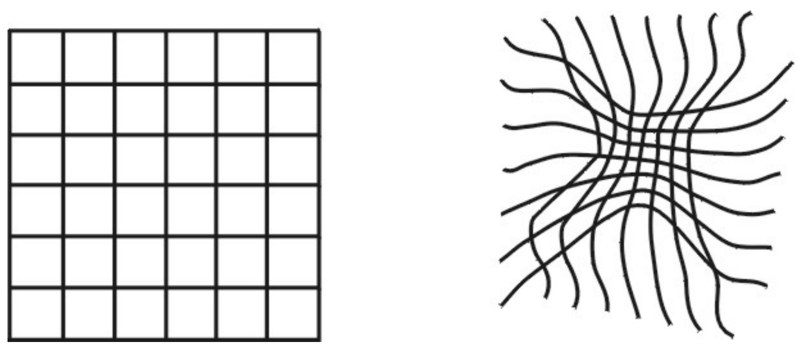

Figure 1. Simple cubic lattice of points in one co-ordinate system (left) maps into a distorted mesh in the other co-ordinate system (right). Choosing the co-ordinate transformation correctly give can control the distortions and produce a mesh tailored to our requirements.

Maxwell's equations as defined above are written in the original Cartesian system. What form does the new set of equations take when expressed in terms of q_1, q_2, q_3? The answer is surprisingly simple. Maxwell's equations in the new system of co-ordinates become

$$\begin{aligned} \nabla_q \times \hat{\mathbf{E}} &= -\mu_0 \hat{\mu} \, \partial \hat{\mathbf{H}} / \partial t, \\ \nabla_q \times \hat{\mathbf{H}} &= +\varepsilon_0 \hat{\varepsilon} \, \partial \hat{\mathbf{E}} / \partial t \end{aligned} \tag{3}$$

where $\hat{\varepsilon}, \hat{\mu}$ are in general tensors, and $\hat{\mathbf{E}}, \hat{\mathbf{H}}$ are renormalized electric and magnetic fields. All four quantities are simply related to the originals. In other words the form of Maxwell's equations is preserved by a co-ordinate transformation: the co-ordinate transformation does not change the fact that we are still solving Maxwell's equations, it simply changes the definition of $\hat{\varepsilon}, \hat{\mu}$.

We define three units vectors, u_1, u_2, u_3, to point along the generalized q_1, q_2, q_3 axes. The length of a line element is given by,

$$ds^2 = dx^2 + dy^2 + dz^2 = Q_{11}dq_1^2 + Q_{22}dq_2^2 + Q_{33}dq_3^2 \\ +2Q_{12}dq_1dq_2 + 2Q_{13}dq_1dq_3 + 2Q_{23}dq_2dq_3 \tag{4}$$

where,

$$Q_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} \tag{5}$$

In particular we shall need the length of a line element directed along one of the three axes,

$$ds_i = Q_i dq_i \tag{6}$$

where for shorthand,

$$Q_i^2 = Q_{ii} \tag{7}$$

To calculate $\nabla \times \mathbf{E}$ consider a small element, small enough that it resembles a parallelepiped (figure 2). In this we assume that the transformation has no singularities such as points or lines where the co-ordinate system suddenly heads off in a different direction.

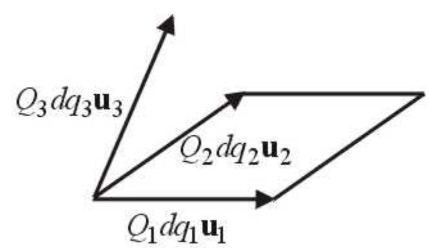

Figure 2. Small element resembling a parallelepiped.

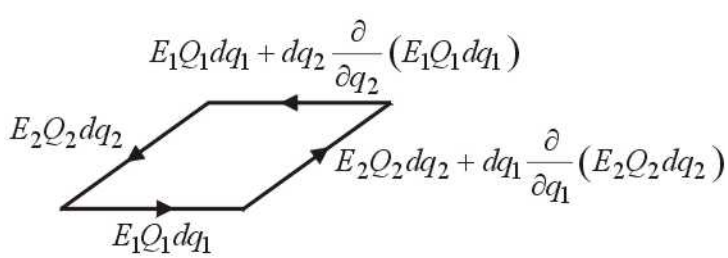

Figure 3. Integration path for finding $\nabla \times \mathbf{E}$.

First we calculate the projection of $\nabla\times\mathbf{E}$ onto the normal to the $\mathbf{u}_1-\mathbf{u}_2$ plane by taking a line integral around the $\mathbf{u}_1-\mathbf{u}_2$ parallelogram and applying Stokes' theorem (figure 3). We define,

$$E_1=\mathbf{E}\cdot\mathbf{u}_1,\quad E_2=\mathbf{E}\cdot\mathbf{u}_2,\quad E_3=\mathbf{E}\cdot\mathbf{u}_3 \tag{8}$$

so that,

$$(\nabla\times\mathbf{E})\cdot(\mathbf{u}_1\times\mathbf{u}_2)\,dq_1Q_1dq_2Q_2=dq_1\frac{\partial}{\partial q_1}(E_2dq_2Q_2)-dq_2\frac{\partial}{\partial q_2}(E_1dq_1Q_1) \tag{9}$$

