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Controlling Mackey–Glass chaos

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The Mackey–Glass equation is the representative example of delay induced chaotic behavior. Here, we propose various control mechanisms so that otherwise erratic solutions are forced to converge to the positive equilibrium or to a periodic orbit oscillating around that equilibrium. We take advantage of some recent results of the delay differential literature, when a sufficiently large domain of the phase space has been shown to be attractive and invariant, where the system is governed by monotone delayed feedback and chaos is not possible due to some Poincaré–Bendixson type results. We systematically investigate what control mechanisms are suitable to drive the system into such a situation and prove that constant perturbation, proportional feedback control, Pyragas control, and state dependent delay control can all be efficient to control Mackey–Glass chaos with properly chosen control parameters. *Published by AIP Publishing.* https://doi.org/10.1063/1.5006922

The Mackey–Glass equation, which was proposed to illustrate nonlinear phenomena in physiological control systems, is a classical example of a simple looking time delay system with very complicated behavior. Here, we use a novel approach for chaos control: we prove that with well-chosen control parameters, all solutions of the system can be forced into a domain where the feedback is monotone, and by the powerful theory of delay differential equations with monotone feedback, we can guarantee that the system is not chaotic any more. We show that this domain decomposition method is applicable with the most common control terms. Furthermore, we propose another chaos control scheme based on state dependent delays.

I. INTRODUCTION

A. The Mackey-Glass equation

$$x'(t) = -\mu x(t) + \frac{px(t-\tau)}{1+x(t-\tau)^n}, \quad \mu, p, n, \tau > 0 \qquad (1.1)$$

was introduced in 1977 to illustrate some nonlinear phenomena arising in physiological control systems.[20] Here, $'$ denotes the temporal derivative of a scalar state variable $x(t)$, and the function $f(\xi) = \frac{p\xi}{1+\xi^n}$ represents a feedback mechanism with time delay τ. The interesting situation is n being large when the function f has a distinctive unimodal shape, and in this paper, we consider only this case (at least $n > 2$). The Mackey–Glass equation provides a benchmark for the application of new techniques for nonlinear delay differential equations as it can generate diverse dynamics, from convergence to oscillations with different characteristics and even chaotic behavior. Despite intensive research over the decades with a number of analytical,[4,17,25] numerical,[2,8,22] and even experimental studies,[1,10] the emergence of such complexity is not fully understood yet.

Recent decades showed a growing interest towards chaos control, and several methods have been proposed and applied.[26] In this paper, we use another strategy, which we think is novel in the context of chaos control: instead of controlling a particular unstable periodic orbit, we drive all solutions into a domain where the system is governed by monotone feedback.[6,15,23,24]

B. The delay differential equation

$$x'(t) = -\mu x(t) + f(x(t-\tau)) \qquad (1.2)$$

with monotone feedback (where $f'(x) < 0$ for all x or $f'(x) > 0$ for all x) has been widely studied in the mathematical literature, and a comprehensive description is available on its global dynamic behaviors for some classes of monotone nonlinearities.[11] There have been some further interesting new developments as well recently.[13,14]

One important result is a Poincaré–Bendixson type theorem of Mallet-Paret and Sell,[21] which implies that in the case of monotone feedback, bound solutions converge either to an equilibrium or to a periodic orbit, and hence, chaotic trajectories are not possible.

The complexity of the Mackey–Glass equation stems from the combination of time delay and the non-monotonicity of the feedback, and in fact, chaotic behavior has been proven for a special class of equations with non-monotone delayed feedback.[16] A domain decomposition method has been proposed for unimodal feedback functions,[25] which provides sufficient conditions such that all solutions eventually enter a domain where f is either increases or decreases, and in this case, the complicated behavior is excluded. In this paper, we take advantage of this idea and propose various schemes that can impose such a situation. After describing the mathematical background in Sec. II, in Sec. III, we propose additive control terms and consider the following equation:

$$x'(t) = -\mu x(t) + \frac{px(t-\tau)}{1+x(t-\tau)^n} + u(t) \qquad (1.3)$$

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with control term $u(t)$. We investigate three typical cases, namely, constant perturbation $u(t)=k$, proportional feedback control $u(t)=kx(t)$, and the delayed feedback controller $u(t)=k[x(t)-x(t-\tau)]$. We shall say that the chaos is controlled if the system shows complicated behavior for $k=0$, but all solutions eventually enter and remains in some monotone domain of f for some $k\neq 0$, in which case convergence to an equilibrium or to a periodic orbit is guaranteed. In Sec. IV, we use a different approach: instead of an additive term, we construct a state dependent delay $\tau=\tau[x(t)]$ in a proper way so that our domain decomposition method is still applicable. It is important to stress that in this case, the form of the controlled equations is of (1.1) instead of (1.3), and the delay itself will be the subject to the control. In Sec. V, we illustrate our control mechanisms with a set of numerical simulations, and we conclude this paper with a summary and discussion of the interpretation of our results.

