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CONVECTION IN ROTATING STARS

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SUMMARY

The influence of rotation on convection in stellar interiors is discussed. It is shown that the character of convective motions can be affected in relatively slowly rotating stars. Although the transport of energy by convection is anisotropic, the requirement that the divergence of total energy flux vanishes means that meridional circulation currents will be set up. These will mix the material of a convective core in a time which is not much different from the time taken by convection in the absence of rotation.

Although rotation does not seriously impede mixing processes in a convective core, the circulation transports angular momentum and changes the law of rotation of the star. The final state of rotation cannot be determined until the form of convection in a rotating star is better understood but it is *possible* that the asymptotic state is closer to one of uniform angular momentum than uniform angular velocity.

1. INTRODUCTION

It is well known that the criterion for the onset of thermal convection is affected by rotation. A detailed discussion for the case of a Boussinesq fluid may be found in Chandrasekhar (1961) and a discussion of the more limited progress made in the case of a compressible fluid may be found in Cowling (1951). The influence of rotation on the onset of convection is rather complicated. Because rotation introduces a preferred direction into the system the effect is anisotropic; convective motions in some directions are strongly inhibited whilst in other directions they are scarcely affected. Although the effect of rotation on fully developed convection is unclear, it seems certain that it will lead to both a reduction and an anisotropy in convective heat transport and mixing processes. Despite this, in most calculations of the interior structure of rotating stars, the influence of rotation on the properties of convective zones has been neglected. This paper is concerned with investigating whether this neglect is serious and particular emphasis is placed on the problem of convective cores. The effect of rotation on convection in stellar envelopes, with particular reference to the solar differential rotation, has previously been discussed by many authors; see Durney (1972) and references therein.

It might be thought that the problem only arises in rapidly rotating stars but this is not true. Convection is affected by rotation if a parameter measuring the importance of rotational forces compared with gravity is comparable not with unity but with the difference between the actual value of PdT/TdP in the convective zone and the adiabatic value. In convective cores, in particular, this difference is very small and this means that rotation affects convection even in quite slowly rotating stars. Because of this and for mathematical simplicity, in what follows we restrict our attention to stars which are rotating so slowly that they may be assumed to be spherically symmetrical. The results obtained should, however, also apply to stars which rotate so rapidly that they are seriously distorted from spherical shape.

Another assumption, which is frequently made, is that convective cores rotate uniformly. There have been arguments that convective zones must rotate as solid bodies, but these are far from convincing, and the assumption is usually made for mathematical simplicity. The main conclusion of this paper is that the ultimate effect of the interference of rotation with convection in convective cores is not to lead to an anisotropic transport of energy but to place strong constraints on the way in which convective zones rotate.

For the simple case of a convective core assumed initially to rotate uniformly, the argument is as follows. The energy transport by convection depends on the value of $\nabla - \nabla_{ad}$ (where $\nabla \equiv PdT/TdP$ and ∇_{ad} is its adiabatic value) and on the polar angle θ between the axis of rotation and the radial direction at the point considered; the latter dependence arises because of the interference of rotation with convection. The simplest solution to the problem of the anisotropic flux of energy would appear to be a slight departure of the surfaces of constant P and T so that $\nabla - \nabla_{ad}$ is also latitude dependent. This possibility is not, however, allowed by the equation of motion which prescribes that surfaces of constant P and T must coincide to a high degree of accuracy; such departures as exist are caused by the convective motions and do not have a simple latitude dependence. It thus appears that an anisotropic convective flux is unavoidable but this means that it (or the sum of it and the radiative flux) cannot be divergence free, as is required by the thermal equation in the absence of energy sources or sinks or of ordered motions. Just as in the case of radiative zones in rotating stars, this flux divergence drives a meridional circulation which mixes material and transports energy through the convective zone and it is found that the mixing time is not greatly different from that produced by convection if the influence of rotation on convection is neglected.

These meridional motions transport angular momentum through the convective core and must modify the initial law of rotation. The way in which the law is modified depends on the precise form of anisotropy of convective motions but it appears plausible that the tendency would be for the regions near the axis of rotation to rotate more rapidly. It has previously been pointed out by Biermann (1958) and Kippenhahn (1963) that convective motions are almost certainly anisotropic, with the preferred direction being the radial direction, even if the effect of rotation on convection is neglected, and that this anisotropy also drives a meridional circulation. Both of these effects must be considered before the law of rotation of convective cores is fully understood.

2. THE MIXING LENGTH THEORY OF CONVECTION

There is not at present a really reliable theory which calculates the energy transport by fully developed convection. Probably the best in stellar conditions is the mixing length theory introduced by Biermann (1932, 1945); for recent accounts see Vitense (1953), Böhm-Vitense (1958). It will prove convenient to use it from time to time in what follows and we therefore summarize some of the basic ideas and formulae of the theory.

