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CONVECTION ENHANCED DIFFUSION FOR PERIODIC FLOWS*

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Abstract. This paper studies the influence of convection by periodic or cellular flows on the effective diffusivity of a passive scalar transported by the fluid when the molecular diffusivity is small. The flows are generated by two-dimensional, steady, divergence-free, periodic velocity fields.

Key words. diffusion, homogenization, convection

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1. Introduction.

The temperature T of a weakly conducting fluid in $\mathbb{R}^2$ satisfies the heat equation

$$\frac{\partial T}{\partial t} = \varepsilon\,\Delta T + \mathbf{u}\cdot\nabla T, \tag{1.1}$$

with $T(0,x,y) = T_0(x,y)$ given. Here $\mathbf{u}(x,y) = \big(u(x,y),\, v(x,y)\big)$ is the fluid velocity, which we assume incompressible,

$$\nabla\cdot\mathbf{u} = 0,$$

and $\varepsilon > 0$ is the molecular diffusivity, which we assume small. We are interested in velocity fields that represent convective flow, for example, in Benard convection. Since $\mathbf{u}$ is incompressible, there is a stream function $H(x,y)$ such that

$$\nabla^{\perp} H = (-H_y, H_x) = \mathbf{u}. \tag{1.2}$$

A typical convective or cellular flow is given by

$$H(x,y) = \sin x \sin y. \tag{1.3}$$

Figure 1.1 shows the stream lines of this periodic flow, which are given by $H(x,y) =$ constant. We are interested in the *effective diffusivity* of the fluid and its behavior as the molecular diffusivity ε tends to zero.

In §2 we briefly review the definition and basic properties of the effective diffusivity. In this introduction, we may simply define it as

$$\sigma_\varepsilon = \lim_{t\uparrow\infty} \frac{1}{t}\int\!\!\int (x^2+y^2)\,T(t,x,y)\,dx\,dy, \tag{1.4}$$

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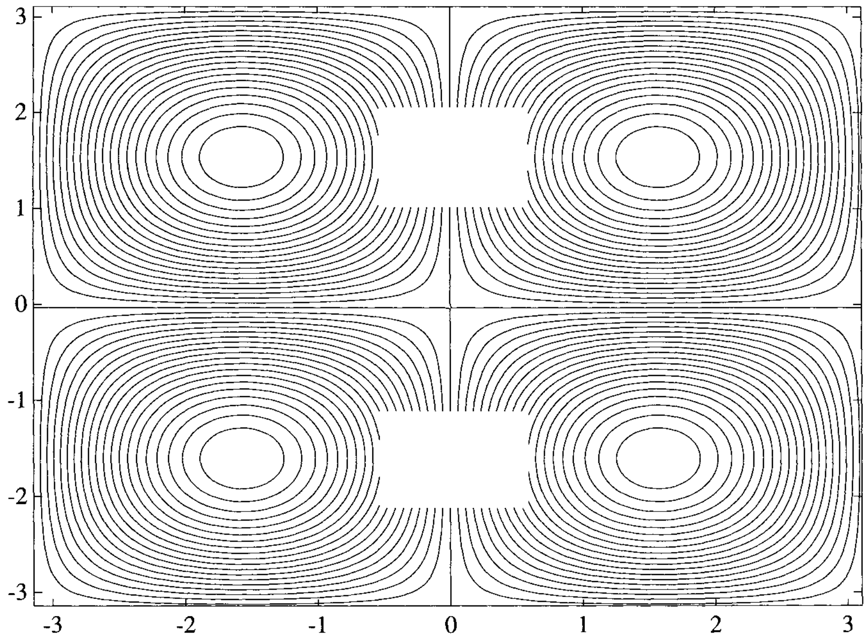

FIG. 1.1. *Cellular flow.*

when the initial function T_0 is the delta function at the origin. With this initial function, $T(t, x, y)$ is the probability density of a test particle diffusing in the flow, and (1.4) states that, when t is large, the mean square displacement of the particle behaves like $\sigma_\varepsilon t$.

We are interested in the behavior of σ_ε as $\varepsilon \to 0$. In [1] Childress showed by a boundary layer analysis that, when H is given by (1.3), then

$$\sigma_\varepsilon \sim c^* \sqrt{\varepsilon} \tag{1.5}$$

as ε tends to zero, and he also characterized the constant c^*. The same problem was reconsidered in [2] and [3], and the constant c^* was evaluated analytically by Soward [4]. The asymptotic relation (1.5) is the simplest example of convection enhanced diffusion because the effective diffusivity σ_ε is much larger than the molecular diffusivity ε. The enhancement is due to the convective flow with the stream function (1.3) (see Fig. 1.1). Flows with stream functions

$$H(x, y) = \sin x \sin y + \delta \cos x \cos y \ , \tag{1.6}$$

with $0 \le \delta \le 1$, are considered in [5], along with discussion of the associated dynamo problem (see Fig. 1.2). In [6] Soward and Childress study diffusion and dynamo action in flows with nonzero mean motion.

Our aim in this paper is to study in detail the effective diffusivity of a passive scalar in a convective flow by variational methods, thus avoiding direct boundary layer analysis. This is important because boundary layer analysis becomes too complicated to be useful when the flow $\mathbf{u}$ is more complex than simple cellular flow or cellular flow with channels (see Fig. 1.2).

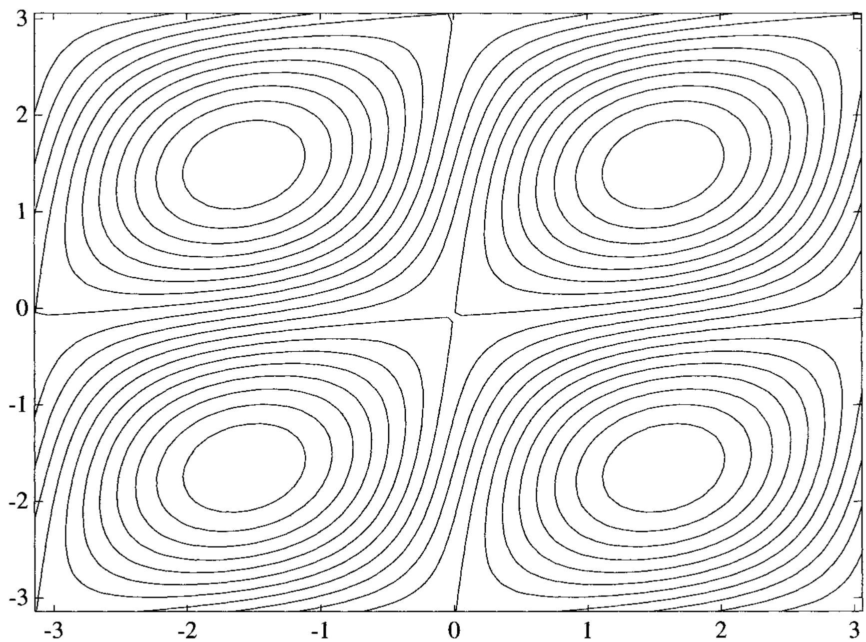

FIG. 1.2. *Cat's-eye flow with* $\delta = 0.2$.

In §2 we review the various definitions of effective diffusivity for periodic flows. In §3 we introduce a Hilbert space formulation for the effective diffusivity. With a simple symmetrization transformation, we can obtain variational principles for the effective diffusivity. The Hilbert space formulation follows the general framework introduced in [7]. The variational principle suitable for upper bounds of the effective diffusivity was noted by Avellaneda and Majda [8]. Another form of this variational principle was given by Cherkaev and Gibiansky and is presented by Milton in [12]. The relations between the various variational principles are analyzed in Appendix A. The variational principle for lower bounds is new and is one of the main contributions in this paper. In §4 we show how to use the variational principles to prove result (1.5), including the characterization of the constant c^*. In §5 we use the variational principles to study the effective diffusivity for cellular flows in point-contact, for which a corner layer theory is developed. In §6 we study the effective diffusivity of cellular flows with open channels, in particular, the cat's-eye flow with stream function (1.6). In §7 we study general periodic flows with zero mean drift. In these problems, we clearly see the power of the variational methods. The only section in which variational methods are not used in an essential way is §8, where we study general periodic flows with nonzero mean drift. In Appendix B, we derive variational principles for time-dependent flows.

We treat only periodic flows in this paper. Convection-enhanced diffusion for random flows is studied in [14]–[16] and in the second part of this work [17].

2. The effective diffusivity. We consider the periodic case [13] and for time-independent flows with mean zero. For d-dimensional flows $\mathbf{u}(\mathbf{x})$ that are incompressible and have mean zero, there exists a skew-symmetric matrix $\mathbf{H} = (H_{ij}(\mathbf{x}))$ such that $\nabla \cdot \mathbf{H} = \mathbf{u}$. The flow $\mathbf{u}$ has the Fourier representation

$$u_p(\mathbf{x}) = \sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}}\, \hat{u}_p(\mathbf{k}) \tag{2.1}$$

and

$$H_{pq}(\mathbf{u}) = \frac{1}{i}\sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{k_p \hat{u}_q(\mathbf{k}) - k_q \hat{u}_p(\mathbf{k})}{|\mathbf{k}|^2}. \tag{2.2}$$

From the fact that $\nabla \cdot \mathbf{u} = \mathbf{0}$, it follows that $\nabla \cdot \mathbf{H} = \mathbf{u}$. Equation (1.1) for T can now be written in divergence form

$$\frac{\partial T}{\partial t} = \nabla \cdot (\varepsilon I + \mathbf{H})\nabla T \tag{2.3}$$

with initial conditions $T(0,\mathbf{x}) = T_0(\mathbf{x})$. To recall the basic facts in homogenization [13], we write (2.3) in the form

$$\frac{\partial T}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x})\frac{\partial T}{\partial x_j}\right), \tag{2.4}$$

where

$$a_{ij}(\mathbf{x}) = \varepsilon\, \delta_{ij} + H_{ij}(\mathbf{x}) \ .$$

Note that the diffusivity matrix (a_{ij}) is not symmetric but that, for $\varepsilon > 0$, the right side of (2.4) is uniformly elliptic. In homogenization, we seek the large time, long-distance behavior of solutions of (2.4). This is expressed in terms of a small parameter $\delta > 0$ by replacing t by t/δ^2 and $\mathbf{x}$ by $\mathbf{x}/\delta$ in (2.4). We then have

$$\frac{\partial T}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{\mathbf{x}}{\delta}\right)\frac{\partial T}{\partial x_j}\right) \tag{2.5}$$

and we assume now that the initial conditions do not depend on δ

$$T(0,\mathbf{x}) = T_0(\mathbf{x}) \tag{2.6}$$

This is equivalent to the statement that the initial data for (2.4) are slowly varying.

For periodic diffusivity coefficients in (2.5) that are uniformly elliptic but not necessarily symmetric, it is not difficult to show [13] that $T(t,\mathbf{x}) = T^\delta(t,\mathbf{x})$, the solution of (2.5), converges to $\overline{T}(t,\mathbf{x})$, the solution of an equation with constant coefficients

$$\frac{\partial \overline{T}}{\partial t} = \sum_{i,j=1}^{d} \bar{a}_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} \ , \tag{2.7}$$
$$\overline{T}(0,\mathbf{x}) = T_0(\mathbf{x}) \ .$$

The convergence is in L^2

$$\sup_{0\le t\le t_0} \int |T^\delta(t,\mathbf{x}) - \overline{T}(t,\mathbf{x})|^2 \, d\mathbf{x} \to 0 \tag{2.8}$$

as $\delta \to 0$, for any $t_0 < \infty$. The effective diffusivity matrix $(\bar{a}_{ij})$ is obtained by solving a cell problem as follows. For each unit vector $\mathbf{e}$, let $\chi = \chi(\mathbf{x};\mathbf{e})$ be the unique (up to a constant) periodic solution of

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{ij}(\mathbf{x})\left(\frac{\partial \chi(\mathbf{x})}{\partial x_j} + e_j\right)\right] = 0. \tag{2.9}$$

Then

$$\bar{a}\mathbf{e}\cdot\mathbf{e} = \langle \mathbf{a}(\nabla\chi + \mathbf{e})\cdot(\nabla\chi + \mathbf{e})\rangle, \tag{2.10}$$

where $\langle\ \rangle$ denotes normalized integration (averaging) over the torus.

The cell problem for the convection-diffusion equation (2.3) has the form

$$\nabla\cdot\big[(\varepsilon I + \mathbf{H})(\nabla\chi + \mathbf{e})\big] = 0, \tag{2.11}$$

which, in view of the relation $\nabla\cdot\mathbf{H} = \mathbf{u}$, is equivalent to

$$\varepsilon\,\Delta\chi + \mathbf{u}\cdot\nabla\chi + \mathbf{u}\cdot\mathbf{e} = 0. \tag{2.12}$$

The effective diffusivity matrix in this case is denoted by σ_ε, as in §1, and (2.10) becomes

$$\sigma_\varepsilon(\mathbf{e}) = \sigma_\varepsilon\mathbf{e}\cdot\mathbf{e} = \sigma_\varepsilon(\mathbf{e}) = \varepsilon\langle(\nabla\chi + \mathbf{e})\cdot(\nabla\chi + \mathbf{e})\rangle. \tag{2.13}$$

We see, therefore, that in the periodic case the small diffusion limit ($\varepsilon \to 0$) of the effective diffusivity σ_ε reduces to the analysis of the singularly perturbed diffusion equation (2.12) on the torus.

The fact that the cell problem (2.9), or (2.11), determines the effective diffusivity can be understood physically from the following. Let $\{\mathbf{e}_j\}$ be a basis of orthogonal unit vectors in $\mathbf{R}^d$, let χ_j be the solution of the cell problem (2.11), and let

$$\mathbf{E}_j = \nabla\chi_j + \mathbf{e}_j. \tag{2.14}$$

Then $\mathbf{E}_j$ is the concentration or heat intensity, and

$$\mathbf{D}_j = (\varepsilon I + \mathbf{H})\mathbf{E}_j \tag{2.15}$$

is the flux. Since $\mathbf{H}$ is skew symemtric, the intensity-flux relationship is similar to that of a Hall medium [14], [15]. From (2.11) and (2.14), we see that

$$\nabla\times\mathbf{E}_j = 0\ ,\quad \nabla\cdot\mathbf{D}_j = 0\ ,\quad \langle\mathbf{E}_j\rangle = \mathbf{e}_j, \tag{2.16}$$

and

$$\sigma_\varepsilon\langle\mathbf{E}_j\rangle = \langle\mathbf{D}_j\rangle. \tag{2.17}$$

Relation (2.15) is the linear constitutive law relating intensity and flux. Relations (2.16) tell us that $\mathbf{E}_j$ is a gradient, that there are no sources or sinks, and that the mean or imposed intensity is a unit vector in the direction $\mathbf{e}_j$. The effective diffusivity σ_ε is defined by (2.17), which is the linear constitutive law relating mean intensity and mean flux. It is generally a nonsymmetric matrix given by

$$\begin{aligned}\sigma_\varepsilon\mathbf{e}_i\cdot\mathbf{e}_j = \sigma_\varepsilon(\mathbf{e}_i,\mathbf{e}_j) &= \langle\mathbf{D}_i\cdot\mathbf{e}_j\rangle \\ &= \langle(\varepsilon I + \mathbf{H})\mathbf{E}_i\cdot\mathbf{E}_j\rangle.\end{aligned} \tag{2.18}$$

In this paper, we require the effective diffusivity matrix to be symmetric, as this makes it easier to apply the variational principles that are introduced in the next section. It is shown in Appendix A.3 that, if the effective diffusivity matrix is the same when the stream function H is changed to $-H$, then it is a symmetric matrix. The same is true in any number of dimensions if $\mathbf{u}$ is changed to $-\mathbf{u}$. If in particular, for two-dimensional periodic flows, the stream functions have one of the following forms of antisymmetry, then the effective diffusivity tensors are symmetric:

(a) Translational antisymmetry: $H(\mathbf{x}+\mathbf{r}) = -H(\mathbf{x})$, for all $\mathbf{x}$ and for some $\mathbf{r}$;

(b) Reflectional antisymmetry with respect to an axis, for example, the x-axis: $H(x_1, x_2) = -H(x_1, -x_2)$, for all x_1, x_2;

(c) 180°-rotational antisymmetry or reflectional antisymmetry with respect to a point, say, the origin: $H(\mathbf{x}) = -H(-\mathbf{x})$ for all $\mathbf{x}$.

There are flows that may not have any of these properties; nevertheless, they have symmetric effective diffusivity tensors, such as shear layer flows. It is not clear as to what are the most general flows that have symmetric effective diffusivity tensors. All flows considered in this paper are either shear layer flows or have one of the above antisymmetries so the effective diffusivity tensors are symmetric.

From the skew symmetry of $\mathbf{H}$ and (2.16), we conclude that (2.18) reduces to (2.13),

$$\begin{aligned} \sigma_\varepsilon(\mathbf{e}_i \cdot \mathbf{e}_i) = \sigma_\varepsilon(\mathbf{e}_i) &= \langle(\varepsilon I + \mathbf{H})\mathbf{E}_i \cdot \mathbf{e}_i\rangle \\ &= \langle(\varepsilon I + \mathbf{H})\mathbf{E}_i \cdot \mathbf{E}_i\rangle \\ &= \varepsilon\langle\mathbf{E}_i \cdot \mathbf{E}_i\rangle. \end{aligned} \tag{2.19}$$

The full diffusivity matrix in the general nonsymmetric case is considered again in Appendix A.

The $\sqrt{\varepsilon}$ behavior of the effective conductivity for the cellular flow (1.5) (see Fig. 1.1) can be understood by the following simple scaling argument. The concentration of the diffusing substance is nonnegligible only in a small neighborhood of the separatrices of the flow. Let δ be the width of this boundary layer around the separatrices. Since the molecular diffusivity is ε, the time to traverse diffusively the boundary layer is $t_D \sim \delta^2/\varepsilon$. The time to go around a flow cell by convection is $t_C \sim 1$, since the flow speed is of order 1 and the flow cell size is of order 1. Convection and diffusion balance to set up the boundary layer so that $t_D \sim t_C$ or $\delta \sim \sqrt{\varepsilon}$, which determines the width of the boundary layer. The effective diffusivity is now estimated by $\sigma_\varepsilon \sim \varepsilon\ \delta^{-2}\ \delta = \sqrt{\varepsilon}$, since in (2.13) the concentration gradient is of order δ^{-1} in the boundary layer and negligible elsewhere.

This simple scaling argument does not consider the stagnation points of the flow near which it slows down. However, the analysis of §4 shows that the stagnation points do not alter the scaling behavior of σ_ε. Only the proportionality constant is affected. An interesting example where the stagnation points of the flow affects the scaling is the following [2]. Consider a one-dimensional array of cellular flows that stick to the lateral walls. Let δ be again the width of the boundary layer near the walls. Here again $t_D \sim \delta^2/\varepsilon$, but since the speed vanishes on the lateral boundaries and is smooth, we have $t_C \sim 1/\delta$, where δ is the speed near the walls. Thus $t_C \sim t_D$ gives $\delta \sim \varepsilon^{1/3}$ and hence $\sigma_\varepsilon \sim \varepsilon\ \delta^{-2}\ \delta = \varepsilon^{2/3}$. We do not treat this case in detail here, but we have given the scaling argument so that the influence of stgnation points and surfaces can be appreciated. More applications of the scaling argument can be found in [16].

3. Hilbert-space formulation and variational principles. In this section, we set $\varepsilon = 1$ and study the cell problem (2.15)–(2.17) that defines the effective diffusivity σ. We give a variational formulation for this problem, which is particularly useful in the asymptotic analysis of σ_ε as $\varepsilon \to 0$. Let $\mathcal{H}$ be the Hilbert space of square integrable, periodic vector functions

$$\mathcal{H} = \{\mathbf{F}(\mathbf{x})\ ,\ \langle|\mathbf{F}|^2\rangle < \infty\}, \tag{3.1}$$

where, as before, $\langle\ \rangle$ denotes integration over the unit period cell (the unit torus). Let $\mathcal{H}_g$ be the subspace of irrotational (gradient) fields. The orthogonal projection onto $\mathcal{H}_g$ is denoted by Γ_g and has an explicit expression in terms of Fourier series. If

$$\mathbf{F}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\,\widehat{\mathbf{F}}(\mathbf{k}), \tag{3.2}$$

then

$$\begin{aligned}\Gamma_g\mathbf{F} &= \nabla\Delta^{-1}\nabla\cdot\mathbf{F}\\ &= \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\mathbf{k}(\mathbf{k}\cdot\widehat{\mathbf{F}}(\mathbf{k})\,)}{|\mathbf{k}|^2} e^{i\mathbf{k}\cdot\mathbf{x}}.\end{aligned} \tag{3.3}$$

Let $\mathcal{H}_0$ be the subspace of constants in $\mathcal{H}$ and Γ_0 orthogonal projection onto it. Clearly,

$$\Gamma_0\mathbf{F} = \langle\mathbf{F}\rangle = \widehat{\mathbf{F}}(0). \tag{3.4}$$

Also, let $\mathcal{H}_c$ be the subspace of divergence free vector functions, with Γ_c its orthogonal projection. Then

$$\begin{aligned}\Gamma_c\mathbf{F} &= -\nabla\times\Delta^{-1}\nabla\times\mathbf{F}\\ &= \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\mathbf{k}\times(\mathbf{k}\times\widehat{\mathbf{F}}(\mathbf{k}))}{|\mathbf{k}|^2} e^{i\mathbf{k}\cdot\mathbf{x}}\\ &= \sum_{\mathbf{k}\neq\mathbf{0}} \left(1 - \frac{\mathbf{k}\mathbf{k}\cdot}{|\mathbf{k}|^2}\right)\widehat{\mathbf{F}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},\end{aligned} \tag{3.5}$$

from which we deduce that

$$\Gamma_0 + \Gamma_g + \Gamma_c = 1 \tag{3.6}$$

or, equivalently, the well-known fact that

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_g \oplus \mathcal{H}_c. \tag{3.7}$$

The cell problem (2.15)–(2.17) (with $\varepsilon = 1$) can be expressed through Γ_g in a very convenient way,

$$\mathbf{E} = \mathbf{e} - \Gamma_g\mathbf{H}\mathbf{E} \tag{3.8}$$

with

$$\sigma(\mathbf{e}) = \langle\mathbf{E}\cdot\mathbf{E}\rangle. \tag{3.9}$$

Here we have written the quadratic form $\sigma\mathbf{e}\cdot\mathbf{e}$ as $\sigma(\mathbf{e})$. The fact that $\mathbf{E}$ satisfying (3.8) also satisfies $\nabla\times\mathbf{E}=\mathbf{0}$ and $\langle\mathbf{E}\rangle=\mathbf{e}$ is clear. Taking divergence of both sides in (3.8) gives

$$\nabla\cdot\mathbf{E}=-\nabla\cdot\mathbf{HE}, \tag{3.10}$$

and hence (2.15) (with $\varepsilon=1$) is satisfied. Note that, in addition to being a convenient way to define $\mathbf{E}$, (3.8) is also a good way to define $\mathbf{E}$ mathematically, since it is an integral equation formulation.

3.1. Variational principle for the upper bound. We want to find a way to express $\sigma(\mathbf{e})$ as the minimum of a functional. However, since $\mathbf{H}$ is skew symmetric, (3.8) is not the Euler equation of a quadratic functional. To obtain a suitable variational formulation, we must first symmetrize (3.8).

Denote $\mathbf{E}$ by $\mathbf{E}^+$; that is, let $\mathbf{E}^+$ satisfy

$$\mathbf{E}^+=\mathbf{e}-\Gamma_g\mathbf{HE}^+ \tag{3.11}$$

and let $\mathbf{E}^-$ satisfy

$$\mathbf{E}^-=\mathbf{e}+\Gamma_g\mathbf{HE}^-. \tag{3.12}$$

Also, let

$$\mathbf{A}=\frac{\mathbf{E}^++\mathbf{E}^-}{2}, \qquad \mathbf{B}=\frac{\mathbf{E}^+-\mathbf{E}^-}{2}. \tag{3.13}$$

Then

$$\mathbf{A}=\mathbf{e}-\Gamma_g\mathbf{HB}, \qquad \mathbf{B}=-\Gamma_g\mathbf{HA}, \tag{3.14}$$

and

$$\begin{aligned}\sigma(\mathbf{e})&=\langle(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})\rangle\\&=\langle(\mathbf{A}\cdot\mathbf{A})\rangle+\langle(\mathbf{B}\cdot\mathbf{B})\rangle.\end{aligned} \tag{3.15}$$

Here we have noted that

$$\begin{aligned}\langle\mathbf{A}\cdot\mathbf{B}\rangle&=\langle(\mathbf{e}-\Gamma_g\mathbf{HB})\cdot\mathbf{B}\rangle\\&=-\langle\Gamma_g\mathbf{HB}\cdot\mathbf{B}\rangle\\&=-\langle\mathbf{B}\cdot\mathbf{H}\Gamma_g\mathbf{B}\rangle\\&=-\langle\mathbf{B}\cdot\Gamma_g\mathbf{HB}\rangle\\&=-\langle\mathbf{B}\cdot(\mathbf{e}-\Gamma_g\mathbf{HB})\rangle\\&=-\langle\mathbf{B}\cdot\mathbf{A}\rangle,\end{aligned}$$

which makes the cross terms in (3.15) vanish. Substituting $\mathbf{B}=-\Gamma_g\mathbf{HA}$ into the first equation in (3.14) and in (3.15), we obtain

$$\mathbf{A}=\mathbf{e}+\Gamma_g\mathbf{H}\Gamma_g\mathbf{HA}, \tag{3.16}$$

$$\begin{aligned}\sigma(\mathbf{e})&=\langle\mathbf{A}\cdot\mathbf{A}\rangle-\langle\mathbf{H}\Gamma_g\mathbf{HA}\cdot\mathbf{A}\rangle\\&=\langle(I-\mathbf{H}\Gamma_g\mathbf{H})\mathbf{A}\cdot\mathbf{A}\rangle.\end{aligned} \tag{3.17}$$

Let

$$\mathbf{K}_H = -\mathbf{H}\Gamma_g\mathbf{H} \tag{3.18}$$

and note that it is a selfadjoint and positive operator

$$\langle \mathbf{K}_H\mathbf{F}\cdot\mathbf{F}\rangle = \langle \Gamma_g\mathbf{H}\mathbf{F}\cdot\Gamma_g\mathbf{H}\mathbf{F}\rangle \geq 0.$$

Thus, $\mathbf{A}$ satisfies

$$\mathbf{A} = \mathbf{e} - \Gamma_g\mathbf{K}_H\mathbf{A} \tag{3.19}$$

and

$$\sigma(\mathbf{e}) = \langle (I+\mathbf{K}_H)\mathbf{A}\cdot\mathbf{A}\rangle. \tag{3.20}$$

Since $\mathbf{K}_H$ is selfadjoint and positive, it is easy to see that

$$\sigma(\mathbf{e}) = \inf_{\substack{\mathbf{F}\in\mathcal{H}\\ \nabla\times\mathbf{F}=0\\ \langle\mathbf{F}\rangle=\mathbf{e}}} \langle (I+\mathbf{K}_H)\mathbf{F}\cdot\mathbf{F}\rangle. \tag{3.21}$$

In fact, the Euler equation for this variational principle is

$$\begin{gathered}\nabla\cdot(I+\mathbf{K}_H)\mathbf{F} = 0,\\ \nabla\times\mathbf{F} = 0, \qquad \langle\mathbf{F}\rangle = \mathbf{e},\end{gathered} \tag{3.22}$$

which is equivalent to (3.19). Note, however, that (3.22) is quite different from the cell problem (2.15), (2.16) (with $\varepsilon = 1$) because $\mathbf{K}_H$ is not a matrix but an operator given by (3.18). Thus, (3.22) is a nonlocal, elliptic cell problem, and the nonlocality is a direct consequence of the symmetrization. The variational principle (3.21) was derived before by a different method in [8]. A more general discussion of variational principles and symmetrization is given in Appendix A.

When the dependence on ε is restored in (3.21), (3.22), we have that

$$\mathbf{K}_H^\varepsilon = -\frac{1}{\varepsilon^2}\mathbf{H}\Gamma_g\mathbf{H} \tag{3.23}$$

and

$$\sigma_\varepsilon(\mathbf{e}) = \inf_{\substack{\mathbf{F}\in\mathcal{H}\\ \nabla\times\mathbf{F}=0,\ \langle\mathbf{F}\rangle=\mathbf{e}}} \varepsilon\langle (I+\mathbf{K}_H^\varepsilon)\mathbf{F}\cdot\mathbf{F}\rangle. \tag{3.24}$$

In two space dimensions, a flow $\mathbf{u}(\mathbf{x})$ that is divergence-free can be expressed in terms of a stream function $H(\mathbf{x})$

$$\mathbf{u}(\mathbf{x}) = \nabla^\perp H(\mathbf{x}) = (-H_y(\mathbf{x}), H_x(\mathbf{x})), \tag{3.25}$$

where $\mathbf{x} = (x,y)$ and then

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} 0 & H(\mathbf{x}) \\ -H(\mathbf{x}) & 0 \end{pmatrix}. \tag{3.26}$$

The simplest bound we can obtain for $\sigma_\varepsilon(\mathbf{e})$, which is, of course, very bad as $\varepsilon \to 0$, comes from (3.24) when we put $\mathbf{F} = \mathbf{e}$ as trial field. Then

$$\begin{aligned}\sigma_\varepsilon &\leq \varepsilon + \langle \Gamma_g\mathbf{H}\mathbf{e}\cdot\mathbf{H}\mathbf{e}\rangle\\ &= \varepsilon + \tfrac{1}{\varepsilon}\langle \mathbf{u}\cdot\mathbf{e}(-\Delta)^{-1}(\mathbf{u}\cdot\mathbf{e})\rangle.\end{aligned} \tag{3.27}$$

Much better bounds and asymptotic limits are obtained in subsequent sections.

The variational principle (3.21) can provide upper bounds, and careful choice of test fields in (3.24) can provide upper bounds for $\sigma_\varepsilon(\mathbf{e})$ that do not become trivial as $\varepsilon \to 0$. To obtain more precise information about $\sigma_\varepsilon(\mathbf{e})$, however, we need lower bounds as well. We describe next how to do this.

3.2. Variational principle for the lower bound. Let us return to the case where $\varepsilon = 1$, since this parameter does not play any role in the calculations that follow and can be reinserted at the end. From general duality considerations, we know from (3.21) that

$$(\sigma(\mathbf{e}))^{-1} = \inf_{\substack{\mathbf{G}\in\mathcal{H}\\ \nabla\cdot\mathbf{G}=0\\ \langle\mathbf{G}\rangle=\mathbf{e}}} \langle (I+\mathbf{K}_H)^{-1}\mathbf{G}\cdot\mathbf{G}\rangle, \tag{3.28}$$

where $(\sigma(\mathbf{e}))^{-1}$ is the inverse of the quadratic form $\sigma(\mathbf{e})$. This variational principle is not useful, however, because $\mathbf{K}_H$ is a nonlocal operator, and, when the ε-scaling is restored, the operator $(I+\mathbf{K}_H^\varepsilon)^{-1}$ is difficult to handle.

To avoid having an operator such as $(I+\mathbf{K}_H)^{-1}$ in the variational expression for $(\sigma(\mathbf{e}))^{-1}$, we proceed as follows. We work in $\mathbb{R}^3$ or $\mathbb{R}^2$ to be able to use simple vector analysis, but there is no loss in generality.[1] Let $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ be an orthonormal basis in $\mathbb{R}^3$. We return to the cell problem (2.15), (2.16), with $\varepsilon = 1$, and write it in the form

$$\begin{aligned} \nabla\cdot(I+\mathbf{H})\mathbf{E}^k &= 0,\\ \nabla\times\mathbf{E}^k &= 0,\\ \langle\mathbf{E}^k\rangle &= \mathbf{e}^k, \qquad k=1,2,3. \end{aligned} \tag{3.29}$$

Let

$$(I+\mathbf{H})\mathbf{E}^k = \sum_l \mathbf{D}^l\sigma_{lk}, \tag{3.30}$$

where σ_{lk} are the matrix elements of $\sigma(\mathbf{e}) = \sigma\mathbf{e}\cdot\mathbf{e}$ given by (2.13) or (2.19) (with $\varepsilon = 1$). If for $l = 1,2,3$, $\mathbf{D}^l$ satisfies

$$\begin{aligned} \nabla\times(I+\mathbf{H})^{-1}\mathbf{D}^l &= 0,\\ \nabla\cdot\mathbf{D}^l &= 0,\\ \langle\mathbf{D}^l\rangle &= \mathbf{e}^l, \end{aligned} \tag{3.31}$$

then $\mathbf{E}^k = \sum_l (I+\mathbf{H})^{-1}\mathbf{D}^l\sigma_{lk}$ satisfies (3.29) and

$$\mathbf{e}^k = \sum_l \langle (I+\mathbf{H})^{-1}\mathbf{D}^l\rangle\sigma_{lk}.$$

Dropping the superscripts, this is equivalent to solving for $\mathbf{D}$ such that

$$\begin{aligned} \nabla\times(I+\mathbf{H})^{-1}\mathbf{D} &= 0,\\ \nabla\cdot\mathbf{D} &= 0,\\ \langle\mathbf{D}\rangle &= \mathbf{e}, \end{aligned} \tag{3.32}$$

[1] In Appendix B, we use differential forms for a similar computation in four dimensions.

and then

$$(3.33)\qquad \begin{aligned}(\sigma(\mathbf{e}))^{-1} &= \langle (I+\mathbf{H})^{-1}\mathbf{D}\cdot\mathbf{e}\rangle\\ &= \langle (I+\mathbf{H})^{-1}\mathbf{D}\cdot\mathbf{D}\rangle.\end{aligned}$$

In two dimensions, the matrix $\mathbf{H}$ has the form (3.26)

$$\mathbf{H} = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}.$$

Therefore

$$(3.34)\qquad (I+\mathbf{H})^{-1} = \frac{1}{1+H^2}(I-\mathbf{H}).$$

In three dimensions, $\mathbf{H}$ has the form

$$(3.35)\qquad \mathbf{H} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Define the vector

$$(3.36)\qquad \mathbf{a} = (a_1, a_2, a_3)$$

and let $a = |\mathbf{a}|$ be the length of $\mathbf{a}$. Then

$$(3.37)\qquad (1+\mathbf{H})^{-1} = \frac{1}{1+a^2}(I+\mathbf{a}\otimes\mathbf{a}-\mathbf{H}).$$

Returning to (3.33), we see that in two dimensions

$$(3.38)\qquad (\sigma(\mathbf{e})\,)^{-1} = \left\langle \frac{1}{1+H^2}\mathbf{D}\cdot\mathbf{D}\right\rangle,$$

while in three dimensions

$$(3.39)\qquad (\sigma(\mathbf{e})\,)^{-1} = \left\langle \frac{1}{1+a^2}(I+\mathbf{a}\otimes\mathbf{a})\mathbf{D}\cdot\mathbf{D}\right\rangle.$$

In both two and three dimensions, problem (3.32) has the form

$$(3.40)\qquad \begin{aligned}\nabla\times(\mathbf{S}-\mathbf{U})\mathbf{D} &= 0,\\ \nabla\cdot\mathbf{D} &= 0,\\ \langle\mathbf{D}\rangle &= \mathbf{e},\end{aligned}$$

where $\mathbf{S}$ is a symmetric, positive definite matrix, and $\mathbf{U}$ is a skew symmetric matrix. We rewrite (3.40) as an integral equation as we did for the cell problem for $\mathbf{E}$, (2.15)–(2.17), with (3.8). As in (3.1), let

$$(3.41)\qquad \mathcal{H}^S = \left\{\mathbf{F}(\mathbf{x})\,,\, \langle|\mathbf{F}|^2\rangle_S < \infty\right\}$$

be the Hilbert space of square integrable, periodic vector functions with inner product

$$(3.42)\qquad \langle\mathbf{F},\mathbf{G}\rangle_S = \langle\mathbf{S}\mathbf{F},\mathbf{G}\rangle.$$

Let

$$\Delta_S = \nabla \times (\mathbf{S}\nabla \times \cdot)\ . \tag{3.43}$$

This is a second-order elliptic operator with bounded inverse Δ_S^{-1}, defined over all square integrable, divergence-free fields $\mathbf{F}$ with $\langle \mathbf{F} \rangle = 0$. Define on $\mathcal{H}^S$ the projection operator

$$\Gamma_c^S = \nabla \times \Delta_S^{-1} \nabla \times (\mathbf{S}\cdot). \tag{3.44}$$

This is indeed a projection operator

$$\begin{aligned}\langle \Gamma_c^S \mathbf{F}, \mathbf{G}\rangle_S &= \langle \mathbf{S}\nabla \times \Delta_S^{-1}\nabla \times (\mathbf{SF}), \mathbf{G}\rangle \\ &= \langle \mathbf{SF}, \nabla \times \Delta_S^{-1}\nabla \times (\mathbf{SG})\rangle \\ &= \langle \mathbf{F}, \Gamma_c^S \mathbf{G}\rangle_S\end{aligned}$$

and

$$\begin{aligned}(\Gamma_c^S)^2\mathbf{F} &= \nabla \times \Delta_S^{-1}\nabla \times (\mathbf{S}\nabla \times \Delta_S^{-1}\nabla \times (\mathbf{SF})) \\ &= \nabla \times \Delta_S^{-1}\nabla \times (\mathbf{SF}) = \Gamma_c^S \mathbf{F}.\end{aligned}$$

Using Γ_c^S, we can now write (3.40) in the form

$$\mathbf{D} = \mathbf{e} - \Gamma_c^S \mathbf{e} + \Gamma_c^S \mathbf{S}^{-1}\mathbf{UD}. \tag{3.45}$$

Clearly, $\mathbf{D}$ satisfies $\langle \mathbf{D} \rangle = \mathbf{e}$ and $\nabla \cdot \mathbf{D} = 0$. We also verify that

$$\begin{aligned}\nabla \times (\mathbf{SD}) &= \nabla \times (\mathbf{Se}) - \nabla \times (\mathbf{S}\Gamma_c^S \mathbf{e}) \\ &\quad + \nabla \times \mathbf{S}\,\Gamma_c^S \mathbf{S}^{-1}\mathbf{UD} \\ &= \nabla \times (\mathbf{UD})\end{aligned}$$

so that $\nabla \times (\mathbf{S} - \mathbf{U})\mathbf{D} = 0$. Thus, (3.45) is equivalent to (3.40).