II. MATHEMATICAL BACKGROUND

Let $C=C([-\tau,0],\mathbb{R})$ denote the Banach space of continuous functions $\phi:[-\tau,0]\to\mathbb{R}$ with the usual sup norm $||\phi||=\max_{-\tau\le s\le 0}|\phi(s)|$. Given its biological interpretation, traditionally, only non-negative solutions of (1.1) are studied, and hence, we restrict our attention to the cone

$$C_+=\{\phi\in C:\phi(s)\ge 0,-\tau\le s\le 0\}$$

and define the corresponding order intervals

$$[\phi,\psi]:=\{\zeta\in C:\psi-\zeta\in C_+,\zeta-\phi\in C_+\}.$$

Every $\phi\in C_+$ determines a unique continuous function $x=x^\phi:[-\tau,\infty)\to\mathbb{R}$, which is differentiable on $(0,\infty)$, and satisfies (1.1) for all $t>0$, and $x(s)=\phi(s)$ for all $s\in[-\tau,0]$. It is easy to see that the cone C_+ is positively invariant, i.e., a solution $x^\phi(t)$ with a non-negative initial function ϕ remains non-negative for all $t\ge 0$. Existence and uniqueness extend to (1.3) too when $u(t)$ has the usually required smoothness; however, non-negativity should be checked in each specific case. The segment $x_t\in C$ of a solution is defined by the relation $x_t(s)=x(t+s)$, where $s\in[-\tau,0]$ and $t\ge 0$, and thus, $x_0=\phi$ and $x_t(0)=x(t)$. The family of maps

$$\Phi:[0,\infty)\times C_+\ni(t,\phi)\mapsto x_t(\phi):=\Phi_t(\phi)\in C_+$$

defines a continuous semiflow on C_+. For any $\xi\in\mathbb{R}$, we write ξ_* for the element of C satisfying $\xi_*(s)=\xi$ for all $s\in[-\tau,0]$. The equilibria ξ_* of (1.1) are given by the solutions of $\mu\xi=f(\xi)$. The trivial equilibrium is 0_*, and in addition, there exists at most one positive equilibrium K_* given by $K=(p/\mu-1)^{1/n}$. Note that $f'(\xi)=p(1-(n-1)\xi^n)(1+\xi^n)^{-2}$, so $f'(0)=p$, and there is a unique $\xi_0=(n-1)^{-1/n}$ such that $f'(\xi_0)=0$. The function f increases on $[0,\xi_0]$, have its maximum $f(\xi_0)=p(n-1)^{1-1/n}n^{-1}$, and decreases on $[\xi_0,\infty)$ with $\lim_{x\to\infty}f(x)=0$. Depending on the parameters, there are three fundamental situations:

(a) if $\mu\ge p$, then only the zero equilibrium exists;
(b) if $\mu<p\le\mu(1+(n-1)^{-1})$, then there is a positive equilibrium K_* on the increasing part of f (i.e., $K\le\xi_0$);
(c) if $p>\mu(1+(n-1)^{-1})$, then there is a positive equilibrium K_* on the increasing part of f (i.e., $K>\xi_0$ or equivalently $\mu<f(\xi_0)/\xi_0$).

It is well known[25] that in case (*a*) all solutions converge to 0 and in case (*b*) all positive solutions converge to K, regardless of the delay. Thus, here we consider only the interesting case (*c*), when the following numbers

$$\beta:=\frac{f(\xi_0)}{\mu},\quad \alpha:=\frac{f(\beta)}{\mu}=\frac{f\left(\frac{f(\xi_0)}{\mu}\right)}{\mu}$$

also play a crucial role in characterizing the nonlinear dynamics of Eq. (1.1). A cornerstone of this paper is the following result, which combines Theorem 3.5 (Röst and Wu[25]) and Theorem 8 (Liz and Röst[17]), ensuring that the long term dynamics is governed by a monotone part of the feedback function.

Theorem II.1. *Let $g(x)=\mu^{-1}f(x)$, and assume $g'(0)>1$ and $K>\xi_0$. Then, if either condition*

$$g^2(\xi_0)>\xi_0 \qquad (L)$$

or

$$h^2(\xi_0)>\xi_0,\quad\text{where}\quad h(x)=(1-e^{-\mu\tau})g(x)+e^{-\mu\tau}K \qquad (T)$$

holds, then every solution eventually enters and remains in the domain where f' is negative, hence converging to K or to a periodic solution oscillating around K.

The assumption of this theorem means that we are in case (*c*). Then, the interval $[\alpha_*,\beta_*]$ is attractive and invariant,[25] and condition (*L*) means $\alpha>\xi_0$. The results relating attractive invariant intervals of the discrete map f to attractive invariant intervals for (1.1) originate from the study by Ivanov and Sharkovsky[7] and recently have been successfully used for other problems as well.[5,18] Note that this condition is independent of τ, and hence, in this situation, chaotic behavior cannot appear by increasing the delay. The delay dependent condition (*T*) is built on earlier works.[4,19]

III. CONTROLLING MACKEY-GLASS CHAOS WITH ADDITIVE TERMS

Our aim is to choose our additive control term $u(t)$ from three common classes, in a way that some analogue of Theorem II.1 holds for (1.3).

A. Constant perturbation control

For any $k\in\mathbb{R}$, we consider

$$x'(t)=-\mu x(t)+\frac{px(t-\tau)}{1+x(t-\tau)^n}+k. \qquad (3.1)$$

Theorem III.1. *Assume that $K>\xi$ but (L) is not satisfied, that is, $g^2(\xi_0)\le\xi_0$ in (1.1). Then, the following statements hold:*

(i) *there is a $k_*<\mu\xi_0$ such that for all $k\ge k_*$, (3.1) has no complicated solution;*

(ii) *there is an explicitly computable* k_1 *such that for* $k < k_1$, *(3.1) has no equilibria and solutions become unfeasible;*

(iii) *for* $k_1 < k < k_2 := \mu\xi_0 - f(\xi_0)$, *there are two positive equilibria* K_1 *and* K_2, *and solutions with initial function* $\phi \in [(K_1 + k/\mu)_*, \xi_{0*}]$ *converge to* K_2;

(iv) *there exists a* k_3 *such that for* $k_2 < k < k_3$, *(3.1) has no complicated solutions.*