In the mixing length theory, elements whose sizes are of the order of mixing length, l, are supposed to move almost adiabatically through a distance l before they are mixed with their surroundings. The value of the mixing length is not prescribed but it is generally supposed to be of order the pressure scale height $H_p(\equiv P/|dP/dr|)$; we shall not need to assume a particular value for l in most of what follows. In the mixing length theory, the typical speed of a convective element is

$$v_{\text{conv}} \approx \frac{g^{1/2} l}{2 H_p^{1/2}} (\nabla - \nabla_{\text{ad}})^{1/2}, \tag{2.1}$$

where g is the local value of the gravitational acceleration, and the convective flux is

$$\begin{aligned} F_{\text{conv}} &\approx \frac{5 P g^{1/2} l^{1/2}}{8 H_p^{3/2}} (\nabla - \nabla_{\text{ad}})^{3/2} \\ &\approx \frac{5 P v_{\text{conv}} l}{4 H_p} (\nabla - \nabla_{\text{ad}}). \end{aligned} \tag{2.2}$$

In different versions of the mixing length theory, the numerical coefficients in (2.1) and (2.2) vary slightly.

There is, in reality, some problem in using the mixing length theory in convective cores. In the mixing length theory, we implicitly assume that the scale of convective motions is small compared with the size of the convective region, whereas in a convective core the pressure scale height is often comparable with the core radius. In this case, a single convection cell may extend throughout the core. Although this would affect the detailed arguments of the present paper, it should not affect the main conclusion, which is that large scale motions will exist in a rotating convective core. If convection in the presence of rotation has a small scale perpendicular to the axis of rotation, the meridional circulation discussed in this paper will arise; otherwise convection itself will transport angular momentum throughout the core.

3. ONSET OF CONVECTION IN THE PRESENCE OF ROTATION

At different stages in this paper it will prove convenient to use both spherical polar coordinates (r, θ, ϕ) and cylindrical polar coordinates (ϖ, ϕ, z). In the discussion of stability criteria, the system (ϖ, ϕ, z) is more convenient.

In the absence of rotation, the condition for the instability of disturbances of all types is the same, being the Schwarzschild criterion

$$\nabla > \nabla_{\text{ad}}. \tag{3.1}$$

This is no longer true in the presence of rotation and the ϕ independent and the ϕ dependent perturbations must be considered separately. The discussion of the ϕ independent perturbations is quite straightforward. These are interchange perturbations of the type shown in Fig. 1 in which two tubes of material symmetrical about the axis of rotation are interchanged, it being assumed that the interchange is so rapid that the changes are adiabatic and each element conserves its angular momentum. The system is unstable if the interchange releases energy. The stability criteria are known as the Solberg–Høiland criteria and they have been discussed recently by Fricke & Smith (1971), for example. An interchange of the type shown in Fig. 1 releases energy and is therefore unstable if and only if

$$\begin{aligned} &\cos^2\theta(\nabla - \nabla_{\text{ad}})\delta z^2 + 2\cos\theta\sin\theta(\nabla - \nabla_{\text{ad}})\,\delta\varpi\delta z \\ &+ \sin^2\theta\left(\nabla - \nabla_{\text{ad}} - \frac{4P\Omega^2}{\rho g^2 \sin^2\theta}\right)\delta\varpi^2 > 0, \end{aligned} \tag{3.2}$$

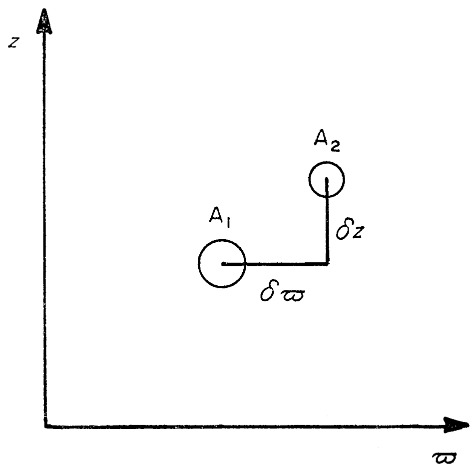

FIG. 1. *Interchange perturbations. An interchange perturbation occurs if a toroidal tube of cross-section A_1 is interchanged with a tube of cross-section A_2.*

where ρ is the density and Ω the (uniform) angular velocity of the star. In obtaining this criterion it has been assumed that the perturbation does not change the gravitational field in the star and that

$$\Omega^2 r^3/GM \ll 1, \tag{3.3}$$

where M is the mass contained within radius r, so that all equilibrium quantities depend on r alone.