The projection operator Γ_c^S takes vector fields $\mathbf{F}$ in $\mathcal{H}^S$ into divergence-free fields that have mean zero. It is therefore analogous to the projection operator Γ_c on $\mathcal{H}$ given by (3.5). It is interesting to look for a characterization of the operator $I - \Gamma_c^S$ that projects into the orthogonal complement of divergence-free fields in $\mathcal{H}^S$. For this purpose, we let

$$\mathbf{F} - \Gamma_c^S \mathbf{F} = \mathbf{G} \tag{3.46}$$

and we note that

$$\langle \mathbf{G} \rangle = \langle \mathbf{F} \rangle\ , \qquad \nabla \cdot \mathbf{G} = \nabla \cdot \mathbf{F}, \tag{3.47}$$

and

$$\nabla \times (\mathbf{SG}) = 0. \tag{3.48}$$

From (3.48), we deduce that

$$\mathbf{G} = \mathbf{S}^{-1}\nabla h \tag{3.49}$$

and from (3.47)

$$\nabla \cdot (\mathbf{S}^{-1}\nabla h) \equiv \tilde{\Delta}_S h = \nabla \cdot \mathbf{F}. \tag{3.50}$$

The elliptic operator $\tilde{\Delta}_S$ has a bounded inverse on zero mean square integrable functions. Thus

$$\nabla h = \nabla \tilde{\Delta}_S^{-1} \nabla \cdot \mathbf{F} + \mathbf{S}\big[\langle \mathbf{F} \rangle - \langle \mathbf{S}^{-1} \nabla \tilde{\Delta}_S^{-1} \nabla \cdot \mathbf{F} \rangle\big], \tag{3.51}$$

and, if we set

$$\Gamma_g^S = \mathbf{S}^{-1} \nabla \tilde{\Delta}_S^{-1} \nabla, \tag{3.52}$$

then

$$\mathbf{G} = \Gamma_g^S \mathbf{F} + \langle \mathbf{F} \rangle - \langle \Gamma_g^S \mathbf{F} \rangle \tag{3.53}$$

and

$$\mathbf{F} = \Gamma_c^S \mathbf{F} + \Gamma_g^S \mathbf{F} + \langle \mathbf{F} \rangle - \langle \Gamma_g^S \mathbf{F} \rangle. \tag{3.54}$$

The operator Γ_g^S is selfadjoint in $\mathcal{H}^S$ and $(\Gamma_g^S)^2 = \Gamma_g^S$, so it is a projection operator. It is, moreover, orthogonal to Γ_c^S, since $\Gamma_c^S \Gamma_g^S = 0$. However, Γ_g^S does not map vector fields to mean zero, curl-free vector fields, but rather to fields annihilated by the operator $\nabla \times (\mathbf{S} \cdot\ )$. Since the mean of $\Gamma_g^S \mathbf{F}$ is not zero, it must be subtracted on the right in (3.54).

We now return to (3.45) and carry out its symmetrization, as we did for (3.8). Let

$$\mathbf{e}^S = \mathbf{e} - \Gamma_c^S \mathbf{e} \tag{3.55}$$

and rewrite (3.45) in the form

$$\mathbf{D} = \mathbf{e}^S + \Gamma_c^S \mathbf{S}^{-1} \mathbf{U} \mathbf{D}. \tag{3.56}$$

With the notation of (3.40), both (3.38) and (3.39) become

$$(\sigma(\mathbf{e}))^{-1} = \langle \mathbf{S}\mathbf{D} \cdot \mathbf{D} \rangle = \langle \mathbf{D} \cdot \mathbf{D} \rangle_S. \tag{3.57}$$

For the symmetrization, let $\mathbf{D}^+$ satisfy (3.56)

$$\mathbf{D}^+ = \mathbf{e}^S + \Gamma_c^S \mathbf{S}^{-1} \mathbf{U} \mathbf{D}^+ \tag{3.58}$$

and $\mathbf{D}^-$ satisfy

$$\mathbf{D}^- = \mathbf{e}^S - \Gamma_c^S \mathbf{S}^{-1} \mathbf{U} \mathbf{D}^{-1}. \tag{3.59}$$

As in (3.13), let

$$\mathbf{A} = \frac{\mathbf{D}^+ + \mathbf{D}^-}{2}, \qquad \mathbf{B} = \frac{\mathbf{D}^+ - \mathbf{D}^-}{2}. \tag{3.60}$$

Then

$$\mathbf{A} = \mathbf{e}^S + \Gamma_c^S \mathbf{S}^{-1} \mathbf{U} \mathbf{B}, \qquad \mathbf{B} = \Gamma_c^S \mathbf{S}^{-1} \mathbf{U} \mathbf{A}, \tag{3.61}$$

and

$$(\sigma(\mathbf{e}))^{-1} = \langle(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}\rangle_S = \langle\mathbf{A}\cdot\mathbf{A}\rangle_S + \langle\mathbf{B}\cdot\mathbf{B}\rangle_S, \tag{3.62}$$

where the cross terms vanish as in (3.15). Substituting $\mathbf{B} = \Gamma_c^S \mathbf{U}\mathbf{A}$ into the first equation in (3.61) and in (3.62), we obtain

$$\mathbf{A} = \mathbf{e}^S + \Gamma_c^S \mathbf{S}^{-1}\mathbf{U}\,\Gamma_c^S \mathbf{S}^{-1}\mathbf{U}\mathbf{A}, \tag{3.63}$$

$$\begin{aligned}(\sigma(\mathbf{e}))^{-1} &= \langle\mathbf{A}\cdot\mathbf{A}\rangle_S + \langle\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}\mathbf{A}\cdot\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}\mathbf{A}\rangle_S \\ &= \langle(I - \mathbf{S}^{-1}\mathbf{U}\Gamma_c^S\mathbf{S}^{-1}\mathbf{U})\mathbf{A}\cdot\mathbf{A}\rangle_S.\end{aligned} \tag{3.64}$$

Put

$$\mathbf{K}_U^S = -\mathbf{S}^{-1}\mathbf{U}\,\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}. \tag{3.65}$$

As with $\mathbf{K}_H$ in (3.18), we note that it is selfadjoint and positive definite in $\mathcal{H}^S$

$$\begin{aligned}\langle\mathbf{K}_U^S\mathbf{F}\cdot\mathbf{F}\rangle_S &= -\langle\mathbf{U}\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}\mathbf{F}\cdot F\rangle \\ &= \langle\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}\mathbf{F}\cdot\mathbf{S}^{-1}\mathbf{U}\mathbf{F}\rangle_S \\ &= \langle\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}F\cdot\Gamma_c^S\mathbf{S}^{-1}\mathbf{U}\mathbf{F}\rangle \geq 0.\end{aligned}$$

In terms of $\mathbf{K}_U^S$, we can write (3.63) for $\mathbf{A}$ in differential form

$$\begin{aligned}\nabla\times[\mathbf{S}(I+\mathbf{K}_U^S)\mathbf{A}] &= 0, \\ \nabla\cdot\mathbf{A} = 0\ , \qquad \langle\mathbf{A}\rangle &= \mathbf{e}.\end{aligned} \tag{3.66}$$

From the differential form of the equation for $\mathbf{A}$ and (3.64), we see that we have the following variational principle for $(\sigma(\mathbf{e}))^{-1}$:

$$(\sigma(\mathbf{e}))^{-1} = \inf_{\substack{\mathbf{G}\in\mathcal{H}^S \\ \nabla\cdot\mathbf{G}=0 \\ \langle\mathbf{G}\rangle=\mathbf{e}}} \langle(I+\mathbf{K}_U^S)\mathbf{G}\cdot\mathbf{G}\rangle_S. \tag{3.67}$$

3.3. Summary for the two-dimensional case. We are particularly interested in the two-dimensional case, where, by (3.34) and (3.40),

$$\mathbf{S} = \frac{1}{1+H^2}I\ , \qquad \mathbf{U} = \frac{1}{1+H^2}\mathbf{H}. \tag{3.68}$$

Since the curl operator in two dimensions can be expressed in terms of the perpendicular gradient

$$\nabla^\perp = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right), \tag{3.69}$$

we have that

$$\begin{aligned}\Delta_S &= \nabla^\perp\cdot\left(\frac{1}{1+H^2}\nabla^\perp\right), \\ \Gamma_c^S &= \nabla^\perp\Delta_S^{-1}\nabla^\perp\cdot\left(\frac{1}{1+H^2}\right), \\ K_U^S &= -\mathbf{H}\Gamma_c^S\mathbf{H},\end{aligned} \tag{3.70}$$

and (3.66) for $\mathbf{A}$ is

$$\nabla^\perp \cdot \left[\frac{1}{1+H^2}(I - \mathbf{H}\Gamma_c^S \mathbf{H})\mathbf{A}\right] = 0, \qquad \nabla \cdot \mathbf{A} = 0, \qquad \langle \mathbf{A} \rangle = \mathbf{e}. \tag{3.71}$$

In the two-dimensional case, we also use the simpler notation

$$\Gamma = \Gamma_g = \nabla\Delta^{-1}\nabla\cdot, \qquad \Gamma^\perp = \Gamma_c = \nabla^\perp\Delta^{-1}\nabla^\perp, \qquad \Delta_H = \Delta_S, \qquad \Gamma_H = \Gamma_g^S, \qquad \Gamma_H^\perp = \Gamma_c^S. \tag{3.72}$$

With this notation and the ε dependence reinserted, the direct and inverse variational principles become

$$\sigma_\varepsilon(\mathbf{e}) = \inf_{\langle \nabla f \rangle = \mathbf{e}} \left\{ \varepsilon \langle \nabla f \cdot \nabla f \rangle + \frac{1}{\varepsilon}\langle \Gamma \mathbf{H} \nabla f \cdot \Gamma \mathbf{H} \nabla f \rangle \right\} \tag{3.73}$$

and

$$(\sigma)_\varepsilon^{-1}(\mathbf{e}) = \inf_{\langle \nabla^\perp g \rangle = \mathbf{e}} \left\{ \frac{1}{\varepsilon}\left\langle \frac{1}{1+\frac{1}{\varepsilon^2}H^2}\nabla^\perp g \cdot \nabla^\perp g \right\rangle + \frac{1}{\varepsilon^3}\left\langle \frac{1}{1+\frac{1}{\varepsilon^2}H^2}\Gamma_{1/\varepsilon H}^\perp \mathbf{H}\nabla^\perp g \cdot \Gamma_{1/\varepsilon H}^\perp \mathbf{H} \nabla^\perp g \right\rangle \right\}, \tag{3.74}$$

where

$$\Gamma_{\varepsilon^{-1}H}^\perp = \nabla^\perp \Delta_{\varepsilon^{-1}H}^{-1} \nabla^\perp \cdot \left(\frac{1}{1+\varepsilon^{-2}H^2}\right), \tag{3.75}$$

$$\Delta_{\varepsilon^{-1}H} = \nabla^\perp \cdot \left(\frac{1}{1+\varepsilon^{-2}H^2}\,\nabla^\perp\right). \tag{3.76}$$

In the following sections, we use the variational principles in the form (3.73) and (3.74).

4. Convection-enhanced diffusion for cellular flows. The cell problem

$$\varepsilon\,\Delta\chi + \mathbf{u} \cdot \nabla\chi + \mathbf{u} \cdot \mathbf{e} = \mathbf{0} \tag{4.1}$$

determines, up to a constant, a periodic function $\chi(x, y)$, $-\pi \le x \le \pi$, $-\pi \le y \le \pi$, and the effective diffusivity is given by

$$\sigma_\varepsilon(\mathbf{e}) = \varepsilon\langle(\nabla\chi + \mathbf{e}) \cdot (\nabla\chi + \mathbf{e})\rangle, \tag{4.2}$$

where $\langle\ \rangle$ is normalized integration over the period cell. The velocity field $\mathbf{u}$ is incompressible, $\nabla \cdot \mathbf{u} = \mathbf{0}$, and comes from a stream function $H(x, y)$

$$\mathbf{u} = (-H_y, H_x) = \nabla^\perp H. \tag{4.3}$$

The stream function $H(x, y) = \sin x \sin y$ gives rise to a cellular flow (see Fig. 1.1), and, when $\mathbf{e} = (1, 0)$ is a unit vector in the x direction, then χ is odd in the x direction

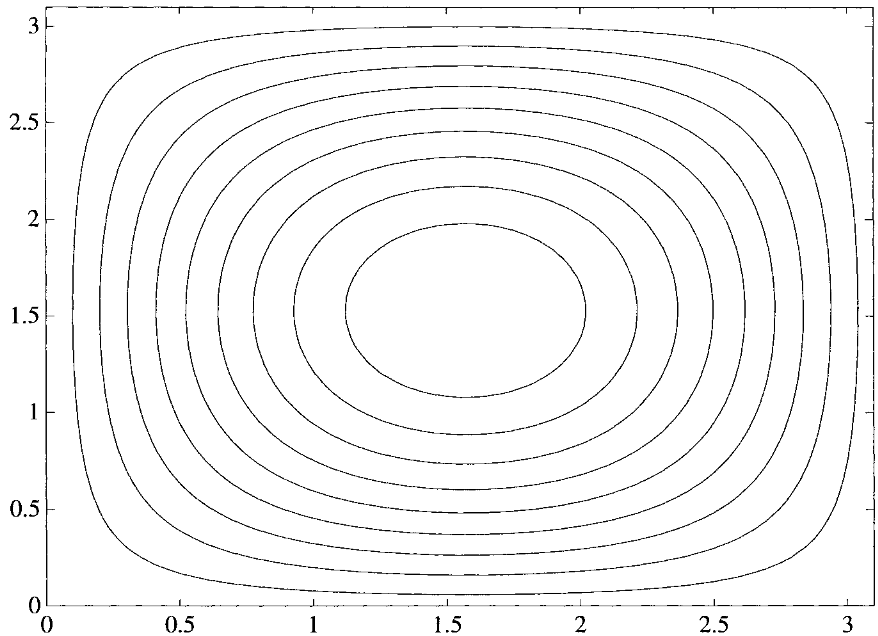

FIG. 4.1. *Quarter cell.*

and even in the y direction. Problem (4.1) can then be restricted to a quarter of the cell (see Fig. 4.1), $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, and, if we define

$$\rho = \chi + x, \tag{4.4}$$

then

$$\varepsilon \, \Delta \rho + \mathbf{u} \cdot \nabla \rho = \mathbf{0}, \tag{4.5}$$

$$\frac{\partial \rho}{\partial y}(x, 0) = \frac{\partial \rho}{\partial y}(x, \pi) = 0, \tag{4.6}$$

$$\rho(0, y) = 0 \ , \qquad \rho(\pi, y) = \pi, \tag{4.7}$$

and

$$\sigma_\varepsilon(\mathbf{e}) = \frac{\varepsilon}{\pi^2} \int_0^\pi \int_0^\pi \left[\left(\frac{\partial \rho}{\partial x} \right)^2 + \left(\frac{\partial \rho}{\partial y} \right)^2 \right] dx \, dy. \tag{4.8}$$

We consider general cellular flows, that is, flows with stream function $H(x, y)$ for which the lines $x = 0$ and $y = 0$ are separatrices, and level lines of $H = 0$. Furthermore, we assume that H is symmetric with respect to the x- and y-axes. Then the quarter cell reduction (4.5)–(4.8) is possible, and we work with it. First, we introduce a new coordinate system $(x, y) \to (H, \theta)$ from the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ to the region $H \geq 0$, $-4 \leq \theta \leq 4$, so that

$$\nabla H \cdot \nabla \theta = 0 \tag{4.9}$$

near the boundary of the rectangle and

$$|\nabla\theta| = |\nabla H| \tag{4.10}$$

on the boundary of the rectangle. There is a unique function $\theta(x, y)$, the circulation or angle variable, satisfying (4.9) and (4.10). It is not defined in all of the rectangle, in general, but only in a region including the boundary of the rectangle. The fact that θ runs over the interval $-4 \le \theta \le 4$ is a normalization condition on the stream function H. We call the coordinates

$$(h, \theta) = \left(\frac{H}{\sqrt{\varepsilon}}, \theta\right) \tag{4.11}$$

the *boundary layer* coordinates. In terms of the boundary layer coordinates, the cell problem (4.5)–(4.7) becomes

$$|\nabla H|^2 \frac{\partial^2 \rho}{\partial h^2} + \sqrt{\varepsilon}\,\Delta H \frac{\partial \rho}{\partial h} + \varepsilon |\nabla\theta|^2 \frac{\partial^2 \rho}{\partial \theta^2} + \varepsilon\,\Delta\theta \frac{\partial \rho}{\partial \theta} + J \frac{\partial \rho}{\partial \theta} = 0, \tag{4.12}$$

where $J = H_y\theta_x - H_x\theta_y = -\nabla^\perp H \cdot \nabla\theta$ is the Jacobian of the map $(x, y) \to (H, \theta)$. Because of (4.10), $|\nabla H|^2 = |J|$ at the boundary, and hence the principal terms as $\varepsilon \to 0$ in (4.12) are

$$\frac{\partial^2 \rho}{\partial h^2} + \frac{\partial \rho}{\partial \theta} = 0, \tag{4.13}$$

with

$$\begin{aligned} \rho(0, \theta) &= 0, & 0 < \theta < 2, \\ \frac{\partial \rho}{\partial h}(0, \theta) &= 0, & 2 < \theta < 4, \\ \rho(0, \theta) &= \pi, & -4 < \theta < -2, \\ \frac{\partial \rho}{\partial h}(0, \theta) &= 0, & -2 < \theta < 0. \end{aligned} \tag{4.14}$$

From (4.8), we obtain

$$\sigma_\varepsilon(\mathbf{e}) \sim \sqrt{\varepsilon}\frac{1}{\pi^2} \int_0^\infty \int_{-4}^4 \left(\frac{\partial \rho}{\partial h}\right)^2 dh\, d\theta. \tag{4.15}$$

The above analysis is essentially due to Childress [1]. In this section, we derive (4.15) using the variational principles of §3. The main difficulty in attempting to justify the asymptotic analysis of Childress is the lack of regularity of χ at the separatrices. This lack of regularity is an essential aspect of convection-enhanced diffusion and not only a technical difficulty. In the variational approach regularity is no longer a problem.

4.1. Upper bound for the effective diffusivity. As in (4.13), (4.14), we fix $\mathbf{e} = \mathbf{e}_1 = (1, 0)$, since the case where $\mathbf{e} = \mathbf{e}_2 = (0, 1)$ is similar. Let

$$\begin{aligned} \mathcal{F}_{BL} = \big\{ f = f(h, \theta),\ h \ge 0,\ -4 \le \theta \le 4,\ f \in C^\infty, \\ f \equiv \text{const for } h \ge N, \text{ for some } N > 0 \big\} \end{aligned} \tag{4.16}$$

and suppose that $f \in \mathcal{F}_{BL}$ also satisfies the boundary conditions

$$\begin{aligned} f(0,\theta) &= 0, && 0<\theta<2,\\ \frac{\partial f}{\partial h}(0,\theta) &= 0, && -2<\theta<0,\ 2<\theta<4,\\ f(0,\theta) &= \pi, && -4<\theta<-2 \end{aligned} \tag{4.17}$$

and the matching conditions on the separatrices

$$\begin{aligned} \int_0^\infty dh\frac{\partial f}{\partial \theta} &= 0, && -2<\theta<0,\ 2<\theta<4,\\ \int_0^\infty dh\int_\infty^h \frac{\partial f}{\partial \theta} &= 0, && 0<\theta<2,\\ \int_0^\infty dh\int_\infty^h \frac{\partial f}{\partial \theta} &= 0, && -4<\theta<-2. \end{aligned} \tag{4.18}$$

The matching conditions (4.18) are also the solvability conditions in evaluating the nonlocal term in the functional, as can be seen in the following estimates. Consider now the variational principle (3.73). We may look for trial fields $\mathbf{F}$ that have the quarter cell symmetry of (4.5)–(4.7). Then the averages in (3.73) can be restricted to a quarter cell, also, and, if $f \in \mathcal{F}_{BL}$, then $\mathbf{F} = \nabla f$ is an admissible trial field.

We now calculate ∇f and $\Gamma\mathbf{H}\,\nabla f$ for $f \in \mathcal{F}_{BL}$ and ε small. We have that

$$f_x = \frac{H_x}{\sqrt{\varepsilon}}\frac{\partial f}{\partial h} + \theta_x\frac{\partial f}{\partial \theta}, \qquad f_y = \frac{H_y}{\sqrt{\varepsilon}}\frac{\partial f}{\partial h} + \theta_y\frac{\partial f}{\partial \theta}. \tag{4.19}$$

Then

$$\begin{aligned} \varepsilon\langle \mathbf{F}\cdot\mathbf{F}\rangle &\sim \frac{\varepsilon}{\pi^2}\int_0^\infty\int_{-4}^4 \frac{1}{\varepsilon}|\nabla H|^2\frac{\sqrt{\varepsilon}}{J}\Big(\frac{\partial f}{\partial h}\Big)^2\,dh\,d\theta\\ &\sim \sqrt{\varepsilon}\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4 \Big(\frac{\partial f}{\partial h}\Big)^2\,dh\,d\theta, \end{aligned} \tag{4.20}$$

since $|\nabla H|^2 \sim |J|$ near $H=0$. Similarly , let $(1/\varepsilon)\Gamma\mathbf{H}\nabla f = \nabla f'$ for some periodic f'; then f' is the solution to the singular Poisson problem

$$\varepsilon\Delta f' = \mathbf{u}\cdot\nabla f \tag{4.21}$$

and

$$\frac{1}{\varepsilon}\langle \Gamma\mathbf{H}\nabla f\cdot\Gamma\mathbf{H}\nabla f\rangle = \varepsilon\langle\nabla f'\cdot\nabla f'\rangle. \tag{4.22}$$

Concerning the energy integral $\varepsilon\langle\nabla f'\cdot\nabla f'\rangle$, to leading order, it is sufficient to solve f' from the dominant terms in (4.21) after the boundary layer rescaling

$$|\nabla H|^2\frac{\partial^2}{\partial h^2}f' \sim J\frac{\partial}{\partial\theta}f, \tag{4.23}$$

which becomes

$$\frac{\partial^2}{\partial h^2}f' \sim \frac{\partial}{\partial\theta}f, \tag{4.24}$$

since $|\nabla H|^2 = J$ on the separatrices. Equation (4.24) is an ordinary differential equation in h and can be solved by direct integration. The matching conditions (4.18) guarantee that the existence of the solution f' to (4.24) in the function space $\mathcal{F}_B L$ satisfying the null boundary conditions. From (4.21), we see that

$$\begin{aligned}\frac{1}{\varepsilon}\langle \Gamma \mathbf{H}\,\nabla f\cdot\Gamma\mathbf{H}\,\nabla f\rangle &\sim \sqrt{\varepsilon}\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left(\int_\infty^h\frac{\partial f}{\partial\theta}\,dh'\right)^2 dh\,d\theta\\ &\sim \sqrt{\varepsilon}\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left(\frac{\partial f'}{\partial h}\right)^2 dh\,d\theta.\end{aligned}\tag{4.25}$$

Since $f \in \mathcal{F}_{BL}$ is identically zero for h large, the h integrals are well defined. Using (4.20) and (4.25) in (4.8), we have

$$\sigma_\varepsilon(\mathbf{e}) \le \varepsilon\langle\nabla f\cdot\nabla f\rangle + \frac{1}{\varepsilon}\langle\Gamma\mathbf{H}\,\nabla f\cdot\Gamma\mathbf{H}\,\nabla f\rangle,$$

and hence

$$\overline{\lim_{\varepsilon\downarrow 0}}\,\frac{1}{\sqrt{\varepsilon}}\sigma_\varepsilon(\mathbf{e}) \le \frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left\{\left(\frac{\partial f}{\partial h}\right)^2+\left(\int_\infty^h\frac{\partial f}{\partial\theta}\,dh'\right)^2\right\}dh\,d\theta.\tag{4.26}$$

Since the left-hand side does not depend on f, we also have

$$\overline{\lim_{\varepsilon\downarrow 0}}\,\frac{1}{\sqrt{\varepsilon}}\sigma_\varepsilon(\mathbf{e}) \le \inf_{f\in\mathcal{F}_{BL}}\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left\{\left(\frac{\partial f}{\partial h}\right)^2+\left(\int_\infty^h\frac{\partial f}{\partial\theta}\,dh'\right)^2\right\}dh\,d\theta.\tag{4.27}$$

4.2. Lower bound for the effective diffusivity. To obtain a lower bound for $\sigma_\varepsilon(\mathbf{e})$, we use the variational principle (3.74)

$$\begin{aligned}(\sigma_\varepsilon(\mathbf{e}))^{-1} = \inf_{\langle\nabla^\perp g\rangle=\mathbf{e}}\frac{1}{\varepsilon}\Bigg\langle&\left\langle\frac{1}{1+\varepsilon^{-2}H^2}\nabla^\perp g\cdot\nabla^\perp g\right\rangle\\ &+\left\langle\frac{1}{1+\varepsilon^{-2}H^2}\frac{1}{\varepsilon}\Gamma^\perp_{\varepsilon^{-1}H}\mathbf{H}\nabla^\perp g\cdot\frac{1}{\varepsilon}\Gamma^\perp_{\varepsilon^{-1}H}\mathbf{H}\nabla^\perp g\right\rangle\Bigg\},\end{aligned}\tag{4.28}$$

where $\Gamma^\perp_{\varepsilon^{-1}H}$ and $\Delta_{\varepsilon^{-1}H}$ are given by (3.75) and (3.76), respectively. Boundary layer trial functions can be constructed by noting that, when $\mathbf{e} = \mathbf{e}_1 = (1,0)$, they arise from $g = \chi - y$, when χ is periodic, so that $\nabla^\perp g = \nabla^\perp\chi + (1,0)$ and $\langle\nabla^\perp g\rangle = (1,0)$. If the space $\mathcal{F}_{BL}$ in (4.16) is denoted more precisely by $\mathcal{F}_{BL}(\mathbf{e}_1)$, then the boundary layer functions for (4.28), with quarter cell symmetry, $\mathcal{F}^\perp_{BL}(\mathbf{e}_1)$, are the same as $-\mathcal{F}_{BL}(\mathbf{e}_2)$. Thus, $\mathcal{F}^\perp_{BL}$ is the same as (4.16), but with the boundary conditions (4.17) replaced by

$$\begin{aligned} g(0,\theta) &= 0, && 2<\theta<4,\\ \frac{\partial g}{\partial h}(0,\theta) &= 0, && 0<\theta<2,\quad -4<\theta<-2,\\ g(0,\theta) &= \pi, && -2<\theta<0\end{aligned}\tag{4.29}$$

and the matching conditions replaced by

$$
\begin{aligned}
&h^2\int_\infty^h dh'\,\frac{1}{(h')^2}\frac{\partial g}{\partial\theta}\to 0\quad\text{as } h\downarrow 0,\qquad 0<\theta<2, -4<\theta<-2,\\
&\int_0^\infty dh\; h^2\int_\infty^h dh'\frac{1}{(h')^2}\frac{\partial g}{\partial\theta}=0,\qquad 0<\theta<2,\\
&\int_0^\infty dh\; h^2\int_\infty^h dh'\frac{1}{(h')^2}\frac{\partial g}{\partial\theta}=0,\qquad -4<\theta<-2.
\end{aligned}
\tag{4.30}
$$

We can now use a trial function $\mathbf{G}=\nabla^\perp g$, with $g\in\mathcal{F}_{BL}^\perp$, in (4.28). Calculations very similar to those for (4.20) and (4.25) now yield the bound

$$
\begin{aligned}
\overline{\lim_{\varepsilon\downarrow 0}}\,(\sigma(\mathbf{e}))^{-1}\sqrt{\varepsilon}\le \inf_{g\in\mathcal{F}_{BL}^\perp}\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\Bigg\{&\frac{1}{h^2}\left(\frac{\partial g}{\partial h}\right)^2\\
&+h^2\left(\int_\infty^h\frac{1}{(h')^2}\frac{\partial g}{\partial\theta}dh'\right)^2\Bigg\}\,dh\,d\theta.
\end{aligned}
\tag{4.31}
$$

4.3. Equality of upper and lower bounds. We must now show that the upper bound (4.27) is equal to the reciprocal of the lower bound (4.31) and that they coincide with the constant in (4.15), obtained by solving (4.13), (4.14). This proves the following theorem.

THEOREM 4.1. *The limit*

$$
\lim_{\varepsilon\downarrow 0}\frac{1}{\sqrt{\varepsilon}}\sigma_\varepsilon(\mathbf{e})=\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left(\frac{\partial\rho}{\partial h}\right)^2 dh\,d\theta
\tag{4.32}
$$

exists and equals the right side.

Proof. We begin with (4.13) and write it in divergence form

$$
\partial\cdot(I_1\pm\mathbf{h})\partial\rho^\pm=\mathbf{0},
\tag{4.33}
$$

where $\rho_+=\rho$, the solution of (4.13), and

$$
\partial=\left(\frac{\partial}{\partial h},\frac{\partial}{\partial\theta}\right),
\tag{4.34}
$$

$$
I_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad \mathbf{h}=\begin{pmatrix}0&h\\-h&0\end{pmatrix}.
\tag{4.35}
$$

Both ρ^+ and ρ^- are to satisfy the boundary conditions (4.14). We define

$$
c^*(\mathbf{e})=\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left(\frac{\partial\rho^+}{\partial h}\right)^2 dh\,d\theta.
\tag{4.36}
$$

We now proceed to symmetrize this problem as we did in §3. Let

$$
A=\frac{\rho^++\rho^-}{2},\qquad B=\frac{\rho^+-\rho^-}{2}.
\tag{4.37}
$$

Then A and B satisfy

$$
\frac{\partial^2 A}{\partial h^2}+\frac{\partial B}{\partial\theta}=0\,,\qquad \frac{\partial^2 B}{\partial h^2}+\frac{\partial A}{\partial\theta}=0\,.
\tag{4.38}
$$

Formally, for now, we note that

$$B = -\int_{\infty}^{h}\int_{\infty}^{h}\frac{\partial A}{\partial \theta}, \tag{4.39}$$

and hence A satisfies

$$\frac{\partial^2 A}{\partial h^2} - \int_{\infty}^{h}\int_{\infty}^{h}\frac{\partial^2 A}{\partial \theta^2} = 0, \tag{4.40}$$

along with the boundary conditions (4.14). Since $\rho^+ = A + B$, we note from (4.36) that

$$\begin{aligned} c^*(\mathbf{e}) &= \frac{1}{\pi^2}\int_0^{\infty}\int_{-4}^{4}\left(\frac{\partial A}{\partial h} + \frac{\partial B}{\partial h}\right)^2 dh\, d\theta \\ &= \frac{1}{\pi^2}\int_0^{\infty}\int_{-4}^{4}\left\{\left(\frac{\partial A}{\partial h}\right)^2 + \left(\int_{\infty}^{h}\frac{\partial A}{\partial \theta}\right)^2\right\} dh\, d\theta, \end{aligned} \tag{4.41}$$

where the cross term vanishes

$$\begin{aligned} \int_0^{\infty}\int_{-4}^{4}\frac{\partial A}{\partial h}\frac{\partial B}{\partial h}\, dh\, d\theta &= -\int_0^{\infty}\int_{-4}^{4}\frac{\partial A}{\partial h}\left(\int_{\infty}^{h}\frac{\partial A}{\partial \theta}\right) dh\, d\theta \\ &= \int_0^{\infty}\int_{-4}^{4} A\frac{\partial A}{\partial \theta}\, dh\, d\theta \\ &= 0. \end{aligned}$$

We now see that the right side of (4.41) is identical with the integral in (4.27) and that (4.40) is the Euler equation for this functional. This identifies the upper bound (4.27) with the constant c^* in (4.36) that comes from the boundary layer problem of Childress (4.13), (4.14). The sense in which (4.40) holds (plus the boundary conditions (4.14)) is precisely as the Euler equation of the variational problem (4.27) in the appropriate Hilbert space defined by the inner product derived from this quadratic form and by the closure of $\mathcal{F}_{BL}$ with this inner product.

To identify the lower bound (4.31) with $(c^*(\mathbf{e}))^{-1}$, we proceed again as in §3. From

$$\partial \cdot (I_1 + \mathbf{h})\partial \rho \doteq 0,$$

we conclude that there is a function $\phi(h, \theta)$ such that

$$c^*(\mathbf{e}_1)\partial^{\perp}\phi = c^*(\mathbf{e}_1)\left(-\frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial h}\right) = (I_1 + \mathbf{h})\partial \rho. \tag{4.42}$$

Thus, since $\partial \cdot \partial^{\perp}\phi = 0$, we have

$$\partial^{\perp} \cdot (I_1 + \mathbf{h})^{-1}\partial^{\perp}\phi = \mathbf{0},$$

which is equivalent to

$$\partial^{\perp} \cdot \frac{1}{h^2}(I_2 - \mathbf{h})\partial^{\perp}\phi = 0 \tag{4.43}$$

with

$$I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.44}$$

We show that ϕ satisfies the dual boundary conditions (4.29) and that

$$(c^*(\mathbf{e}_1))^{-1} = \frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 \frac{1}{h^2}\left(\frac{\partial\phi}{\partial h}\right)^2 dh\, d\theta. \tag{4.45}$$

We prove (4.45) first. From (4.15), we have that

$$\begin{aligned}
c^*(\mathbf{e}_1) &= \frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 \left(\frac{\partial\rho}{\partial h}\right)^2 dh\, d\theta \\
&= \frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 \partial\rho\cdot(I_1+\mathbf{h})\partial\rho\, dh\, d\theta \\
&= (c^*(\mathbf{e}_1))^2\frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 (I_1+\mathbf{h})^{-1}\,\partial^\perp\phi\cdot\partial^\perp\phi\, dh\, d\theta \\
&= (c^*(\mathbf{e}_1))^2\frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 \frac{1}{h^2}\left(\frac{\partial\phi}{\partial h}\right)^2 dh\, d\theta,
\end{aligned}$$

which is the same as (4.45).