Proof. After using the change of variable $y = x - \frac{k}{\mu}$, (3.1) reads as

$$y'(t) = -\mu y(t) + p\frac{y(t-\tau) + \dfrac{k}{\mu}}{1 + \left(y(t-\tau) + \dfrac{k}{\mu}\right)^n}.$$

That is

$$y'(t) = -\mu y(t) + f_k(y(t-\tau)) \tag{3.2}$$

with $f_k(\xi) = f\left(\xi + \frac{k}{\mu}\right)$, and thus, adding the constant perturbation k has the same effect as shifting the graph of f by k/μ. Note that we are interested only in non-negative solutions $x(t)$ of (3.1), that is, $y(t) \geq -k/\mu$, and we call such solutions feasible. Let $\hat{\xi}_0 = \xi_0 - \frac{k}{\mu}$, $\hat{\beta} = \frac{f_k(\hat{\xi}_0)}{\mu}$, $\hat{\alpha} = \frac{f_k(\hat{\beta})}{\mu}$. Clearly, $f_k'(\hat{\xi}_0) = 0$ and $\hat{\beta} = \beta$, that is, $\hat{\alpha} = f_k(f(\xi_0))$.

(i) For $k > 0$, the graph of f is shifted to the left and solutions remain positive, and we also have $\hat{\alpha} > 0$, with $\liminf_{t\to\infty} y(t) \geq \hat{\alpha}$ (analogously to Theorem 3.5 from Röst and Wu[25]). At $k \geq \mu\xi_0$, $\hat{\xi}_0 \leq 0 < \hat{\alpha}$ and by continuity, the relation $\hat{\xi}_0 < \hat{\alpha}$ must hold on some interval $(k_*, \mu\xi_0)$ as well.

(ii) We shift the graph of f to the right until equilibria are destroyed. In the critical case $k = k_1$, f is tangential to $\mu\xi$, so first we find the unique $\xi_\mu > 0$ such that $\mu = f'(\xi_\mu) = \frac{p(1+(1-n)\xi_\mu^n)}{(1+\xi_\mu^n)^2}$. This is a quadratic equation in ξ_μ^n, and taking its positive root, we find

$$\xi_\mu = \left(\frac{-2\mu - p(n-1) + \sqrt{4p\mu n + p^2(n-1)^2}}{2\mu}\right)^{\frac{1}{n}}.$$

When the graph is shifted by k_1/μ, the tangent line of the shifted graph f_{k_1} with slope μ is exactly the line $\mu\xi$, so we must have $f(\xi_\mu) = \mu(\xi_\mu - k_1/\mu)$, which gives $k_1 = f(\xi_\mu) - \mu\xi_\mu$. For $k < k_1$, $f_k(\xi) < \mu\xi$ holds on $[-k/\mu, \infty)$, where f_k is defined. For a solution $y(t)$, let $v(t) := y(t) + \int_{t-\tau}^{t} f_k(y(s))ds$, then $v'(t) = -\mu y(t) + f(y(t)) < -\min_{\xi\geq -k/\mu}[\mu\xi - f_k(\xi)] < 0$. This means that $v(t)$ becomes smaller than $-k/\mu$ in finite time, but due to $y(t) < v(t)$, each solution $y(t)$ becomes unfeasible.

(iii) For $k_1 < k < 0$, there are always two equilibria of (3.2); now, we are looking for another critical value k_2 that separates the cases when the larger equilibrium is on the decreasing part of f_k from when both are on the increasing part. The critical case is characterized by one of the equilibria being $\hat{\xi}_0$, that is, $\mu\hat{\xi}_0 = f_k(\hat{\xi}_0) = f(\xi_0)$, and $k_2 = \mu(\xi_0 - f(\xi_0))$ follows. For $k_1 < k < k_2$, there are two equilibria, $\hat{K}_1$ and $\hat{K}_2 < \hat{\xi}_0$. It is easy to see that there are initial functions ϕ with $\phi(0) = -k/\mu$, $\phi(\theta)$ small for $\theta < 0$ such that the derivative of the solution is negative at zero, and thus, unfeasible solutions exist. To avoid such situations, we restrict our attention to the interval $[\hat{K}_{1*}, \hat{\xi}_{0*}]$, where f_k is monotone increasing. For solutions with segments from this interval, $y(t) = \hat{K}_1$ implies $y'(t) \geq -\mu K_1 + f_k(\hat{K}_1) = 0$, and $y(t) = \hat{\xi}_0$ implies $y'(t) \leq -\mu\hat{\xi}_0 + f_k(\hat{\xi}_0) < 0$; therefore, this interval is invariant. Now, we can apply Proposition 2 from Röst and Wu[25] to show that all solutions in this interval converge to $\hat{K}_2$. Transforming back to variable x, we obtain (iii).

(iv) First notice that $f_{k_2}(f_{k_2}(\hat{\xi}_0)/\mu) = \mu\hat{\xi}_0$. Our goal is to show that

$$\begin{aligned} D(k) :&= f_k(f_k(\hat{\xi}_0)/\mu) - \mu\hat{\xi}_0 = f_k(f(\xi_0)/\mu) - \mu\xi_0 + k \\ &= f((f(\xi_0) + k)/\mu) - \mu\xi_0 + k > 0 \end{aligned}$$

in an interval (k_2, k_3), and then, an analogue of Theorem II.1 provides the result. Differentiating with respect to k gives

$$D'(k) = f'((f(\xi_0) + k)/\mu)/\mu + 1,$$

and evaluating at $k_2 = \mu\xi_0 - f(\xi_0)$, we arrive at

$$D(k_2) = 0, \quad D'(k_2) = f'(\xi_0)/\mu + 1 = 1 > 0.$$

□

B. Proportional feedback control

In this subsection, we consider $u(t) = kx(t)$. The rearrangement

$$x'(t) = -(\mu - k)x(t) + \frac{px(t-\tau)}{1 + x(t-\tau)^n} \tag{3.3}$$

shows that the control has no effect on the key properties of the nonlinearity in (1.1).