From (3.2), it can be seen that, if $\nabla > \nabla_{ad}$, interchanges in the z direction are possible (as is obvious intuitively) but that interchanges in the ϖ direction are only possible if

$$\nabla > \nabla_{ad} + \frac{4P\Omega^2}{\rho g^2 \sin^2 \theta}. \tag{3.4}$$

At the equator (where $\sin \theta = 1$), ϖ interchanges are in the radial direction, which is the direction in which we expect convection to transport energy. Clearly interchanges there will be seriously affected by rotation if

$$4P\Omega^2/\rho g^2(\nabla - \nabla_{ad}) \gtrsim 1. \tag{3.5}$$

Criterion (3.5) can be rewritten, by using $dP/dr = -\rho g$, as

$$4\Omega^2 H_p/g(\nabla - \nabla_{ad}) \gtrsim 1. \tag{3.6}$$

If, in addition, we assume that the mixing length theory of convection is valid and use equation (2.1) we can write (3.6) as

$$\Omega^2 l^2/v_{conv}^2 \gtrsim 1, \tag{3.7}$$

which has an appealing simplicity. It states that interchanges will be significantly affected by rotation if the lifetime of a convective element ($\sim l/v_{conv}$) exceeds the rotation period of the star; this means that Coriolis force can have a significant effect on an element. This sounds like a general result which must transcend the uncertainties of the mixing length theory of convection.

Before discussing non-axisymmetric disturbances, we should perhaps discuss the validity of considering the interchange of complete toroidal rings. Such a coherent disturbance is presumably only possible if sound propagates many times around a toroidal ring in its lifetime. Thus we require to order of magnitude

$$r^2/(P/\rho) \ll l^2/v_{\mathrm{conv}}^2,$$

or

$$r^2/H_{\mathrm{p}}g \ll l^2/v_{\mathrm{conv}}^2. \tag{3.8}$$

If once again (2.1) is used, (3.8) becomes

$$\nabla - \nabla_{\mathrm{ad}} \ll H_{\mathrm{p}}^2/r^2. \tag{3.9}$$

This criterion will certainly be satisfied in a convective core, where H_{p} is comparable in size with r or greater and $\nabla - \nabla_{\mathrm{ad}} \ll 1$, but it may well be violated in convective envelopes.

Above we have considered only the axisymmetric perturbations which may not be the most important instabilities. It is not possible to obtain, in a closed form, a criterion which expresses the influence of rotation on all possible $m \neq 0$ disturbances. It is however possible to study stability against very localized perturbations, which essentially means perturbations of wavelength small compared to a scale height. This was done by Cowling (1951). In the present notation, he showed that a uniformly rotating star would be unstable if

$$\nabla > \nabla_{\mathrm{ad}} + \frac{4P\Omega^2}{\rho g^2} \frac{k_z^2}{k_\phi^2 + (k_\varpi \cos\theta - k_z \sin\theta)^2}, \tag{3.10}$$

where k_ϖ, k_ϕ, k_z are wavenumbers in the ϖ, ϕ and z directions ($k_\phi = m/r$).

This criterion can be seen to include those for $m = 0$ as a special case; in particular (3.4) can be obtained by putting $k_\phi = 0$ and $k_z = \infty$. Although the implications of (3.10) are more complicated than those of the $m = 0$ criterion, it is clear that only a small set of perturbations will be seriously affected by rotation if $4P\Omega^2/\rho g^2(\nabla - \nabla_{\mathrm{ad}}) \ll 1$ but that a significant number will be affected once (3.5) is satisfied. As it is believed that the perturbations which are most effective in carrying energy are comparable in size with H_{p}, an extension of (3.10) to non-localized disturbances is really needed. Cowling expressed the view that the effect of rotation on long wavelength disturbances would not be very different from the effect on localized disturbances but this has not been proved rigorously. He also pointed out that some of the disturbances which are allowed by (3.10) involve significant motions perpendicular to the rotation axis so that it is not easy to decide how anisotropic the flow of energy will be. We shall have something further to say about non-axisymmetric disturbances later but at present we will simply adopt (3.5) as the condition for rotation to affect convection seriously.

A recent discussion of thermal instabilities in rapidly rotating systems is by Busse (1970). He has studied the onset of convection in a sphere using the Boussinesq approximation and has found that marginally stable modes are aligned along the axis of rotation.

Criterion (3.5), or equivalently (3.6) or (3.7) can be satisfied quite easily in a convective core of a star which is not rotating extremely rapidly, since $\nabla - \nabla_{\mathrm{ad}}$ is very small in convective cores. A crude estimate of $\nabla - \nabla_{\mathrm{ad}}$ at a typical point in the convective core of a star of $10\,M_\odot$ gives a value between 10^{-5} and 10^{-6}. If we calculate $4\Omega^2 H_{\mathrm{p}}/g$ at a corresponding point for a surface rotation speed of 100 km s^{-1} we obtain $4\Omega^2 H_{\mathrm{p}}/g \approx 5 \times 10^{-5}$, so that the criterion for significant influence of rotation on convection is comfortably satisfied and it would probably also be satisfied for a surface speed of 30 km s^{-1}. With the values chosen, the maximum value of $\Omega^2 r^3/GM$ is $2{\cdot}5 \times 10^{-3}$ so that (3.3) is satisfied and the star can be assumed spherical. In addition, with a typical value of $H_p{}^2/r^2$ not much less than unity, (3.9) is easily satisfied.