To prove that ϕ, defined by (4.42), satisfies the boundary conditions (4.29), we write (4.42) in component form

$$\begin{aligned}
-c^*\frac{\partial\phi}{\partial\theta} &= \frac{\partial\rho}{\partial h} + h\frac{\partial\rho}{\partial\theta}, \\
c^*\frac{\partial\phi}{\partial h} &= -h\frac{\partial\rho}{\partial h}.
\end{aligned} \tag{4.46}$$

From the second relation, we obtain

$$\begin{aligned}
c^*\phi &= c(\theta) - \int_\infty^h h\frac{\partial\rho}{\partial h}, \\
&= c(\theta) - h\rho + \int_\infty^h \rho,
\end{aligned}$$

where $c(\theta)$ is a periodic function. Using the first relation in (4.46), we obtain

$$\begin{aligned}
-c^*\frac{\partial\phi}{\partial\theta} &= -c'(\theta) + h\frac{\partial\rho}{\partial\theta} + \int_\infty^h \frac{\partial\rho}{\partial\theta} \\
&= -c'(\theta) + h\frac{\partial\rho}{\partial\theta} - \int_\infty^h \frac{\partial^2\rho}{\partial h^2} \\
&= -c'(\theta) + h\frac{\partial\rho}{\partial\theta} + \frac{\partial\rho}{\partial h},
\end{aligned}$$

from which we conclude that $c'(\theta) = 0$ and hence $c(\theta) \equiv c$, a constant. Now, on the sides $2 < \theta < 4$ and $-2 < \theta < 0$, we have that

$$\frac{\partial}{\partial\theta}\int_\infty^0 \rho\, dh = -\int_\infty^0 \frac{\partial^2\rho}{\partial h^2}\, dh = -\frac{\partial\rho(0,\theta)}{\partial h} = 0$$

by (4.14). Thus we may choose the constant c to equal

$$c = -\int_\infty^0 \rho\, dh\,, \qquad 2 < \theta < 4$$

and then

$$\phi(0,\theta) = 0 \quad \text{on} \;\; 2 < \theta < 4$$

It remains to show that

$$\phi(0,\theta) = \pi \quad \text{on} \quad -2 < \theta < 0 \,. \tag{4.47}$$

For this purpose, we note that, on $-2 < \theta < 0$,

$$\begin{aligned} \phi(0,\theta) &= \phi(0,\theta) - \phi(0,\theta+4) \\ &= -\int_\theta^{\theta+4} \frac{\partial \phi}{\partial \theta}(0,\theta) \\ &= \frac{1}{c^*}\int_\theta^{\theta+4} \frac{\partial \rho}{\partial h}(0,\theta) \\ &= \frac{1}{c^*}\int_0^{2} \frac{\partial \rho}{\partial h}(0,\theta) \\ &\quad -\frac{1}{c^*}\int_{-4}^{-2} \frac{\partial \rho}{\partial h}(0,\theta). \end{aligned}$$

However,

$$\begin{aligned} c^* &= \frac{1}{\pi^2}\int_0^\infty \int_{-4}^4 \Big(\frac{\partial \rho}{\partial h}\Big)^2 \, dh\, d\theta \\ &= -\frac{1}{\pi^2}\int_{-4}^4 \rho(0,\theta)\frac{\partial \rho(0,\theta)}{\partial h} \quad \text{(from (4.13))} \\ &= -\frac{1}{\pi}\int_{-4}^{-2} d\theta \, \frac{\partial \rho(0,\theta)}{\partial h} \quad \text{(from (4.14))}, \end{aligned}$$

and hence (4.47) follows.

We now return to (4.43)–(4.45) and symmetrize it so that $(c^*(\mathbf{e}_1))^{-1}$ is given by a variational principle. We let $\phi = \phi^+$ and define ϕ^- by

$$\partial^\perp \cdot \frac{1}{h^2}(I_2 + \mathbf{h})\partial^\perp \phi^- = 0, \tag{4.48}$$

where both ϕ^+ and ϕ^- satisfy the boundary conditions (4.29). We define again A and B by

$$A = \tfrac{1}{2}(\phi^+ + \phi^-), \qquad B = \tfrac{1}{2}(\phi^+ - \phi^-)$$

and find that

$$\begin{aligned} h^2\frac{\partial}{\partial h}\left(\frac{1}{h^2}\frac{\partial A}{\partial h}\right) + \frac{\partial B}{\partial \theta} &= 0, \\ h^2\frac{\partial}{\partial h}\left(\frac{1}{h^2}\frac{\partial B}{\partial h}\right) + \frac{\partial A}{\partial \theta} &= 0. \end{aligned}$$

Thus

$$B = -\int_\infty^h h^2 \int_\infty^h \frac{1}{h^2}\frac{\partial A}{\partial \theta},$$

and

$$h^2\frac{\partial}{\partial h}\left(\frac{1}{h^2}\frac{\partial A}{\partial h}\right)-\int_\infty^h h^2\int_\infty^h\frac{1}{h^2}\frac{\partial^2 A}{\partial\theta^2}=0, \tag{4.49}$$

while, from (4.45),

$$\begin{aligned}(c^*(\mathbf{e}_1))^{-1}&=\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\frac{1}{h^2}\left(\frac{\partial A}{\partial h}+\frac{\partial B}{\partial h}\right)^2 dh\,d\theta\\ &=\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\frac{1}{h^2}\left[\left(\frac{\partial A}{\partial h}\right)^2+\left(\frac{\partial B}{\partial h}\right)^2\right]dh\,d\theta\\ &=\frac{1}{\pi^2}\int_0^\infty\int_{-4}^4\left[\frac{1}{h^2}\left(\frac{\partial A}{\partial h}\right)^2+h^2\left(\int_\infty^h\frac{1}{h^2}\frac{\partial A}{\partial\theta}\right)^2\right]dh\,d\theta.\end{aligned} \tag{4.50}$$

This is precisely the right-hand side of (4.31), since (4.49) is just the Euler equation for that quadratic functional. This proves that the limit (4.32) exists and equals c^*.

5. Corner layer theory: Nonoverlapping eddies in point-contact. The effective conductivity of a two-component conductor with checkerboard geometry is equal to the square root of the product of the component conductivities. If, for example, the conductivity of the black squares is 1 and the conductivity of the white squares ε, then the effective conductivity is $\sqrt{\varepsilon}$. Conductors with random checkerboard geometries can also be studied. Now each square has conductivity ε with probability p and conductivity 1 with probability $1-p$, independently of other squares. Kozlov [10] studied this problem by variational methods and found that there are the following three regimes: when $1>p>p_c$, the poorly conducting material prevails, and the effective conductivity is $O(\varepsilon)$; when $1-p_c>p>0$, the normally conducting material prevails, and the effective conductivity is $O(1)$; when $p_c>p>1-p_c$, the checkerboard configuration prevails, and the effective conductivity for this intermediate regime is $O(\sqrt{\varepsilon})$. The critical probability $p_c\approx 0.59\ldots$ is equal to the critical probability for the site percolation problem.

In this section, we study convection-diffusion problems for a two-dimensional periodic checkerboard configuration that consists of eddies with stream function $H=\sin x\sin y$, for example, and still fluid, $H=0$, alternatively from cell to cell, as in Fig. 5.1. The molecular diffusivity is ε. Using variational methods, we develop a corner layer theory that includes the boundary layer theory treated in §4 as a limiting case. We have also studied the random checkerboard configuration for convection-diffusion problems. Our results are parallel to those of Kozlov [10] and are presented in an upcoming paper.

Corner layers arise because eddies have in contact only a point instead of an edge (i.e., a separatrix); for example, if we take away every other vortex in the cellular flow $H=\sin x\sin y$ and change the sign of every other remaining vortex. The resulting periodic array of vortices are in contact only at the corners and have the 180°-rotational antisymmetry with respect to the origin and consequently a symmetric effective flux tensor. The contact angle is equal to $\pi/2$ (see Fig. 5.1). For these flows, the corners, rather than the separatrices, control the effective diffusivity.

Before analyzing the problem with positive contact angle, let us modify the flow near the corner as follows. Let us regularize the streamlines near the corner so that they have well-defined tangent at contact point and therefore zero contact angle (see Fig. 5.2). Let t denote the tangential coordinate and s the normal coordinate. Now

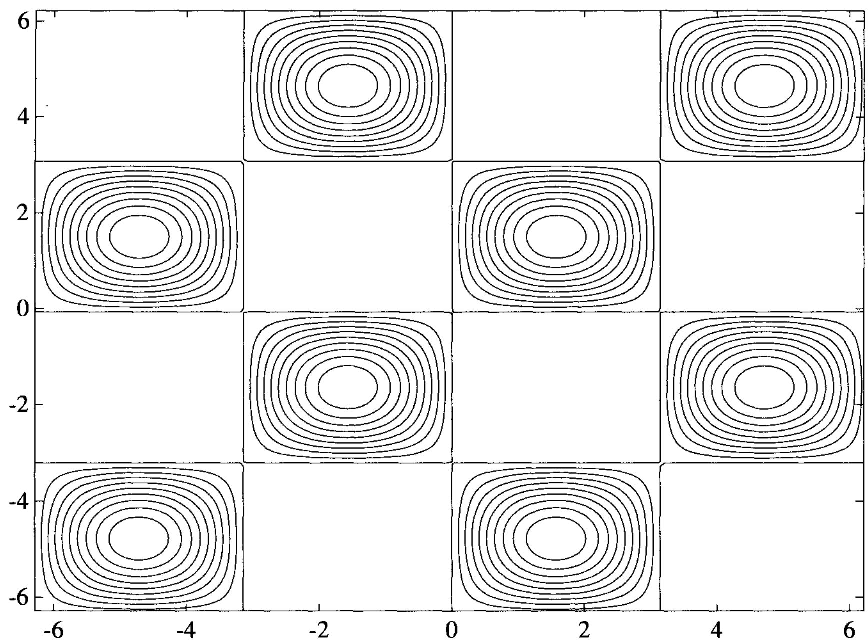

FIG. 5.1. *Nonoverlapping eddies in point-contact.*

assume that the streamlines near the contact point are asymptotically defined by $s - t^{1+\gamma}$ = constant. Here γ is the degree of the vanishing of the contact angle approaching the contact point. When two separatrices collapse, γ is infinite, and the situation is back to cellular flows treated in §4. When γ is zero, the contact angle is positive.

It turns out that the specific shape of the separatrices is not important. Only their asymptotic form near the contact point matters. When sufficiently close to the contact point, we may assume, without loss of generality, that the boundaries of eddies (that is, the separatrices) are defined exactly by $s = \pm|t|^{1+\gamma}$, and the stream function has the form

$$H = \begin{cases} u_0(s - |t|^{1+\gamma}) & \text{when } s \geq |t|^{1+\gamma}, \\ 0 & \text{when } |s| \leq |t|^{1+\gamma}, \\ u_0(s + |t|^{1+\gamma}) & \text{when } s \leq -|t|^{1+\gamma}. \end{cases} \tag{5.1}$$

We assume that the velocity at the separatrices $u_0 \neq 0$ is different from zero in the following. This assumption makes the flow discontinous and somewhat unrealistic. The case when the velocity is zero at the separatrices can also be studied, but gives rise to different scalings, depending on how fast the velocity vanishes.

As with cellular flow, particles away from the boundary are nearly trapped in stable closed orbits. However, unlike the cellular flow case, particles that stay near the boundary and eventually exit are almost trapped again in the adjacent vacant cells, except for those that exit from near the contact point. They can travel with the flow near the boundary of the adjacent vortices and exit again. Note that the narrow gap near the contact point creates a large concentration gradient and hence large diffusive flux.

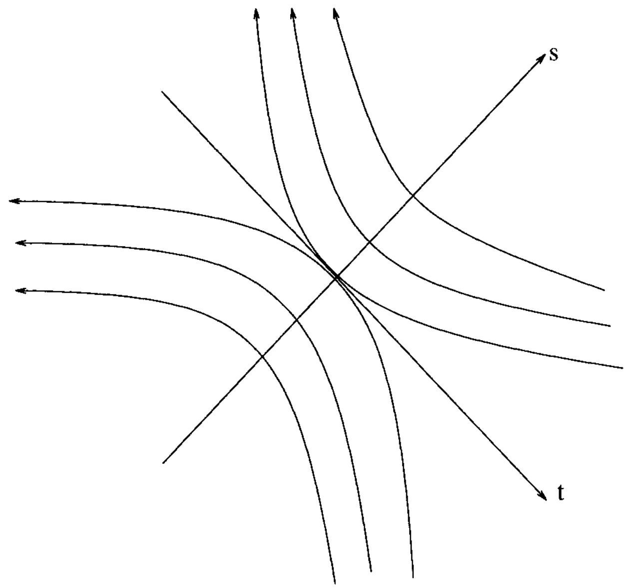

FIG. 5.2. *Corner flow.*

Let us define scaled variables

$$\tilde{t} = t/\varepsilon^{\alpha}, \quad \tilde{s} = s/\varepsilon^{\alpha(1+\gamma)},$$
$$\tilde{\theta} = \theta/\varepsilon^{\alpha}, \quad h = H/\varepsilon^{\alpha(1+\gamma)}, \tag{5.2}$$

where θ is the circulation variable defined as in §4. Here $\alpha = 1/(1+2\gamma)$ by the following scaling argument. The velocity u_0 at the contact point is not zero, so we let the time it takes to pass the corner be $O(\varepsilon^{\beta})$. The time it takes to diffuse across the narrow gap between vortices is $\varepsilon^{2\beta(1+\gamma)}/\varepsilon$. These two timescales should be of the same order, and thus $\beta = 1/(1+2\gamma)$. The scaling of time should be the same as that of the tangential coordinate t; thus $\alpha = \beta = 1/(1+2\gamma)$. Since $\gamma > 0$, the scale of the normal coordinate is smaller than that of the tangential coordinate. Therefore concentration gradients are $O(1/(\varepsilon^{\alpha(1+\gamma)}))$, and σ_ε is proportional to

$$\begin{aligned} &\varepsilon \times \text{the area of corner layer} \times \text{the square of concentration gradient} \\ &\sim \varepsilon \times (\varepsilon^{\alpha} \times \varepsilon^{\alpha(1+\gamma)}) \times \left(\frac{1}{\varepsilon^{\alpha(1+\gamma)}}\right)^2 \\ &\sim \varepsilon^{(1/2)(1+1/(1+2\gamma))} \end{aligned} \tag{5.3}$$

after substituting $\alpha = 1/(1+2\gamma)$. The power of ε in σ_ε ranges from $\frac{1}{2}$ to 1 as γ ranges from infinity to zero. Using the variational principles, we justify this scaling argument and prove the following result.

THEOREM 5.1. *For a checkerboard flow with stream function* (5.1) *near corners, the effective conductivity behaves like*

$$\sigma_\varepsilon \sim c^* \varepsilon^{(1/2)(1+1/(1+2\gamma))},$$

where c^* is a constant that can be computed explicitly.

The proof of Theorem 5.1 is given in the following three sections. We refer to $(\tilde{t}, \tilde{s})$ for $|s| \leq |t|^{1+\gamma}$ and $(h, \tilde{\theta})$ for $|s| \geq |t|^{1+\gamma}$ as the corner layer variables. The period cell is $[-\pi, \pi]^2$.

5.1. Upper bound for the effective diffusivity. For the upper bound, we again use the direct variational principle and choose trial functions according to our scaling argument given above. The class of corner layer trial functions for the upper bound is denoted by $\mathcal{C}$, and f belongs to it if it is piecewise smooth and (a) for some $N_0 > 0$, it satisfies the far field boundary conditions

$$f = \begin{cases} \pi & \text{for } \tilde{s} \geq N_0^{1+\gamma} \text{ and } \tilde{s} \geq |\tilde{t}|^{1+\gamma}, \\ 0 & \text{for } \tilde{s} \leq -N_0^{1+\gamma} \text{ and } -\tilde{s} \geq |\tilde{t}|^{1+\gamma}. \end{cases}$$

Each $f \in \mathcal{C}$ is associated with a corner region C, defined by $\{(\tilde{t}, \tilde{s}) \mid |\tilde{t}| \leq N_0, |\tilde{s}| \leq N_0^{1+\gamma}\}$, an eddy region E excluding C and a vacant region V excluding C. The corner region $C = C(N_0, \varepsilon)$ depends on N_0, which may differ for different f. The period cell $[-\pi, \pi]^2$ is the union of the regions C, E, and V. We split the region C into $C_e \cup C_v$, where C_e and C_v are intersections of C with the eddies and the vacant cells, i.e., $C_e = \{|\tilde{s}| \geq |\tilde{t}|^{1+\gamma}\}$ and $C_v = \{|\tilde{s}| \leq |\tilde{t}|^{1+\gamma}\}$;

(b) $f|_C$ is a function of the corner layer variables and is piecewise smooth in $|\tilde{s}| \geq |\tilde{t}|^{1+\gamma}$ and $|\tilde{s}| \leq |\tilde{t}|^{1+\gamma}$;

(c) The matching condition on the separatrices are specified later. When ε is small, we choose $N_0 = N_0(\varepsilon) \uparrow \infty$, while $\varepsilon^\alpha N_0 \downarrow 0$ as $\varepsilon \downarrow 0$, for some $\alpha > 0$, to be determined later; we define the corner region C using this $N_0(\varepsilon)$. We can then discuss a common corner region C, eddy region E, and vacant region V for all $f \in \mathcal{C}$ where C, E, and V depend only on ε. For every $f \in \mathcal{C}$, $f\big|_E = \pi$ if $H > 0$, $f\big|_E = 0$ if $H < 0$, and the profile of f restricted to the vacant cells $f|_{V \cup C_v}$ is determined later. The entire profile of f in the period cell is shown schematically in Fig. 5.3. The functions f are normalized so that $\langle \nabla f \rangle \to (1, 0)$, as $\varepsilon \downarrow 0$.

The functional in the direct variational principle (3.73) for the upper bound has two terms, the local one $\varepsilon\langle \mathbf{F} \cdot \mathbf{F} \rangle$ and the nonlocal one $1/\varepsilon\langle \Gamma \mathbf{HF} \cdot \Gamma \mathbf{HF} \rangle$. To estimate them, we break the integral over the period cell $\langle \cdot \rangle$ into the integrals over the regions C, E, and V and write $\langle \cdot \rangle = \langle \cdot \rangle_C + \langle \cdot \rangle_E + \langle \cdot \rangle_V$.

Let us consider the local integral $\varepsilon\langle \mathbf{F} \cdot \mathbf{F} \rangle$. First, $\langle \mathbf{F} \cdot \mathbf{F} \rangle_E = 0$ by the far field boundary conditions (a). Second, $\mathbf{F}\big|_{V \cup C_v}$ can be chosen so that

$$\varepsilon\langle \mathbf{F} \cdot \mathbf{F} \rangle_V = o\left(\frac{\varepsilon\varepsilon^\alpha}{\varepsilon^{\alpha(1+\gamma)}}\right) \quad \text{as } N_0 \uparrow \infty,\ \varepsilon \downarrow 0 \quad \text{while } \varepsilon^\alpha N_0 \downarrow 0.$$

To see this, choose $f\big|_{V \cup C_v}$ to be smooth so that $f\big|_{V'}$ for every $V' \subset V$ is independent of ε and N_0 if V' is. Then the principal contribution to $\langle \mathbf{F} \cdot \mathbf{F} \rangle_V$ comes from the tiny region $\delta > 0$ fixed, where $\mathbf{F} = \nabla f$ is most singular due to the merging of two

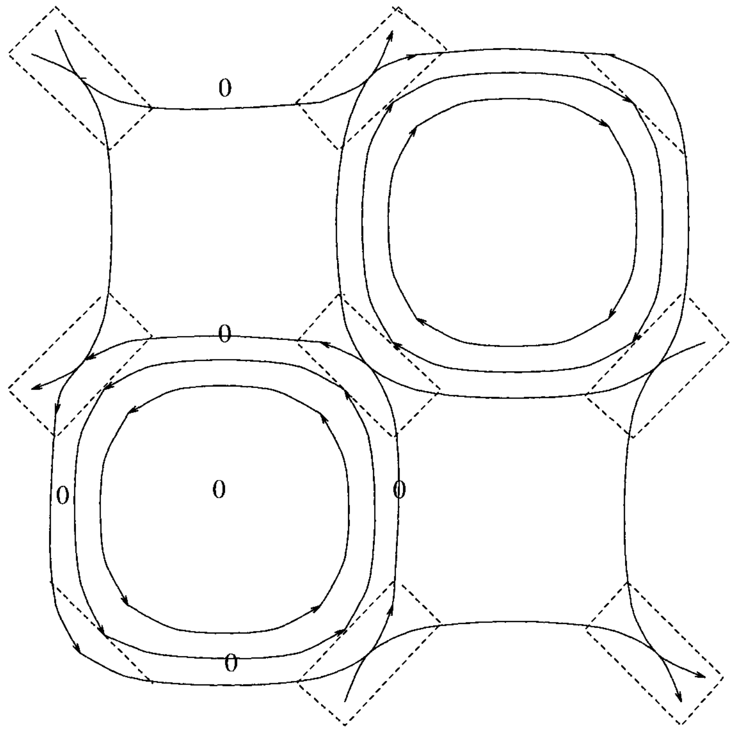

FIG. 5.3. *Direct corner layer function.*

separatrices, $|s| = \pm|t|^{1+\gamma}$, and the far field boundary conditions (a). Thus, $\langle \mathbf{F}\cdot\mathbf{F}\rangle_V$ is of order

$$
\begin{aligned}
\frac{2\varepsilon}{\pi^2}\int_{N_0\varepsilon^\alpha}^{\delta}\int_{-|t|^{1+\gamma}}^{|t|^{1+\gamma}}\left(\frac{\pi}{t^{1+\gamma}}\right)^2 ds\,dt &= \frac{\varepsilon\varepsilon^\alpha}{\varepsilon^{\alpha(1+\gamma)}}\int_{N_0}^{\delta\varepsilon^{-\alpha}}\frac{1}{\tilde t^{1+\gamma}}\,d\tilde t\\
&\sim \frac{\varepsilon\varepsilon^\alpha}{\varepsilon^{\alpha(1+\gamma)}}\frac{1}{N_0^\gamma}\quad \text{as } \varepsilon\downarrow 0\\
&= o\left(\frac{\varepsilon\varepsilon^\alpha}{\varepsilon^{\alpha(1+\gamma)}}\right)\quad \text{as } N_0\uparrow\infty\\
&\qquad\qquad\text{if}\quad \gamma>0.
\end{aligned}
\tag{5.4}
$$

We note that the last identity in (5.1) does not hold for $\gamma = 0$, and this limiting case is analyzed later, where a logarithmic factor $\log 1/\varepsilon$ appears. Third, for $\langle \mathbf{F}\cdot\mathbf{F}\rangle_C$, a simple calculation gives

$$
\varepsilon\langle \mathbf{F}\cdot\mathbf{F}\rangle_C \sim \frac{1}{\pi^2}\frac{\varepsilon\varepsilon^\alpha}{\varepsilon^{\alpha(1+\gamma)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde t\,d\tilde s\left(\frac{\partial}{\partial\tilde s}f\right)^2 \quad \text{as } \varepsilon\downarrow 0,
\tag{5.5}
$$

since derivatives with respect to $\tilde s$ and h dominate those with respect to $\tilde t$ and $\tilde\theta$ as $\varepsilon\to 0$. In summary, we have

$$
\begin{aligned}
\varepsilon\langle \mathbf{F}\cdot\mathbf{F}\rangle &\sim \varepsilon\langle \mathbf{F}\cdot\mathbf{F}\rangle_C\\
&\sim \frac{\varepsilon^{1-\alpha\gamma}}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde t\,d\tilde s\left(\frac{\partial}{\partial\tilde s}f\right)^2.
\end{aligned}
\tag{5.6}
$$

We next consider the nonlocal term $(1/\varepsilon)\langle \Gamma \mathbf{HF} \cdot \Gamma \mathbf{HF} \rangle$. Let $(1/\varepsilon)\Gamma\mathbf{HF} = \nabla \tilde{f}$ for some periodic $\tilde{f}$. Then

$$\varepsilon \Delta \tilde{f} = \mathbf{u} \cdot \nabla f \tag{5.7}$$

and

$$\frac{1}{\varepsilon}\langle \Gamma \mathbf{HF} \cdot \Gamma \mathbf{HF} \rangle = \varepsilon \langle \nabla \tilde{f} \cdot \nabla \tilde{f} \rangle. \tag{5.8}$$

The right-hand side of (5.7) has zero mean

$$\begin{aligned} \langle \mathbf{u} \cdot \nabla f \rangle &= \langle \mathbf{u} \cdot \mathbf{F} \rangle \\ &= \langle \mathbf{u} \cdot \mathbf{e} \rangle + \langle \mathbf{u}(\mathbf{F} - \mathbf{e}) \rangle \\ &= 0 \end{aligned}$$

by $\langle \mathbf{u} \cdot \mathbf{e} \rangle = 0$, integrating the second term by parts. Hence (5.7) is solvable, and $\tilde{f}$ exists. As in the case of cellular flows, to leading order, it is enough to solve the following approximate equation for f' with the null far field boundary conditions:

$$\begin{aligned} \frac{\varepsilon}{\varepsilon^{2\alpha(1+\gamma)}} \frac{\partial^2}{\partial \tilde{s}^2} f' &= -\frac{1}{\varepsilon^\alpha} u_o^2 \left(\frac{\partial}{\partial \tilde{t}} f + (1+\gamma)\tilde{t}^\gamma \frac{\partial}{\partial \tilde{s}} f \right) \quad \text{in } C_e, \\ \frac{\varepsilon}{\varepsilon^{2\alpha(1+\gamma)}} \frac{\partial^2}{\partial \tilde{s}^2} f' &= 0 \quad \text{in } C_v. \end{aligned} \tag{5.9}$$

With $\alpha = (1/1 + 2\gamma)$, (5.9) becomes

$$\frac{\partial^2}{\partial \tilde{s}^2} f' = -u_0^2 \left(\frac{\partial}{\partial \tilde{t}} f + (1+\gamma)\tilde{t}^\gamma \frac{\partial}{\partial \tilde{s}} f \right) P \quad \text{in } C_e, \quad \frac{\partial^2}{\partial \tilde{s}^2} f' = 0 \quad \text{in } C_v, \tag{5.10}$$

with f' is continuous across the separatrices $\tilde{s} = \pm |\tilde{t}|^{1+\gamma}$. In (5.9), only the dominant term of the Laplacian in the corner layer coordinates appears, and α is chosen so that the diffusive flux is balanced by the convective flux. For (5.9) to be a valid approximation to the leading order of the energy integral $\varepsilon \langle \nabla \tilde{f} \cdot \nabla \tilde{f} \rangle$, it is actually required that (5.9) can be solved by a solution f' with the first derivative continuous across the separatrices. Thus, an additional matching condition must be imposed on the trial function f, which is, in view of the second equation of (5.10),

$$\begin{aligned} &\int_{\pm\infty}^{\tilde{s}=\pm|\tilde{t}|^{1+\gamma}} u_o^2 \left(\frac{\partial}{\partial \tilde{t}} f + (1+\gamma)\tilde{t}^\gamma \frac{\partial}{\partial \tilde{s}} f \right) \\ &= \frac{1}{|\tilde{t}|^{1+\gamma}} \int_{\pm\infty}^{\tilde{s}=\pm|\tilde{t}|^{1+\gamma}} \int_{\pm\infty}^{\tilde{s}'} u_o^2 \left(\frac{\partial}{\partial \tilde{t}} f + (1+\gamma)\tilde{t}^\gamma \frac{\partial}{\partial \tilde{s}} f \right). \end{aligned} \tag{5.11}$$

With this, f' can be solved continuously up to the first derivative, it has the following two parts:

$$f' = \begin{cases} f'_e & \text{in } C_e, \\ f'_v & \text{in } C_v. \end{cases}$$

Here f'_e satisfies the first equation of (5.10) and the far field boundary conditions in the definition of $\mathcal{C}$, and

$$\begin{aligned} f'_v = f'_v(\tilde{s}) &\text{ is a linear function that matches} \\ &\text{the values of } f'_e \text{ on the separatrices.} \end{aligned} \tag{5.12}$$

We then have

$$\frac{1}{\varepsilon}\langle \Gamma\mathbf{HF}\cdot\Gamma\mathbf{HF}\rangle \tag{5.13}$$
$$= \varepsilon\langle\nabla\tilde f\cdot\nabla\tilde f\rangle$$
$$\sim \varepsilon\langle\nabla f'\cdot\nabla f'\rangle_C \tag{5.14}$$
$$\sim \frac{1}{\pi^2}\varepsilon^{1-\alpha\gamma}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde t\, d\tilde s\left(\frac{\partial}{\partial \tilde s}f\right)^2\right\}.$$

From the upper bounds for both the local term and nonlocal term in the direct variational principle, we obtain the upper bound for the effective diffusivity σ_ε,

$$\overline{\lim_{\varepsilon\downarrow 0}}\frac{1}{\varepsilon^{(1/2)(1+(1/1+2\gamma))}}\sigma_\varepsilon \le \frac{1}{\pi^2}\inf_{f\in C}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde t\, d\tilde s\left[\left(\frac{\partial}{\partial \tilde s}f\right)^2+\left(\frac{\partial}{\partial \tilde s}f'\right)^2\right]\right\}, \tag{5.15}$$

with f' defined in (5.10). When $f' = f = \rho$ in (5.10), (5.10) is called the corner layer equation

$$\begin{aligned} &\frac{\partial^2}{\partial \tilde s^2}\rho = -u_o^2\left(\frac{\partial}{\partial \tilde t}\rho + (1+\gamma)\tilde t^{\gamma}\frac{\partial}{\partial \tilde s}\rho\right) && \text{in } C_e,\\ &\frac{\partial^2}{\partial \tilde s^2}\rho = 0 && \text{in } C_v, \end{aligned} \tag{5.16}$$

with ρ and its first derivative continuous across the separatrices $\tilde s = \pm|\tilde t|^{1+\gamma}$. Equation (5.16) is complemented by the boundary conditions

$$\rho = \begin{cases} \pi & \text{for } h>0,\ \tilde\theta = \pm\infty,\\ 0 & \text{for } h<0,\ \tilde\theta=\pm\infty \end{cases} \tag{5.17}$$

and

$$\rho = \begin{cases} \pi & \text{for } h=\infty,\\ 0 & \text{for } h=-\infty. \end{cases} \tag{5.18}$$

The correct weak form of (5.16) is given by (5.32).

5.2. Lower bound for the effective diffusivity. To estimate σ_ε from below, we use the inverse variational principle (3.74). Let us define a class of corner layer trial functions for the lower bound, denoted by $\mathcal{C}^\perp$ as follows. A function $g\in\mathcal{C}^\perp$ if it satisfies the following conditions:

(a) Far field boundary conditions: There exists a positive number $N_0>0$ such that

$$g = \begin{cases} \pi & \text{for } \tilde t \ge N_0,\\ 0 & \text{for } \tilde t\le -N_0. \end{cases}$$

As for the upper bound, we can associate with each $g\in\mathcal{C}^\perp$ a corner layer region $\{(\tilde t,\tilde s)\mid |\tilde t|\le N_0,\ |\tilde s|\le N_0^{1+\gamma}\}$, an eddy region E, and a vacant region V. The period cell $[-\pi,\pi]^2$ is the union of C, E, and V;

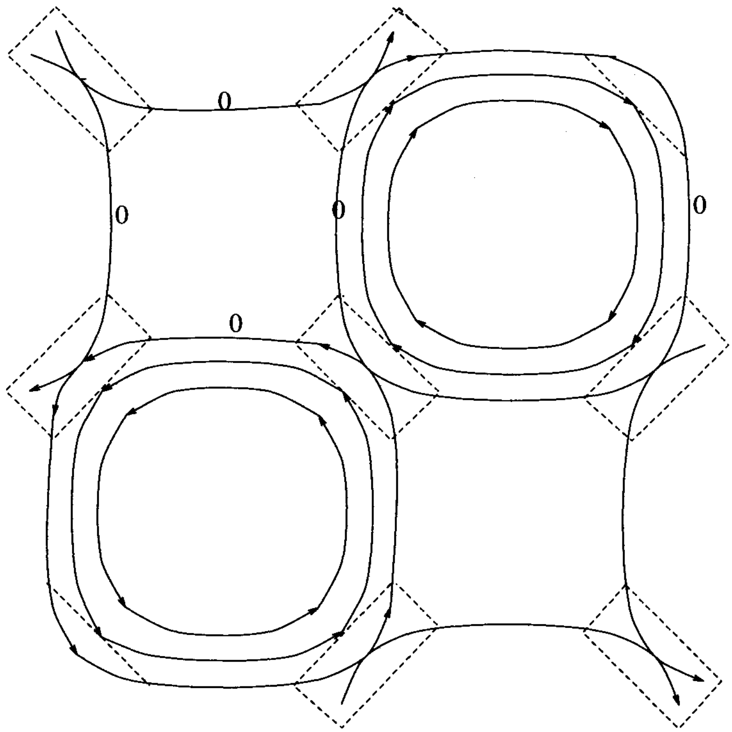

FIG. 5.4. *Dual corner layer function.*

(b) $g|_C$ is a function of the corner layer variables, which is piecewise smooth in $|\tilde{s}| \geq |\tilde{t}|^{1+\gamma}$ and $|\tilde{s}| < |\tilde{t}|^{1+\gamma}$ and continuous everywhere;

(c) $g = g(\tilde{t})$ for $|\tilde{s}| \leq |\tilde{t}|^{1+\gamma}$.

For every $g \in \mathcal{C}^{\perp}$, $g|_V = \pi$, if $t > 0$ and $g|_V = 0$, if $t < 0$. The profile of g restricted to region E, which is not covered by the definition of $\mathcal{C}^{\perp}$, is specified later. We note that the conditions on g are formulated so that $\langle \nabla^{\perp} g \rangle = \mathbf{e}_1$. The overall profile of g is shown schematically in Fig. 5.4.