With $w = \mu - k$, Theorem II.1 can be directly applied.

Theorem III.2. *Assume* $\alpha < \xi_0 < K$. *Then, the following holds:*

(i) *there is a* $k_* < 0$ *such that for* $k \in (\mu - f(\xi_0)/\xi_0, k_*)$, *(3.3) has no complicated solutions;*

(ii) *if* $\mu - p < k < \mu - f(\xi_0)/\xi_0$, *then all solutions converge to* $(p/(\mu - k) - 1)^{1/n}$;

(iii) *if* $k \leq \mu - p$, *then all solutions converge to 0;*

(iv) *if* $k > \mu$, *all solutions converge to infinity.*

Proof. (i) For a given k, let

$$\begin{aligned} \tilde{\beta} &= \frac{f(\xi_0)}{\mu - k} = \frac{p(n-1)^{\frac{n-1}{n}}}{n(\mu - k)}, \\ \tilde{\alpha} &= \frac{f(\tilde{\beta})}{\mu - k} = \frac{p^2(n-1)^{\frac{n-1}{n}} n^n (\mu - k)^n}{n(\mu - k)^2\left(n^n(\mu - k)^n + p^n(n-1)^{n-1}\right)}, \end{aligned} \tag{3.4}$$

and $\tilde{g} = f/(\mu - k)$. Notice that $k = \mu - f(\xi_0)/\xi_0$ means that $\tilde{\alpha} = \tilde{\beta}$ and $\tilde{g}^2(\xi_0) = \xi_0$. Hence, to apply (*L*) to (3.3), we want to show that

$$\tilde{\alpha}\frac{1}{\xi_0} = \frac{p^2(n-1)n^{n-1}(\mu-k)^{n-2}}{\left(n^n(\mu-k)^n + p^n(n-1)^{n-1}\right)} > 1$$

for some k. For simplicity, we write $w = \mu - k$, and let

$$S(w) = \frac{p^2(n-1)n^{n-1}w^{n-2}}{\left(n^n w^n + p^n(n-1)^{n-1}\right)}$$

and $w_0 = \frac{p(n-1)}{n}$. It is easy to check that $S(w_0) = 1$. Furthermore,

$$S'(w) = p^2(n-1)^{\frac{n-1}{n}} n^n \frac{p^n(n-1)^{n-1}(n-2)w^{n+1} - 2n^n w^{2n+1}}{nw^4\left(n^n w^n + p^n(n-1)^{n-1}\right)^2},$$

hence

$$S'(w_0) = p^2(n-1)^{\frac{n-1}{n}} n^n \frac{\left(\frac{p(n-1)}{n}\right)^{n+1}(n-2n)}{nw^4\left(n^n w^n + p^n(n-1)^{n-1}\right)^2} < 0$$

and $S'(\hat{w}) = 0$ only for $\hat{w} = \frac{p(n-1)}{n}\sqrt[n]{\frac{n-2}{2n-2}} < w_0$. These, together with the facts $S(0) = 0$ and $S'(w) > 0$ for $w \in (0, \hat{w})$, imply the existence of a unique $w_* < w_0$, satisfying $S(w_*) = 1$. That is for $(\mu - k) \in (w_*, w_0)$, every solution enters the interval $[\tilde{\alpha}, \tilde{\beta}]$, where f monotonically decreases, prohibiting the existence of chaotic solutions. Shifting back, $(\mu - k) \in (w_*, w_0)$ is equivalent to $k \in (\mu - w_0, \mu - w_*)$, and with $k_* = \mu - w_*$ and noting that $w_0 = f(\xi_0)/\xi_0$, we conclude (i). To see (ii) and (iii), notice that in these cases, (3.3) falls in the cases of (b) and (c) as described in Sec. II, and thus, Proposition 3.2 and Proposition 3.1 from Röst and Wu[25] give the result. To check (iv), from $x'(t) > (k - \mu)x(t)$, convergence to infinity is clear for $k > \mu$. □

Next, we give a delay dependent result.

Theorem III.3. *Assume that $K > \xi_0$ and (L) does not hold for $\tilde{g}$ with some k. Then, for sufficiently small delay, (T) holds for $\tilde{h}$. Furthermore, the smaller the delay, the larger the range of k that enables chaos control.*

Proof. The first statement is obvious, since as $\tau \to 0$, (*T*) becomes $K > \xi_0$ regardless of k. For the second statement, note that the control parameter does not change ξ_0, but K becomes $\tilde{K}$. Fix all the parameters but τ such that $\tilde{K} > \xi_0$, and let $w = \mu - k$ and denote by $\tilde{h}_\tau$ the function in condition (*T*) corresponding to Eq. (3.3), belonging to a given τ. We show that if $\tau_1 < \tau_2$, then $\tilde{h}^2_{\tau_1}(\xi_0) > \tilde{h}^2_{\tau_2}(\xi_0)$. Since $\tilde{g}(\xi_0) > \tilde{K}$, we have $\tilde{h}_{\tau_2}(\xi_0) > \tilde{h}_{\tau_1}(\xi_0) > \tilde{K}$, and for $\xi > \tilde{K}$, $\tilde{g}(x) < \tilde{K}$ implies $\tilde{h}_{\tau_2}(\xi) < \tilde{h}_{\tau_1}(\xi)$. Together with the monotone decreasing property of $\tilde{h}$ for $\xi > \tilde{K}$, we find

$$\begin{aligned}\tilde{h}^2_{\tau_1}(\xi_0) &= \tilde{h}_{\tau_1}(\tilde{h}_{\tau_1}(\xi_0)) > \tilde{h}_{\tau_2}(\tilde{h}_{\tau_1}(\xi_0)) > \tilde{h}_{\tau_2}(\tilde{h}_{\tau_2}(\xi_0)) \\ &= \tilde{h}^2_{\tau_2}(\xi_0).\end{aligned}$$