4. THE STRUCTURE OF ROTATING CONVECTIVE CORES

Having indicated that the flow of energy by convection is likely to be anisotropic both in the sense that it will be latitude dependent and that at most points the flow will not be in the radial direction, it is necessary to ask what effect this has on the structure of the convective core. The first naive idea is that, as the flow of energy would like to be spherically symmetrical, small deviations between the surfaces of constant P and constant T are produced, so that $\nabla - \nabla_{ad}$ varies with latitude and the isotropic flow of energy is restored. It is, however, easy to see that this cannot be true by considering the dynamical equation.

If the inertial forces due to the convective motions are neglected, the equation of hydrostatic equilibrium in a uniformly rotating star is

$$\operatorname{grad} P = \rho \operatorname{grad} \Psi, \tag{4.1}$$

where

$$\Psi = \Phi + \tfrac{1}{2}\Omega^2 \varpi^2 \tag{4.2}$$

and Φ is the gravitational potential. It can be deduced immediately from equation (4.1) that surfaces of constant P, ρ and Ψ are coincident as are the surfaces of constant T. This implies that ∇ is constant on Ψ surfaces, or in a slowly rotating star on spheres, to a high degree of approximation.

In fact there will be variations of $\nabla - \nabla_{ad}$ around a sphere which will be comparable with $\nabla - \nabla_{ad}$ itself. It is easy to understand this if we first consider convection in a non-rotating star, as according to the mixing length theory rising and falling elements, which move essentially adiabatically, have 'temperature gradients' ∇_{ad} following their motions instead of the mean gradient ∇. The result can also be verified by considering the inertial terms due to the convective motions in equation (4.1). To a first approximation

$$\rho v_{conv}{}^2/l \approx \delta(\nabla - \nabla_{ad}), \tag{4.3}$$

where $\delta(\nabla - \nabla_{ad})$ is its variation from its mean value. Using the expression (2.1) for v_{conv}, (4.3) becomes

$$\delta(\nabla - \nabla_{ad}) \approx \frac{l}{4H_p}(\nabla - \nabla_{ad}). \tag{4.4}$$

Although (4.4) appears to contradict our earlier statement that $\nabla - \nabla_{ad}$ is constrained to be constant on spheres, the contradiction is not really serious. In order to produce a symmetrical flow of energy by convection, $\nabla - \nabla_{ad}$ would have to vary in a specific latitude dependent way determined by the manner in which rotation affects convection, whereas the variations in $\nabla - \nabla_{ad}$ due to convection itself are irregular fluctuations of order $\nabla - \nabla_{ad}$ on the characteristic scale of a mixing length.*

If it is accepted that $\nabla - \nabla_{ad}$ cannot vary on a sphere so as to produce a spherical outflow of energy, it is necessary to ask what are the consequences of a latitude dependent convective flux. Ideally a theory of fully developed convection in the

* As mentioned in Section 2, this statement will need to be modified if a single convection cell extends from pole to equator.

presence of rotation would give the exact dependence of flux on latitude but such a theory does not exist at present. In the past, there has been considerable dispute about the effect of rotation on the flow of energy by convection, as opposed to its effect on the criterion for the onset of convection. The early studies were confined to the axisymmetric perturbations which will carry energy preferentially in the direction of the axis of rotation. It is easy to picture these large scale interchanges, which are similar to convective rolls observed in experiments on convection in rotating liquids, but that is not enough to demonstrate that they are the disturbances which carry energy most efficiently. Cowling (1951) studied non-axisymmetric disturbances in an attempt to discover whether convection would indeed be more efficient parallel to the axis of rotation. He showed, as can be seen from criterion (3.10) and from the corresponding expressions for the velocity components at marginal stability, that there are non-axisymmetric disturbances which are not stabilized by rotation and which involve significant motions perpendicular to the axis of rotation. As he did not study large amplitude motions, he was unable to reach a definite conclusion about the latitude dependence of energy transport but he suggested that energy transport perpendicular to the axis of rotation might be less efficient than transport parallel to the axis. Unlike the interchange modes, it is not easy to visualize the large scale structure of non-axisymmetric disturbances.

To discuss further the structure of a rotating convective core, we consider the consequences of an energy transport by convection which is extremely anisotropic and in which there is no energy transport except in the direction of the axis of rotation. The expression for the convective flux is purely illustrative and is certainly not meant to be taken seriously but a study of it may enable some conclusions to be drawn about less anisotropic fluxes. Suppose then that the convective flux has the form

$$\mathbf{F}_{\text{conv}} = \frac{L_{\text{conv}}}{\pi r^2} (\hat{\mathbf{z}} . \hat{\mathbf{r}})^2 \hat{\mathbf{z}}, \tag{4.5}$$

where $\hat{\mathbf{z}}$ and $\hat{\mathbf{r}}$ are unit vectors in the z and r directions; the energy transport is always in the z direction, is greatest at the pole and vanishes at the equator and the expression is normalized so that L_{conv} is the total convective luminosity.