Let us consider the local term in the inverse variational principle $(1/\varepsilon) \cdot \langle (1/1 + (1/\varepsilon^2)H^2)\mathbf{G} \cdot \mathbf{G} \rangle$. We break the integral into three parts,

$$\frac{1}{\varepsilon}\left\langle \frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle = \frac{1}{\varepsilon}\left\langle \frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_C + \frac{1}{\varepsilon}\left\langle \frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_E + \frac{1}{\varepsilon}\left\langle \frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_V .$$

First, $(1/\varepsilon)\langle(1/1 + (1/\varepsilon^2)H^2)\mathbf{G} \cdot \mathbf{G}\rangle_V = 1/\varepsilon\langle \mathbf{G} \cdot \mathbf{G}\rangle_V = 0$ by the far field boundary conditions (a). Second, we can choose $N_0 = N_0(\varepsilon) \uparrow \infty$, $\delta = N_0\varepsilon^{\alpha} \downarrow 0$ as $\varepsilon \downarrow 0$ such that $g|_V$ is a boundary layer function for the lower bound and the boundary layer theory developed in §4 applies. We have

$$\frac{1}{\varepsilon}\left\langle \frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_V = O\left(\frac{1}{\sqrt{\varepsilon}}\right). \tag{5.19}$$

Third, we split the integral over region C into regions C_e and C_v,

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_C &= \frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_{C_e} \\ &\quad+\frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_{C_v} \\ &= \frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_{C_e}+\frac{1}{\varepsilon}\langle\mathbf{G}\cdot\mathbf{G}\rangle_{C_v}.\end{aligned} \tag{5.20}$$

Since $g=g(\tilde{t})$ in C_v, for the second term we have

$$\begin{aligned}\frac{1}{\varepsilon}\langle\mathbf{G}\cdot\mathbf{G}\rangle_{C_v} &\sim \frac{1}{\pi^2}\frac{\varepsilon^{\alpha(1+\gamma)}}{\varepsilon\,\varepsilon^{\alpha}}\int_{-N_0}^{N_0}d\tilde{t}\int_{-|\tilde{t}|^{1+\gamma}}^{|\tilde{t}|^{1+\gamma}}d\tilde{s}\left(\frac{\partial g}{\partial\tilde{t}}\right)^2 \\ &\sim \frac{1}{\pi^2}\frac{1}{\varepsilon^{(1/2)(1+1/(1+2\gamma))}}\int_{-\infty}^{\infty}d\tilde{t}\int_{-|\tilde{t}|^{1+\gamma}}^{|\tilde{t}|^{1+\gamma}}d\tilde{s}\left(\frac{\partial g}{\partial\tilde{t}}\right)^2.\end{aligned} \tag{5.21}$$

Here we have used the far field boundary condition (a) and $\alpha=1/(1+2\gamma)$. For the first term,

$$\frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_{C_e}\sim\frac{1}{\pi^2}\frac{\varepsilon\,\varepsilon^{\alpha}}{\varepsilon^{3\alpha(1+\gamma)}}\int\int_{|\tilde{s}|\ge|\tilde{t}|^{1+\gamma}}d\tilde{t}\,d\tilde{s}\,\frac{1}{h^2}\left(\frac{\partial g}{\partial\tilde{s}}\right)^2. \tag{5.22}$$

With $\alpha=1/(1+2\gamma)$, the right-hand side of (5.22) can be further reduced to

$$\frac{1}{\pi^2}\frac{(1}{\varepsilon^{(1/2)(1+1/(1+2\gamma))}}\int\int_{|\tilde{s}|\ge|\tilde{t}|^{1+\gamma}}d\tilde{t}\,d\tilde{s}\,\frac{1}{h^2}\left(\frac{\partial g}{\partial\tilde{s}}\right)^2. \tag{5.23}$$

Since $1/\varepsilon^{(1/2)(1+1/(1+2\gamma))}\gg1/\sqrt{\varepsilon}$ if $0<\gamma<\infty$, we conclude that the integration over region C gives the dominant contribution and we summarize the estimate on the local term by combining (5.21) and (5.23) to obtain

$$\begin{aligned}\frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle &\sim \frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle_C \\ &\lesssim \frac{1}{\pi^2}\frac{1}{\varepsilon^{(1/2)(1+1/(1+2\gamma))}}\left\{\int\int_{|\tilde{s}|<|\tilde{t}|^{1+\gamma}}d\tilde{t}\,d\tilde{s}\left(\frac{\partial g}{\partial\tilde{t}}\right)^2\right. \\ &\quad\left.+\int\int_{|\tilde{s}|\ge|\tilde{t}|^{1+\gamma}}d\tilde{t}\,d\tilde{s}\,\frac{1}{h^2}\left(\frac{\partial g}{\partial\tilde{s}}\right)^2\right\}.\end{aligned} \tag{5.24}$$

We consider next the nonlocal term in the inverse variational principle, which is

$$\frac{1}{\varepsilon^3}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\Gamma^{\perp}_{(1/\varepsilon)H}\mathbf{H}\mathbf{G}\cdot\Gamma^{\perp}_{(1/\varepsilon)H}\mathbf{H}\mathbf{G}\right\rangle.$$

To estimate it, we write $(1/\varepsilon)\Gamma^{\perp}_{(1/\varepsilon)H}\mathbf{H}\nabla^{\perp}g=\nabla^{\perp}\tilde{g}$ for some periodic function $\tilde{g}$. Then

$$\varepsilon\nabla^{\perp}\cdot\frac{1}{1+(1/\varepsilon^2)H^2}\nabla^{\perp}\tilde{g}=\nabla^{\perp}\cdot\frac{\mathbf{H}}{1+(1/\varepsilon^2)H^2}\nabla^{\perp}g. \tag{5.25}$$

As before, to leading order, it is sufficient to consider only the dominant terms in corner layer coordinates

$$\frac{\partial}{\partial \tilde{s}}\frac{1}{h^2}\frac{\partial}{\partial \tilde{s}}g' = -\frac{u_o^2}{h^2}\left(\frac{\partial}{\partial \tilde{t}}g + (1+\gamma)\tilde{t}^{\gamma}\frac{\partial}{\partial \tilde{s}}g\right) \quad \text{in } C_e. \tag{5.26}$$

Equation (5.26) is equivalent to

$$\frac{\partial g'}{\partial \tilde{s}} = \mp h^2 \int_{\pm\infty}^{\tilde{s}} \frac{u_o^2}{h^2}\left(\frac{\partial}{\partial \tilde{t}}g + (1+\gamma)\tilde{t}^{\gamma}\frac{\partial}{\partial \tilde{s}}g\right) \quad \text{in } C_e, \tag{5.27}$$

where the different signs are taken for $h > 0$ and $h < 0$, respectively. To ensure that the normal derivative of g' is continuous across the separatrices, an additional matching condition is needed, which is

$$h^2 \int_{\pm\infty}^{\tilde{s}} \frac{u_o^2}{h^2}\left(\frac{\partial}{\partial \tilde{t}}g + (1+\gamma)\tilde{t}^{\gamma}\frac{\partial}{\partial \tilde{s}}g\right) = 0 \tag{5.28}$$

on the separatrices. In summary, we have the lower bound

$$\begin{aligned} &\overline{\lim_{\varepsilon\downarrow 0}}(\sigma)^{-1}\varepsilon^{(1/2)(1+1/(1+2\gamma))} \\ &\le \frac{1}{\pi^2}\inf_{g\in C^{\perp}}\left\{\int\int_{|\tilde{s}|\le|\tilde{t}|^{1+\gamma}} d\tilde{t}\,d\tilde{s}\left[\left(\frac{\partial g}{\partial \tilde{t}}\right)^2 + \left(\frac{\partial g'}{\partial \tilde{t}}\right)^2\right]\right. \\ &\quad \left. + \int\int_{|\tilde{s}|\ge|\tilde{t}|^{1+\gamma}} d\tilde{t}\,d\tilde{s}\frac{1}{h^2}\left[\left(\frac{\partial}{\partial \tilde{s}}g\right)^2 + \left(\frac{\partial}{\partial \tilde{s}}g'\right)^2\right]\right\} \end{aligned} \tag{5.29}$$

with g' and g related by (5.26).

When $g' = g = \phi$, (5.26) is called the dual corner layer equation,

$$\frac{\partial}{\partial \tilde{s}}\frac{1}{h^2}\frac{\partial}{\partial \tilde{s}}\phi = -\frac{u_o^2}{h^2}\left(\frac{\partial}{\partial \tilde{t}}\phi + (1+\gamma)\tilde{t}^{\gamma}\frac{\partial}{\partial \tilde{s}}\phi\right) \quad \text{in } C_e. \tag{5.30}$$

The dual boundary conditions are

$$\phi = \begin{cases} \pi & \text{for } \tilde{\theta} = \infty, \\ 0 & \text{for } \tilde{\theta} = -\infty. \end{cases} \tag{5.31}$$

5.3. Equality of upper and lower bounds. We show how to compute the constant in Theorem 5.1 in terms of the solution of the corner layer problem.

THEOREM 5.2. *The limit of the effective diffusivity is given by*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{(1/2)(1+(1/1+2\gamma))}}\sigma_\varepsilon = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde{t}\,d\tilde{s}\left(\frac{\partial}{\partial \tilde{s}}\rho\right)^2,$$

where ρ is the solution of the corner layer problem (5.16).

We use the saddlepoint variational principle to establish the reciprocity of the upper and lower bounds. We closely follow Appendix A.2, which is different from the method we used for cellular flows in §2.

We begin with the forward and backward corner layer equations in divergence form,

$$\partial \cdot (\mathbf{I}_2 \pm \mathbf{h})\,\partial\rho^{\pm} = 0, \tag{5.32}$$

where $\rho^+ = \rho$, the solution of the corner layer problem, and

$$\partial = (\partial_{\tilde{t}}, \partial_{\tilde{s}}) = \left(\frac{\partial}{\partial \tilde{t}}, \frac{\partial}{\partial \tilde{s}}\right), \tag{5.33}$$

$$\mathbf{I}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.34}$$

$$\mathbf{h} = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}, \tag{5.35}$$

with

$$h = \begin{cases} u_0(\tilde{s} - |\tilde{t}|^{1+\gamma}) & \text{when } \tilde{s} \geq |\tilde{t}|^{1+\gamma}, \\ 0 & \text{when } |\tilde{s}| \leq |\tilde{t}|^{1+\gamma}, \\ u_0(\tilde{s} + |\tilde{t}|^{1+\gamma}) & \text{when } \tilde{s} \leq -|\tilde{t}|^{1+\gamma}. \end{cases} \tag{5.36}$$

Set

$$\begin{aligned} \mathbf{E}^+ &= \partial \rho^+, \qquad \mathbf{E}^- = \partial \rho^-, \\ \mathbf{D}^+ &= (\mathbf{I}_2 + \mathbf{h})\, \partial \rho^+ = (\mathbf{I}_2 + \mathbf{h})\mathbf{E}^+, \\ \mathbf{D}^- &= (\mathbf{I}_2 - \mathbf{h})\, \partial \rho^- = (\mathbf{I}_2 - \mathbf{h})\mathbf{E}^-. \end{aligned} \tag{5.37}$$

Then, in terms of $\mathbf{E}^\pm$ and $\mathbf{D}^\pm$, (5.32) is equivalent to

$$\partial \cdot \mathbf{D}^\pm = 0, \qquad \partial^\perp \cdot \mathbf{E}^\pm = 0, \tag{5.38}$$

and the boundary conditions (5.17), (5.18) play a similar role to the mean field conditions.

Let us define

$$\begin{aligned} \mathbf{E} &= \tfrac{1}{2}(\mathbf{E}^+ + \mathbf{E}^-), \\ \mathbf{E}' &= \tfrac{1}{2}(\mathbf{E}^+ - \mathbf{E}^-), \\ \mathbf{D} &= \tfrac{1}{2}(\mathbf{D}^+ + \mathbf{D}^-), \\ \mathbf{D}' &= \tfrac{1}{2}(\mathbf{D}^+ - \mathbf{D}^-). \end{aligned} \tag{5.39}$$

They satisfy

$$\begin{aligned} \partial \cdot \mathbf{D}' &= \partial \cdot \mathbf{D} = 0, \\ \partial^\perp \cdot \mathbf{E}' &= \partial^\perp \cdot \mathbf{E} = 0 \end{aligned} \tag{5.40}$$

and

$$\begin{aligned} \mathbf{D}' &= \mathbf{I}_2\mathbf{E}' + \mathbf{h}\mathbf{E}, \\ \mathbf{D} &= \mathbf{I}_2\mathbf{E} + \mathbf{h}\mathbf{E}', \end{aligned} \tag{5.41}$$

or in matrix form

$$\begin{pmatrix} -\mathbf{D}' \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} -\mathbf{I}_2 & -\mathbf{h} \\ \mathbf{h} & \mathbf{I}_2 \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{E} \end{pmatrix}. \tag{5.42}$$

Let c^* denote the quantity of interest

$$
\begin{aligned}
c^* &\equiv \frac{1}{\pi^2}\int\int \left(\mathbf{I}_1\partial\rho^+\right)^2 \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde{t}\,d\tilde{s}\left(\frac{\partial}{\partial\tilde{s}}\rho^+\right)^2 \\
&= \langle \mathbf{I}_2\mathbf{E}^+\cdot\mathbf{E}^+\rangle \\
&= \tfrac{1}{2}\langle \mathbf{I}_2\mathbf{E}^+\cdot\mathbf{E}^+\rangle + \tfrac{1}{2}\langle \mathbf{I}_2\mathbf{E}^-\cdot\mathbf{E}^-\rangle \\
&= \tfrac{1}{2}\langle \mathbf{D}^+\cdot\mathbf{E}^+\rangle + \tfrac{1}{2}\langle \mathbf{D}^-\cdot\mathbf{E}^-\rangle \\
&= \tfrac{1}{2}\langle \mathbf{D}^+\cdot\mathbf{E}^-\rangle + \tfrac{1}{2}\langle \mathbf{D}^-\cdot\mathbf{E}^+\rangle,
\end{aligned} \tag{5.43}
$$

where

$$
\langle \mathbf{F}\cdot\mathbf{G}\rangle = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{F}\cdot\mathbf{G}\,d\tilde{t}\,d\tilde{s}, \tag{5.44}
$$

and $\mathbf{E}^+$, $\mathbf{E}^-$ satisfy the same direct boundary conditions. The last equality in (5.43) then follows from integration by parts, in view of (5.38). Representation (5.43) is equivalent to

$$
\begin{aligned}
c^* &= -\langle \mathbf{D}'\cdot\mathbf{E}'\rangle + \langle \mathbf{D}\cdot\mathbf{E}\rangle \\
&= \left\langle \begin{pmatrix} -\mathbf{I}_2 & -\mathbf{h} \\ \mathbf{h} & \mathbf{I}_2 \end{pmatrix}\begin{pmatrix}\mathbf{E}' \\ \mathbf{E}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{E}' \\ \mathbf{E}\end{pmatrix}\right\rangle,
\end{aligned} \tag{5.45}
$$

which is a symmetric, indefinite form. The constant c^* is given by the saddlepoint variational principle

$$
c^* = \inf_{\substack{\mathbf{F}=\partial f \\ f\in\mathcal{C}}}\ \sup_{\substack{\mathbf{F}'=\partial f' \\ f'\in\mathcal{C}_0}} \left\langle \begin{pmatrix} -\mathbf{I}_2 & -\mathbf{h} \\ \mathbf{h} & \mathbf{I}_2 \end{pmatrix}\begin{pmatrix}\mathbf{F}' \\ \mathbf{F}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{F}' \\ \mathbf{F}\end{pmatrix}\right\rangle, \tag{5.46}
$$

where $\mathcal{C}$ is the space of direct corner layer functions with the direct boundary conditions, and $\mathcal{C}_0$ is the space of direct corner layer functions that are the difference of functions in $\mathcal{C}$ and hence have null direct boundary conditions.

We eliminate the supremum by solving the corresponding Euler equation

$$
\partial\cdot\mathbf{I}_2\partial f' + \partial\cdot\mathbf{h}\,\partial f = 0. \tag{5.47}
$$

With (5.47) holding, (5.46) is equivalent to

$$
c^* = \inf_{\substack{\mathbf{F}=\partial f \\ f\in\mathcal{C}}} \left\{\langle \mathbf{I}_2\mathbf{F}'\cdot\mathbf{F}'\rangle + \langle \mathbf{I}_1\mathbf{F}\cdot\mathbf{F}\rangle\right\}. \tag{5.48}
$$

More explicitly, (5.47) is equivalent to

$$
\begin{aligned}
&\frac{\partial^2}{\partial\tilde{s}^2}f' = 0, && \text{for } |\tilde{s}| \le |\tilde{t}|^{1+\gamma}, \\
&\frac{\partial^2}{\partial\tilde{s}^2}f' = -u_o^2\left(\frac{\partial}{\partial\tilde{t}}f + (1+\gamma)\tilde{t}^{\gamma}\frac{\partial}{\partial\tilde{s}}f\right), && \text{for } |\tilde{s}| \ge |\tilde{t}|^{1+\gamma},
\end{aligned}
$$

with $f' \in \mathcal{C}_0$, which is (5.10). Thus the right-hand side of (5.48) is identical to the upper bound (5.15).

Now, let $\mathbf{E}^{\pm}$ be scaled by a factor of c^*, then, in view of the quadratic nature of (5.43), we have

$$(c^*)^{-1} = \langle \mathbf{I}_2\mathbf{E}^+ \cdot \mathbf{E}^+ \rangle = \tfrac{1}{2}\langle \mathbf{D}^+ \cdot \mathbf{E}^- \rangle + \tfrac{1}{2}\langle \mathbf{D}^- \cdot \mathbf{E}^+ \rangle, \tag{5.49}$$

where $\mathbf{D}^{\pm}$ are still related to $\mathbf{E}^{\pm}$ via (5.37). Representation (5.49) is equivalent to

$$\begin{aligned}(c^*)^{-1} &= -\langle \mathbf{D}' \cdot \mathbf{E}' \rangle + \langle \mathbf{D} \cdot \mathbf{E} \rangle \\ &= -\langle \mathbf{D}' \cdot \mathbf{E}' \rangle_{h\neq 0} + \langle \mathbf{D} \cdot \mathbf{E} \rangle_{h\neq 0} \\ &\quad -\langle \mathbf{D}' \cdot \mathbf{D}' \rangle_{h=0} + \langle \mathbf{D} \cdot \mathbf{D} \rangle_{h=0}\end{aligned} \tag{5.50}$$

from (5.41).

What boundary conditions do $\mathbf{D}'$ and $\mathbf{D}$ or, equivalently, $\mathbf{D}^{\pm}$, satisfy after the contraction? From (5.37), it follows that for $|\tilde{s}| \geq |\tilde{t}|^{1+\gamma}$

$$\begin{aligned} c^*\partial_{\tilde{s}}\phi^+ &= -h\,\partial_{\tilde{s}}\rho^+, \\ c^*\partial_{\tilde{t}}\phi^+ &= -h\,\partial_{\tilde{t}}\rho^+ + \partial_{\tilde{s}}\rho^+ \end{aligned} \tag{5.51}$$

and

$$\begin{aligned} c^*\,\partial_{\tilde{s}}\phi^- &= h\,\partial_{\tilde{s}}\rho^-, \\ c^*\,\partial_{\tilde{t}}\phi^- &= h\,\partial_{\tilde{t}}\rho^- + \partial_{\tilde{s}}\rho^- \end{aligned} \tag{5.52}$$

if $\rho^{\pm}$ satisfy the direct boundary conditions. The following equalities follow easily from (5.51) and the boundary condition (5.17):

$$\begin{aligned} \left[\phi^+\right]_{\tilde{\theta}=-\infty}^{\tilde{\theta}=\infty} &= \int_{h=h_o} (d\tilde{t}\,\partial_{\tilde{t}} + d\tilde{s}\,\partial_{\tilde{s}})\phi^+ \\ &= -\frac{1}{c^*}h_o\int_{h=h_o}^{\infty} (d\tilde{t}\,\partial_{\tilde{t}} + d\tilde{s}\,\partial_{\tilde{s}})\rho^+ + \frac{1}{c^*}\int_{h=h_o} d\tilde{t}\,\partial_{\tilde{s}}\,\rho^+ \\ &= \frac{1}{c^*}\int_{h=h_o} d\tilde{t}\partial_{\tilde{s}}\,\rho^+. \end{aligned} \tag{5.53}$$

On the other hand, from (5.16), we have

$$\begin{aligned} 0 &= -\int_{-\infty}^{\infty} d\tilde{\theta}\int_{h_o} dh\,\partial_{\tilde{\theta}}\,\rho^+ \\ &= -\int\int_{h\geq h_o} d\tilde{t}\,d\tilde{s}u_o^2\left(\frac{\partial}{\partial\tilde{t}}\rho^+ + (1+\gamma)\tilde{t}^{\gamma}\frac{\partial}{\partial\tilde{s}}\rho^+\right) \\ &= \int\int_{h\geq h_o} d\tilde{t}\,d\tilde{s}\frac{\partial^2}{\partial\tilde{s}^2}\rho^+ \\ &= \int_{h=h_o} d\tilde{t}\left[\partial_{\tilde{s}}\,\rho^+\right]_{h_o}^{\infty} \\ &= c^*\pi - \int_{h=h_o} d\tilde{t}\,\partial_{\tilde{s}}\,\rho^+, \end{aligned} \tag{5.54}$$

since

$$\int_{h=\infty} d\tilde{t}\,\frac{\partial}{\partial \tilde{s}}\rho^{+} = \pi\left(\frac{1}{\pi^{2}}\int_{h=\infty} d\tilde{t}\;\rho^{+}\frac{\partial}{\partial \tilde{s}}\rho^{+}\right) = c^{*}\pi,$$

following the definition of c^*, the boundary conditions and the energy identity of the direct corner layer problem. Therefore,

$$[\phi^{+}]_{\tilde{\theta}=-\infty}^{\tilde{\theta}=\infty} = \pi, \tag{5.55}$$

and the dual boundary conditions are satisfied for

$$\begin{aligned}[\phi^{+}]_{\tilde{s}_o}^{\tilde{s}} &= \int_{\tilde{s}_o}^{\tilde{s}} d\tilde{s}\;\partial_{\tilde{s}}\,\phi^{+}\\ &= -\frac{1}{c^{*}}\int_{\tilde{s}_o}^{\tilde{s}} h\partial_{\tilde{s}}\,\rho^{+}\\ &= -\frac{1}{c^{*}}u_o(\tilde{s}-\tilde{s}_o)\rho^{+} + \frac{u_o}{c^{*}}\int_{\tilde{s}_o}^{\tilde{s}} d\tilde{s}\;\rho^{+},\end{aligned} \tag{5.56}$$

which converges to zero as $\tilde{t}$ approaches infinity by the direct boundary conditions (5.17). The boundary conditions of ϕ^+ for $h<0$ can be similarly derived.

Let us invert relation (5.42) and express $\mathbf{E}'$ and $\mathbf{E}$ in terms of $\mathbf{D}'$ and $\mathbf{D}$

$$\begin{aligned}\mathbf{E}' &= \frac{1}{h^{2}}\mathbf{I}_1\mathbf{D}' - \frac{1}{h^{2}}\mathbf{h}\mathbf{D},\\ \mathbf{E} &= \frac{1}{h^{2}}\mathbf{I}_1\mathbf{D} - \frac{1}{h^{2}}\mathbf{h}\mathbf{D}',\end{aligned} \tag{5.57}$$

or, in matrix form,

$$\begin{pmatrix}-\mathbf{E}'\\ \mathbf{E}\end{pmatrix} = \begin{pmatrix}-\frac{1}{h^2}\mathbf{I}_1 & \frac{1}{h^2}\mathbf{h}\\ -\frac{1}{h^2}\mathbf{h} & \frac{1}{h^2}\mathbf{I}_1\end{pmatrix}\begin{pmatrix}\mathbf{D}'\\ \mathbf{D}\end{pmatrix}, \tag{5.58}$$

where $\mathbf{I}_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, and it is understood that when $|\tilde{s}| \le |\tilde{t}|^{1+\gamma}$, $h\equiv 0$ and

$$\mathbf{I}_2\mathbf{E}' = \mathbf{D}' = \mathbf{I}_2\mathbf{D}', \qquad \mathbf{I}_2\mathbf{E} = \mathbf{D} = \mathbf{I}_2\mathbf{D} \tag{5.59}$$

from (5.41). Again, (5.50) is a symmetric, indefinite form in view of (5.58) and (5.59) and admits a saddlepoint variational formulation

$$\begin{aligned}(c^{*})^{-1} = \inf_{\substack{\mathbf{G}=\partial^{\perp}g\\ g\in\mathcal{C}^{\perp}}}\ \sup_{\substack{\mathbf{G}'=\partial^{\perp}g'\\ g'\in\mathcal{C}_0^{\perp}}} &\left\{\left\langle\begin{pmatrix}-\frac{1}{h^2}\mathbf{I}_1 & \frac{1}{h^2}\mathbf{h}\\ -\frac{1}{h^2}\mathbf{h} & \frac{1}{h^2}\mathbf{I}_2\end{pmatrix}\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\right\rangle_{h\neq 0}\right.\\ &\left.+\left\langle\begin{pmatrix}-\mathbf{I}_1 & 0\\ 0 & \mathbf{I}_2\end{pmatrix}\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\right\rangle_{h=0}\right\}.\end{aligned} \tag{5.60}$$

Here $\mathcal{C}^{\perp}$ is the space of the dual corner layer functions with the dual boundary conditions and $\mathcal{C}_0^{\perp}$ is the space of the dual corner layer functions with null dual boundary conditions. We eliminate the supremum by solving the corresponding Euler equation

$$\begin{aligned}\partial^{\perp}\cdot\frac{1}{h^{2}}\mathbf{I}_1\mathbf{G}' - \partial^{\perp}\cdot\frac{1}{h^{2}}\mathbf{h}\mathbf{G} = 0 \quad &\text{for } |\tilde{s}| \ge |\tilde{t}|1+\gamma,\\ \partial^{\perp}\cdot\mathbf{I}_1\mathbf{G}' = 0 \quad &\text{for } |\tilde{s}| \le |\tilde{t}|^{1+\gamma}.\end{aligned} \tag{5.61}$$

Using (5.61) in (5.60), we obtain

$$(c^*)^{-1} = \inf_{\substack{\mathbf{G}=\nabla^\perp g \\ g\in C^\perp}} \left\{ \langle \mathbf{I}_2\mathbf{G}'\cdot\mathbf{G}'\rangle_{h=0} + \langle \mathbf{I}_2\mathbf{G}\cdot\mathbf{G}\rangle_{h=0} + \left\langle \frac{1}{h^2}\mathbf{I}_1\mathbf{G}'\cdot\mathbf{G}'\right\rangle_{h\neq 0} + \left\langle \frac{1}{h^2}\mathbf{I}_1\mathbf{G}\cdot\mathbf{G}\right\rangle_{h\neq 0} \right\}, \tag{5.62}$$

which is exactly the right-hand side of (5.29).

We have therefore identified c^* with the constant in Theorem 5.1.

5.4. Limiting cases. There are two interesting limiting cases in the corner layer problem. In one, $\gamma \downarrow 0$, and, in the other, $\gamma \uparrow \infty$. Note that $\frac{1}{2} < \frac{1}{2}(1+1/(1+2\gamma)) < 1$ for $\gamma > 0$ and $\lim_{\gamma\to\infty} \frac{1}{2}(1+1/(1+2\gamma)) = \frac{1}{2}$. The edge-contact situation of $H = \sin x \sin y$ can be thought of as point-contact with infinite degree of contact (i.e., $\gamma = \infty$), and the $\sqrt{\varepsilon}$ asymptotic behavior (but not the constant factor c^*, since the boundary conditions are different) is recovered in the limit $\gamma \uparrow \infty$.

The preceding analysis breaks down when $\gamma \downarrow 0$, as was noted before. The case where $\gamma = 0$ is the one in which two separatrices meet at the contact point, which is a stagnation point at a positive angle. Therefore, it requires a separate treatment. For simplicity, we assume that the separatrices meet at a positive angle $= \pi/2$ and that the flow near the corner is similar to that of cellular flows. As we see in the following analysis, in addition to ε, a $\log 1/\varepsilon$ factor appears. Contrary to what we might guess from previous analysis, the corner layer scaling involved here is $\sqrt{\varepsilon}$ and not $\varepsilon = \lim_{\gamma\downarrow 0} \varepsilon^{1/(1+2\gamma)}$. This is because of the presence of the stagnation point at the corner. As a result, it always takes order 1 time for a particle to pass around the corner, regardless of how short the traveling distance. The small molecular diffusivity ε then builds up a $\sqrt{\varepsilon}$ corner layer, which gives an order ε contribution to the effective diffusivity, while the region outside of the corner layer gives contribution of order $\varepsilon \log 1/\varepsilon$. These facts follow from the construction of suitable trial functions and the estimate of the variational principles. A similar argument also handles the case where the contact point is not a stagnation point, provided that we work with the corner layer of order ε, and the result is similar to the following theorem.

THEOREM 5.3. *If $\gamma = 0$, then there exist positive constants c_1^* and c_2^* such that*

$$c_1^* \varepsilon \log \frac{1}{\varepsilon} \le \sigma_\varepsilon \le c_2^* \varepsilon \log \frac{1}{\varepsilon}.$$

We have not been able to show that $c_1^* = c_2^*$ and determine this constant. The actual value of the angle is not important, since it affects only the constants. Although the tangent at the corner is no longer well defined, we still use t as the "tangential" coordinate whose axis is parallel to $(1,-1)$ and s as the "normal" coordinate whose axis is parallel to $(1,1)$ (see Fig. 5.5). We now turn to the proof of Theorem 5.3.

Upper bound. Consider trial functions f defined as in the direct corner layer functions $\mathcal{C}$ except that the corner layer scaling $\varepsilon^\alpha \times \varepsilon^{\alpha(1+\gamma)}$ is replaced by $\sqrt{\varepsilon} \times \sqrt{\varepsilon}$. We decompose the period cell into the regions C, E, and V as before. For the local term $\varepsilon\langle \nabla f \cdot \nabla f\rangle$ in the direct quadratic functional, it is easy to see that the corner layer region C gives a contribution only of order ε, while

$$\varepsilon\langle \nabla f \cdot \nabla f\rangle_V = O\left(\varepsilon \int_1^{\sqrt{\varepsilon}} \left(\frac{1}{t}\right)^2 t\, dt\right) = O\left(\varepsilon \log \frac{1}{\varepsilon}\right), \tag{5.63}$$

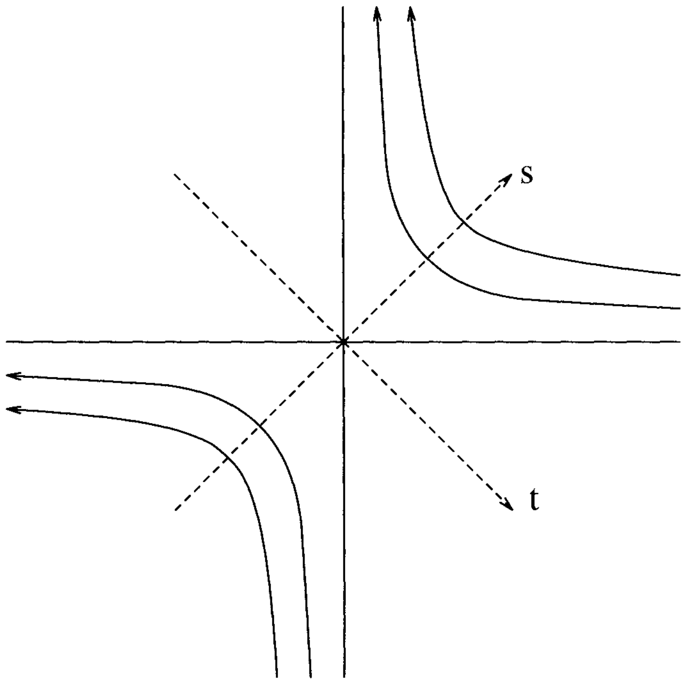

FIG. 5.5. *Limiting corner flow.*

since $\nabla f = O(1/t)$ and the area element is $t\,dt$. Thus

$$\varepsilon\langle\nabla f\cdot\nabla f\rangle \sim \varepsilon\langle\nabla f\cdot\nabla f\rangle_V = O\left(\varepsilon\log\frac{1}{\varepsilon}\right). \tag{5.64}$$

Next, we can estimate the nonlocal term $(1/\varepsilon)\langle\Gamma\mathbf{H}\,\nabla f\cdot\Gamma\mathbf{H}\,\nabla f\rangle$ in the following way. Let $(1/\varepsilon)\Gamma\mathbf{H}\,\nabla f = \nabla\tilde{f}$ for some periodic function $\tilde{f}$ or, equivalently,

$$\varepsilon\Delta\tilde{f} = \mathbf{u}\cdot\nabla f, \tag{5.65}$$

so that

$$\frac{1}{\varepsilon}\langle\Gamma\mathbf{H}\,\nabla f\cdot\Gamma\mathbf{H}\,\nabla f\rangle = \varepsilon\langle\nabla\tilde{f}\cdot\nabla\tilde{f}\rangle. \tag{5.66}$$

We claim that the right-hand side of (5.66) is of order ε. This is because of the scale invariance of the energy integral

$$\varepsilon\langle\nabla h\cdot\nabla h\rangle,$$

where h is an arbitrary nice function, and the convection operator is $\mathbf{u}\cdot\nabla$. More precisely, let us define scaled variables $\tilde{x}$ and $\tilde{y}$ in the corner layer by

$$x = \tilde{x}\sqrt{\varepsilon}\,, \qquad y = \tilde{y}\sqrt{\varepsilon}.$$

In terms of $\tilde{x}$ and $\tilde{y}$, (5.65) becomes

$$\tilde{\Delta}\tilde{f} \approx -\tilde{x}\frac{\partial}{\partial\tilde{x}}f + \tilde{y}\frac{\partial}{\partial\tilde{y}}f. \tag{5.67}$$

Therefore

$$\varepsilon\langle\nabla\tilde{f}\cdot\nabla\tilde{f}\rangle = \varepsilon\langle\tilde{\nabla}\tilde{f}\cdot\tilde{\nabla}\tilde{f}\rangle_{\sim} \le c\varepsilon\langle\tilde{\nabla}f\cdot\tilde{\nabla}f\rangle_{\sim} = O(\varepsilon), \tag{5.68}$$

where $\langle\cdot\rangle_{\sim} = \int\int_C(\cdot)d\tilde{x}\,d\tilde{y}$, and $\tilde{\nabla}$, $\tilde{\Delta}$ are the gradient and Laplacian with respect to $\tilde{x}$, $\tilde{y}$, respectively. From (5.64) and (5.68), we conclude that

$$\sigma_\varepsilon \le c_2^*\varepsilon\log\frac{1}{\varepsilon} \tag{5.69}$$

for some constant c_2^*.

Lower bound. Let us construct trial functions g in the following way. We define an arbitrary outer layer whose scale, say $\sqrt[4]{\varepsilon}$, is larger than that of the corner layer, which is $\sqrt{\varepsilon}$. We denote the outer region by U, the complementary region in the vacant cells by V, and the complementary region in the eddies by E. In the outer region U, let g satisfy the same far field boundary conditions in the definition of $\mathcal{C}^\perp$ and

$$g\big|_C = \frac{\pi}{2}, \qquad g\big|_U = g_\varepsilon(t). \tag{5.70}$$

In the eddies, g is a boundary layer function. From this, we know that the contribution of the eddies to the inverse variational principle is $O(1/\sqrt{\varepsilon})$. Now let us consider the contribution of vacant cells to the local term. We have

$$\frac{1}{\varepsilon}\left\langle\frac{1}{1+(1/\varepsilon^2)H^2}\nabla^\perp g\cdot\nabla^\perp g\right\rangle_V = \frac{1}{\varepsilon}\langle\nabla^\perp g\cdot\nabla^\perp g\rangle_V, \tag{5.71}$$

since $H = 0$ in the vacant cell. The right-hand side of (5.71) is, by the choice of g,

$$\frac{1}{\varepsilon}\int_{\sqrt{\varepsilon}}^{\sqrt[4]{\varepsilon}}(g_\varepsilon')^2\,t\,dt, \tag{5.72}$$

since $\nabla g = 0$ elsewhere and $t\,dt$ is the area element. The minimum of (5.72) can be achieved by g_ε that satisfies

$$(g_\varepsilon' t)' = 0 \quad \text{with the far field boundary conditions.} \tag{5.73}$$

The solution of (5.73) is

$$g_\varepsilon = \begin{cases} 2\pi\left(\dfrac{1}{2}-\dfrac{\log t}{\log\varepsilon}\right)+\dfrac{\pi}{2} & \text{when } \sqrt[4]{\varepsilon} > t > \sqrt{\varepsilon}, \\ -2\pi\left(\dfrac{1}{2}-\dfrac{\log|t|}{\log\varepsilon}\right)+\dfrac{\pi}{2} & \text{when } -\sqrt[4]{\varepsilon} < t < -\sqrt{\varepsilon}. \end{cases} \tag{5.74}$$

The energy integral for (5.74) is $O(1/\varepsilon\log(1/\varepsilon))$. Hence

$$(\sigma_\varepsilon)^{-1} \le \frac{1}{c_1^*}\cdot\frac{1}{\varepsilon\log(1/\varepsilon)}, \tag{5.75}$$

where c_1^* is a constant, and this with (5.69) proves Theorem 5.3.