The conclusion is that for $\tau_1 < \tau_2$, if $\tilde{h}^2_{\tau_2}(\xi_0) > \xi_0$ holds, then $\tilde{h}^2_{\tau_1}(\xi_0) > \xi_0$ also holds, and thus, if k is a good control for some delay (in the sense that (*T*) holds), it is a good control for all smaller delays as well. The consequence is that for smaller delays, we always have a larger range of k such that (*T*) still holds.

□

C. Pyragas control

A popular control mode is $u(t) = k(x(t-\tau)) - x(t)$, and with such a term, (1.1) becomes

$$x'(t) = -(\mu + k)x(t) + \frac{px(t-\tau)}{1 + x(t-\tau)^n} + kx(t-\tau),$$

that is

$$x'(t) = -(\mu + k)x(t) + F_k(x(t-\tau)) \tag{3.5}$$

with $F_k(\xi) = f(\xi) + k\xi$. Notice that while the Pyragas control changes the shape of the nonlinearity, it does not change the equilibria of the system.

Theorem III.4. *Assume $K > \xi_0$ and $g^2(\xi_0) < \xi_0$. Then, for $k > \frac{p(n-1)^2}{4n}$, all solutions of (3.5) converge to K.*

Proof. (i) A straightforward calculation shows that the function $f'(\xi) = \frac{p(1-(n-1)\xi^n)}{(1+\xi^n)^2}$ has a minimum when $\xi^n = \frac{n+1}{n-1}$: let $b(u) = \frac{p(1-(n-1)u)}{(1+u)^2}$, then $b'(u) = \frac{p(n(u-1)-u-1)}{(u+1)^3}$, and $b'(u) = 0$ exactly at $u = (n+1)/(n-1)$. Therefore, $f'(\xi) \geq \frac{-np}{(\frac{n+1}{n-1}+1)^2} = \frac{-p(n-1)^2}{4n}$, with equality at that point. Hence, if $k > \frac{p(n-1)^2}{4n}$, then $F'_k(\xi) = f'(\xi) + k > 0$, and in this case, (3.5) is governed by positive monotone feedback. Since $F_k(\xi) < (\mu + k)\xi$ for $\xi > K$, it is easy to see that any $[0_*, L_*]$ interval is invariant whenever $L > K$, and the same proof as Proposition 3.2 from Röst and Wu[25] ensures that all positive solutions converge to K.

□

When $k < 0$, there is a $\breve{\xi}$ such that $F_k(\breve{\xi}) < 0$, and then, solutions with initial functions satisfying $\phi(0) = 0$ and $\phi(-\tau) = \breve{\xi}$ immediately become negative. Since the nonnegative cone is not invariant any more, here we do not discuss Pyragas control with negative k.

When $0 < k < \frac{p(n-1)^2}{4n}$, then $F_k(\xi)$ has a bimodal shape, with local extrema $q_1 < q_2$. The numbers q_1 and q_2 can be found as the solutions of $f'(\xi) = -k$, which is quadratic in ξ^n, so it is possible to find them explicitly, similar to case (ii) of Theorem III.1. It is natural to try to apply an analogue of the (*L*) condition in the bimodal case too, forcing all solutions into the domain (q_1, q_2), where F is monotone decreasing. Nevertheless, the required conditions $q_1 < \alpha_k$ and $\beta_k < q_2$ become analytically intractable, and one can find parameter settings when they fail when k being near either zero or $\frac{p(n-1)^2}{4n}$. Another possibility is to force solutions to the increasing part of F_k, thus expecting convergence to K again, so we may require $\alpha_k > q_2$, that is, $F_k(F_k(q_1)/(\mu + k)) > (\mu + k)q_2$, but again that seems too involved to find a simply interpretable condition.

IV. STATE DEPENDENT DELAY CONTROL

From Theorem II.1, it is clear that chaos can be controlled by decreasing the delay to a small quantity, since as $\tau \to 0$, condition (T) becomes $K > \xi_0$, and hence, for sufficiently small τ, (T) is satisfied. However, it may be impossible or very expensive to permanently keep τ small, and thus, here we explore how can we establish chaos control when we modify the delay only temporarily, depending on the current state. Thus, we consider equation

$$x'(t) = -\mu x(t) + \frac{px(t - r(x(t)))}{1 + x(t - r(x(t)))^n} \tag{4.1}$$

with state dependent delay $r(x(t))$, where one can interpret $r(x(t)) = \tau - k(x(t))$ with baseline delay τ and delay control $k(x(t))$. It is reasonable to assume $k(x(t)) \geq 0$ and $k(x(t)) < \tau$, and then $r(x(t)) \in (0, \tau]$. We say that a solution is slowly oscillatory, if $x(t) - K$ has at most one sign change on each time interval of length τ.

Theorem IV.1. *Assume $K > \xi_0$ and let $\hat{K} < \xi_0$ be defined by $f(\hat{K}) = f(K)$. Let $\tau_* := \min\left\{\tau, \frac{K - \xi_0}{f(\xi_0)}, \frac{\xi_0 - \hat{K}}{f(\xi_0)}\right\}$, and $\zeta = (\tau - \tau_*)(f(\xi_0) - \mu\xi_0)$.*

The state dependent delay function is defined as follows:

$$r(x) = \tau \quad \text{for} \quad x \geq \xi_0 + \zeta;$$

$$r(x) = \tau_* \quad \text{for} \quad x \leq \xi_0;$$

$r(x)$ is the C^2-smooth and monotone on $[\xi_0, \xi_0 + \zeta]$ with $r'(x) \leq (f(\xi_0) - \mu\xi_0)^{-1}$.