What is quite clear is that, if there is an anisotropic flow of energy by convection, div $\mathbf{F}_{\text{conv}}$ does not vanish. The inclusion of energy transport by radiation does not correct for this as, with the star assumed spherical, the radiative transport does not vary with latitude. The anisotropic flow of energy by convection is not consistent with the surfaces of constant temperature being spheres and a meridional circulation must be set up to balance div $\mathbf{F}_{\text{conv}}$. Although the cause is quite different, this circulation resembles that driven by rotation in radiative zones of non-spherical stars.

The value of div $\mathbf{F}_{\text{conv}}$, from purely dimensional considerations, is likely to be of order $L_{\text{conv}}/4\pi r^3$ and using the particular expression (4.5)

$$\text{div } \mathbf{F}_{\text{conv}} = -2 \cos \theta \cos 2\theta L_{\text{conv}}/\pi r^3. \tag{4.6}$$

The radial component of the circulation velocity can then be calculated from the thermal equation

$$\text{div } \mathbf{F}_{\text{conv}} = -\rho c_v \left[\frac{dT}{dr} - (\gamma - 1) \frac{T}{\rho} \frac{d\rho}{dr}\right] v_{\text{circ}, r}, \tag{4.7}$$

where γ is the ratio of specific heats and c_v the specific heat at constant volume of the material. This leads to

$$v_{\mathrm{circ},\, r} = \frac{\nabla_{\mathrm{ad}} H_{\mathrm{p}}}{(\nabla - \nabla_{\mathrm{ad}})\, P} \operatorname{div} \mathbf{F}_{\mathrm{conv}}. \tag{4.8}$$

Using equation (4.7) and taking $\gamma = 5/3$, this becomes

$$v_{\mathrm{circ},\, r} = -\frac{4 H_{\mathrm{p}} L_{\mathrm{conv}}}{5\pi P r^3 (\nabla - \nabla_{\mathrm{ad}})} \cos\theta \cos 2\theta. \tag{4.9}$$

The equation of continuity

$$\operatorname{div} \rho \mathbf{v} = 0 \tag{4.10}$$

can now be used to obtain an expression for the θ component of the circulation velocity. If (4.9) is used, we obtain

$$v_{\mathrm{circ},\, \theta} = \frac{1}{5\pi\rho r} \frac{d}{dr} \left[\frac{H_{\mathrm{p}} L_{\mathrm{conv}} \rho}{r(\nabla - \nabla_{\mathrm{ad}})}\right] \sin 2\theta \cos\theta. \tag{4.11}$$

Thus we conclude that there will be a large scale circulation in the convective core which, with the presumed predominance of energy transport in the z direction, has the form shown in Fig. 2*. Motion is inwards at the pole and outwards near

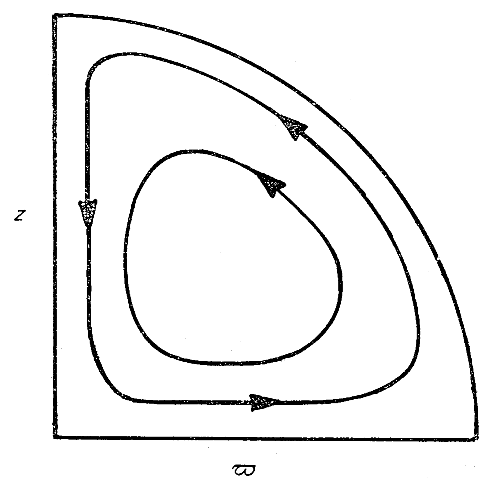

FIG. 2. *Circulation pattern. The stream lines of the circulation have the qualitative form shown. The arrows indicate the direction of flow. The exact behaviour of circulation near the centre of the core depends on account being taken of nuclear energy release and it must be stressed that an extreme law of energy transport by convection has been assumed.*

the equator. If, initially, we ignore the overall change of $\nabla - \nabla_{\mathrm{ad}}$ caused by the inhibition of convection by rotation and if we use the mixing length theory, we see that

$$v_{\mathrm{circ},\, r}/v_{\mathrm{conv}} \approx 4 \cos\theta \cos 2\theta l/r \approx l/r. \tag{4.12}$$

It is then easy to see that the time taken by circulation to travel through the core of the star is comparable with the mixing time due to convection if rotation is

* In this discussion, we have neglected the energy generation in the core but it is a simple matter to include it.