6. Periodic arrays of eddies and channels. In this section, we study advection-diffusion in the steady velocity field

$$\mathbf{u} = (-H^\delta_y, H^\delta_x), \ H^\delta = \sin x \sin y + \delta \cos x \cos y, \ \delta > 0. \tag{6.1}$$

Here $\delta \cos x \cos y$ is a small periodic perturbation that preserves the structure of critical points of the stream function $\sin x \sin y$. The periodicity of the perturbation together with the instability of the separatrices creates periodic open channels in the vicinity of the separatrices of $\sin x \sin y$. The width of the channels is of order δ. The streamlines H^δ = constant form a periodic array of oblique cat's-eyes separated by open channels carrying finite fluid flux of order δ. Transport takes place both in thin boundary layers and within the channels, and the parameter $\delta/\sqrt{\varepsilon}$ measures the relative influence of the two. If $\delta = \beta\sqrt{\varepsilon}$ with $\beta \gg 1$, then advection in the channels dominates diffusion. This occurs when, for example, $\delta = \delta(\varepsilon) = a\varepsilon^\alpha$, $0 \le \alpha < \frac{1}{2}$, $a < 1$, so that $\beta = a\varepsilon^{\alpha-1/2} \overset{\varepsilon \downarrow 0}{\longrightarrow} \infty$.

The streamline structure is like that of Fig. 1.2. There are two types of streamlines: those in the channels

$$-\delta < H_\delta < \delta \tag{6.2}$$

and those in the eddies

$$\delta < |H_\delta| \le 1. \tag{6.3}$$

These streamlines are separated by separatrices defined by $H_\delta = \pm\delta$. The flow structure is no longer isotropic and has two eigendirections: one parallel to the channel, $\mathbf{e} = 1/\sqrt{2}(1,1)$, and the other, $\mathbf{e}_\perp = 1/\sqrt{2}(-1,1)$, orthogonal to the channel. Because of symmetry, the cell problem (4.1) can be reduced to one-quarter-period enclosed by the dotted lines in Fig. 6.1.

The behavior of the effective diffusivity (4.2) as ε tends to zero was first analyzed by Childress and Soward [5], who obtained asymptotic solutions for $\beta \gg 1$ using the Wiener–Hopf technique. Surprisingly, their asymptotic method gives reliable values of the effective diffusivity down to $\beta \approx 1.5$. Here we recover their results by our variational methods.

THEOREM 6.1 (Special cat's-eye). *For $H^\delta = \sin x \sin y + \delta \cos x \cos y$, $\sqrt{\varepsilon} \ll \delta \ll 1$, we have*

$$\sigma_\varepsilon(\mathbf{e}_\perp) \sim \varepsilon/\delta, \qquad \sigma_\varepsilon(\mathbf{e}) \sim \frac{\delta^3}{3\varepsilon} \quad \textit{as } \varepsilon \downarrow 0.$$

In particular, if $\delta = a\varepsilon^\alpha$, $0 \le \alpha < \frac{1}{2}$, we have

$$\sigma_\varepsilon(\mathbf{e}_\perp) \sim \frac{1}{a}\varepsilon^{1-\alpha}, \qquad \sigma_\varepsilon(\mathbf{e}) \sim \frac{a^3}{3}\varepsilon^{3\alpha-1}.$$

This theorem can be understood by a scaling argument in the following manner. The channels provide a very efficient vehicle in which a diffusing particle can take a long flight. The eddies are trapping regions, except in the $\sqrt{\varepsilon}$-boundary layer. In the $\mathbf{e}$ direction, the time the particle stays in one channel is $O(\beta^2)$, since this time is proportional to the reciprocal of the diffusion coefficient ε multiplied by the width of the channel squared, $(\sqrt{\varepsilon}\beta)^2$. The distance traveled in the direction $\mathbf{e}_\perp$ during this time is also $O(\beta^2)$. Therefore, the effective diffusivity should be $O(\beta^2)$ times the proportion of the time the particle spends in the channels, which is proportional to channel's width $\beta\sqrt{\varepsilon}$. It ends with a $O(\beta^3\sqrt{\varepsilon})$ effective diffusivity. In the $\mathbf{e}_\perp$ direction, the trapping of the eddies is active, while the channels do not help. Since $\beta \gg 1$, the boundary layers are essentially separated, the timescale involved is again $O(\beta^2)$, and the stepsize is $O(1)$ due to the boundary layers. The effective diffusivity is then $O(1/\beta^2)$ times the channel's width $\beta\sqrt{\varepsilon}$, which is $O(\sqrt{\varepsilon}/\beta)$.

In the following analysis, we take the limit $\varepsilon \downarrow 0$ first, keeping β fixed, and then we consider the asymptotics of $\beta \uparrow \infty$. In addition, (a) when passing to the limit $\varepsilon \downarrow 0$, with β fixed, different boundary layers overlap in the channel. The boundary layer type of trial functions used in the case of $H = \sin x \sin y$ are still appropriate, except that we must patch them in the channel region. This eventually gives $\sigma_\varepsilon(\mathbf{e})$, $\sigma_\varepsilon(\mathbf{e}_\perp) = O(\sqrt{\varepsilon})$.

(b) For the $\beta \uparrow \infty$ asymptotics, we must estimate the numerical constants $c^*, c^*_\perp$ multiplying $\sqrt{\varepsilon}$. As β gets larger, the channel region becomes dominant, and we are able to capture the dependence of c^* and $c^*_\perp$ on β.

We now continue with the analysis that leads to Theorem 6.1.

6.1. The asymptotic behavior of $\sigma_\varepsilon(\mathbf{e}_\perp)$. The upper bound for $\sigma_\varepsilon(\mathbf{e}_\perp)$ is obtained as follows. The boundary layer theory of eddies in §4 tells us that the trial functions f for the upper bound should be constant at least in the interior of each eddy. To specify our ansatz in the channel, let us first define in the channel

$$\begin{aligned}[f]_h(\theta) &= f(-\delta,\theta) - f(\delta,\theta),\\ [f]_\theta(h) &= \text{ the difference of } f \text{ along a streamline in half a period.}\end{aligned} \tag{6.4}$$

We consider a trial function f such that

$$[f]_h = \frac{\sqrt{2}}{2}\pi\,, \qquad [f]_\theta = 0 \quad \text{in the channel,} \tag{6.5}$$

and f assumes constant values in each eddy since we are concerned with the $\beta \gg 1$ limit. Condition (6.5) ensures that f satisfies the mean field condition $\langle \nabla f\rangle = \mathbf{e}$ as ε tends to zero. Thus

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e}_\perp)/\sqrt{\varepsilon} \le \inf_{\substack{[f]_h=(\sqrt{2}/2)\pi\\ [f]_\theta=0}} \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \left\{ \left(\frac{\partial}{\partial h} f\right)^2 + \left(\int_\infty^h \frac{\partial}{\partial \theta} f\right)^2 \right\}. \tag{6.6}$$

Since we are looking at the direction perpendicular to the channel, the diffusive energy integral should dominate, and the appropriate trial functions are $f = f(h)$. Set $h' = h/\beta$, $-1 \le h' \le 1$. Then we have

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e}_\perp)/\sqrt{\varepsilon} \le \inf_{\substack{[f]_h=(\sqrt{2}/2)\pi\\ \partial f/\partial\theta \equiv 0}} \frac{1}{\beta\pi^2}\int_{-2}^{2} d\theta \int_{-1}^{1} dh' \left(\frac{\partial}{\partial h'} f\right)^2. \tag{6.7}$$

The minimum in (6.7) is achieved by a linear function of h', $f = \frac{1}{2}\sqrt{2}/2\pi h'$, and the right side of (6.7) becomes

$$\frac{1}{\beta\pi^2}\left(\frac{1}{2}\frac{\sqrt{2}}{2}\pi\right)^2 2 \times 4 = \frac{1}{\beta}$$

after substitution.

The lower bound for $\sigma_\varepsilon(\mathbf{e}_\perp)$ is as follows. Let g be a boundary layer function and satisfy

$$[g]_h = 0 \quad \text{and} \quad [g]_\theta = \sqrt{2}\,\pi \quad \text{in the channel.} \tag{6.8}$$

Then (6.8) guarantees that g generates the correct mean field $\langle \nabla^\perp g \rangle = \mathbf{e}_\perp$ as ε tends to zero. After substitution, we have, to principal order as $\beta \downarrow \infty$,

$$\left(\overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e}_\perp)/\sqrt{\varepsilon} \right)^{-1} \leq \inf_{\substack{[g]_h=0\\ [g]_\theta=\sqrt{2}\pi\\ \text{in the channel}}} \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \left\{ \frac{1}{h^2}\left(\frac{\partial}{\partial h} g\right)^2 + h^2\left(\int_\infty^h \frac{1}{h^2}\frac{\partial}{\partial \theta} g\right)^2 \right\}. \tag{6.9}$$

Consider $g = g(\theta)$. The right side of (6.9) restricted to this particular class of trial functions can then be minimized by $g = (\sqrt{2}\pi/4)\theta$ in the channel; then it becomes

$$\frac{\beta}{\pi^2}\int_{-2}^{2} d\theta \int_{-1}^{1} dh' \left(\frac{\sqrt{2}\pi}{4}\right)^2 = \beta$$

after substitution. It does not matter how we choose g in the boundary layer since it only affects the $O(\beta)$ correction.

Combining the upper and lower bounds, we have

$$\lim_{\varepsilon\downarrow 0} \frac{\sigma_\varepsilon(\mathbf{e}_\perp)}{\sqrt{\varepsilon}} \sim \frac{1}{\beta} \quad \text{as } \beta \uparrow \infty.$$

6.2. The asymptotic behavior of $\sigma_\varepsilon(\mathbf{e})$. For the upper bound for $\sigma_\varepsilon(\mathbf{e})$ consider trial functions f, which are boundary layer functions in the eddies, that satisfy the matching condition on the separatrices

$$\int_0^\infty dh \, \frac{\partial}{\partial \theta} f = 0, \tag{6.10}$$

or equivalently

$$\int_0^\infty dh \; f = \text{constant independent of } \theta \tag{6.11}$$

and

$$[f]_h = 0\,, \qquad [f]_\theta = \sqrt{2}\,\pi \quad \text{in the channel.} \tag{6.12}$$

Like (6.8), (6.12) ensures that f generates the correct mean field $\langle \nabla f \rangle = \mathbf{e}$ in the limit $\varepsilon \downarrow 0$. As with (6.6), we have

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e})/\sqrt{\varepsilon} \leq \inf_{\substack{[f]_h=0\\ [f]_\theta=\sqrt{2}\pi}} \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \left\{ \left(\frac{\partial}{\partial h} f\right)^2 + \left(\int_\infty^h \frac{\partial}{\partial \theta} f\right)^2 \right\}. \tag{6.13}$$

We are looking at the direction parallel to the channel in the large β limit, so, clearly, the convective energy integral will dominate. Therefore, the appropriate trial functions should be in the form $f = f(\theta)$, which makes the the first term of (6.13), the diffusive energy integral, vanish, and we have

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e})/\sqrt{\varepsilon} \leq \inf_{\substack{[f]_h=0\\ [f]_\theta=\sqrt{2}\pi}} \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \left(\int_\infty^h \frac{\partial}{\partial \theta} f\right)^2. \tag{6.14}$$

The right side of (6.14) is minimized by a linear function in θ: $f = \frac{1}{4}\sqrt{2}\pi\theta$ in the channel. Then

$$\begin{aligned} \overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e})/\sqrt{\varepsilon} &\le \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \left(\frac{\sqrt{2}\pi}{4}\right)^2 h^2 \\ &= \frac{1}{3}\beta^3 \end{aligned} \tag{6.15}$$

to principal order as $\beta \uparrow \infty$.

For the lower bound for $\sigma_\varepsilon(\mathbf{e})$, consider the trial functions g, satisfying

$$[g]_h = \frac{\pi}{\sqrt{2}}, \qquad [g]_\theta = 0 \quad \text{in the channel} \tag{6.16}$$

so that $\langle \nabla^\perp g\rangle = \mathbf{e}$ in the limit. Consider $g = g(h)$, since we are looking at the perpendicular direction. The inverse variational principle becomes

$$\left(\overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e})/\sqrt{\varepsilon}\right)^{-1} \le \inf_{\substack{[g]_h=\pi/\sqrt{2}\\ [g]_\theta=0\\ \text{in the channel}}} \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \frac{1}{h^2}\left(\frac{\partial}{\partial h} g\right)^2. \tag{6.17}$$

The right side of (6.17) is minimized by $g = (\pi/\sqrt{2})\,(1/2\beta^3)\,h^3$ and, after substitution,

$$\begin{aligned} \overline{\lim_{\varepsilon\downarrow 0}}\, \sigma_\varepsilon(\mathbf{e})/\sqrt{\varepsilon} &= \frac{1}{\pi^2}\int_{-2}^{2} d\theta \int_{-\beta}^{\beta} dh \frac{1}{h^2}\left(\frac{\pi}{\sqrt{2}}\frac{1}{2\beta^3}\right)^2 (3h^2)^2 \\ &= \frac{3}{\beta^3} \end{aligned} \tag{6.18}$$

to principal order as $\beta \uparrow \infty$. Combining the upper and lower bounds (6.15), (6.18), we have

$$\lim_{\varepsilon\downarrow 0} \sigma_\varepsilon(\mathbf{e})/\sqrt{\varepsilon} \sim \frac{\beta^3}{3} \quad \text{as } \beta \uparrow \infty.$$

Clearly, the above analysis also works when $\varepsilon \downarrow 0$ and $\beta \uparrow \infty$ simultaneously, such as $\delta = a\varepsilon^\alpha, 0 \le \alpha < \frac{1}{2}$. The opposite asymptotic limit $\beta \downarrow 0$ corresponds to a channel perturbation of cellular boundary layers and can also be studied by variational methods. The leading term of σ_ε is $O(\sqrt{\varepsilon})$ and comes from the boundary layer theory. The next correction term is a power of β and depends on the direction. This problem has not been analyzed in detail.

7. General periodic flows with a zero mean drift. The stream function $H = \sin x \sin y$ is a Morse function (i.e., its critical points are not degenerate), but is not generic in the sense that it assumes the same value zero at the four saddlepoints. Generically, as a consequence of Morse's lemma (see Milnor [18]), we have the following theorem.

THEOREM 7.1 (Existence of channels). *Let H be a Morse function on the torus T^2 and $c_1, c_2, \ldots, c_n$ its saddlepoint values. If $c_i \ne c_j$, for $i \ne j$, then there exists some k's such that*

$H^{-1}(c_k - \delta, c_k)$: the collection of streamlines defined by $H =$ constant in $(c_k - \delta, c_k)$

or

$H^{-1}(c_k, c_k+\delta)$: *the collection of streamlines defined by* $H =$ *constant in* $(c_k, c_k+\delta)$

is an open channel regardless of how small δ *is.*

Theorem 7.1 is actually true for any compact two-surface without boundary except the two-sphere. It implies the existence of open channels for stream functions that are Morse functions and that have distinct saddlepoint values. We call such stream functions generic. In other words, channels always exist for generic stream functions. However, genericity is not a necessary condition for channels to exist. For example, the cat's-eye flow discussed in the previous section is not generic but nevertheless contains channels.

If channels do not exist, then the flow consists only of eddies and separatrices. Not every separatrix enhances particle diffusion. The important sets of separatrices are those that are not of the trivial homotopy type, equivalently, do not "separate" the torus. Any closed curve of the trivial homotopy type necessarily hits one of those nonseparating separatrices that form a web on the torus and induce boundary layers near them. In this case, our boundary layer theory developed in §4 can be applied to those nonseparating separatrices, and the effective diffusivity σ_ε is of order $\sqrt{\varepsilon}$, the constant factor can be calculated from the reduced variational principles in which the boundary conditions should, due to lack of symmetry, be replaced by matching conditions across the separatrix. This is all for the nongeneric case of no-channel flows.

Generically, channels exist. The channels are all periodic and are of the same homotopy type. In other words, all streamlines are periodic and have the same asymptotic slope or rotation number. Without loss of generality, we can assume that the rotation number is zero by making the following linear change of coordinates:

$$(x,y) \to (px+qy, rx+sy), \qquad \begin{vmatrix} p & q \\ r & s \end{vmatrix} = 1, \tag{7.1}$$

where p, q, r, s are integers and where q/p is the rotation number. After this transformation, the periodic channel structure resembles the one in Fig. 7.1.

We know from the cat's-eye flow analysis that, in the direction $\mathbf{e}$ parallel to the asymptotic slope, $\sigma_\varepsilon(\mathbf{e})$ is $O(1/\varepsilon)$, and, in the perpendicular direction $\mathbf{e}_\perp$, $\sigma_\varepsilon(\mathbf{e}_\perp)$ is $O(\varepsilon)$. The constant factor can also determined as was done in §6. In the special cat's-eye flow (see Fig. 1.2), two identical channels appear in a period cell, going in opposite directions, making the mean flow flux zero, while the rotation number is 1. In general, we have an even number $2n$ of channels, half of them going in one direction, say $(1,0)$, the others going in the opposite direction, $(-1,0)$. Let us first state a general two-channel result.

THEOREM 7.2 (Two-channel cat's-eye theory). *Let* δ *be the flow flux, equal to* $\frac{1}{2}[H]_\perp$ *with* $\sqrt{\varepsilon} \ll \delta \ll 1$. *Then*

$$\sigma_\varepsilon(\mathbf{e}) \sim c^* \frac{1}{\varepsilon}, \qquad \sigma_\varepsilon(\mathbf{e}_\perp) \sim c_\perp^* \varepsilon,$$

where

$$c^* = c_1 \frac{\delta^3}{\oint d\theta} = \frac{c_1}{2} \frac{[H^3]_\perp}{\oint d\theta}, \qquad c_\perp^* = c_2 \frac{\oint d\theta}{\delta} = 2c_2 \frac{\oint d\theta}{[H]_\perp}.$$

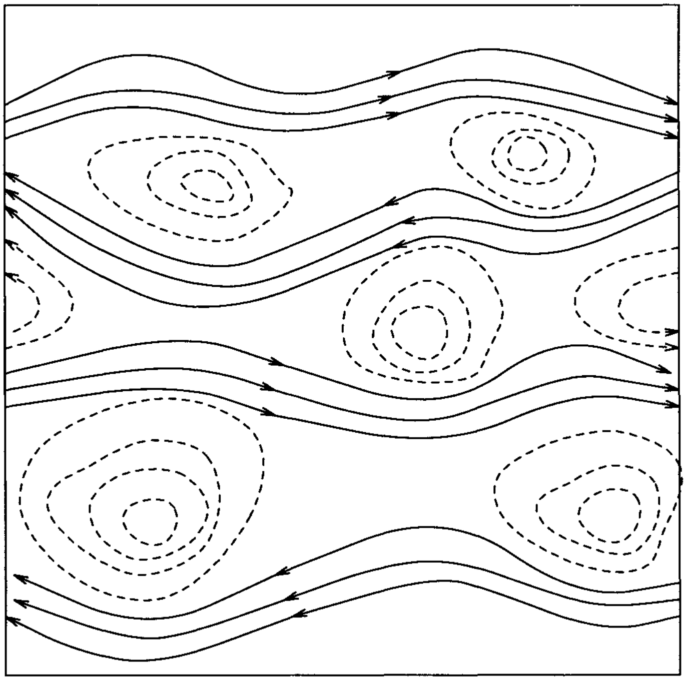

FIG. 7.1. *Multichannel flow.*

Here $[\cdot]_\perp$ *is the absolute difference of the function across the channel, and* c_1, c_2 *are constants independent of the flow structure;* $\oint d\theta$ *denotes flow circulation over a cycle in the channel.*

The proof of this statement is a slight modification of the theorem for cat's-eye flow in §6. The effect of the eddies can be seen by comparing this result with that for shear layer flows (see Fig. 7.2), which is considered next.

For shear layer flows, the effective diffusivity in either direction can be computed exactly using the inverse variational principle.

THEOREM 7.3 (Shear layer). *If* $\mathbf{u} = (u(y), 0)$, *then*

$$\sigma_\varepsilon(\mathbf{e}) = \varepsilon + \frac{1}{\varepsilon}\langle H^2\rangle, \qquad \mathbf{e} = (1,0)$$

and

$$\sigma_\varepsilon(\mathbf{e}_\perp) = \varepsilon, \qquad \mathbf{e}_\perp = (0,1).$$

Proof. From the inverse variational principle (3.74) for $\sigma_\varepsilon(\mathbf{e})$,

$$(7.2)\qquad \begin{aligned}(\sigma)_\varepsilon^{-1}(\mathbf{e}) = \inf_{\langle\nabla^\perp g\rangle = \mathbf{e}} \Bigg\{ &\frac{1}{\varepsilon}\left\langle \frac{1}{1+\frac{1}{\varepsilon^2}H^2}\nabla^\perp g\cdot\nabla^\perp g\right\rangle \\ &+ \frac{1}{\varepsilon^3}\left\langle\frac{1}{1+\frac{1}{\varepsilon^2}H^2}\Gamma^\perp_{\frac{1}{\varepsilon}H}\mathbf{H}\nabla^\perp g\cdot\Gamma^\perp_{\frac{1}{\varepsilon}H}\mathbf{H}\nabla^\perp g\right\rangle\Bigg\}.\end{aligned}$$

We obtain the Euler equation

$$(7.3)\qquad \nabla^\perp\cdot\left(\frac{1}{1+\frac{1}{\varepsilon^2}H^2} - \frac{1}{\varepsilon^2}\mathbf{H}\Gamma^\perp_{\frac{1}{\varepsilon}H}\mathbf{H}\right)\nabla^\perp g = 0,$$

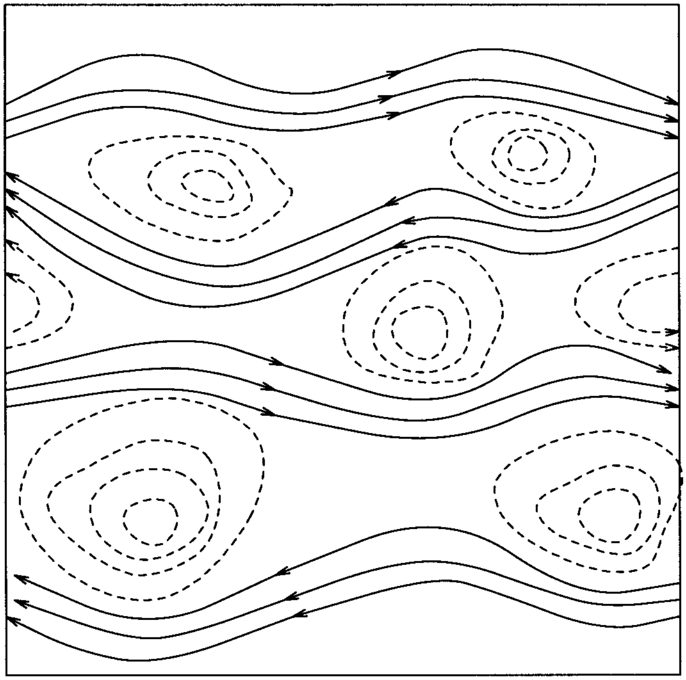

FIG. 7.2. *Two-channel shear layer flow.*

which can be solved exactly with a function $g = g(y)$. Equation (7.3) reduces to

$$\nabla^{\perp} \cdot \left(\frac{1}{1 + \frac{1}{\varepsilon^2} H^2} \right) \nabla^{\perp} g = 0, \tag{7.4}$$

and the second term in (7.3) simply drops out. Equation (7.4) is easily solved by taking

$$\nabla^{\perp} g = (-g', 0) = \left(\frac{1}{1 + \frac{1}{\varepsilon^2}\langle H^2 \rangle} \left(1 + \frac{1}{\varepsilon^2} H^2 \right), 0 \right), \tag{7.5}$$

which satisfies the mean field condition $\langle \nabla^{\perp} g \rangle = \mathbf{e}$. Substituting (7.5) into (7.2), we have

$$(\sigma)_{\varepsilon}^{-1}(\mathbf{e}) = \frac{1}{\varepsilon + \frac{1}{\varepsilon}\langle H^2 \rangle}$$

or

$$\sigma_{\varepsilon}(\mathbf{e}) = \varepsilon + \frac{1}{\varepsilon}\langle H^2 \rangle. \tag{7.6}$$

It is also easy to see that $\sigma_{\varepsilon}(\mathbf{e}_{\perp}) = \varepsilon$. In particular, for two-channel shear layer flow (see Fig. 7.2),

$$\sigma_{\varepsilon}(\mathbf{e}) \sim \varepsilon + \frac{\delta^2}{\varepsilon}, \qquad \sigma_{\varepsilon}(\mathbf{e}_{\perp}) = \varepsilon. \tag{7.7}$$

Thus, in view of Theorems 7.2 and 7.3, we conclude that the effect of eddies in open channel flows is to enhance σ_{ε} in the perpendicular direction by a factor δ^{-1}

and to diminish σ_ε in the parallel direction by a factor δ. Let us also state a general multichannel cat's-eye result.

THEOREM 7.4 (Multichannel cat's-eye theory). *If $2n$ periodic channels exist and their contributions to σ_ε, as in the two-channel theory, are $c^*(i), c^*_\perp(i), i = 1, \ldots, 2n$, then*

$$\sigma_\varepsilon(\mathbf{e}) \sim \frac{c^*}{\varepsilon}, \qquad \sigma_\varepsilon(\mathbf{e}_\perp) \sim c^*_\perp \varepsilon,$$

where c^* is the arithmetic mean of $c^*(i)$ and $c^*_\perp$ is the harmonic mean of $c^*_\perp(i), i = 1, \ldots, 2n$.

This result is analagous to what happens in conductivity problems. The proof of Theorem 7.4 is an extension of the argument given in the theorem for cat's-eye flows.

For shear layer flows, $\sigma_\varepsilon(\mathbf{e}_\perp) = \varepsilon$ and $\langle H^2 \rangle$ in formula (7.6) for $\sigma_\varepsilon(\mathbf{e})$ accounts for its enhancement, which increases with the correlation of flow directions in adjacent channels, since particles can take bigger flights. However, the flow direction must alternate from channel to adjacent channel to sandwich eddies between them, while maintaining the consistency of the flow structure. The total effect of multichannels cat's-eye flows on $\sigma_\varepsilon(\mathbf{e})$ is simply the sum of that of individual channel contribution.

8. Periodic flows with a nonzero mean drift. What happens if the mean drift is not zero? In this section, we consider particle dispersion in periodic flows with nonzero mean drifts. Such problems arise in the diffusion of contaminants in saturated porous media (e.g., see [9]) and in the diffusion of particles deposited as sediment in convective flows, which is treated in [6] for small mean drifts using boundary layer techniques. Bhattacharya, Gupta, and Walker [9] analyze the case with mean drifts that are not small, as do Majda and McLaughlin [25]. Bhattacharya et al. [26] make several observations, which are essentially Lemmas 8.2 and 8.3, below, and then apply them to a class of simple flows to obtain extremal diffusivity, that is, $\sigma_\varepsilon = O(\varepsilon)$ or $O(1/\varepsilon)$. We reformulate their observations and apply them to general periodic flows with nonzero mean drifts. Variational methods for flows with a nonzero mean drift are a special case of the variational principles for time-dependent flows that are discussed in Appendix B. Hou and Xin [19] and Weinan [20] study the homogenization of the advective transport equations without diffusion under the hyperbolic scaling, and they obtain various effective equations, depending on the rotation number, ergodicity, and the stagnation points of the flows. It is interesting to compare their results to the ones we obtain in this section under the diffusive scaling with vanishing diffusion.

We write the flows in the form $\mathbf{c} + \mathbf{u}$, where $\mathbf{c}$ is a constant vector and $\langle \mathbf{u} \rangle = 0$. As before, $\mathbf{u}$ is an incompressible, $\nabla \cdot \mathbf{u} = 0$, periodic vector field of period 2π in two dimensions, and we assume that it is smooth: $\mathbf{u} \in C^r(\mathbf{T}^2)$, $r \geq 0$. According to a generalization of the classical theorem of Poincare by Moser and described in [20], when stagnation points occur, we have that (i) the asymptotic direction of the streamlines is parallel to $\mathbf{c}$, and (ii) when considered on the plane $\mathbf{R}^2$, let the set of closed streamlines be the eddies and the rest the channels, then c_1 and c_2, the components of $\mathbf{c}$, are commensurate if and only if the flow has a periodic streamline in the channels, when embedded in the torus $\mathbf{T}^2$. When c_1 and c_2 are incommensurate, any single streamline starting from inside the channels is dense in the channels.

It follows that the rotation number is defined in the channels and is independent of the streamlines. The behavior of the streamlines in the channels is completely characterized by $\mathbf{c}$, as long as we know the structure of the channels or equivalently the structure of the eddies. Furthermore, we can decompose $\mathbf{T}^2$ into the sum of invariant sets: $\mathbf{T}^2 = \sum_{i=1}^N U_i$, ($N$ might be ∞) such that $\mathbf{c}+\mathbf{u}$ restricted to U_i is either completely integrable or ergodic, for all $i = 1, \ldots, N$. An invariant region U_i is an ergodic region only when it is a channel and the rotation number is irrational. Complete integrability means that the circulation variable θ exists and that (H, θ) form a coordinate system.

The cell problem is

$$\varepsilon\Delta\chi + (\mathbf{c}+\mathbf{u})\cdot\nabla\chi + \mathbf{u}\cdot\mathbf{e} = 0, \tag{8.1}$$

and the effective diffusivity is given by

$$\sigma_\varepsilon(\mathbf{e}) = \varepsilon + \varepsilon\langle\nabla\chi\cdot\nabla\chi\rangle. \tag{8.2}$$

We rewrite the cell problem (8.1) in the form

$$\nabla\cdot(\varepsilon + \mathbf{H} + \Delta^{-1}\mathbf{c}\cdot\nabla)\nabla\chi + \mathbf{u}\cdot e = 0, \tag{8.3}$$

where

$$\mathbf{H} = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}$$

and where H is the stream function with $\langle H\rangle = 0$, and $\nabla^\perp H = \mathbf{u}$. In terms of the projection operator Γ and with $\tilde{\mathbf{E}} = \nabla\chi$, we have

$$\varepsilon\tilde{\mathbf{E}} + \Gamma\mathbf{H}\Gamma\tilde{\mathbf{E}} + \Gamma\Delta^{-1}\mathbf{c}\cdot\nabla\tilde{\mathbf{E}} + \Gamma\mathbf{H}\cdot\mathbf{e} = 0 \tag{8.4}$$

and

$$\sigma_\varepsilon(\mathbf{e}) = \varepsilon + \varepsilon\langle\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}\rangle. \tag{8.5}$$

8.1. A decomposition of the Hilbert space and its applications. Let

$$G = \Gamma\mathbf{H}\Gamma + \Gamma\Delta^{-1}\mathbf{c}\cdot\nabla$$

and denote by $\mathcal{H}_g$ the Hilbert space of mean-zero curl-free fields with $\langle\cdot\rangle$ as inner product. Then G: $\mathcal{H}_g \to \mathcal{H}_g$, is bounded and skew adjoint. Furthermore, we have the following lemma.

LEMMA 8.1. *G is a compact, skew-adjoint operator.*

Proof. For $\mathbf{F} \in \mathcal{H}_g$,

$$\begin{aligned} G\mathbf{F} &= \Gamma\mathbf{H}\Gamma\mathbf{F} + \Gamma\Delta^{-1}\mathbf{c}\cdot\nabla\mathbf{F} \\ &= \nabla\Delta^{-1}\mathbf{u}\cdot\mathbf{F} + \Gamma\Delta^{-1}\mathbf{c}\cdot\nabla\mathbf{F}. \end{aligned} \tag{8.6}$$

Since one derivative is gained by applying G, it is compact.

Denote the nullspace of G in $\mathcal{H}_g$ by $\mathcal{N}$. Then the Hilbert space $\mathcal{H}_g$ has the decomposition

$$\mathcal{H}_g = \mathcal{N} \oplus \mathcal{N}^\perp,$$

where $\mathcal{N}^\perp = \overline{(\text{Range}\, G)}$. The effective diffusivity $\sigma_\varepsilon(\mathbf{e})$ can now be expressed as

$$\sigma_\varepsilon(\mathbf{e}) = \varepsilon + \varepsilon\langle(\nabla\chi)_{\mathcal{N}}\cdot(\nabla\chi)_{\mathcal{N}}\rangle + \varepsilon\langle(\nabla\chi)_{\mathcal{N}^\perp}\cdot(\nabla\chi)_{\mathcal{N}^\perp}\rangle. \tag{8.7}$$

LEMMA 8.2 (Bhattacharya, Gupta, and Walker [9]). *If* $\Gamma\mathbf{H}\cdot\mathbf{e}$ *has a nonzero component in* $\mathcal{N}$, *then*

$$\frac{c'}{\varepsilon} \le \sigma_\varepsilon(\mathbf{e}) \le \frac{c''}{\varepsilon} \quad \textit{as } \varepsilon \downarrow 0$$

for some positive numbers c' *and* c''.

Proof. Equation (8.4) can be decomposed into components in $\mathcal{N}$ and $\mathcal{N}^\perp$,

$$\begin{aligned} \varepsilon\tilde{\mathbf{E}}_{\mathcal{N}^\perp} + G\tilde{\mathbf{E}}_{\mathcal{N}^\perp} + (\Gamma\mathbf{H}\cdot\mathbf{e})_{\mathcal{N}^\perp} &= 0, \\ \varepsilon\tilde{\mathbf{E}}_{\mathcal{N}} + (\Gamma\mathbf{H}\cdot\mathbf{e})_{\mathcal{N}} &= 0. \end{aligned} \tag{8.8}$$

If $(\Gamma\mathbf{H}\cdot\mathbf{e})_{\mathcal{N}} \ne 0$, then $\langle\tilde{\mathbf{E}}_{\mathcal{N}}\cdot\tilde{\mathbf{E}}_{\mathcal{N}}\rangle \sim 1/\varepsilon^2$, and $\sigma_\varepsilon(\mathbf{e}) \ge c'/\varepsilon$, for some $c' > 0$. However, from the variational principle, we know that $\sigma_\varepsilon(\mathbf{e}) \le c''/\varepsilon$, for some $c'' > 0$. This completes the proof.

The following lemma tells us when singular perturbations do not arise.

LEMMA 8.3 (Bhattacharya, Gupta, and Walker [9]). *If* $\Gamma\mathbf{H}\cdot\mathbf{e} \in \operatorname{Range} G$, *that is, there exists* $\tilde{\mathbf{F}}$ *in* $\mathcal{H}_g$ *such that* $G\tilde{\mathbf{F}} = \Gamma\mathbf{H}\cdot\mathbf{e}$, *then*

$$\varepsilon \le \sigma_\varepsilon(\mathbf{e}) \le c\varepsilon \quad \textit{as } \varepsilon \downarrow 0$$

for some $c > 1$.