Then, solutions of (4.1) eventually enter the domain where f' is negative, and slowly oscillatory complicated solutions cannot exist.

Proof. The existence and uniqueness of solutions have been discussed in the study by Krisztin and Arino.[12] Since $\tau_* \leq r(x(t)) \leq \tau$, we can deduce that $[\alpha_*, \beta_*]$ is attractive and invariant analogously to the constant delay case Theorem 3.5 from Röst and Wu.[25] For solutions in this interval, $|x'(t)| < f(\xi_0)$ holds. Now, we claim that positive solutions always go beyond ξ_0, i.e., $\limsup_{t\to\infty} x(t) > \xi_0$. Assume the contrary, then there is a solution $x(t) > 0$ such that $x(t) < \xi_0 + \epsilon$ holds for all $t > t_0$ with some $0 < \epsilon < K - \xi_0$. Define

$$z(t) = x(t) + \int_{t - r(x(t))}^{t} f(x(s))ds.$$

Then, $z(t) < \xi_0 + \epsilon + \tau f(\xi_0)$, but $z'(t) = -\mu x(t) + f(x(t)) > \min_{\xi \in [\alpha, \xi_0]}(f(\xi) - \mu\xi) > 0$ for all $t > 0$, which is a contradiction. Hence, for any positive solution, there is a t_* such that $x(t_*) > \xi_0$. Next, we show that for all $t \geq t_*$, $x(t) > \xi_0$ also holds. Assuming the contrary, there exists a t^* such that $x(t^*) = \xi_0$ and $x'(t^*) \leq 0$. Note that

$$\xi_0 = x(t^*) = x(t^* - r(\xi_0)) + \int_{t^* - r(\xi_0)}^{t^*} x'(s)ds$$
$$> x(t^* - r(\xi_0)) - r(\xi_0)f(\xi_0),$$

so $x(t - r(\xi_0)) < \xi_0 + r(\xi_0)f(\xi_0) < K$. Similarly, $x(t - r(\xi_0)) > \xi_0 - r(\xi_0)f(\xi_0) > \hat{K}$. But then, $x'(t^*) = -\mu\xi_0 + f(x(t - r(\xi_0))) \geq \mu(K - \xi_0) > 0$, a contradiction. We conclude that solutions enter the domain where $f' < 0$ and remain there. To apply the Poincaré–Bendixson type results of Krisztin–Arino,[12] we need to confirm the increasing property of $t \mapsto t - r(x(t))$, cf. condition (H2) of Ref. 12. This is equivalent to $r'(x)x'(t) < 1$, which obviously holds outside $(\xi_0, \xi_0 + \zeta)$. Within $(\xi_0, \xi_0 + \zeta)$, $x'(t) < f(\xi_0) - \mu\xi_0$ is valid, and hence, one can find a C^2-smooth $r(x)$ such that $r(\xi_0) = \tau_*$, $r(\xi_0 + \zeta) = \tau$, and meanwhile, $r'(x) \leq (f(\xi_0) - \mu\xi_0)^{-1}$.

Then, we can apply Theorem 8.1. of Krisztin and Arino,[12] and thus, slowly oscillatory solutions converge to K or to a periodic orbit. □

Remark IV.2. *Some recent results of Kennedy,[9] which have not been published yet, suggest that Theorem IV.1. can be extended from slowly oscillatory solutions to all solutions.*

While the control scheme in this theorem may seem complicated, what it really means is that when a solution approaches ξ_0 from above, we decrease the delay in a way that the solution will turn back before reaching ξ_0, hence forcing it to stay in the domain where $f' < 0$. In particular, $k(x) = 0$ for $x \geq \xi_0 + \zeta$ and $k(x) = \tau - \tau_*$ for $x \leq \xi_0$, and some intermediate control $k(x)$ is applied when the solution is in the interval $(\xi_0, \xi_0 + \zeta)$. For such an equation with state dependent delay, the Poincaré–Bendixson type theorem was proven only to the subset of slowly oscillatory solutions, and hence, at the current state-of-the-art of the theory, we cannot say more, but the applicability of this control scheme will be illustrated in Sec. V, (see also Fig. 3).

V. APPLICATIONS, SIMULATIONS, AND DISCUSSION

We investigated a number of possible mechanisms so that with a well-chosen control parameter, an otherwise chaotic Mackey–Glass system is forced to show regular behavior. The Mackey–Glass equation was used to model the rate of change of circulating red blood cells, and most of our results have a meaningful interpretation in this context. For example, $u(t) = k$ with $k > 0$ may represent medical replacement of blood cells at a constant rate, or $u(t) = kx(t)$ with negative k may represent the increased destruction rate of blood cells, which can be achieved by administration of antibodies.[3] Our approach is different from typical chaos control methods since our strategy is to choose a control such that all solutions will be attracted to a domain where the feedback function is monotone, and then, some Poincaré–Bendixson type results exclude the possibility of chaotic behavior. By applying this domain decomposition method, which is based on the study by Röst and Wu,[25] instead of stabilizing a particular orbit, we push the full dynamics into a non-chaotic regime.