neglected. Thus, if the two times are τ_{circ} and τ_{conv} and noting that convective mixing is a random walk process

$$\left.\begin{aligned} \tau_{\text{circ}} &\approx r/v_{\text{circ},\, r}, \\ \tau_{\text{conv}} &\approx (l/v_{\text{conv}})\,(r/l)^2 = r^2/lv_{\text{conv}} \approx r/v_{\text{circ},\, r}. \end{aligned}\right\} \qquad (4.13)$$

Thus the mixing of material in a convective core is not affected seriously by rotation even if it assumed that rotation causes convection to take the form (4.5). The mode of mixing is different but the time scale is similar. In fact the relation between the circulation speed and the convective velocity in the absence of rotation will not be quite as given in (4.12) because the partial suppression of convection by rotation means that $\nabla - \nabla_{\text{ad}}$ must be rather greater in the rotating star. However, it is unlikely that τ_{circ} and τ_{conv} differ by more than an order of magnitude and they will both remain very much smaller than stellar evolution times in most cases.

The increase in $\nabla - \nabla_{\text{ad}}$ produced by rotation will not have a significant effect on the overall structure of the convective core or the entire star and mixing by convection and circulation will normally ensure that the chemical composition of the core is homogeneous. It therefore appears that the effect of slow rotation on the structure of convective cores can safely be ignored. However, in saying this we are ignoring the transport of angular momentum by circulation and the effect that this has on the rotation law of the core. So far we have assumed that the convective core rotates uniformly but in the following section we discuss that assumption.

5. THE ROTATION LAW OF CONVECTIVE CORES

Although most, but not all, authors have assumed that convective cores rotate uniformly, there is considerable uncertainty about this point. It is often argued that the effect of viscosity must be to lead to uniform rotation. Although ordinary viscosity (molecular or radiative) is too small to change the rotation law of a star, motions in a convective core may produce a turbulent viscosity which causes the core to rotate uniformly. The problem is not however this simple as we shall see by considering the possible states of steady motion of a viscous fluid. In a steady motion the viscous force must vanish. If the coefficient of viscosity, μ, is assumed to be a scalar, the viscous force has the form

$$f_{\text{visc}} = \frac{4}{3}\,\mu \text{ grad div } \mathbf{v} - \mu \text{ curl curl } \mathbf{v} + \text{grad } \mu \times \text{curl } \mathbf{v} + 2(\text{grad } \mu . \nabla)\mathbf{v} - \frac{2}{3} \text{ grad } \mu \text{ div } \mathbf{v}. \qquad (5.1)$$

If we consider first the case of constant μ, which is certainly not valid in a star, and assume that the motion is purely one of rotation about the z axis, then

$$\mathbf{v} = (0, v_\phi\,(\varpi, z), 0), \qquad (5.2)$$

$$\text{div } \mathbf{v} \equiv 0, \qquad (5.3)$$

and the condition for the viscous force to vanish is

$$\text{curl curl } \mathbf{v} = 0. \qquad (5.4)$$

One possibility is that curl $\mathbf{v} = 0$, which is only possible if

$$v_\phi = C_1/\varpi, \tag{5.5}$$

where C_1 is a constant. Otherwise equation (5.4) can be written

$$\frac{\partial^2 v_\phi}{\partial z^2} + \frac{\partial}{\partial \varpi}\left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi v_\phi)\right\} = 0. \tag{5.6}$$

One solution of (5.6) is

$$v_\phi = C_2 \varpi, \tag{5.7}$$

which is solid body rotation. In general there are other solutions including for example

$$v_\phi = C_3 \varpi z \tag{5.8}$$

in which the two hemispheres rotate in an opposite sense but in a convective core in which $\nabla \approx \nabla_{\text{ad}}$ this solution is not possible. For $\nabla \equiv \nabla_{\text{ad}}$, the equation of motion demands that v_ϕ depends on ϖ alone (see e.g. Mestel (1965)) and this must be almost eaxctly true in a real convective core.

It is clear that solid body rotation is not the only steady state of rotation of a fluid of constant viscosity, although it might be argued that it is the only reasonable one. Against this it can be argued that, if viscosity causes an arbitrary law of rotation to tend towards the state (5.5) in which the velocity is singular on the axis of rotation, the fluid cannot foresee that it is heading towards a singular state. There exists the possibility that solution (5.5), which represents constant angular momentum per unit mass is the valid law except very close to the axis of rotation where there might be a thin column in solid body rotation. Indeed, if we suppose that energy is largely carried by interchanges in which each element conserves its own angular momentum until it merges with its surroundings, it is quite natural to suppose that the role of convection is to lead to an equalization of angular momentum rather than angular velocity. However, as Cowling (1951) stressed, elements in non-axisymmetric perturbations do not conserve their angular momentum.