or,

$$(\nabla\times\mathbf{E})\cdot(\mathbf{u}_1\times\mathbf{u}_2)\,Q_1Q_2=\frac{\partial\hat{E}_2}{\partial q_1}-\frac{\partial\hat{E}_1}{\partial q_2}=\left(\nabla_q\times\hat{\mathbf{E}}\right)^3 \tag{10}$$

where we use the conventional superscript notation for contravariant components of a vector. We have defined,

$$\hat{E}_1=Q_1E_1,\quad \hat{E}_2=Q_2E_2,\quad \hat{E}_3=Q_3E_3 \tag{11}$$

Note that the right-hand side of equation (10) is simple 'component 3' of curl evaluated in the new co-ordinate system. Now applying .Maxwell,

$$(\nabla\times\mathbf{E})\cdot(\mathbf{u}_1\times\mathbf{u}_2)\,Q_1Q_2=-\mu_0\mu\frac{\partial\mathbf{H}}{\partial t}\cdot(\mathbf{u}_1\times\mathbf{u}_2)\,Q_1Q_2 \tag{12}$$

We write $\mathbf{H}$ in terms of the contravariant components,

$$\mathbf{H}=H^1\mathbf{u}_1+H^2\mathbf{u}_2+H^3\mathbf{u}_3 \tag{13}$$

which in turn can be expressed in terms of the covariant components,

$$\mathbf{g}^{-1}\begin{bmatrix}H^1\\H^2\\H^3\end{bmatrix}=\begin{bmatrix}\mathbf{u}_1\cdot\mathbf{u}_1 & \mathbf{u}_1\cdot\mathbf{u}_2 & \mathbf{u}_1\cdot\mathbf{u}_3\\ \mathbf{u}_2\cdot\mathbf{u}_1 & \mathbf{u}_2\cdot\mathbf{u}_2 & \mathbf{u}_2\cdot\mathbf{u}_3\\ \mathbf{u}_3\cdot\mathbf{u}_1 & \mathbf{u}_3\cdot\mathbf{u}_2 & \mathbf{u}_3\cdot\mathbf{u}_3\end{bmatrix}\begin{bmatrix}H^1\\H^2\\H^3\end{bmatrix}=\begin{bmatrix}H_1\\H_2\\H_3\end{bmatrix} \tag{14}$$

where the first part defines $\mathbf{g}$, and

$$H_1=\mathbf{H}\cdot\mathbf{u}_1,\quad H_2=\mathbf{H}\cdot\mathbf{u}_2,\quad H_3=\mathbf{H}\cdot\mathbf{u}_3 \tag{15}$$

Inverting **g** gives

$$H^i=\sum_{j=1}^{3}g^{ij}H_j\ . \tag{16}$$

Substituting equation (13) into (12) gives

$$
\begin{aligned}
(\nabla\times\mathbf{E})\cdot(\mathbf{u}_1\times\mathbf{u}_2)Q_1Q_2 &= -\mu_0\mu\frac{\partial\mathbf{H}}{\partial t}\cdot(\mathbf{u}_1\times\mathbf{u}_2)Q_1Q_2 \\
&= -\mu_0\mu\sum_{j=1}^{3}g^{3j}\frac{\partial H_j}{\partial t}\mathbf{u}_3\cdot(\mathbf{u}_1\times\mathbf{u}_2)Q_1Q_2
\end{aligned}
\tag{17}
$$

Define

$$\hat{\mu}^{ij} = \mu g^{ij}\left|\mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3)\right|Q_1Q_2Q_3\left(Q_iQ_j\right)^{-1} \tag{18}$$

and

$$\hat{H}_j = Q_jH_j \tag{19}$$

so that,

$$(\nabla\times\mathbf{E})\cdot(\mathbf{u}_1\times\mathbf{u}_2)Q_1Q_2 = -\mu_0\sum_{j=1}^{3}\hat{\mu}^{3j}\frac{\partial\hat{H}_j}{\partial t} \tag{20}$$

Hence on substituting from equation (10),

$$\left(\nabla_q\times\hat{\mathbf{E}}\right)^i = -\mu_0\sum_{j=1}^{3}\hat{\mu}^{ij}\frac{\partial\hat{H}_j}{\partial t} \tag{21}$$

and by symmetry between E and H fields,

$$\left(\nabla_q\times\hat{\mathbf{H}}\right)^i = +\varepsilon_0\sum_{j=1}^{3}\hat{\varepsilon}^{ij}\frac{\partial\hat{E}_j}{\partial t} \tag{22}$$

where

$$\hat{\varepsilon}^{ij} = \varepsilon g^{ij}\left|\mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3)\right|Q_1Q_2Q_3\left(Q_iQ_j\right)^{-1} \tag{23}$$

Note that these expressions simplify considerably if the new co-ordinate system is orthogonal, e.g. cylindrical or spherical, when

$$g^{ij}\left|\mathbf{u}_1\cdot(\mathbf{u}_2\times\mathbf{u}_3)\right| = \delta_{ij} \tag{24}$$

The orthogonal version is quoted in the main text.