For $u(t) = k$, clearly $k > 0$ helps the cell population, and as Theorem III.1 shows, with sufficiently large k, chaos can always be controlled regardless of the delay. A somewhat counterintuitive part of Theorem III.1 is that for some negative k, it is possible to force the system to converge to a positive equilibrium; however, one has to be careful as the system will collapse if k is below the threshold k_1. This is illustrated in Fig. 1, where on the left, we can see how $k > 0$

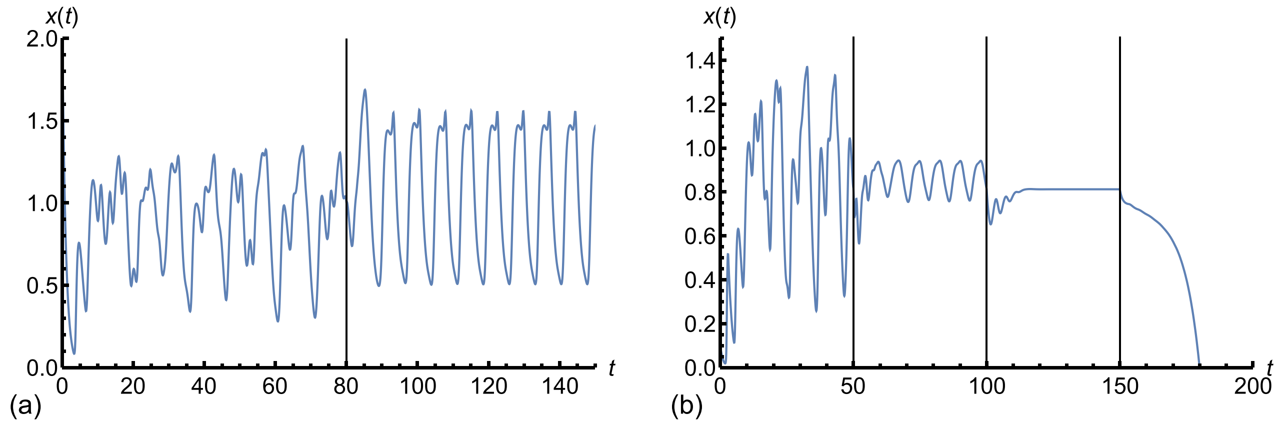

FIG. 1. Constant perturbation control: illustrations to Theorem III.1. On the left, a numerical solution to (3.1) is plotted. For $0 \leq t < 80$, there is no control ($k = 0$), and the solution is irregular. At $t = 80$, we switch on the constant control with $k = 0.39$. The initial function was $2 + 0.02 \sin t$. On the right, for $0 \leq t < 50$, $k = 0$, and the solution is irregular. At $t = 50$, k was decreased to -0.48, and the solution becomes periodic. At $t = 100$, k is set to -0.62, and the solution converges to K. From $t = 150$, we use $k = -0.69$, and the solution reaches 0 in finite time. The initial function was $-1.2t + 0.1e^t$. In both cases, the parameters were set to $\mu = 1$, $\tau = 3$, $p = 2$, and $n = 9.65$.

controls a chaotic solution into a periodic one, while in the right, we can observe that decreasing $k < 0$ first regulate the system into periodic behavior, then to convergence, and finally to collapse (i.e., hitting zero in finite time).

The proportional feedback control $u(t) = kx(t)$ again helps the population when $k > 0$, and when it fully compensates the baseline mortality ($\kappa > \mu$), the population grows and unbound (Theorem III.2, (iv)). Yet, controlling chaos is best achieved with $k < 0$, when the destruction of cells is increased, and then, with a fine tuning of k, the dynamics can be made regular (Theorem III.2, (i) and (ii)), which is shown on the left panel of Fig. 2. If cell destruction is too high, the population goes extinct (Theorem III.2, (iii)). Theorem III.3 gives a delay dependent result, showing that even if the condition (L) fails, chaos control can be achieved by satisfying (T). We showed that the smaller the delay, the easier the control, in the sense that we can pick k from a larger range to satisfy (T). On the right panel of Fig. 2, we illustrated this delay dependent feature: when we switched on the control, we decreased the delay temporarily to show that with this smaller delay, it is a good control, but when we reset the delay at some time later, the delay dependent condition (T) fails and the solution goes back to the irregular mode with the same control.

We also used the popular Pyragas control $u(t) = k(x(t-\tau) - x(t))$. The conclusion of our Theorem III.4 is that for positive k, the unimodal shape of the nonlinearity turns into a bimodal shape, and when k is large enough (our theorem explicitly tells us how large), the nonlinearity is transformed into a monotone feedback, as the control term overwhelms the original unimodality. Once we achieved monotonicity, we can use the results from the study by Röst and Wu[25] to prove that solutions converge to the positive equilibrium. Figure 3 (left) shows how such regulation occurs as we increase k. For negative k, the non-negative cone is not invariant anymore, so we do not consider this possibility. Let us remark that the control of Mackey–Glass chaos has been experimentally observed with Pyragas-type control,[10] and