A combination of two arguments led Gough & Lynden-Bell (1968) to suggest that the correct law of rotation of a convection zone is no rotation at all. They argued as above that elements would tend to exchange angular momentum and to produce zones with uniform angular momentum per unit mass, h, and they noted that this would lead to the singularity on the axis of rotation. They also noted that Weiss (1966), studying the effect of convection on magnetic fields, showed that magnetic fields tend to be expelled from convection zones. Using an analogy between the equations of hydrodynamics and magnetohydrodynamics, they argued that vorticity might also be expelled by convection and that the true final state for a rotating convective region was one of zero rotation. An experiment, which they performed, appeared to confirm their suggestion but this result has not been obtained in a later investigation (Strittmatter, Illingworth & Freeman 1970) in which it was suggested that the result of Gough & Lynden-Bell resulted from their initial conditions.

The argument which has been given above concerning the possible steady laws of rotation must be modified if the viscosity though scalar is a function of position; for example we might consider $\mu(r)$. If this is done, the only law of rotation which is possible independent of the form of the viscosity is solid body rotation. Once the form of the viscosity is known, it is possible to find other rotation laws for which the viscous force vanishes and one of these approximates to a state of uniform h. However, such a rotation law does not have v_ϕ a function of ϖ alone, which means that it cannot be a true steady state of convective core.

We will now consider what happens to an initially uniformly rotating convective core if the convective energy transport is given by the law (4.5) and if the meridional circulation given by (4.9), (4.11) results. Such an anisotropic energy transport is unrealistic but the discussion should indicate in an exaggerated way what may happen if convection is strongest in the z direction. Convection of the type (4.5) would produce an effective tensor viscosity which, if the rotation were not already uniform, would lead to a core rotating on cylinders but would not force uniform rotation. We can now ask what effect the meridional circulation will have on the rotation law.

To do this we use a neat argument due to Kippenhahn (1964). If the core is rotating uniformly, h increases with r along any radius vector. Near the surface of the core the circulation velocity v_θ is carrying high h material towards the axis of the star whilst near to the centre low h material is being carried away from the axis. It is easy to see that the net effect is to transport angular momentum towards the axis. Thus, as mass must be conserved in the motion,

$$\left|\int_0^{r_0} \rho v_\theta r\, dr\right| = \left|\int_{r_0}^{r_c} \rho v_\theta r\, dr\right|, \tag{5.9}$$

where r_c is the radius of the convective core and r_0 the radius at which v_θ vanishes. Then, comparison of the transport of angular momentum in the two regions gives

$$\left|\int_0^{r_0} \rho v_\theta h r\, dr\right| < \left|\int_{r_0}^{r_c} \rho v_\theta h r\, dr\right|, \tag{5.10}$$

so that angular momentum is transported towards the axis. If the law of energy transport being considered really were correct, it would appear that, as angular momentum was carried towards the axis of the star, convection would mix angular momentum parallel to the rotation axis, so that the core would continue to rotate on cylinders to a first approximation. There would then be a tendency away from a state of uniform angular velocity towards one of uniform angular momentum. If the effect of convection on rotation were to enhance convection on the equator relative to the poles, the transport of angular momentum by circulation would be in the opposite direction; such an effect in the solar atmosphere related to the observed equatorial acceleration has been proposed by several authors including Weiss (1965) and Durney (1972).

It is impossible to discuss the ultimate effect of the meridional circulation on the rotation law without discussing the effect of non-uniform rotation on convection and, of course, there is no need to start by considering uniform rotation if that has no permanence. It is once again easy to obtain the interchange criteria. The criterion equivalent to (3.4), which is applicable on the equator ($\sin\theta = 1$), was obtained by Walén (1946) and is

$$\nabla > \nabla_{\text{ad}} + \frac{2P\Omega}{\rho g^2 \varpi}\frac{\partial h}{\partial \varpi}. \tag{5.11}$$

It can be seen from (5.11) that rotation always exerts a stabilizing influence on equatorial interchanges provided that h increases outwards but that the stabilizing

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influence vanishes in the case of uniform h. It can readily be shown that this is true for arbitrary interchanges. Although this rotation law cannot be valid right to the axis, it once again appears as an attractive possibility for the asymptotic law through much of the core, if the initial effect of rotation is to reduce convection near to the equator and to drive meridional circulation which carries angular momentum towards the rotation axis.

The main problem once again is that of the non-axisymmetric modes. These were considered by Cowling (1951), who also rederived the criteria for the interchange modes. He showed that, if rotation was non-uniform, it was no longer possible to obtain meaningful criteria for non-axisymmetric modes by a local analysis. He was not able to obtain completely general results but he did show that not only do there exist some perturbations which are not seriously affected by rotation but that non-uniform rotation can introduce new shear flow instabilities. It was not however clear that such instabilities would lead to convective energy transport and Cowling tentatively concluded that non-uniform rotation would have an inhibiting effect on energy transport by non-axisymmetric modes.