Proof. The direct variational principle for σ_ε is a special case of the one for time-dependent flows with $\partial/\partial t$ replaced by $-\mathbf{c}\cdot\nabla$ (cf. Appendix B), that is,

$$\sigma_\varepsilon(\mathbf{e}) = \inf_{\substack{\nabla\times\mathbf{F}=0 \\ \langle\mathbf{F}\rangle=\mathbf{e}}} \left\{\varepsilon\langle\mathbf{F}\cdot\mathbf{F}\rangle + \frac{1}{\varepsilon}\langle\Gamma\mathbf{H}'\mathbf{F}\cdot\Gamma\mathbf{H}'\mathbf{F}\rangle\right\}, \tag{8.9}$$

where

$$\mathbf{H}' = \mathbf{H} + \Delta^{-1}\mathbf{c}\cdot\nabla. \tag{8.10}$$

We first show that $\Gamma\mathbf{H}\cdot\mathbf{e} \in \operatorname{Range} G$ is equivalent to the existence of $\mathbf{F}$ such that $\langle\Gamma\mathbf{H}'\mathbf{F}\cdot\Gamma\mathbf{H}'\mathbf{F}\rangle = 0$, which is equivalent to

$$\begin{aligned} \nabla\cdot\Gamma\mathbf{H}'\mathbf{F} &= \nabla\cdot\Gamma(\mathbf{H} + \Delta^{-1}\mathbf{c}\cdot\nabla)\mathbf{F} \\ &= \mathbf{u}\cdot\mathbf{F} + \mathbf{c}\cdot\tilde{\mathbf{F}} = 0, \end{aligned} \tag{8.11}$$

where $\tilde{\mathbf{F}} = \mathbf{F} - \mathbf{e}$, or where

$$(\mathbf{c}+\mathbf{u})\cdot\tilde{\mathbf{F}} + \mathbf{u}\cdot\mathbf{e} = 0\ . \tag{8.12}$$

However, $\Gamma\mathbf{H}\cdot\mathbf{e} \in \operatorname{Range} G \Leftrightarrow$ there exists $\tilde{\mathbf{F}} \in \mathcal{H}_g$ such that

$$-\Gamma\mathbf{H}\cdot\mathbf{e} = \nabla\Delta^{-1}\mathbf{u}\cdot\tilde{\mathbf{F}} + \Gamma\Delta^{-1}\mathbf{c}\cdot\nabla\tilde{\mathbf{F}} \tag{8.13}$$

or

$$-\mathbf{u}\cdot\mathbf{e} = \mathbf{u}\cdot\tilde{\mathbf{F}} + \mathbf{c}\cdot\tilde{\mathbf{F}} = (\mathbf{c}+\mathbf{u})\cdot\tilde{\mathbf{F}}, \tag{8.14}$$

which is (8.12). Therefore, the nonlocal term in (8.9) vanishes and

$$\sigma_\varepsilon(\mathbf{e}) \le c\varepsilon \quad \text{for some } c > 0 \quad \text{as } \varepsilon \downarrow 0. \tag{8.15}$$

Since $\sigma_\varepsilon(\mathbf{e}) \ge \varepsilon$, (8.15) leads to the conclusion of the lemma.

The converses of Lemmas 8.2 and 8.3 also hold.

LEMMA 8.4. *If* $\Gamma\mathbf{H}\cdot\mathbf{e}$ *does not have a component in* $\mathcal{N}$*, then*

$$\sigma_\varepsilon(\mathbf{e}) = O(1/\varepsilon)\,.$$

Proof. By the assumption, $\Gamma\mathbf{H}\cdot\mathbf{e}\in\mathcal{N}^\perp = \overline{\operatorname{Range} G}$. Moreover, there exists $\mathbf{F}$, with $\nabla\times\mathbf{F}=0$, $\langle\mathbf{F}\rangle=\mathbf{e}$, $\langle\mathbf{F}\cdot\mathbf{F}\rangle<\infty$, such that, for arbitrarily small δ, the nonlocal term in (8.9) is

$$\frac{1}{\varepsilon}\langle\Gamma\mathbf{H}'\mathbf{F}\cdot\Gamma\mathbf{H}'\mathbf{F}\rangle\le\delta,$$

and hence the conclusion.

LEMMA 8.5. *If* $\Gamma\mathbf{H}\cdot\mathbf{e}$ *is not in* Range G*, then*

$$\sigma_\varepsilon(\mathbf{e})\gg\varepsilon\quad \textit{as } \varepsilon\downarrow 0.$$

Proof. Let us assume that $\Gamma\mathbf{H}\cdot\mathbf{e}$ does not have a component in $\mathcal{N}$; otherwise Lemma 8.2 applies, and the conclusion is obviously true. From Lemma 8.4, it follows that, to avoid $1/\varepsilon$ behavior, in (8.8),

$$G\tilde{\mathbf{E}}_{\mathcal{N}^\perp}+(\Gamma\mathbf{H}\cdot\mathbf{e})_{\mathcal{N}^\perp}\downarrow 0,$$

as $\varepsilon\downarrow 0$. By the assumption of the lemma and the compactness of G, $\langle\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}\rangle$ is not bounded as $\varepsilon\downarrow 0$, and hence the conclusion of the lemma.

The gap between Lemmas 8.4 and 8.5 is when $\Gamma\mathbf{H}\cdot\mathbf{e}\in\overline{\operatorname{Range} G}$ but not in $\operatorname{Range} G$. In this case, $\varepsilon\ll\sigma_\varepsilon(\mathbf{e})\ll 1/\varepsilon$. If this occurs when $\mathbf{c}=0$, then various boundary layers and corner layers arise, and their effects on the effective diffusivity have been discussed in previous sections. It is shown in the following sections that the flow is rarely in this gap when $\mathbf{c}$ is not zero.

8.2. A characterization of $\mathcal{N}$ and $\mathcal{N}^\perp$. Each $\mathbf{F}\in\mathcal{H}_g$ can be written as $\mathbf{F}=\nabla f$ for some periodic function or the limit of a sequence of such gradients. Furthermore,

$$\begin{aligned}\mathbf{F}\in\mathcal{N}\ &\Leftrightarrow \Gamma\mathbf{H}\nabla f+\Gamma\Delta^{-1}c\cdot\nabla\nabla f=0\\ &\Leftrightarrow(\mathbf{u}+\mathbf{c})\cdot\nabla f=0\end{aligned}\tag{8.16}$$

or, equivalently, f is constant along every streamline of $\mathbf{c}+\mathbf{u}$, and $\mathcal{N}$ is the closure of the set of fields that is the gradient of such functions. Let us state this as a lemma.

LEMMA 8.6. $\mathcal{N}=\overline{\{\nabla f\mid f \textit{ is constant along every streamline of } \mathbf{c}+\mathbf{u}\}}$.

The main result of this section is a characterization of $\mathcal{N}^\perp$.

LEMMA 8.7. *We have*

$$\mathcal{N}^\perp=\overline{\{\nabla g\mid \textstyle\int_\gamma\Delta g\,dt=0,\ \textit{for every nonergodic streamline } \gamma \textit{ in every region } U_i.\}},$$

where t *is the time associated with the streamline* γ *under the flow* $\mathbf{c}+\mathbf{u}$.

Proof. It suffices to consider $\mathbf{E}\in\mathcal{N}^\perp$, $\mathbf{F}\in\mathcal{N}$ of the form $\mathbf{E}=\nabla g$, $\mathbf{F}=\nabla f$, for some smooth g and f. Then

$$\begin{aligned}0=\langle\mathbf{E}\cdot\mathbf{F}\rangle&=\int\!\!\int_{\mathbf{T}^2}dx\,dy\,\nabla f\cdot\nabla g\\ &=-\int\!\!\int_{\mathbf{T}^2}dx\,dy\,f\,\Delta g.\end{aligned}\tag{8.17}$$

If γ is a nonergodic streamline, then consider a sequence of $f_n = f_n(J)$, where J is an action variable (i.e., $\nabla J \in \mathcal{N}$) that is defined in a neighborhood of γ, such that

$$f_n \overset{n\uparrow\infty}{\to} \delta_{J_0}(J),$$

the Dirac delta function concentrated on J_0, and $J_0 = J$ defines γ. We have

$$-\int\int_{\mathbf{T}^2} dx\, dy\, f_n\, \Delta g \overset{n\uparrow\infty}{\to} -c\int_\gamma dt\, \Delta g, \tag{8.18}$$

where c equals $\frac{\partial(x,y)}{\partial(J,t)}\big|_\gamma$, which is constant on streamlines, since both $dx\, dy$ and $dJ\, dt$ are invariant for the flow. Thus

$$\int_\gamma dt\, \Delta g = 0\ . \tag{8.19}$$

On the other hand, if $\gamma \in U_i$ in which $\mathbf{c} + \mathbf{u}$ is ergodic, then it does not matter what we choose for $g\big|_{U_i}$. This completes the proof.

Actually, $\int_\gamma dt\, \Delta g = 0$ for every ∇g in the range of G and every nonergodic streamline γ, since

$$\int_\gamma dt\, \nabla \cdot \Gamma\mathbf{H}\Gamma\nabla f \tag{8.20}$$
$$= \int_\gamma dt\, (\mathbf{c}+\mathbf{u})\cdot\nabla f$$
$$= \int_\gamma ds \frac{\partial}{\partial s} f \tag{8.21}$$
$$= 0.$$

8.3. Flows with stagnation points. We analyze the behavior of the effective diffusivity when $\mathbf{c}+\mathbf{u}$ has stagnation points (see Figs. 8.1–8.3).

First, we establish the following general result.

THEOREM 8.8. *If the flow* $\mathbf{c}+\mathbf{u}$ *has periodic orbits of the trivial homotopic type, then*

$$\frac{c'}{\varepsilon} \le \sigma_\varepsilon(\mathbf{e}) \le \frac{c''}{\varepsilon} \quad \text{as } \varepsilon \downarrow 0 \quad \text{when } \mathbf{e} \not\perp \mathbf{c}$$

for some positive numbers c' *and* c''.

Proof. In view of Lemma 8.2, it suffices to prove $\Gamma\mathbf{H}\cdot\mathbf{e} \notin \mathcal{N}^\perp$. Let γ be one of the periodic orbits of the trivial homotopic type. Obviously, $\int_\gamma dt(\mathbf{c}+\mathbf{u}) = 0$, since this integral is the displacement after a cycle. Now consider

$$\int_\gamma \Delta(\Delta^{-1}\nabla\cdot\mathbf{H}\cdot\mathbf{e}) = \int_\gamma \Delta(\Delta^{-1}(\mathbf{u}\cdot\mathbf{e}))$$
$$= \int_\gamma \mathbf{u}\cdot\mathbf{e} \tag{8.22}$$
$$= -\int_\gamma dt\, \mathbf{c}\cdot\mathbf{e}$$
$$\neq 0$$

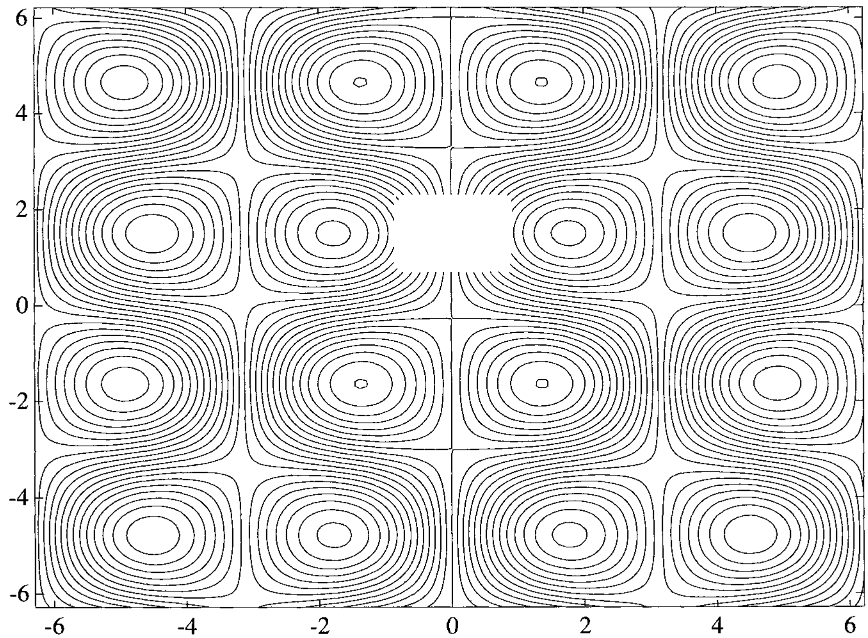

FIG. 8.1. *Cellular flow with drift* $\mathbf{c} = (0, .2)$.

if $\mathbf{c}$ is not periodicular to $\mathbf{e}$. Thus

$$\nabla\Delta^{-1}\nabla\cdot\mathbf{H}\cdot\mathbf{e} = \Gamma\mathbf{H}\cdot\mathbf{e} \notin \mathcal{N}^{\perp} \tag{8.23}$$

by Lemma 8.7. This completes the proof.

The condition in Theorem 8.8 seems to be generic whenever $\mathbf{c}+\mathbf{u}$ has stagnation points (see Figs. 8.1–8.3). For example, if some of those stagnation points are elliptic points, then there are always periodic orbits of trivial homotopic type around those elliptic stagnation points. Theorem 8.8 can be generalized to higher-dimensional spaces.

THEOREM 8.9. *Let* $\mathbf{c}+\mathbf{u} \in C^{\infty}(T^n)$. *If there exists a bounded domain* D *invariant under* $\mathbf{c}+\mathbf{u}$ *viewed as dynamical system on* $\mathbb{R}^n$, *then*

$$\frac{c'}{\varepsilon} \le \sigma_{\varepsilon}(\mathbf{e}) \le \frac{c''}{\varepsilon} \quad \text{as } \varepsilon \downarrow 0 \quad \text{if } \mathbf{e} \not\perp \mathbf{c}$$

for some positive numbers c' *and* c''.

Proof. Let $M = \int_D d^n\mathbf{x}$ be the "mass" of the fluid volume D. It is finite, since D is bounded. Define the center of mass for D by

$$\mathbf{q}_D(t) = \int_D d^n\mathbf{x}\,\mathbf{X}(t,\mathbf{x})/M, \tag{8.24}$$

where $\mathbf{X}(t,\mathbf{x})$ is the flow generated by $\mathbf{c}+\mathbf{u}$ and $\mathbf{X}(0,\mathbf{x}) = \mathbf{x}$. The invariance of D and incompressibility of $\mathbf{c}+\mathbf{u}$ shows us that

$$\frac{d}{dt}\mathbf{q}_D(t) = 0 \quad \forall\, t. \tag{8.25}$$

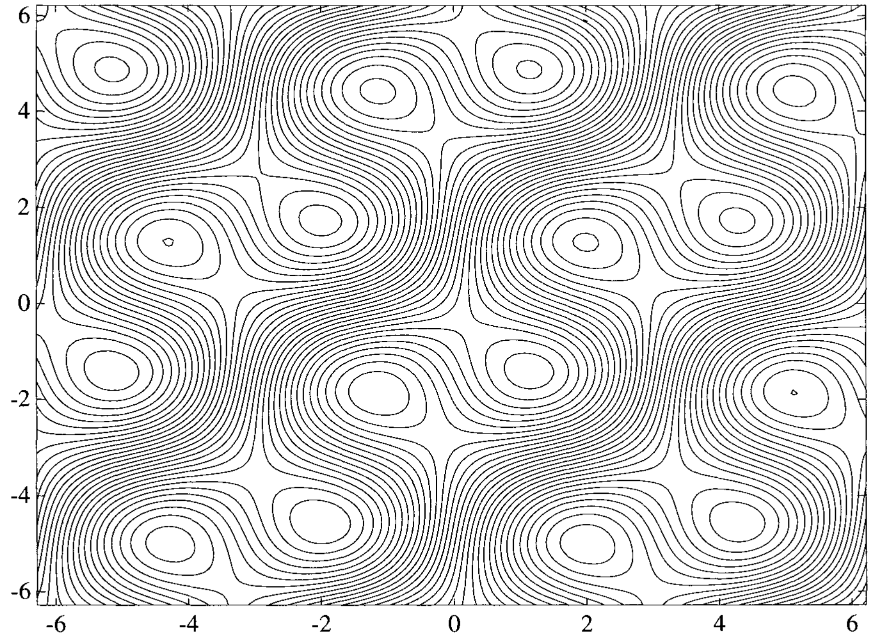

FIG. 8.2. *Cellular flow with drift* $\mathbf{c} = 0.2(1,2)$.

However,

$$\left.\frac{d}{dt}\right|_{t=0} \mathbf{q}_D(t) = \frac{1}{M}\int_D d^n\mathbf{x}(\mathbf{c}+\mathbf{u}(\mathbf{x})) = 0, \tag{8.26}$$

and thus

$$\int_D d^n\mathbf{x}\mathbf{u}\cdot\mathbf{e} = -\left(\int_D d^n\mathbf{x}\right)\mathbf{c}\cdot\mathbf{e} = -M\mathbf{c}\cdot\mathbf{e}. \tag{8.27}$$

On the other hand, $\int_D d^n\mathbf{x}\mathbf{u}\cdot\mathbf{e} = 0$ is a necessary condition for $\Gamma\mathbf{H}\cdot\mathbf{e} \in \mathcal{N}^\perp$, if D is invariant. Therefore, $\Gamma\mathbf{H}\cdot\mathbf{e}$ has a nonzero component in $\mathcal{N}$. With the help of Lemma 8.2, the theorem is proved.

What about $\sigma_\varepsilon(\mathbf{e}_\perp)$, $\mathbf{e}_\perp \perp \mathbf{c}$? In view of the results for cat's-eye flows, the following theorem is intuitively clear.

THEOREM 8.10. *If the slope of* $\mathbf{c}$ *is rational, then*

$$\varepsilon < \sigma_\varepsilon(\mathbf{e}_\perp) < c\varepsilon \quad \text{as } \varepsilon \downarrow 0$$

for some $c > 1$.

Proof. By the result of Moser [20], mentioned in the beginning of this section, rationality of $\mathbf{c}$ implies rationality of the rotation number of the channels and the streamlines in the channels, which implies the periodicity of the streamlines. Without loss of generality, we can assume that the rotation number is zero by considering a linear change of coordinates (7.1) on $\mathbf{T}^2$. Then we can simply assume that $\mathbf{c} = (1,0)$.

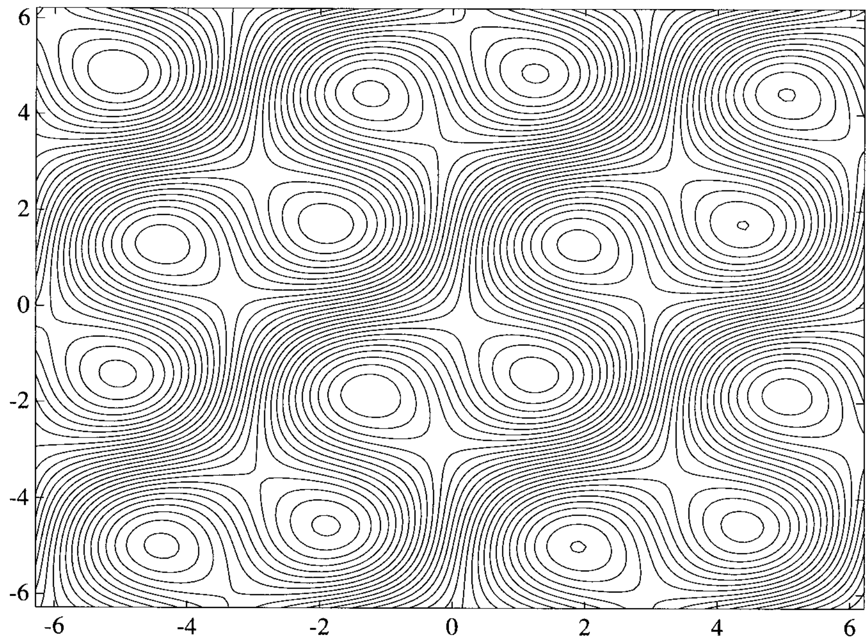

FIG. 8.3. *Cellular flow with drift* $\mathbf{c} = 0.2(1, \pi/2)$.

By Lemma 8.3, it is sufficient to prove that $\Gamma\mathbf{H}\cdot\mathbf{e} \in \text{Range } G$, which is equivalent to the existence of $\tilde{\mathbf{F}} \in \mathcal{H}_g$ such that (see (8.14))

$$(\mathbf{c}+\mathbf{u})\cdot\tilde{\mathbf{F}} = \mathbf{u}\cdot\mathbf{e}. \tag{8.28}$$

However, since $\mathbf{e}\perp\mathbf{c}$, (8.28) is equivalent to the existence of $\mathbf{F}$, $\langle\mathbf{F}\rangle = \mathbf{e}$ such that

$$(\mathbf{c}+\mathbf{u})\cdot\mathbf{F} = 0. \tag{8.29}$$

The existence of $\mathbf{F}$ satisfying (8.29) is clear for flows with periodic channels.

When the rotation number is irrational, we have the following upper bound.

THEOREM 8.11. *If the slope of* $\mathbf{c}$ *is irrational, then*

$$\sigma_\varepsilon(\mathbf{e}_\perp) = O(1/\varepsilon)\,.$$

Proof. Since the rotation number ρ is irrational, the subspaces $\mathcal{N}$ and $\mathcal{N}^\perp$ are completely determined by eddies in view of Lemmas 8.6 and 8.7, and the only non-ergodic streamlines are in eddies. It is easy to see that

$$\int_\gamma dt\,\nabla\cdot\Gamma\mathbf{H}'\cdot\mathbf{e} = \int_\gamma dt\,(\mathbf{c}+\mathbf{u})\cdot\mathbf{e} = 0 \tag{8.30}$$

for every closed streamline γ, every $\mathbf{e}$. Let $\mathbf{e}=\mathbf{e}_\perp$, and, since $\mathbf{c}\perp\mathbf{e}_\perp$, we have

$$\int_\gamma dt\,\mathbf{u}\cdot\mathbf{e}_\perp = \int_\gamma dt\,\nabla\cdot\Gamma\mathbf{H}\cdot\mathbf{e}_\perp = 0. \tag{8.31}$$

By the characterization of $\mathcal{N}^\perp$ in Lemma 8.7, $\Gamma\mathbf{H}\cdot\mathbf{e}_\perp \in \mathcal{N}^\perp$, and Lemma 8.4 implies the theorem.

The precise asymptotic behavior of the effective diffusivity for flows with eddies and an ergodic channel is not clear and is the subject of a future study.

8.4. Flows with no stagnation points. Now, we consider the case where $\mathbf{c}+\mathbf{u}$ does not have any stagnation points (see Figs. 8.4 and 8.5).

The following theorem [21] is known in the theory of dynamical systems on the torus $\mathbf{T}^2$.

THEOREM 8.12 (Kolmogorov–Denjoy). *There exists a coordinate transformation in $C^r(\mathbf{T}^2)$such that the trajectories in the new coordinate system are straight lines and the system has the form*

$$\frac{d\xi}{dt} = c_1 v\ , \qquad \frac{d\eta}{dt} = c_2 v, \tag{8.32}$$

where $(c_1, c_2) = \mathbf{c}$ and v is some positive C^{r-1} function.

Here $\rho = c_2/c_1$ is the rotation number of the dynamical system generated by $\mathbf{c}+\mathbf{u}$. Instead of the original system, we may study the transformed one and assume that $\langle v\rangle = 1$ for simplicity, so that

$$\mathbf{u} = (v-1)\mathbf{c}. \tag{8.33}$$

Note that the transformed flow cannot be incompressible in the new coordinates unless it is a shear layer flow in the new coordinates system $v = v(s)$, $s = c_1\eta - c_2\xi$. However, this does not hinder us from using Lemma 8.3, since solvability of (8.12) in one set of coordinates implies solvability in another.

For rational rotation numbers, we have the following theorem.

THEOREM 8.13. *Let ρ be a rational number. Then we have*

$$\frac{c'}{\varepsilon} \le \sigma_\varepsilon(\mathbf{e}) \le \frac{c''}{\varepsilon} \quad \text{as } \varepsilon \downarrow 0 \quad \text{for } \mathbf{e} \not\perp (1,\rho),$$
$$\varepsilon \le \sigma_\varepsilon(\mathbf{e}_\perp) \le c\varepsilon \quad \text{as } \varepsilon \downarrow 0 \quad \text{for } \mathbf{e}_\perp \perp (1,\rho)$$

for some positive c, c', and c'', unless

$$\int_\gamma dt = \text{constant independent of } \gamma,$$

in which case, the system can be transformed to

$$\frac{dx}{dt} = c_1, \qquad \frac{dy}{dt} = c_2,$$

and $\sigma_\varepsilon(\mathbf{e}) = O(\varepsilon)$ as $\varepsilon \downarrow 0$ for all $\mathbf{e}$.

Proof. We want to show that

$$\int_\gamma \mathbf{u}\cdot\mathbf{e}\, dt \neq 0 \quad \text{for some } \gamma \tag{8.34}$$

and then apply Lemmas 8.2 and 8.7 to show that $c'/\varepsilon \le \sigma_\varepsilon(\mathbf{e}) \le c''/\varepsilon$. Since all orbits are periodic with rational rotation number ρ, we have

$$\int_\gamma (\mathbf{c}+\mathbf{u})\cdot\mathbf{e}\, dt = \mathbf{c}\cdot\mathbf{e} \tag{8.35}$$

and

$$\begin{aligned} \int_\gamma \mathbf{u}\cdot\mathbf{e}\, dt &= \mathbf{c}\cdot\mathbf{e} - \int_\gamma \mathbf{c}\cdot\mathbf{e}\, dt \\ &= \mathbf{c}\cdot\mathbf{e}\left(1 - \int_\gamma dt\right) \\ &\neq 0, \end{aligned} \tag{8.36}$$

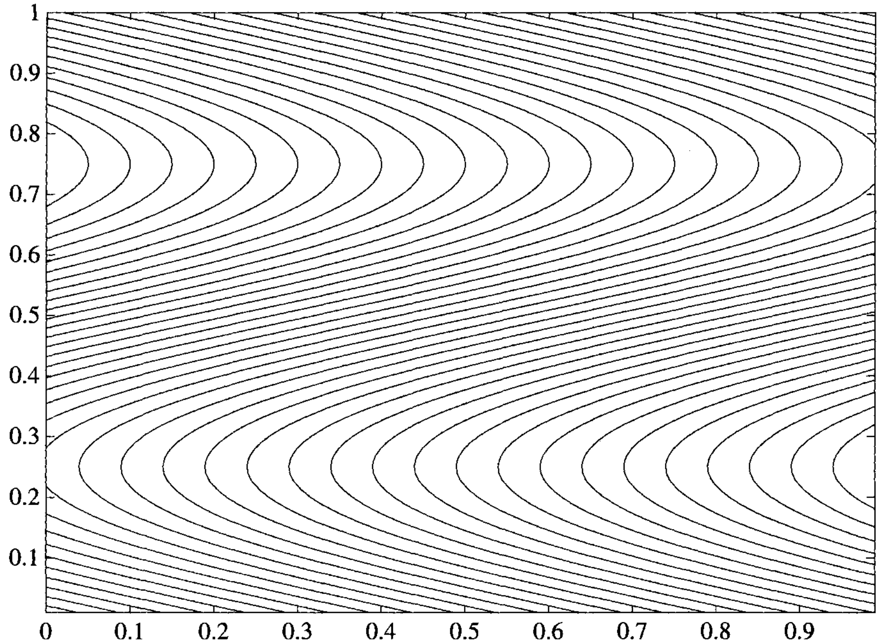

FIG. 8.4. *Shear layer flow with drift* $\mathbf{c} = (0, 0.3)$.

unless (i) $\mathbf{e} = \mathbf{e}_\perp \perp \mathbf{c}$ and (ii) $\int_\gamma dt =$ constant independent of γ.

If (ii) is true, then the system can be further transformed to

$$\frac{dx}{dt} = c_1 \;, \qquad \frac{dy}{dt} = c_2, \tag{8.37}$$

which obviously does not enhance the diffusion process, and we have

$$\sigma_\varepsilon(\mathbf{e}) = O(\varepsilon) \quad \text{as}\, \varepsilon \downarrow 0 \quad \text{for all}\, \mathbf{e}\ . \tag{8.38}$$

If (i) occurs, we want to show that

$$(\mathbf{c} + \mathbf{u}) \cdot \nabla f + \mathbf{u} \cdot \mathbf{e}_\perp = 0 \tag{8.39}$$

is solvable. Actually, $\mathbf{u} \cdot \mathbf{e}_\perp = (v - 1)\mathbf{c} \cdot \mathbf{e}_\perp \equiv 0$. Therefore $f \equiv 0$ is a solution. It follows from Lemma 8.3 that

$$\sigma_\varepsilon(\mathbf{e}_\perp) = O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0. \tag{8.40}$$

Shear layer flows with a nonzero perpendicular drift are examples where the condition

$$\int_\gamma dt = \text{constant} \quad \text{independent of } \gamma$$

in Theorem 8.13 holds, and therefore no enhancement occurs (see Fig. 8.4). To see this, let us consider the flow with $\mathbf{u} = (\cos 2\pi y, 0)$ and $\mathbf{c} = c(0, 1)$, $c > 0$.

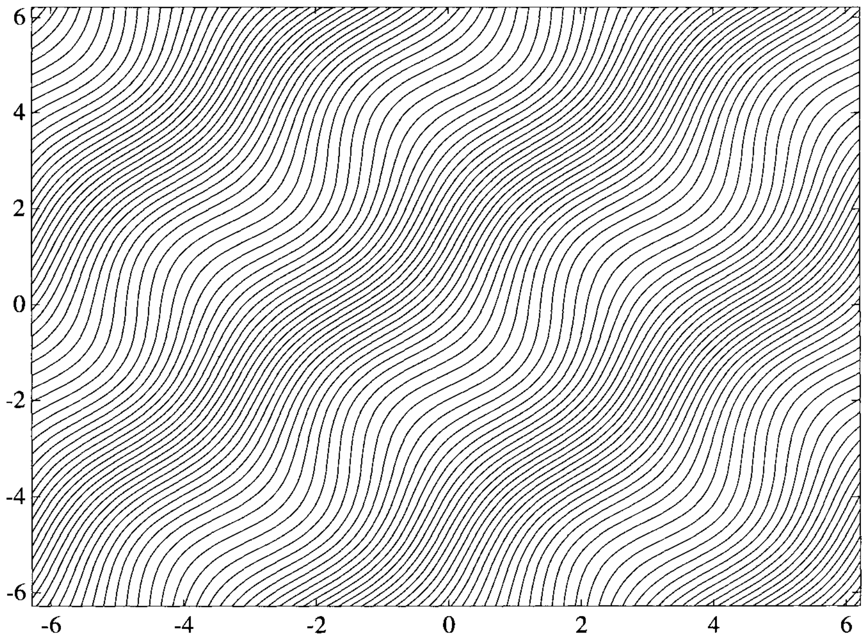

FIG. 8.5. *Cellular flow with drift* $\mathbf{c} = 1.1(1, \pi/2)$.

The cell problem (8.1) becomes

$$\varepsilon\Delta\chi + u(y)\frac{\partial}{\partial x}\chi + c\frac{\partial}{\partial y}\chi + \mathbf{u}\cdot\mathbf{e} = 0. \tag{8.41}$$

For $\mathbf{e} = \mathbf{e}_2 = (0, 1)$, $\mathbf{u}\cdot\mathbf{e} = 0$; thus $\chi = 0$ is the solution of (8.41), and we have $\sigma_\varepsilon(\mathbf{e}_2) = \varepsilon$. For $\mathbf{e} = \mathbf{e}_1 = (1, 0)$, (8.41) can be solved by a function $\chi = \chi(y)$ whose derivative is

$$\frac{1}{1+\frac{c^2}{4\pi^2\varepsilon^2}}\left\{-\frac{1}{2\pi\varepsilon}\sin 2\pi y - \frac{c}{4\pi^2\varepsilon^2}\cos 2\pi y\right\} \tag{8.42}$$

and

$$\sigma_\varepsilon(\mathbf{e}_1) = \varepsilon + \frac{\varepsilon}{2(4\pi^2\varepsilon^2 + c^2)}, \tag{8.43}$$

which is of order ε when c is not zero. To see the enormous effect of the drift $\mathbf{c} = c(0, 1)$ on the effective diffusivity for shear layer flows, we can compare (8.43) with formula (7.6).

If the rotation number ρ is an irrational number, then the flow is ergodic, the space $\mathcal{N}$ is trivial, and we have $\sigma_\varepsilon = O(1/\varepsilon)$. Actually, there is almost surely no enhancement, as can be seen from the following theorem.

THEOREM 8.14. *Assume that* (1) *there exists* c, $\delta > 0$, *such that*

$$\min_p |\rho - p/q| \geq \frac{c}{q^{2+\delta}} \quad \forall \text{integer } p, q$$

and (2) $r \geq 3 + \delta$. *Then*

$$\sigma_\varepsilon(\mathbf{e}) = O(\varepsilon) \quad \text{as } \varepsilon \downarrow 0 \quad \forall \mathbf{e}.$$

Proof. Consider the transformed system

$$\frac{dx}{dt} = c_1 v, \quad \frac{dy}{dt} = c_2 v, \quad v > 0, \tag{8.44}$$

as before. We claim that $v\mathbf{c} \cdot \nabla g + f = 0$, for any $f \in C^{\gamma-1}$, is always solvable if the rotation number satisfies the Diophantine inequality. Dividing the equation by v, we have $\mathbf{c} \cdot \nabla g + f/v = 0$. Writing g and f in terms of Fourier series, we have

$$\begin{aligned} g &= \sum_{\mathbf{m}} g_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}}, \qquad g_{\mathbf{m}} = 0 \quad \text{if } \mathbf{m} = (m_1, m_2) = (0,0), \\ f/v &= \sum_{\mathbf{m}} c_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}} \in C^{r-1}. \end{aligned} \tag{8.45}$$

Then

$$g_{\mathbf{m}} = c_{\mathbf{m}}/(m_1 + \rho m_2). \tag{8.46}$$

By assumption (1) in Theorem 8.14, however,

$$|m_1 + \rho m_2| = |m_2| \left| \frac{m_1}{m_2} + \rho \right| \geq \frac{c}{m_2^{1+\delta}} \Rightarrow |g_{\mathbf{m}}| \leq \frac{c_{\mathbf{m}}}{c} m_2^{1+\delta}, \tag{8.47}$$

and we know that $\sum_{\mathbf{m}} (c_{\mathbf{m}} |\mathbf{m}|^{r-1})^2 < \infty$. Therefore $\sum_{\mathbf{m}} (g_{\mathbf{m}} |\mathbf{m}|)^2 < \infty$ if $r > 3 + \delta$. This completes the proof of the theorem.

It is easy to see that a coordinate transformation affects only the constant coefficient but not the asymptotics; therefore, if the transformed flow is constant streaming, which obviously does not enhance the effective diffusivity, then the effective diffusivity for the original flow is order ε. The Diophantine condition in Theorem 8.14 is also a sufficient condition under which a flow can be transformed to constant streaming.

A number ρ is "normally approximated" by rational numbers if it satisfies the Diophantine inequality

$$\min_p |\rho - p/q| \geq \frac{c}{q^{2+\delta}}. \tag{8.48}$$

The set of normally approximated numbers has full measure, as can be shown in the following manner. Consider

$$A_q = \left\{ \rho : \min_p \left| \rho - \frac{p}{q} \right| < \frac{c}{q^{2+\delta}} \right\}. \tag{8.49}$$

Then measure $(A_q) \leq 2c/q^{1+\varepsilon}$, which implies that $\sum_q$ measure $(A_q) < \infty$, and the assertion follows from the Borel–Cantelli lemma (see [21]).

The exceptional cases where enormous enhancement might occur, not covered by Theorem 8.14, are discontinous flows or flows with nearly rational rotation numbers, that include rational rotation numbers as a special, trivial case.

8.5. A theorem concerning general time-dependent, nonballistic flows. If, instead of $\Gamma\mathbf{H}\Gamma + \Gamma\Delta^{-1}\mathbf{c}\cdot\nabla$, let

$$G = \Gamma\mathbf{H}\Gamma + \Gamma\Delta^{-1}\frac{\partial}{\partial t},$$

then, as in §8.1, the Hilbert space $\mathcal{H}_g$ of time-dependent, mean-zero, curl-free fields can be decomposed

$$\mathcal{H}_g = \mathcal{N} \oplus \mathcal{N}^{\perp}$$

with $\mathcal{N}$ the nullspace of G and $\mathcal{N}^{\perp}$ the complementary space of $\mathcal{N}$ in $\mathcal{H}_g$, which is also equal to $\overline{(\text{Range}\, G)}$. As for Lemma 8.4, it is also easy to deduce Lemma 8.15.