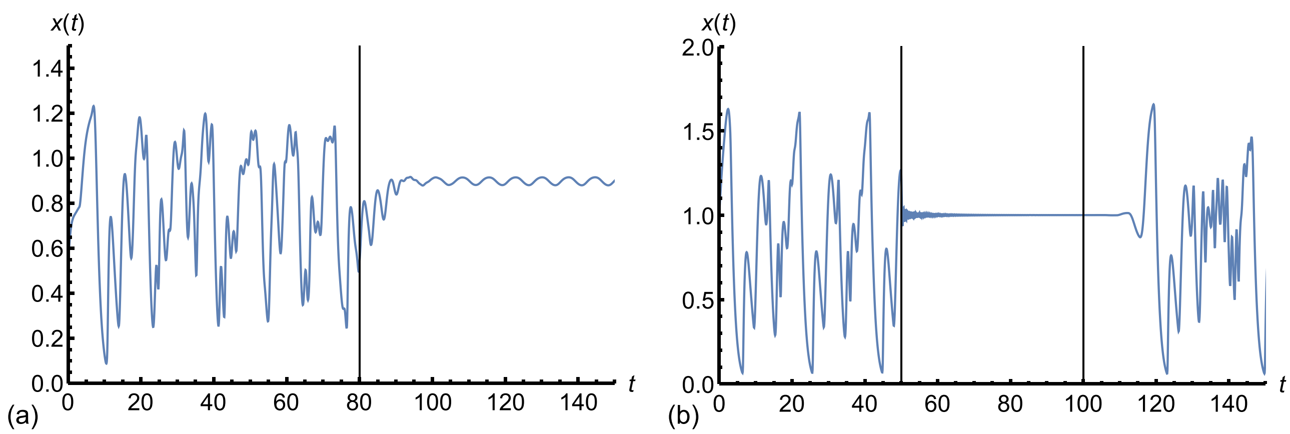

FIG. 2. Proportional feedback control: illustrations to Theorems III.2 and III.3. Numerical solutions to (3.3) are plotted. On the left for $0 \leq t < 80$, there is no control ($k = 0$), and the solution is irregular. At $t = 80$, we switch on the proportional control with $k = -0.507$. The other parameters were $n = 20$, $\mu = 1.275$, $\tau = 3.11$, and $p = 2$. With these parameters, the condition in (i) of Theorem III.2 is satisfied, and the solution converges to a regular oscillation. The initial function was $0.5 + 0.01 \cos 2t$. On the right, for $0 \leq t < 50$, $\tau = 3$ and $k = 0$. At $t = 50$, to illustrate Theorem III.3, τ is decreased to 0.125 and k is set to -0.022, so (T) holds and the solution behaves regularly. From $t = 100$, $\tau = 3$, and while k is still -0.022, now (T) fails with this larger delay, and the solution becomes irregular again. The other parameters were $n = 27.9$, $\mu = 0.97$, and $p = 2$. The initial function was $1 + 0.1t$.

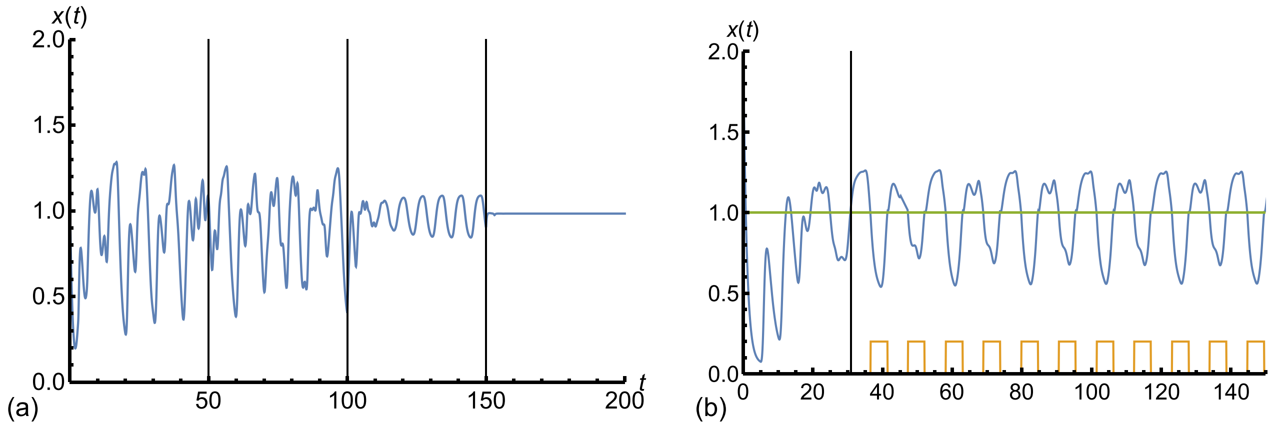

FIG. 3. Left: Pyragas control illustration to Theorem III.4. The numerical solution to (3.5) is plotted. For $0 \le t < 50$, there is no control ($k = 0$), and the solution is irregular. At $t = 50$, we switch on the Pyragas control with $k = 0.08$, which is too small to cease the irregularity of the solution, which becomes periodic after $t = 100$ when the control was increased to $k = 0.95$. Finally, the solution converges K after $t = 150$ when the control increased further to $k = 3.9$, when the condition of Theorem III.4 holds. The other parameters were set to $\mu = 1.08$, $\tau = 3$, $p = 2$, and $n = 9.65$. The initial function was $1 + 0.1e^{-t}$. Right: State dependent delay control. A numerical solution to (4.1) is plotted. We switch on the delay function scheme at $t = 31$, which drives the solution to a periodic orbit. The horizontal line shows the equilibrium, and it is also the boundary for delay reduction, where for the sake of simplicity, we used a step function for $r[x(t)]$: the delay is 5 for $x > K$ and 4 for $x < K$. In the lower part of the graph, it is shown when the delay control is on or off. Parameter values are $n = 6$, $p = 2$, and $\mu = 1$, and the initial function is 2_*.

here, our results give an analytic explanation how and why this happens.

Finally, we considered a very different type of control, taking advantage of some results from the theory of state dependent delays. While it is clear from Theorem (II.1) that chaos can be eliminated when the delay is sufficiently small, in Theorem (IV.1), we constructed a state dependent delay function that allows us to construct a delay control scheme where the delay is reduced only in a part of the phase space. This is illustrated in the right panel of Fig. 3, where we applied delay reduction only in the region $x < K$, and this was sufficient to drive the irregular solution into periodic behavior.

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