Before concluding this discussion of the possible effect of convection on the rotation laws of convective regions, it is necessary to mention some very important work by Biermann (1958) and Kippenhahn (1963, 1964), in which they were mainly trying to explain the observed equatorial acceleration of the Sun. In their work they neglected the influence of rotation on convection, though they mentioned that it needed to be studied, and they concentrated on the effect of convection on rotation. They first pointed out that, although energy transport by convection is not latitude dependent, if the effect of rotation on convection is ignored, at any point the convective motions are anisotropic. This suggests that the turbulent viscosity due to convection must be a tensor rather than a scalar. They used a form for the viscous stress tensor first derived by Wasiutynski (1946)* and assumed that, in spherical polar coordinates, the viscosity was a diagonal tensor with

$$\mu_{\theta\theta} = \mu_{\phi\phi} = s\mu_{rr}. \tag{5.12}$$

With such a tensor viscosity, solid body rotation is no longer a possible steady state. Biermann and Kippenhahn showed that viscous forces only vanish if the fluid rotates on spheres with the rotation law

$$\Omega \propto r^{-2(1-s)}. \tag{5.13}$$

This law, for $s = 0$ which implies that viscosity only acts in the radial direction, has constant angular momentum along any radius vector which is to be expected.

Although the viscous forces vanish only if Ω has the form (5.11), it is impossible to have equilibrium in an adiabatic zone unless Ω is $\Omega(\varpi)$, as has already been mentioned. Biermann (1958) concluded that a circulation must be driven by the failure to satisfy the equation of motion and that this must occur on a dynamical timescale. Kippenhahn (1963) calculated the first order change in the rotation law due to the circulation and he showed that equatorial acceleration could only be obtained with $s > 1$. No argument was given for the appropriate value of s. A recent discussion of this topic is by Köhler (1970). He considered an anisotropic and spatially dependent viscosity, which was smaller in the radial direction, and found an equatorial acceleration in good agreement with observations of the Sun.

* A more recent and accessible derivation is by Elasässer (1966).

A complete discussion of the mutual interaction of convection and rotation must combine the considerations of Kippenhahn with those of the earlier part of this paper. The natural anisotropy of convective motions combined with those produced by the effects of rotation will lead to some effective tensor viscosity. Several authors including Mestel (1965) have mentioned the desirability of considering a latitude dependent s but the actual form of the viscosity is likely to be more complicated than (5.12). It seems certain that some meridional circulation will be driven, either on a dynamical or a thermal time scale. If the largest components of viscosity act in the z direction or the r direction, it appears that the effect of the circulation will be to try to equalize angular momentum through the convective region, rather than to lead to solid body rotation.

6. CONCLUSIONS

In this paper we have discussed some of the interactions between rotation and convection in stellar interiors. In particular, we have discussed two questions:

(i) What effect does rotation have on the character of convection and on mixing processes in a convective region?

(ii) What is likely to be the ultimate law of rotation in a convective region? In fact, it is artificial to separate these two problems.

With regard to (i) our conclusion is as follows. Although rotation causes the flow of energy due to convection to be anisotropic the total energy flux cannot be anisotropic in a spherical star. As a result a meridional circulation is set up and the speed of the circulation and its character are such that material is mixed through the convective region in a time comparable with that taken by convection in a non-rotating star. If it were not for the transport of angular momentum by the circulation, the interaction of rotation with convection would have negligible effect on the overall properties of a star.

The circulation does transport angular momentum and in an initially uniformly rotating star its effect is to make regions of the core near to the rotation axis rotate more rapidly than regions away from the axis. The ultimate state *may* be one in which the bulk of the core has constant angular momentum per unit mass, except very near to the rotation axis, but this certainly has not been proved.

One reason why our discussion is far from complete is that any ultimate steady state of rotation will be determined by the turbulent viscosity as well as by the circulation. Because convective motions are anisotropic, the turbulent viscosity is a tensor and this is even true when rotation does not interfere with convection. At present we have no clear understanding of the form of the viscosity tensor when the natural anisotropy of convective motions is modified by the influence of rotation and this is the major problem requiring solution.

Although we have referred mainly to convective cores, much of what we have to say also applies to convective envelopes. The influence of rotation on convection is, however, generally less in convective envelopes unless the stars are rotating extremely rapidly, because the value of $\nabla - \nabla_{ad}$ is much higher than in typical cores. In other ways the discussion may be better adapted to envelopes. We have discussed convection and turbulent viscosity in a way which suggests that the size of typical convective elements is very small compared to the size of the convection zone. Whilst this is true in convective envelopes, it is probably not true in convective cores. This will have the effect of smearing out some of the conclusions which we have reached and should deal with the threatened singularity on the axis of rotation.

The paper can finally be summarized in one sentence. In calculating the structure of rotating stars with convective cores it is probably not necessary to take account of an anisotropic flow of energy due to convection but it is important to worry about whether the correct law of rotation is being used.

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