LEMMA 8.15. $\Gamma\mathbf{H}\cdot\mathbf{e}$ *does not have a component in* $\mathcal{N}$ (*i.e.*, $\Gamma\mathbf{H}\cdot\mathbf{e}\in\mathcal{N}^{\perp}$) *if and only if*

$$\sigma_\varepsilon(\mathbf{e}) = O\left(\frac{1}{\varepsilon}\right) \quad \textit{as } \varepsilon \downarrow 0.$$

Before applying Lemma 8.15, let us define the notion of "ballstic" and "nonballistic" motions. An orbit $\mathbf{x}(t), d\mathbf{x}(t)/dt = \mathbf{u}(\mathbf{x},t)$ is called "ballistic" in the direction $\mathbf{e}$ if

$$\limsup_{t\uparrow\infty} \frac{|\mathbf{x}(t)\cdot\mathbf{e}|}{t} \geq c \tag{8.50}$$

for some positive c; otherwise, it is called nonballistic in the direction $\mathbf{e}$. A flow is called nonballistic in the direction $\mathbf{e}$ if *almost all* orbits are nonballistic in that direction. The following theorem is a direct application of Lemma 8.15.

THEOREM 8.16. *If the flow generated by* $\mathbf{u}(\mathbf{x},t)$ *is nonballistic in the direction* $\mathbf{e}$, *then*

$$\sigma_\varepsilon(\mathbf{e}) = O\left(\frac{1}{\varepsilon}\right) \quad \textit{as } \varepsilon \downarrow 0.$$

Proof. It is sufficient to show that

$$\int_0^1 dt \int d\mathbf{x}\,\mathbf{e}\cdot f\mathbf{u}(\mathbf{x},t) = 0$$

for every $\mathbf{F} = \nabla f \in \mathcal{N}$. Since both f and $\mathbf{u}$ are time-periodic, we have

$$\begin{aligned} &\int_0^1 dt \int d\mathbf{x}\,\mathbf{e}\cdot f\mathbf{u}(\mathbf{x},t) \\ &= \lim_{N\uparrow\infty}\frac{1}{N}\int_0^N dt\int d\mathbf{x}\,\mathbf{e}\cdot f\mathbf{u}(\mathbf{x},t) \\ &= \lim_{N\uparrow\infty}\frac{1}{N}\int_0^N dt\int d\mathbf{x}'\,\mathbf{e}\cdot f\mathbf{u}(X(\mathbf{x}',t),t), \end{aligned} \tag{8.51}$$

where $X(\mathbf{x}',t)$ is the flow

$$\frac{dX(\mathbf{x}',t)}{dt} = \mathbf{u}(X(\mathbf{x}',t),t), \qquad X(\mathbf{x}',0) = \mathbf{x}'. \tag{8.52}$$

The last equality of (8.51) is due to the incompressibility of $\mathbf{u}$. It is easy to see that a characterization of the space $\mathcal{N}$ similar to that in Lemma 8.8 holds for time-dependent flows and f is constant along every streamline if $\nabla f \in \mathcal{N}$, i.e., $f(X(\mathbf{x}',t),t) = f(\mathbf{x}',0)$. Thus, after interchange of spatial and temporal integrals, (8.51) becomes

$$(8.53)\quad \left|\lim_{N\uparrow\infty}\int d\mathbf{x}'\, f\,\frac{1}{N}\int_0^N dt\,\mathbf{e}\cdot\mathbf{u}\right| \le \int d\mathbf{x}'\,|f|\,\limsup_{N\uparrow\infty}\left(\frac{|\mathbf{e}\cdot X(\mathbf{x}',N)|}{N}\right) = 0$$

by the definition of nonballistic flows.

Orbits in an open channel are clearly ballistic, and they result in $O(1/\varepsilon)$ effective diffusivity, as stated in the theorems of §7. Together with previous results on flows with open channels, Theorem 8.16 indicates that ballistic flows are the only ones that lead to $O(1/\varepsilon)$ asymptotic behavior of the effective diffusivity. Theorem 8.16 also holds for nonballistic flows that are temporally random. As a comparison, ballistic motion in flows with nonzero mean drifts may not enhance the effective diffusivity as shown in Theorem 8.14. Zhikov [23] makes an observation similar to Theorem 8.16 for two-dimensional steady flows that do not have nontrivial contours. According to the results in §7, the effective diffusivity for these flows is of order $\sqrt{\varepsilon}$ generally.

Appendix A. Relations between different variational principles for nonsymmetric diffusivities. Homogenization theory as described in §2 is valid quite generally, even when the conductivity or diffusivity matrix (a_{ij}) (cf.(2.4)) is complex-valued. The complex effective conductivity can be characterized by a saddlepoint variational principle. A key observation of Gibiansky and Cherkaev (see [12]) is that the saddlepoint variational principle can be converted, via Legendre transforms, into a Dirichlet-type variational principle. Milton [12] generalized the formulation of Gibiansky and Cherkaev to nonselfadjoint problems, such as the conductivity problem when a magnetic field is present, including the Hall effect. Milton's extension procedure is equivalent to our symmetrization procedure. In this section, we use their idea to derive a variational principle similar to that of Gibiansky–Cherkaev–Milton, except that the variation is under different constraints. Then we use the duality relation to derive a dual variational principle under a dual constraint and study the connection between these variational principles and those developed in §3. In §A.2 we show how to derive our general variational principles for the full flux tensor directly from a pair of saddlepoint variational principles.

A.1. Derivation of the variational principles of § 3 by a partial Legendre transformation. Consider the forward and backward cell problems ((2.11) in §2), with $\varepsilon = 1$,

$$(A.1)\qquad \nabla\cdot(\mathbf{I}+\mathbf{H})\mathbf{E}^+ = 0\ ,\quad \nabla\times\mathbf{E}^+ = 0\ ,\quad \langle\mathbf{E}^+\rangle = \mathbf{e},$$

$$(A.2)\qquad \nabla\cdot(\mathbf{I}-\mathbf{H})\mathbf{E}^- = 0\ ,\quad \nabla\times\mathbf{E}^- = 0,\quad \langle\mathbf{E}^-\rangle = \mathbf{e}.$$

Let $\mathbf{D}^+ = (\mathbf{I}+\mathbf{H})\mathbf{E}^+$, $\mathbf{D}^- = (\mathbf{I}-\mathbf{H})\mathbf{E}^-$ be the fluxes for the forward and backward problems, respectively, and define

$$(A.3)\qquad \mathbf{D}' = \tfrac{1}{2}(\mathbf{D}^+ - \mathbf{D}^-)\ ,\qquad \mathbf{D} = \tfrac{1}{2}(\mathbf{D}^+ + \mathbf{D}^-),$$

$$(A.4)\qquad \mathbf{E}' = \tfrac{1}{2}(\mathbf{E}^+ - \mathbf{E}^-),\qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{E}^+ + \mathbf{E}^-).$$

Then $\mathbf{E}'$, $\mathbf{E}$, and $\mathbf{D}'$, $\mathbf{D}$ are related by

$$\mathbf{D}' = \mathbf{E}' + \mathbf{H}\mathbf{E}, \tag{A.5}$$
$$\mathbf{D} = \mathbf{E} + \mathbf{H}\mathbf{E}' \tag{A.6}$$

or, in the matrix form,

$$\begin{pmatrix} \mathbf{D}' \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{H} \\ \mathbf{H} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{E} \end{pmatrix}. \tag{A.7}$$

The cell problems (A.1) and (A.2) are equivalent to (A.7) with

$$\nabla \cdot \mathbf{D}' = \nabla \cdot \mathbf{D} = 0, \tag{A.8}$$
$$\nabla \cdot \mathbf{E}' = \nabla \cdot \mathbf{E} = 0 \tag{A.9}$$

under the constraints

$$\langle \mathbf{E}' \rangle = 0, \tag{A.10}$$

$$\langle \mathbf{E} \rangle = \mathbf{e}. \tag{A.11}$$

Note that the matrix $\left(\begin{smallmatrix} I & H \\ H & I \end{smallmatrix}\right)$ is not symmetric. Following the Gibiansky and Cherkaev idea of performing a partial Legendre transform, let us rewrite (A.5) as

$$\mathbf{E}' = \mathbf{D}' - \mathbf{H}\mathbf{E}\ . \tag{A.12}$$

Then (A.6) becomes

$$\mathbf{D} = \mathbf{H}\mathbf{D}' + (\mathbf{I} - \mathbf{H}^2)\mathbf{E}, \tag{A.13}$$

and in matrix form, (A.12) and (A.13) are equivalent to

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & -\mathbf{H} \\ \mathbf{H} & \mathbf{I} - \mathbf{H}^2 \end{pmatrix} \begin{pmatrix} \mathbf{D}' \\ \mathbf{E} \end{pmatrix}. \tag{A.14}$$

Now the matrix is symmetric and positive definite as a result of this transformation.

The effective diffusivity is given by

$$\begin{aligned} \sigma(\mathbf{e}) &= \langle \mathbf{E}^+ \cdot \mathbf{E}^+ \rangle \\ &= \langle \mathbf{D}^+ \cdot \mathbf{e} \rangle \\ &= \tfrac{1}{2}\langle \mathbf{D}^+ \cdot \mathbf{e} \rangle + \tfrac{1}{2}\langle \mathbf{D}^- \cdot \mathbf{e} \rangle \\ &= \tfrac{1}{2}\langle \mathbf{D}^+ \cdot \mathbf{E}^+ \rangle + \tfrac{1}{2}\langle \mathbf{D}^- \cdot \mathbf{E}^- \rangle \\ &= \langle \mathbf{D}' \cdot \mathbf{E}' \rangle + \langle \mathbf{D} \cdot \mathbf{E} \rangle \\ &= \left\langle \begin{pmatrix} \mathbf{I} & -\mathbf{H} \\ \mathbf{H} & \mathbf{I} - \mathbf{H}^2 \end{pmatrix} \begin{pmatrix} \mathbf{D}' \\ \mathbf{E} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{D}' \\ \mathbf{E} \end{pmatrix} \right\rangle. \end{aligned} \tag{A.15}$$

Since

$$\begin{pmatrix} \mathbf{I} & -\mathbf{H} \\ \mathbf{H} & \mathbf{I} - \mathbf{H}^2 \end{pmatrix}$$

is symmetric and positive definite, we have the following variational formulation for $\sigma(\mathbf{e})$:

$$\sigma(\mathbf{e}) = \inf_{\substack{\nabla\times\mathbf{F}=0\\ \langle\mathbf{F}\rangle=\mathbf{e}}} \inf_{\substack{\nabla\cdot\mathbf{G}'=0\\ \langle\mathbf{G}'\rangle=\langle\mathbf{HF}\rangle}} \left\langle \begin{pmatrix} \mathbf{I} & -\mathbf{H} \\ \mathbf{H} & \mathbf{I}-\mathbf{H}^2 \end{pmatrix} \begin{pmatrix} \mathbf{G}' \\ \mathbf{F} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{G}' \\ \mathbf{F} \end{pmatrix} \right\rangle. \tag{A.16}$$

The constraint $\langle\mathbf{G}'\rangle = \langle\mathbf{HF}\rangle$ comes from (A.5) and $\langle\mathbf{E}'\rangle = 0$ for the original problems. More explicitly, we have

$$\begin{aligned}\sigma(\mathbf{e}) = \inf_{\substack{\nabla\times\mathbf{F}=0\\ \langle\mathbf{F}\rangle=\mathbf{e}}} \inf_{\substack{\nabla\cdot\mathbf{G}'=0\\ \langle\mathbf{G}'\rangle=\langle\mathbf{HF}\rangle}} &\big\{\langle\mathbf{G}'\cdot\mathbf{G}'\rangle - 2\langle\mathbf{HF}\cdot\mathbf{G}'\rangle \\ &+ \langle\mathbf{F}\cdot\mathbf{F}\rangle + \langle\mathbf{HF}\cdot\mathbf{HF}\rangle\big\}.\end{aligned} \tag{A.17}$$

Let us fix $\mathbf{F}$ and perform the minimization on $\mathbf{G}'$. The resulting Euler equation is

$$\nabla\times(\mathbf{G}' - \mathbf{HF}) = 0. \tag{A.18}$$

Equation (A.18) can be solved using the projection operator $\Gamma^\perp = \nabla^\perp\Delta^{-1}\nabla^\perp\cdot$, denoted by Γ_c in §3 (cf. (3.72)),

$$\mathbf{G}' = \langle\mathbf{HF}\rangle + \Gamma^\perp\mathbf{HF}. \tag{A.19}$$

Substituting (A.19) into (A.17) and observing that $\Gamma = \nabla\Delta^{-1}\nabla\cdot = \mathbf{I} - \Gamma^\perp - \langle\cdot\rangle$, we obtain

$$\sigma(\mathbf{e}) = \inf_{\substack{\nabla\times\mathbf{F}=0\\ \langle\mathbf{F}\rangle=\mathbf{e}}} \big\{\langle\mathbf{F}\cdot\mathbf{F}\rangle + \langle\Gamma\mathbf{HF}\cdot\Gamma\mathbf{HF}\rangle\big\}, \tag{A.20}$$

which is the direct variational principle (3.73) with $\varepsilon = 1$.

To derive our inverse variational principle, let us consider the dual forward and backward cell problems, with $\varepsilon = 1$ again,

$$\nabla\times(\mathbf{I}+\mathbf{H})^{-1}\mathbf{D}^+ = 0\ , \quad \nabla\cdot\mathbf{D}^+ = 0\ , \quad \langle\mathbf{D}^+\rangle = \mathbf{e}, \tag{A.21}$$

$$\nabla\times(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D}^- = 0\ , \quad \nabla\cdot\mathbf{D}^- = 0\ , \quad \langle\mathbf{D}^-\rangle = \mathbf{e}. \tag{A.22}$$

Set $\mathbf{E}^+ = (\mathbf{I}+\mathbf{H})^{-1}\mathbf{D}^+$, $\mathbf{E}^- = (\mathbf{I}-\mathbf{H})^{-1}\mathbf{D}^-$ and define $\mathbf{D}'$, $\mathbf{D}$ and $\mathbf{E}'$, $\mathbf{E}$ as before and related as in (A.5) and (A.6). The dual cell problems (A.21), (A.22) are equivalent to (A.7) with (A.8) and (A.9) under the constraints

$$\langle\mathbf{D}'\rangle = 0, \tag{A.23}$$

$$\langle\mathbf{D}\rangle = \mathbf{e}. \tag{A.24}$$

Again, we do the partial Legendre transform to (A.6)

$$\mathbf{E} = \mathbf{D} - \mathbf{HE}'\ . \tag{A.25}$$

Then (A.5) becomes

$$\begin{aligned}\mathbf{D}' &= \mathbf{E}' + \mathbf{H}(\mathbf{D} - \mathbf{HE}') \\ &= \mathbf{HD} + (\mathbf{I} - \mathbf{H}^2)\mathbf{E}',\end{aligned} \tag{A.26}$$

and in matrix form (A.25) and (A.26) are equivalent to

$$\begin{pmatrix}\mathbf{D}'\\ \mathbf{E}\end{pmatrix} = \begin{pmatrix}\mathbf{I}-\mathbf{H}^2 & \mathbf{H}\\ -\mathbf{H} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{E}'\\ \mathbf{D}\end{pmatrix}, \tag{A.27}$$

in which the matrix is symmetric and positive definite. The inverse effective diffusivity is given by

$$\begin{aligned}(\sigma)^{-1}(\mathbf{e}) &= \langle \mathbf{D}'\cdot\mathbf{E}'\rangle + \langle \mathbf{D}\cdot\mathbf{E}\rangle\\ &= \left\langle \begin{pmatrix}\mathbf{I}-\mathbf{H}^2 & \mathbf{H}\\ -\mathbf{H} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{E}'\\ \mathbf{D}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{E}'\\ \mathbf{D}\end{pmatrix}\right\rangle.\end{aligned} \tag{A.28}$$

Since this quadratic functional (A.28) is symmetric and positive definite, we have the following variational formulation for $(\sigma)^{-1}(\mathbf{e})$:

$$(\sigma)^{-1}(\mathbf{e}) = \inf_{\substack{\nabla\cdot\mathbf{G}=0\\ \langle\mathbf{G}\rangle=\mathbf{e}}}\ \inf_{\substack{\nabla\times\mathbf{F}'=0\\ \langle(1+H^2)\mathbf{F}'\rangle=-\langle\mathbf{HG}\rangle}} \left\langle \begin{pmatrix}\mathbf{I}-\mathbf{H}^2 & \mathbf{H}\\ -\mathbf{H} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{F}'\\ \mathbf{G}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{F}'\\ \mathbf{G}\end{pmatrix}\right\rangle, \tag{A.29}$$

where the constraint for $\mathbf{F}'$ comes from (A.26), $\langle\mathbf{HD}\rangle + \langle(\mathbf{I}-\mathbf{H}^2)\mathbf{E}'\rangle = 0$, by (A.23). More explicitly, we have

$$\begin{aligned}(\sigma)^{-1}(\mathbf{e}) = \inf_{\substack{\nabla\cdot\mathbf{G}=0\\ \langle\mathbf{G}\rangle=\mathbf{e}}}\ \inf_{\substack{\nabla\times\mathbf{F}'=0\\ \langle(1+H^2)\mathbf{F}'\rangle=-\langle\mathbf{HG}\rangle}} &\big\{\langle\mathbf{F}'\cdot\mathbf{F}'\rangle + \langle H^2\mathbf{F}'\cdot\mathbf{F}'\rangle\\ &+ 2\langle\mathbf{HG}\cdot\mathbf{F}'\rangle + \langle\mathbf{G}\cdot\mathbf{G}\rangle\big\}.\end{aligned} \tag{A.30}$$

Let $\mathbf{G}$ be fixed and perform the minimization on $\mathbf{F}'$. The Euler equation is

$$\nabla\cdot(1+H^2)\mathbf{F}' + \nabla\cdot\mathbf{HG} = 0. \tag{A.31}$$

We can solve (A.31) in the following way. Write $(1+H^2)\mathbf{F}' = -\mathbf{HG} + \nabla^\perp\chi$, where χ is a periodic function. This will satisfy the constraint, and χ must solve

$$0 = \nabla\times\mathbf{F}' = \nabla^\perp\cdot\mathbf{F}' = -\nabla^\perp\cdot\frac{\mathbf{H}}{1+H^2}\mathbf{G} + \nabla^\perp\cdot\frac{1}{1+H^2}\nabla^\perp\chi \tag{A.32}$$

or

$$\nabla^\perp\chi = \Gamma_H^\perp\mathbf{HG}, \tag{A.33}$$

where $\Gamma_H^\perp = \nabla^\perp\Delta_H^{-1}\nabla^\perp\cdot(1/1+H^2)$ (cf. (3.70) and (3.71) with $\varepsilon = 1$) and

$$\mathbf{F}' = -\frac{\mathbf{H}}{1+H^2}\mathbf{G} + \frac{1}{1+H^2}\Gamma_H^\perp\mathbf{HG}. \tag{A.34}$$

We now substitute (A.34) into (A.30) and consider each term separately. For the first term, we have

$$\langle\mathbf{F}'\cdot\mathbf{F}'\rangle = \left\langle\frac{1}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\cdot\frac{1}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\right\rangle;$$

for the second term,

$$\begin{aligned}\langle H^2\mathbf{F}'\cdot\mathbf{F}'\rangle &= \left\langle\frac{H^2}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\cdot\frac{1}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\right\rangle\\ &= \left\langle\frac{1}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\cdot(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\right\rangle\\ &\quad - \left\langle\frac{1}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\cdot\frac{1}{1+H^2}(\mathbf{I}-\Gamma_H^\perp)\mathbf{HG}\right\rangle;\end{aligned}$$

for the third term,

$$
\begin{aligned}
2\langle \mathbf{HG} \cdot \mathbf{F}' \rangle &= -2\Big\langle \mathbf{HG} \cdot \frac{1}{H^2}(\mathbf{I} - \Gamma_H^{\perp})\mathbf{HG} \Big\rangle \\
&= -2\Big\langle (\mathbf{I} - \Gamma_H^{\perp})\mathbf{HG} \cdot \frac{1}{1+H^2}(\mathbf{I} - \Gamma_H^{\perp})\mathbf{HG} \Big\rangle,
\end{aligned} \tag{A.35}
$$

where we use the fact that $\Gamma_H^{\perp}$ is a projection operator that is selfadjoint with respect to the inner product weighted with $(1 + H^2)^{-1}$. For the fourth term, we have

$$
\begin{aligned}
\langle \mathbf{G} \cdot \mathbf{G} \rangle &= \Big\langle \frac{1}{1+H^2}\mathbf{G} \cdot \mathbf{G} \Big\rangle + \Big\langle \frac{H^2}{1+H^2}\mathbf{G} \cdot \mathbf{G} \Big\rangle \\
&= \Big\langle \frac{1}{1+H^2}\mathbf{G} \cdot \mathbf{G} \Big\rangle + \Big\langle \frac{1}{1+H^2}\mathbf{HG} \cdot \mathbf{HG} \Big\rangle.
\end{aligned}
$$

When we add these terms, we obtain

$$
(\sigma)^{-1}(\mathbf{e}) = \inf_{\substack{\nabla \cdot \mathbf{G}=0 \\ \langle \mathbf{G} \rangle = \mathbf{e}}} \Big\{ \Big\langle \frac{1}{1+H^2}\mathbf{G} \cdot \mathbf{G} \Big\rangle + \Big\langle \frac{1}{1+H^2}\Gamma_H^{\perp}\mathbf{HG} \cdot \Gamma_H^{\perp}\mathbf{HG} \Big\rangle \Big\}, \tag{A.36}
$$

which is our inverse variational principle.

A.2. Derivation of the variational principles of § 3 from a saddlepoint variational principle. Our variational principles can be derived directly from a pair of saddlepoint variational principles. This is actually closer in spirit to our original approach in §3. At the end of §2 we noted that the full effective flux tensor, defined by (2.17), is not symmetric. We now give variational formulations for the full effective flux tensor

$$
\sigma(\mathbf{e}_1, \mathbf{e}_2) = \langle \mathbf{D}_{\mathbf{e}_1}^{+} \cdot \mathbf{e}_2 \rangle \quad \forall \mathbf{e}_1, \mathbf{e}_2,
$$

where $\mathbf{e}_1, \mathbf{e}_2$ are unit vectors,

$$
\mathbf{D}_{\mathbf{e}_1}^{+} = (\mathbf{I} + \mathbf{H})\mathbf{E}_{\mathbf{e}_1}^{+}, \tag{A.37}
$$

and $\mathbf{E}_{\mathbf{e}_1}^{+}$ is the solution to the forward cell problem in the direction $\mathbf{e}_1$

$$
\nabla \cdot (\mathbf{I} + \mathbf{H})\mathbf{E}_{\mathbf{e}_1}^{+} = 0 \ , \quad \nabla \times \mathbf{E}_{\mathbf{e}_1}^{+} = 0 \ , \quad \langle \mathbf{E}_{\mathbf{e}_1}^{+} \rangle = \mathbf{e}_1. \tag{A.38}
$$

The effective diffusivity

$$
\sigma(\mathbf{e}) = \sigma(\mathbf{e}_1, \mathbf{e}_2) \quad \text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}
$$

is the symmetric part of the effective flux tensor. Define the backward cell problem in the direction $\mathbf{e}_2$ by

$$
\nabla \cdot (\mathbf{I} - \mathbf{H})\mathbf{E}_{\mathbf{e}_2}^{-} = 0 \ , \quad \nabla \times \mathbf{E}_{\mathbf{e}_2}^{-} = 0 \ , \quad \langle \mathbf{E}_{\mathbf{e}_2}^{-} \rangle = \mathbf{e}_2 \tag{A.39}
$$

and let

$$
\mathbf{D}_{\mathbf{e}_2}^{-} = (\mathbf{I} - \mathbf{H})\mathbf{E}_{\mathbf{e}_2}^{-}. \tag{A.40}
$$

Define also

$$\text{(A.41)}\qquad \begin{aligned} \mathbf{E}'_{12} &= \tfrac{1}{2}(\mathbf{E}^+_{\mathbf{e}_1} - \mathbf{E}^-_{\mathbf{e}_2}),\\ \mathbf{E}_{12} &= \tfrac{1}{2}(\mathbf{E}^+_{\mathbf{e}_1} + \mathbf{E}^-_{\mathbf{e}_2}),\\ \mathbf{D}'_{12} &= \tfrac{1}{2}(\mathbf{D}^+_{\mathbf{e}_1} - \mathbf{D}^-_{\mathbf{e}_2}),\\ \mathbf{D}_{12} &= \tfrac{1}{2}(\mathbf{D}^+_{\mathbf{e}_1} + \mathbf{D}^-_{\mathbf{e}_2}). \end{aligned}$$

Then, from (A.37) and (A.40),

$$\text{(A.42)}\qquad \begin{aligned} \mathbf{D}'_{12} &= \mathbf{E}'_{12} + \mathbf{H}\mathbf{E}_{12},\\ \mathbf{D}_{12} &= \mathbf{E}_{12} + \mathbf{H}\mathbf{E}'_{12}, \end{aligned}$$

and the cell problems (A.38) and (A.39) are equivalent to

$$\text{(A.43)}\qquad \nabla \times \mathbf{E}'_{12} = \nabla \times \mathbf{E}_{12} = 0,$$

$$\text{(A.44)}\qquad \nabla \cdot \mathbf{D}'_{12} = \nabla \cdot \mathbf{D}_{12} = 0$$

along with (A.42) and subject to the mean field constraints

$$\langle \mathbf{E}'_{12} \rangle = \frac{\mathbf{e}_1 - \mathbf{e}_2}{2}, \qquad \langle \mathbf{E}_{12} \rangle = \frac{\mathbf{e}_1 + \mathbf{e}_2}{2}.$$

The effective flux tensor is given by

$$\text{(A.45)}\qquad \begin{aligned} \sigma(\mathbf{e}_1, \mathbf{e}_2) &= \langle \mathbf{D}^+_{\mathbf{e}_1} \cdot \mathbf{e}_2 \rangle\\ &= \tfrac{1}{2}\langle \mathbf{D}^+_{\mathbf{e}_1} \cdot \mathbf{e}_2 \rangle + \tfrac{1}{2}\langle \mathbf{D}^-_{\mathbf{e}_2} \cdot \mathbf{e}_1 \rangle\\ &= \tfrac{1}{2}\langle \mathbf{D}^+_{\mathbf{e}_1} \cdot \mathbf{E}^-_{\mathbf{e}_2} \rangle + \tfrac{1}{2}\langle \mathbf{D}^-_{\mathbf{e}_2} \cdot \mathbf{E}^+_{\mathbf{e}_1} \rangle\\ &= \tfrac{1}{4}\langle (\mathbf{D}^+_{\mathbf{e}_1} + \mathbf{D}^-_{\mathbf{e}_2})(\mathbf{E}^+_{\mathbf{e}_1} + \mathbf{E}^-_{\mathbf{e}_2}) \rangle\\ &\quad \tfrac{1}{4}\langle (\mathbf{D}^+_{\mathbf{e}_1} - \mathbf{D}^-_{\mathbf{e}_2})(\mathbf{E}^+_{\mathbf{e}_1} - \mathbf{E}^-_{\mathbf{e}_2}) \rangle\\ &= \langle \mathbf{D}_{12} \cdot \mathbf{E}_{12} \rangle - \langle \mathbf{D}'_{12} \cdot \mathbf{E}'_{12} \rangle\\ &= \left\langle \begin{pmatrix} -\mathbf{I} & -\mathbf{H} \\ \mathbf{H} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{E}'_{12} \\ \mathbf{E}_{12} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}'_{12} \\ \mathbf{E}_{12} \end{pmatrix} \right\rangle. \end{aligned}$$

We note that the last expression in (A.45) is a symmetric, indefinite functional whose Euler equations are (A.44) via (A.42). Therefore

$$\text{(A.46)}\qquad \sigma(\mathbf{e}_1, \mathbf{e}_2) = \inf_{\substack{\langle \mathbf{F} \rangle = (\mathbf{e}_1 + \mathbf{e}_2)/2 \\ \nabla \times \mathbf{F} = 0}} \sup_{\substack{\nabla \times \mathbf{F}' = 0 \\ \langle \mathbf{F}' \rangle = (\mathbf{e}_1 - \mathbf{e}_2)/2}} \left\langle \begin{pmatrix} -\mathbf{I} & -\mathbf{H} \\ \mathbf{H} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{F}' \\ \mathbf{F} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{F}' \\ \mathbf{F} \end{pmatrix} \right\rangle.$$

The Euler equation for the supremum is

$$\text{(A.47)}\qquad \nabla \cdot \mathbf{F}' + \nabla \cdot \mathbf{H}\mathbf{F} = \mathbf{0},$$

and hence

$$\text{(A.48)}\qquad \mathbf{F}' = \frac{\mathbf{e}_1 - \mathbf{e}_2}{2} - \Gamma \mathbf{H}\mathbf{F}.$$

When (A.48) is substituted into (A.46), we obtain our general variational principle

$$\text{(A.49)}\qquad \sigma(\mathbf{e}_1,\mathbf{e}_2) = \inf_{\substack{\nabla\times\mathbf{F}=0\\ \langle\mathbf{F}\rangle=(\mathbf{e}_1+\mathbf{e}_2)/2}} \left\{ \langle \mathbf{F}\cdot\mathbf{F}\rangle + \langle \Gamma\mathbf{H}\mathbf{F}\cdot\Gamma\mathbf{H}\mathbf{F}\rangle - \langle\mathbf{H}\mathbf{F}\rangle\cdot(\mathbf{e}_1-\mathbf{e}_2) - \left|\frac{\mathbf{e}_1-\mathbf{e}_2}{2}\right|^2 \right\}.$$

When $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}$, (A.49) is identical to the direct variational principle (A.20) of the previous section and of §3.

To get an inverse variational principle, we note that

$$\text{(A.50)}\qquad (\sigma)^{-1}(\mathbf{e}_1,\mathbf{e}_2) = \langle \mathbf{D}_{12}\cdot\mathbf{E}_{12}\rangle - \langle \mathbf{D}'_{12}\cdot\mathbf{E}'_{12}\rangle,$$

provided that $\mathbf{E}'_{12}$, $\mathbf{E}_{12}$, $\mathbf{D}'_{12}$, and $\mathbf{D}_{12}$ satisfy (A.42), (A.43), and (A.44), subject to the mean field conditions

$$\langle\mathbf{D}'_{12}\rangle = \frac{\mathbf{e}_1-\mathbf{e}_2}{2}, \qquad \langle\mathbf{D}_{12}\rangle = \frac{\mathbf{e}_1+\mathbf{e}_2}{2}.$$

Let us invert (A.42) as follows:

$$\text{(A.51)}\qquad \mathbf{E}'_{12} = \frac{1}{1+H^2}(\mathbf{D}'_{12} - \mathbf{H}\mathbf{D}_{12}), \qquad \mathbf{E}_{12} = \frac{1}{1+H^2}(\mathbf{D}_{12} - \mathbf{H}\mathbf{D}'_{12}).$$

As before, we have the saddlepoint variational principle

$$\text{(A.52)}\qquad (\sigma)^{-1}(\mathbf{e}_1,\mathbf{e}_2) = \inf_{\substack{\nabla\cdot\mathbf{G}=0\\ \langle\mathbf{G}\rangle=(\mathbf{e}_1+\mathbf{e}_2)/2}} \ \sup_{\substack{\nabla\cdot\mathbf{G}'=0\\ \langle\mathbf{G}'\rangle=(\mathbf{e}_1-\mathbf{e}_2)/2}} \left\langle \frac{1}{1+H^2}\begin{pmatrix} -\mathbf{I} & \mathbf{H}\\ -\mathbf{H} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\right\rangle.$$

Eliminating the supremum by solving the corresponding Euler equation, we obtain

$$\nabla\times\frac{1}{1+H^2}\mathbf{G}' - \nabla\times\frac{1}{1+H^2}\mathbf{H}\mathbf{G} = 0,$$

and hence

$$\text{(A.53)}\qquad \mathbf{G}' = \frac{\mathbf{e}_1-\mathbf{e}_2}{2} - \Gamma_H^{\perp}\left(\frac{\mathbf{e}_1-\mathbf{e}_2}{2}\right) + \Gamma_H^{\perp}\mathbf{H}\mathbf{G},$$

where $\Gamma_H^{\perp}$ is defined by (3.70) and (3.72). Using (A.53) in (A.52), we can obtain our general inverse variational principle

$$\text{(A.54)}\qquad \begin{aligned}(\sigma)^{-1}(\mathbf{e}_1,\mathbf{e}_2) = \inf_{\substack{\nabla\cdot\mathbf{G}=0\\ \langle\mathbf{G}\rangle=(\mathbf{e}_1+\mathbf{e}_2)/2}} \Bigg\{ &\left\langle\frac{1}{1+H^2}\mathbf{G}\cdot\mathbf{G}\right\rangle + \left\langle\frac{1}{1+H^2}\Gamma_H^{\perp}\mathbf{H}\mathbf{G}\cdot\Gamma_H^{\perp}\mathbf{H}\mathbf{G}\right\rangle\\ &+ 2\left\langle\frac{1}{1+H^2}\mathbf{H}\mathbf{G}\cdot\left(\frac{\mathbf{e}_1-\mathbf{e}_2}{2} - \Gamma_H^{\perp}\left(\frac{\mathbf{e}_1-\mathbf{e}_2}{2}\right)\right)\right\rangle\\ &- \left\langle\frac{1}{1+H^2}\left(\frac{\mathbf{e}_1-\mathbf{e}_2}{2} - \Gamma_H^{\perp}\left(\frac{\mathbf{e}_1-\mathbf{e}_2}{2}\right)\right)^2\right\rangle\Bigg\}.\end{aligned}$$

When $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}$, (A.54) is identical to (A.36) of the previous section.

A.3. The symmetry of the full effective flux tensor. Let σ^- be the effective flux tensor associated with the flow $-\mathbf{H}$, instead of $\mathbf{H}$. In view of (A.46) and (A.49), σ^- admits also variational formulations

$$\sigma^-(\mathbf{e}_1,\mathbf{e}_2) = \inf_{\substack{\langle\mathbf{F}\rangle=(\mathbf{e}_1+\mathbf{e}_2)/2\\ \nabla\times\mathbf{F}=0}} \sup_{\substack{\nabla\times\mathbf{F}'=0\\ \langle\mathbf{F}'\rangle=(\mathbf{e}_1-\mathbf{e}_2)/2}} \left\langle \begin{pmatrix} -\mathbf{I} & \mathbf{H}\\ -\mathbf{H} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{F}'\\ \mathbf{F}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{F}'\\ \mathbf{F}\end{pmatrix}\right\rangle \tag{A.55}$$

and

$$\sigma^-(\mathbf{e}_1,\mathbf{e}_2) = \inf_{\substack{\nabla\times\mathbf{F}=0\\ \langle\mathbf{F}\rangle=(\mathbf{e}_1+\mathbf{e}_2)/2}} \Big\{\langle\mathbf{F}\cdot\mathbf{F}\rangle + \langle\Gamma\mathbf{H}\mathbf{F}\cdot\Gamma\mathbf{H}\mathbf{F}\rangle + \langle\mathbf{H}\mathbf{F}\rangle\cdot(\mathbf{e}_1-\mathbf{e}_2) - \left|\frac{\mathbf{e}_1-\mathbf{e}_2}{2}\right|^2\Big\}. \tag{A.56}$$

Clearly,

$$\sigma^-(\mathbf{e}_1,\mathbf{e}_2) = \sigma(\mathbf{e}_2,\mathbf{e}_1). \tag{A.57}$$

The symmetry of σ, that is, $\sigma(\mathbf{e}_1,\mathbf{e}_2) = \sigma(\mathbf{e}_1,\mathbf{e}_2)$, is equivalent to the statement that the effective flux tensor is independent of the sign of the stream matrix $\mathbf{H}$. Several situations lead to the symmetry of the effective flux tensor for two-dimensional, periodic flows: (a) Translational antisymmetry of H in the sense that there is a vector $\mathbf{r}$ such that $H(\mathbf{x}+\mathbf{r}) = -H(\mathbf{x})$ for all $\mathbf{x}\in\mathbf{R}^2$.

The symmetry of the effective flux tensor follows easily from this translational antisymmetry of H in view of the transformation $\mathbf{x}\to\mathbf{x}+\mathbf{r}$, $\mathbf{F}(\mathbf{x})\to\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{x}+\mathbf{r})$. This brings (A.49) to

$$\inf_{\substack{\nabla\times\mathbf{G}=0\\ \langle\mathbf{G}\rangle=(\mathbf{e}_1+\mathbf{e}_2)/2}} \Big\{\langle\mathbf{G}\cdot\mathbf{G}\rangle + \langle\Gamma\mathbf{H}\mathbf{G}\cdot\Gamma\mathbf{H}\mathbf{G}\rangle + \langle\mathbf{H}\mathbf{G}\rangle\cdot(\mathbf{e}_1-\mathbf{e}_2) - \left|\frac{\mathbf{e}_1-\mathbf{e}_2}{2}\right|^2\Big\}, \tag{A.58}$$

which is equivalent to (A.56);

(b) Reflectional antisymmetry of H with respect to an axis, say, x-axis, in the sense that $H(x,-y) = -H(x,y)$ for all $\mathbf{x}=(x,y)\in\mathbf{R}^2$.

Write the trial fields $\mathbf{F}$ and $\mathbf{F}'$ in (A.46) as the gradient of periodic functions f and f' plus the mean fields $(\mathbf{e}_1+\mathbf{e}_2)/2$ and $(\mathbf{e}_1-\mathbf{2}_2)/2$, respectively, and consider the transformation

$$\mathbf{F}\to\mathbf{G} = \nabla\Big(g+\frac{x-y}{2}\Big), \qquad \mathbf{F}'\to\mathbf{G}' = \nabla\Big(g'+\frac{-x-y}{2}\Big),$$

where

$$g(x,y) = f(x,-y), \qquad g'(x,y) = -f'(x,-y).$$

This transformation maps (A.46) into

$$\inf_{\substack{\langle\mathbf{G}\rangle=(\mathbf{e}_1-\mathbf{e}_2)/2\\ \nabla\times\mathbf{G}=0}} \sup_{\substack{\nabla\times\mathbf{G}'=0\\ \langle\mathbf{G}'\rangle=(-\mathbf{e}_1-\mathbf{e}_2)/2}} \left\langle \begin{pmatrix} -\mathbf{I} & \mathbf{H}\\ -\mathbf{H} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{G}'\\ \mathbf{G}\end{pmatrix}\right\rangle,$$

which is equal to $\sigma^-(-\mathbf{e}_2,\mathbf{e}_1)$. Using the relation (A.57), we have

$$\sigma(\mathbf{e}_1,\mathbf{e}_2) = \sigma^-(-\mathbf{e}_2,\mathbf{e}_1) = \sigma(\mathbf{e}_1,-\mathbf{e}_2) = -\sigma(\mathbf{e}_1,\mathbf{e}_2),$$

that is, $\sigma(\mathbf{e}_1, \mathbf{e}_2) = 0$. Similarly, we have $\sigma(\mathbf{e}_2, \mathbf{e}_1) = 0$. In other words, the antisymmetry of H with respect to the x-axis leads not only to the symmetry of the effective flux tensor, but also to the statement that $\mathbf{e}_1$ and $\mathbf{e}_2$ are the eigenvectors of the tensor. The same conclusion holds for any H that is reflectionally antisymmetric with respect to the y-axis. Generally, if the stream function H has the reflectional antisymmetry with respect to a vector $\mathbf{e}$, then the effective flux tensor is symmetric, and $\mathbf{e}$ and its perpendicular direction are the eigendirections of the tensor;

(c) The 180°-rotational antisymmetry of H with respect to a point, say, the origin in the sense that $H(-x,,-y) = -H(x,y)$ for all $\mathbf{x} = (x,y) \in \mathbf{R}^2$.

Consider the transformation

$$\mathbf{F} \to \mathbf{G} = \nabla\left(g + \frac{x+y}{2}\right), \qquad \mathbf{F}' \to \mathbf{G}' = \nabla\left(g' + \frac{x-y}{2}\right),$$

where

$$g(x,y) = -f(-x,-y), \qquad g'(x,y) = -f'(-x,-y).$$

Note that $\mathbf{G}(\mathbf{x}) = \mathbf{F}(-\mathbf{x})$. This transformation maps (A.46) into

$$\inf_{\substack{\langle \mathbf{G}\rangle = (\mathbf{e}_1+\mathbf{e}_2)/2 \\ \nabla\times\mathbf{G}=0}} \quad \sup_{\substack{\nabla\times\mathbf{G}'=0 \\ \langle \mathbf{G}'\rangle = (\mathbf{e}_1-\mathbf{e}_2)/2}} \left\langle \begin{pmatrix} -\mathbf{I} & \mathbf{H} \\ -\mathbf{H} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{G}' \\ \mathbf{G} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{G}' \\ \mathbf{G} \end{pmatrix} \right\rangle,$$

which is $\sigma^-(\mathbf{e}_1, \mathbf{e}_2)$, and the symmetry of the effective flux follows immediately.

A special class of flows that have symmetric effective flux tensor are shear layer flows for which the cell problems can be solved exactly as follows. The cell problem for $\mathbf{u}(\mathbf{x}) = (u(y), 0)$ in the direction $\mathbf{e}_1$ is

$$\text{(A.59)} \qquad \Delta\chi_1 + u(y)\frac{\partial}{\partial x}\chi_1 + u(y) = 0,$$

which reduces to

$$\text{(A.60)} \qquad \frac{\partial^2}{\partial y^2}\chi_1 + u(y) = 0$$

when the ansatz $\chi_1 = \chi_1(y)$ is chosen. Thus

$$\chi_1(y) = \int_0^y dy'\ H(y').$$

The effective flux

$$\sigma(\mathbf{e}_1, \mathbf{e}_2) = \langle (I + \mathbf{H})\nabla\chi_1 \cdot \mathbf{e}_2 \rangle = \left\langle \frac{\partial \chi_1}{\partial y} \right\rangle = 0.$$

On the other hand, the solution χ_2 to the cell problem in the direction $\mathbf{e}_2$ is trivially zero, and

$$\sigma(\mathbf{e}_2, \mathbf{e}_1) = \langle (I + \mathbf{H})\nabla\chi_2 \cdot \mathbf{e}_1 \rangle = 0.$$

Thus, we have

$$\sigma(\mathbf{e}_1, \mathbf{e}_2) = \sigma\mathbf{e}_2 \cdot \mathbf{e}_1 = 0.$$

Therefore, for the shear layer flows in the x- or y-directions, the effective flux tensors are symmetric, and $\mathbf{e}_1$, $\mathbf{e}_2$ are the eigenvectors of the tensors.

Appendix B. Variational principles for time-dependent flows. In this section, we derive various variational principles for the effective diffusivity in time-dependent flows by two different methods. Let us consider two-dimensional space-time periodic flows $\mathbf{u} = \mathbf{u}(x, y, t)$ that are incompressible, i.e., $\nabla \cdot \mathbf{u} = 0$. The space-time cell problem is

$$\frac{\partial}{\partial t}\chi = \varepsilon\Delta\chi + \mathbf{u}\cdot\nabla\chi + \mathbf{u}\cdot\mathbf{e}, \tag{B.1}$$

and the effective diffusivity is given by

$$\sigma_\varepsilon(\mathbf{e}) = \varepsilon + \varepsilon \fint \langle \nabla\chi \cdot \nabla\chi \rangle, \tag{B.2}$$

where $\fint$ denotes temporal average over a time period and $\langle\cdot\rangle$ for spatial average over a spatial period. In the derivation of the variational principles, we set $\varepsilon = 1$.

B.1. Variational principles from a nonlocal space-time cell formulation. Equation (B.1) can be put into divergence form

$$\nabla \cdot (\mathbf{I} + \mathbf{H} - \Delta^{-1}\partial_t)\nabla\chi + \nabla\cdot\mathbf{He} = 0 \tag{B.3}$$

or

$$\nabla\cdot(\mathbf{I} + \mathbf{H} - \Delta^{-1}\partial_t)\mathbf{E}^+ = 0\ , \quad \nabla\times\mathbf{E}^+ = 0\ , \quad \langle\mathbf{E}^+\rangle = \mathbf{e} \tag{B.4}$$

with

$$\sigma(\mathbf{e}) = \fint \langle \mathbf{E}^+ \cdot \mathbf{E}^+ \rangle.$$

Here

$$\mathbf{H} = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} \quad \text{and} \quad \nabla^\perp H = \mathbf{u}\ .$$

Consider the forward and backward cell problems

$$\nabla\cdot(\mathbf{I} + \mathbf{H} - \Delta^{-1}\partial_t)\mathbf{E}^+ = 0\ , \quad \nabla\times\mathbf{E}^+ = 0\ , \quad \langle\mathbf{E}^+\rangle = \mathbf{e},$$

$$\nabla\cdot(\mathbf{I} - \mathbf{H} + \Delta^{-1}\partial_t)\mathbf{E}^- = 0\ , \quad \nabla\times\mathbf{E}^- = 0\ , \quad \langle\mathbf{E}^-\rangle = \mathbf{e}.$$

Define the (nonlocal) fluxes by

$$\mathbf{D}^+ = (\mathbf{I} + \mathbf{H} - \Delta^{-1}\partial_t)\mathbf{E}^+,$$

$$\mathbf{D}^- = (\mathbf{I} - \mathbf{H} + \Delta^{-1}\partial_t)\mathbf{E}^-$$

and set

$$\begin{aligned} \mathbf{E}' &= \tfrac{1}{2}(\mathbf{E}^+ - \mathbf{E}^-), \\ \mathbf{E} &= \tfrac{1}{2}(\mathbf{E}^+ + \mathbf{E}^-), \\ \mathbf{D}' &= \tfrac{1}{2}(\mathbf{D}^+ - \mathbf{D}^-), \\ \mathbf{D} &= \tfrac{1}{2}(\mathbf{D}^+ + \mathbf{D}^-). \end{aligned}$$

Then we have

$$\begin{aligned} \mathbf{D}' &= \mathbf{E}' + \mathbf{H}'\mathbf{E}, \\ \mathbf{D} &= \mathbf{E} + \mathbf{H}'\mathbf{E}', \end{aligned} \tag{B.5}$$

where $\mathbf{H}' = \mathbf{H} - \Delta^{-1}\partial_t$ is a skew symmetric operator with respect to the space-time inner product. The original cell problem (B.4) is now transformed into

$$\nabla \times \mathbf{E}' = \nabla \times \mathbf{E} = 0, \tag{B.6}$$

$$\nabla \cdot \mathbf{D}' = \nabla \cdot \mathbf{D} = 0 \tag{B.7}$$

along with relations (B.5) and the mean field conditions $\langle \mathbf{E}' \rangle = 0$, $\langle \mathbf{E} \rangle = \mathbf{e}$.

The effective diffusivity can be expressed in terms of $\mathbf{E}'$, $\mathbf{E}$, $\mathbf{D}'$, and $\mathbf{D}$ as follows:

$$\begin{aligned} \sigma(\mathbf{e}) &= \frac{1}{2}⨍ \langle \mathbf{D}^+ \cdot \mathbf{e} \rangle + \frac{1}{2}⨍ \langle \mathbf{D}^- \cdot \mathbf{e} \rangle \\ &= \frac{1}{2}⨍ \langle \mathbf{D}^+ \cdot \mathbf{E}^- \rangle + \frac{1}{2}⨍ \langle \mathbf{D}^- \cdot \mathbf{E}^+ \rangle \\ &= ⨍ \langle \mathbf{D} \cdot \mathbf{E} \rangle - ⨍ \langle \mathbf{D}' \cdot \mathbf{E}' \rangle. \end{aligned} \tag{B.8}$$

Using (B.5), we can write (B.8) in the form

$$\sigma(\mathbf{e}) = ⨍ \left\langle \begin{pmatrix} -\mathbf{I} & -\mathbf{H}' \\ \mathbf{H}' & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{E} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}' \\ \mathbf{E} \end{pmatrix} \right\rangle, \tag{B.9}$$

which is a symmetric, indefinite functional whose Euler equations are (B.7) via (B.5). Therefore, $\sigma(\mathbf{e})$ comes from a saddlepoint variational principle that is

$$\sigma(\mathbf{e}) = \inf_{\substack{\nabla \times \mathbf{F} = 0 \\ \langle \mathbf{F} \rangle = \mathbf{e}}} \sup_{\substack{\nabla \times \mathbf{F}' = 0 \\ \langle \mathbf{F}' \rangle = 0}} ⨍ \left\langle \begin{pmatrix} -\mathbf{I} & -\mathbf{H}' \\ \mathbf{H}' & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{F}' \\ \mathbf{F} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{F}' \\ \mathbf{F} \end{pmatrix} \right\rangle. \tag{B.10}$$

We can eliminate the supremum by solving the corresponding Euler equation

$$\nabla \cdot \mathbf{F}' + \nabla \cdot \mathbf{H}'\mathbf{F} = 0.$$

Using projection operator, the solution has the form

$$\mathbf{F}' = -\Gamma \mathbf{H}'\mathbf{F}, \tag{B.11}$$

and, substituting (B.11) into (B.10), we have

$$\sigma(\mathbf{e}) = \inf_{\substack{\nabla \times \mathbf{F} = 0 \\ \langle \mathbf{F} \rangle = \mathbf{e}}} ⨍ \left\{ \langle \mathbf{F} \cdot \mathbf{F} \rangle + \langle \Gamma \mathbf{H}'\mathbf{F} \cdot \Gamma \mathbf{H}'\mathbf{F} \rangle \right\}, \tag{B.12}$$

which is the (direct) variational principle for the upper bound.

To obtain a reciprocal variational principle, we note that

$$(\sigma)^{-1}(\mathbf{e}) = ⨍ \langle \mathbf{D} \cdot \mathbf{E} \rangle - ⨍ \langle \mathbf{D}' \cdot \mathbf{E}' \rangle \tag{B.13}$$

if $\mathbf{D}'$, $\mathbf{D}$, $\mathbf{E}'$, and $\mathbf{E}$ satisfy (B.5), (B.6), and (B.7), subject to the mean field constraints $\langle \mathbf{D}' \rangle = 0$, $\langle \mathbf{D} \rangle = \mathbf{e}$. Inverting the relation (B.5), we have

$$\begin{aligned} \mathbf{E}' &= (\mathbf{I} - (\mathbf{H}')^2)^{-1}(\mathbf{D}' - \mathbf{H}'\mathbf{D}), \\ \mathbf{E} &= (\mathbf{I} - (\mathbf{H}')^2)^{-1}(\mathbf{D} - \mathbf{H}'\mathbf{D}'). \end{aligned} \tag{B.14}$$

Note that $-(\mathbf{H}')^2$ is nonnegative. In terms of $\mathbf{D}'$ and $\mathbf{D}$ via (B.14), (B.13) is a symmetric, indefinite functional whose Euler equations are (B.6). Therefore

$$(\sigma)^{-1}(\mathbf{e}) = \inf_{\substack{\nabla\cdot\mathbf{G}=0 \\ \langle\mathbf{G}\rangle=\mathbf{e}}} \sup_{\substack{\nabla\cdot\mathbf{G}'=0 \\ \langle\mathbf{G}'\rangle=0}} \fint \left\langle (\mathbf{I} - (\mathbf{H}')^2)^{-1} \begin{pmatrix} -\mathbf{I} & \mathbf{H}' \\ -\mathbf{H}' & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{G}' \\ \mathbf{G} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{G}' \\ \mathbf{G} \end{pmatrix} \right\rangle. \tag{B.15}$$

We can eliminate the supremum by solving the corresponding Euler equations to establish the inverse variational principle for the lower bound of $\sigma(\mathbf{e})$. However, this variational principle seems useless because the operator $(\mathbf{I} - (\mathbf{H}')^2)^{-1}$ is difficult to work with.

In the next section, we derive different variational principles that are easier to use.

B.2. Variational principles from a local, augmented, space-time cell formulation. This approach is based on the following simple observation. If, instead of (B.1), we consider

$$-\frac{\partial}{\partial t}\chi' = \Delta\chi' + \mathbf{u}\cdot\nabla\chi' + \mathbf{u}\cdot\mathbf{e} \tag{B.16}$$

with $\chi' = \chi'(x, y, t)$ space-time periodic, then the effective diffusivity is again given by

$$\sigma(\mathbf{e}) = 1 + \fint \langle \nabla\chi' \cdot \nabla\chi' \rangle. \tag{B.17}$$

This can be readily seen, since the right-hand side of (B.17) has a variational formulation similar to (B.12) with $\mathbf{H}'$ replaced by $\mathbf{H} + \Delta^{-1}\partial_t$. The infima are the same since both trial fields $\mathbf{F}(x, y, t)$ and $\mathbf{F}(x, y, -t)$ are admissible.

Consider now the following extended coordinate space (x, y, t, w) and an extended cell problem:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{\chi} &= \Delta\tilde{\chi} + \mathbf{u}\cdot\nabla\tilde{\chi} + \mathbf{u}\cdot\mathbf{e} \qquad \text{when } 0 < w \leq \tfrac{1}{2}, \\ -\frac{\partial}{\partial t}\tilde{\chi} &= \Delta\tilde{\chi} + \mathbf{u}\cdot\nabla\tilde{\chi} + \mathbf{u}\cdot\mathbf{e} \qquad \text{when } -\tfrac{1}{2} < w \leq 0, \end{aligned} \tag{B.18}$$

where (B.18) is periodized in w with period 1. The function $\tilde{\chi} = \tilde{\chi}(x, y, t, w)$ is simply

$$\begin{aligned} &\chi \text{ defined by } (B.1) && \text{when } 0 < w \leq \tfrac{1}{2}, \\ &\chi' \text{ defined by } (B.16) && \text{when } -\tfrac{1}{2} < w \leq 0. \end{aligned}$$

Let us introduce the following notation:

(a) the extended gradient: $\tilde{\nabla} = (\nabla, \partial_t, \partial_w)$;

(b) the extended intensity: $\tilde{\mathbf{E}} = \tilde{\nabla}\tilde{\chi} + \tilde{\mathbf{e}}$ with $\tilde{\mathbf{e}} = (\mathbf{e}, 0, 0)$;

(c) the extended velocity: $\mathbf{u}(x, y, w, t) = (\mathbf{u}, \pm 1, 0)$ depending on the sign of w, mod 1;

(d) the extended average: $\langle\langle \cdot \rangle\rangle = \int_0^1 dw f \langle \cdot \rangle$.

Note that $\tilde{u}$ is incompressible in the extended space (x, y, t, w) and has zero mean. Thus there exists a periodic skew symmetric matrix $\widetilde{\mathbf{H}}$, such that $\tilde{\nabla} \cdot \widetilde{\mathbf{H}} = \tilde{\mathbf{u}}$. In fact,

$$\widetilde{\mathbf{H}} = \begin{pmatrix} \mathbf{H} & 0 \\ 0 & \mathbf{L} \end{pmatrix},$$

where

$$\mathbf{H} = \begin{pmatrix} 0H & \\ -H & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix},$$

where $L = L(w)$ is a piecewise linear sawtooth function defined by

$$L(w) = \begin{cases} \frac{1}{2}w & \text{when} \quad 0 < w \le \frac{1}{2}, \\ -\frac{1}{2}w & \text{when} \quad -\frac{1}{2} < w \le 0 . \end{cases}$$

With this notation, (B.18) can be put into divergence form

$$\tilde{\nabla} \cdot (\mathbf{I} + \widetilde{\mathbf{H}})\widetilde{\mathbf{E}}^+ = 0, \qquad \widetilde{\mathbf{E}}^+ = \tilde{\nabla}\tilde{\chi} + \tilde{\mathbf{e}}, \tag{B.19}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\mathbf{I}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that $\mathbf{I} + \mathbf{I}'$ is the identity matrix in the extended space, which is denoted by $\tilde{\mathbf{I}}$, i.e., $\tilde{\mathbf{I}} = \mathbf{I} + \mathbf{I}'$.

The effective diffusivity is given by

$$\sigma(\mathbf{e}) = \langle\langle \mathbf{I}\widetilde{\mathbf{E}}^+ \cdot \widetilde{\mathbf{E}}^+ \rangle\rangle.$$

Since the extended space is four-dimensional, it is not convenient to use gradient and curl, and we use differential forms to interpret (B.19). The field $\widetilde{\mathbf{E}}^+$ is a 1-form such that $d\widetilde{\mathbf{E}}^+ = 0$, $\langle\langle \widetilde{\mathbf{E}}^+ \rangle\rangle = \tilde{\mathbf{e}}$, and

$$d{*}(\mathbf{I} + \widetilde{\mathbf{H}})\widetilde{\mathbf{E}}^+ = 0 , \tag{B.20}$$

where d is the exterior derivative and $*$ is the Hodge star operator on the four-dimensional torus (see [24]).

Next, we carry out the symmetrization procedure as before by considering the following forward and backward problems:

$$\text{(B.21)}\qquad \begin{aligned} &d{*}(\mathbf{I}+\widetilde{\mathbf{H}})\widetilde{\mathbf{E}}^{+}=0, \quad d\widetilde{\mathbf{E}}^{+}=0, \quad \langle\!\langle \widetilde{\mathbf{E}}^{+}\rangle\!\rangle=\tilde{\mathbf{e}},\\ &d{*}(\mathbf{I}-\widetilde{\mathbf{H}})\widetilde{\mathbf{E}}^{-}=0, \quad d\widetilde{\mathbf{E}}^{-}=0, \quad \langle\!\langle \widetilde{\mathbf{E}}^{-}\rangle\!\rangle=\tilde{\mathbf{e}}. \end{aligned}$$

Let

$$\text{(B.22)}\qquad \begin{aligned} \widetilde{\mathbf{D}}^{+}&={*}(\mathbf{I}+\widetilde{\mathbf{H}})\widetilde{\mathbf{E}}^{+},\\ \widetilde{\mathbf{D}}^{-}&={*}(\mathbf{I}-\widetilde{\mathbf{H}})\widetilde{\mathbf{E}}^{-} \end{aligned}$$

and

$$\text{(B.23)}\qquad \begin{aligned} \widetilde{\mathbf{E}}'&=\tfrac{1}{2}(\widetilde{\mathbf{E}}^{+}-\widetilde{\mathbf{E}}^{-}),\\ \widetilde{\mathbf{E}}&=\tfrac{1}{2}(\widetilde{\mathbf{E}}^{+}+\widetilde{\mathbf{E}}^{-}),\\ \widetilde{\mathbf{D}}'&=\tfrac{1}{2}(\widetilde{\mathbf{D}}^{+}-\widetilde{\mathbf{D}}^{-}),\\ \widetilde{\mathbf{D}}&=\tfrac{1}{2}(\widetilde{\mathbf{D}}^{+}+\widetilde{\mathbf{D}}^{-}). \end{aligned}$$

Note that (B.22) are local relations in the extended space. The relations between $\widetilde{\mathbf{E}}'$, $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{D}}'$, and $\widetilde{\mathbf{D}}$ are

$$\text{(B.24)}\qquad \begin{aligned} \widetilde{\mathbf{D}}'&={*}(\mathbf{I}\widetilde{\mathbf{E}}'+\widetilde{\mathbf{H}}\widetilde{\mathbf{E}}),\\ \widetilde{\mathbf{D}}&={*}(\mathbf{I}\widetilde{\mathbf{E}}+\widetilde{\mathbf{H}}\widetilde{\mathbf{E}}') \end{aligned}$$

or in matrix form

$$\text{(B.25)}\qquad \begin{pmatrix} -\widetilde{\mathbf{D}}' \\ \widetilde{\mathbf{D}} \end{pmatrix} = {*}\begin{pmatrix} -\mathbf{I} & -\widetilde{\mathbf{H}} \\ \widetilde{\mathbf{H}} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \widetilde{\mathbf{E}}' \\ \widetilde{\mathbf{E}} \end{pmatrix},$$

which is a symmetric, indefinite form. We have that

$$\begin{aligned} \sigma(\mathbf{e}) &= \tfrac{1}{2}\langle\!\langle(-{*})\widetilde{\mathbf{D}}^{+}\cdot\tilde{\mathbf{e}}\rangle\!\rangle+\tfrac{1}{2}\langle\!\langle(-{*})\widetilde{\mathbf{D}}^{-}\cdot\tilde{\mathbf{e}}\rangle\!\rangle,\\ &= \tfrac{1}{2}\langle\!\langle(-{*})\widetilde{\mathbf{D}}^{+}\cdot\widetilde{\mathbf{E}}^{-}\rangle\!\rangle+\tfrac{1}{2}\langle\!\langle(-{*})\widetilde{\mathbf{D}}^{-}\cdot\widetilde{\mathbf{E}}^{+}\rangle\!\rangle,\\ &= \langle\!\langle(-{*})\widetilde{\mathbf{D}}\cdot\widetilde{\mathbf{E}}\rangle\!\rangle-\langle\!\langle(-{*})\widetilde{\mathbf{D}}'\cdot\widetilde{\mathbf{E}}'\rangle\!\rangle. \end{aligned}$$

The minus sign is due to the identity ${*}{*}\mathbf{E}=-\mathbf{E}$ for 1-forms in four dimensions. As before, we have a saddlepoint variational principle in view of (B.25)

$$\text{(B.26)}\qquad \sigma(\mathbf{e}) = \inf_{\substack{d\widetilde{\mathbf{F}}=0\\ \langle\!\langle \mathbf{F}\rangle\!\rangle=\tilde{\mathbf{e}}}} \ \sup_{\substack{d\widetilde{\mathbf{F}}=0\\ \langle\!\langle \mathbf{F}\rangle\!\rangle=0}} \left\langle\!\!\left\langle \begin{pmatrix} -\mathbf{I} & -\widetilde{\mathbf{H}} \\ \widetilde{\mathbf{H}} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \widetilde{\mathbf{F}}' \\ \widetilde{\mathbf{F}} \end{pmatrix}\cdot\begin{pmatrix} \widetilde{\mathbf{F}}' \\ \widetilde{\mathbf{F}} \end{pmatrix}\right\rangle\!\!\right\rangle.$$

We can eliminate the supremum by solving the corresponding Euler equation

$$\text{(B.27)}\qquad d{*}\mathbf{I}\widetilde{\mathbf{F}}'+d{*}\widetilde{\mathbf{H}}\widetilde{\mathbf{F}}=0$$

in the following manner. Set $\widetilde{\mathbf{F}}'=df$ with f is a periodic function in the extended space since $\langle\!\langle\widetilde{\mathbf{F}}'\rangle\!\rangle=0$. Then f satisfies

$$\text{(B.28)}\qquad (d{*}\mathbf{I}d)f=-d{*}\widetilde{\mathbf{H}}\widetilde{\mathbf{F}}.$$

The left-hand side of (B.28) is simply the spatial Laplacian Δ over the spatial period that is invertible. Thus

$$f = -(d{*}\mathbf{I}d)^{-1} d{*}\widetilde{\mathbf{H}}\widetilde{\mathbf{F}}. \tag{B.29}$$

When (B.29) is substituted into (B.26), we have

$$\begin{aligned} \sigma(\mathbf{e}) &= \inf_{\substack{d\widetilde{\mathbf{F}}=0 \\ \langle\langle \widetilde{\mathbf{F}}\rangle\rangle = \tilde{\mathbf{e}}}} \langle\langle \mathbf{I}\widetilde{\mathbf{F}}' \cdot \widetilde{\mathbf{F}}'\rangle\rangle + \langle\langle \mathbf{I}\widetilde{\mathbf{F}} \cdot \widetilde{\mathbf{F}}\rangle\rangle \\ &= \inf_{\substack{d\widetilde{\mathbf{F}}=0 \\ \langle\langle \widetilde{\mathbf{F}}\rangle\rangle = \tilde{\mathbf{e}}}} \langle\langle \mathbf{I}\widetilde{\mathbf{F}} \cdot \widetilde{\mathbf{F}}\rangle\rangle + \langle\langle (d{*}\mathbf{I}d)^{-1} d{*}\widetilde{\mathbf{H}}\widetilde{\mathbf{F}} \cdot d{*}\widetilde{\mathbf{H}}\widetilde{\mathbf{F}}\rangle\rangle. \end{aligned} \tag{B.30}$$

It is not hard to see that, after some algebra, (B.30) is the same as (B.12).

To obtain an inverse variational principle, we note that

$$(\sigma)^{-1}(\mathbf{e}) = \langle\langle (-{*})\widetilde{\mathbf{D}} \cdot \widetilde{\mathbf{E}}\rangle\rangle - \langle\langle (-{*})\widetilde{\mathbf{D}}' \cdot \widetilde{\mathbf{E}}'\rangle\rangle \tag{B.31}$$

with $\langle\langle \widetilde{\mathbf{D}}'\rangle\rangle = 0$ and $\langle\langle \widetilde{\mathbf{D}}\rangle\rangle = *\tilde{\mathbf{e}}$. We now invert (B.25), which is a local operation, and we have

$$\begin{aligned} \widetilde{\mathbf{E}}' &= -\frac{1}{1+H^2}\mathbf{I}{*}\widetilde{\mathbf{D}}' + \left(\frac{1}{1+H^2}\mathbf{H} + \frac{1}{L}\mathbf{J}'\right){*}\widetilde{\mathbf{D}}, \\ \widetilde{\mathbf{E}} &= -\frac{1}{1+H^2}\mathbf{I}{*}\widetilde{\mathbf{D}} + \left(\frac{1}{1+H^2}\mathbf{H} + \frac{1}{L}\mathbf{J}'\right){*}\widetilde{\mathbf{D}}', \end{aligned} \tag{B.32}$$

where

$$\mathbf{J}' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

As before, we have a saddlepoint variational principle for (B.31),

$$\begin{aligned} &(\sigma)^{-1}(\mathbf{e}) \\ &= \inf_{\substack{d\widetilde{\mathbf{G}}=0 \\ \langle\langle \widetilde{\mathbf{G}}\rangle\rangle = *\tilde{\mathbf{e}}}} \ \sup_{\substack{d\widetilde{\mathbf{G}}'=0 \\ \langle\langle \widetilde{\mathbf{G}}'\rangle\rangle = 0}} \left\langle\!\left\langle \begin{pmatrix} -\frac{\mathbf{I}}{1+H^2} & \frac{1}{1+H^2}\mathbf{H} + \frac{1}{L}\mathbf{J}' \\ -\frac{1}{1+H^2}\mathbf{H} - \frac{1}{L}\mathbf{J}' & \frac{\mathbf{I}}{1+H^2} \end{pmatrix} \begin{pmatrix} *\widetilde{\mathbf{G}}' \\ *\widetilde{\mathbf{G}} \end{pmatrix} \cdot \begin{pmatrix} *\widetilde{\mathbf{G}}' \\ *\widetilde{\mathbf{G}} \end{pmatrix} \right\rangle\!\right\rangle. \end{aligned} \tag{B.33}$$

The Euler equation for the supremum is

$$d\frac{1}{1+H^2}\mathbf{I}{*}\widetilde{\mathbf{G}}' - d\left(\frac{1}{1+H^2}\mathbf{H} + \frac{1}{L}\mathbf{J}'\right){*}\widetilde{\mathbf{G}} = 0\ , \tag{B.34}$$

and, when (B.34) holds, (B.33) can be simplified to

$$(\sigma)^{-1}(\mathbf{e}) = \inf_{\substack{d\widetilde{\mathbf{G}}=0 \\ \langle\langle \widetilde{\mathbf{G}}\rangle\rangle = *\tilde{\mathbf{e}}}} \left\langle\!\left\langle \frac{1}{1+H^2}\mathbf{I}{*}\widetilde{\mathbf{G}} \cdot *\widetilde{\mathbf{G}} \right\rangle\!\right\rangle + \left\langle\!\left\langle \frac{1}{1+H^2}\mathbf{I}{*}\widetilde{\mathbf{G}}' \cdot *\widetilde{\mathbf{G}}' \right\rangle\!\right\rangle. \tag{B.35}$$

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REFERENCES

[1] S. Childress, *Alpha-effect in flux ropes and sheets*, Phys. Earth Planet Internat., 20 (1979), pp. 172–180.

[2] M. N. Rosenbluth, H. L. Berk, I. Doxas, and W. Horton, *Effective diffusion in laminar convective flows*, Phys. Fluids, 30 (1987), pp. 2636–2647.

[3] B. Shraiman, *Diffusive transport in a Rayleigh–Benard convection cell*, Phys. Rev. A, 36 (1987), p. 261.

[4] A. M. Soward, *Fast dynamo action in steady flow*, J. Fluid Mech., 180 (1987), pp. 267–295.

[5] S. Childress and A. M. Soward, *Scalar transport and alpha-effect for a family of cat's-eye flows*, J. Fluid Mech., 205 (1989), pp. 99–133.

[6] A. M. Soward and S. Childress, *Large magnetic Reynolds number dynamo action in a spatially periodic flow with mean motion*, Philos. Trans. Roy. Soc. London Ser. A, 331 (1990), pp. 649–733.

[7] K. Golden and G. C. Papanicolaou, *Bounds for effective parameters of heterogeneous media by analytic continuation*, Comm. Math. Phys., 90 (1983), pp. 473–491.

[8] M. Avellaneda and A. J. Majda, *An integral representation and bounds on the effective diffusivity in passive advection by laminar and turbulent flows*, Comm. Math. Phys., 138 (1991), pp. 339–391.

[9] R. N. Bhattacharya, V. K. Gupta, and H. F. Walker, *Asymptotics of solute dispersion in periodic porous media*, SIAM J. Appl. Math., 49 (1989), pp. 86–98.

[10] S. M. Kozlov, *Geometric aspects of averaging*, Russian Math. Surveys, 44 (1989), pp. 91–144.

[11] A. Bensoussan, J. L. Lions, and G. C. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.

[12] G. W. Milton, *On characterizing the set of possible effective tensors of composites: the variational method and the translational method*, Comm. Pure Appl. Math., 43 (1990), pp. 63–125.

[13] ———, *Classical Hall effect in two-dimensional composites: A characterization of the set of realizable effective conductivity tensors*, Phys. Rev. B, 38 (1988), pp. 296–303.

[14] A. V. Gruzinov, M. B. Isichenko, and Ya. L. Kalda, *Two-dimensional turbulent diffusion*, Soviet Phys. JETP, 70 (1990), pp. 263–269.

[15] M. B. Isichenko and J. Kalda, *Statistical topography* II. 2D *transport of a passive scalar*, J. Nonlinear Sci., 1 (1991), pp. 375–396.

[16] M. B. Isichenko, *Percolation, statistical topography and transport in random media*, Rev. Modern Phys., 64 (1992), pp. 961–1043.

[17] A. Fannjiang and G. Papanicolaou, *Convection enhanced diffution for random flows*, manuscript.

[18] J. Milnor, *Morse Theory*, Princeton University Press, Princeton, NJ, 1970.

[19] T. Hou and X. Xin, *Homogenization of linear transport equations with oscillatory vector fields*, SIAM J. Appl. Math., 52 (1992), pp. 32–45.

[20] Weinan E, *Homogenization of Linear and Nonlinear Transport Equations*, preprint, 1991.

[21] Ya Sinai, *Introduction to Ergodic Theory*, Princeton University Press, Princeton, NJ, 1977.

[22] K. Oelschlager, *Homogenization of a diffusion process in a divergence-free random field*, Ann. Probab. 16 (1988), pp. 1084–1126.

[23] V. V. Zhikov, *Remarks on the problem of residual diffusion*, Comm. Moscow Math. Soc., 44 (1989), pp. 194–195.

[24] C. von Westenholz, *Differential Forms in Mathematical Physics*, North-Holland, Amsterdam, 1981.

[25] A. J. Majda and R. M. McLaughlin, *The effect of mean flows on enhanced diffusivity in transport by incompressible periodic velocity fields*, Stud. Appl. Math., 89 (1993), pp. 245–279.

[26] M. Avellaneda, S. Torquato, and I. C. Kim, *Diffusion and geometric effects in passive advection by random arrays of vortices*, Phys. Fluids A, 3 (1991), pp. 1880–1891.