TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 11, November 2011, Pages 5789–5825
S 0002-9947(2011)05107-X
Article electronically published on June 6, 2011

CONVENIENT CATEGORIES OF SMOOTH SPACES

JOHN C. BAEZ AND ALEXANDER E. HOFFNUNG

Abstract. A 'Chen space' is a set X equipped with a collection of 'plots', i.e., maps from convex sets to X, satisfying three simple axioms. While an individual Chen space can be much worse than a smooth manifold, the category of all Chen spaces is much better behaved than the category of smooth manifolds. For example, any subspace or quotient space of a Chen space is a Chen space, and the space of smooth maps between Chen spaces is again a Chen space. Souriau's 'diffeological spaces' share these convenient properties. Here we give a unified treatment of both formalisms. Following ideas of Penon and Dubuc, we show that Chen spaces, diffeological spaces, and even simplicial complexes are examples of 'concrete sheaves on a concrete site'. As a result, the categories of such spaces are locally Cartesian closed, with all limits, all colimits, and a weak subobject classifier. For the benefit of differential geometers, our treatment explains most of the category theory we use.

1. Introduction

Algebraic topologists have become accustomed to working in a category of spaces for which many standard constructions have good formal properties: mapping spaces, subspaces and quotient spaces, limits and colimits, and so on. In differential geometry the situation is quite different, since the most popular category, that of finite-dimensional smooth manifolds, lacks almost all these features. So, researchers are beginning to seek a 'convenient category' of smooth spaces in which to do differential geometry.

In this paper we study two candidates: Chen spaces and diffeological spaces. But before we start, it is worth recalling the lesson of algebraic topology in a bit more detail. Dissatisfaction arose when it became clear that the category of topological spaces suffers from a defect: there is generally no way to give the set $C(X, Y)$ of continuous maps from a space X to a space Y a topology such that the natural map

$$\begin{array}{ccc} C(X \times Y, Z) & \to & C(X, C(Y, Z)) \\ f & \mapsto & \tilde{f} \end{array}$$
$$\tilde{f}(x)(y) = f(x, y)$$

is a homeomorphism. In other words, this category fails to be Cartesian closed. This led to the search for a better framework, or as Brown [6] put it, a "convenient category".

Steenrod's paper "A convenient category of topological spaces" [41] popularized the idea of restricting attention to spaces with a certain property to obtain a

Received by the editors September 13, 2008 and, in revised form, October 13, 2009.
2000 *Mathematics Subject Classification.* Primary 58A40; Secondary 18F10, 18F20.

Cartesian closed category. It was later realized that by adjusting this property a bit, we can also make quotient spaces better behaved. The resulting category, with compactly generated spaces as objects, and continuous maps as morphisms, has now been widely adopted in algebraic topology [30]. This shows that it is perfectly possible, and at times quite essential, for a discipline to change the category that constitutes its main subject of inquiry. Something similar happened in algebraic geometry when Grothendieck invented schemes as a generalization of algebraic varieties.

Now consider differential geometry. Like the category of topological spaces, the category of smooth manifolds fails to be Cartesian closed. Indeed, if X and Y are finite-dimensional smooth manifolds, the space of smooth maps $\mathcal{C}^\infty(X, Y)$ is hardly ever the same sort of thing. It is a kind of *infinite-dimensional* manifold, but making the space of smooth maps between *these* into an infinite-dimensional manifold becomes more difficult. It can be done [23, 31], but there are still many spaces on which we can do differential geometry that do not live in the resulting Cartesian closed category. The simplest examples are manifolds with boundary, or more generally manifolds with corners. There are also many formal properties one might want, which are lacking: for example, a subspace or quotient space of a manifold is rarely a manifold, and the category of manifolds does not have limits and colimits.

In 1977, Chen defined a simple notion that avoids all these problems [9]. A 'Chen space' is a set X equipped with a collection of 'plots', i.e., maps $\varphi\colon C \to X$ where C is any convex subset of any Euclidean space $\mathbb{R}^n$, obeying three simple axioms. Despite a superficial resemblance to charts in the theory of manifolds, plots are very different: we should think of a plot in X as an *arbitrary* smooth map to X from a convex subset of a Euclidean space of *arbitrary* dimension. So instead of ensuring that Chen spaces look nice locally, plots play a different role: they determine which maps between Chen spaces are smooth. Given a map $f\colon X \to Y$ between Chen spaces, f is 'smooth' if and only if for any plot in X, say $\varphi\colon C \to X$, the composite $f\varphi\colon C \to Y$ is a plot in Y.

In 1980, Souriau introduced another category of smooth spaces: 'diffeological spaces' [39]. The definition of these closely resembles that of Chen spaces: the only difference is that the domain of a plot can be any *open* subset of $\mathbb{R}^n$, instead of any convex subset. As a result, Chen spaces and diffeological spaces have many similar properties. So, in what follows, we use 'smooth space' to mean *either* Chen space *or* diffeological space. We shall see that:

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.
- Every smooth space has a natural topology, and smooth maps between smooth spaces are automatically continuous.
- Any subset of a smooth space becomes a smooth space in a natural way, and the inclusion of this subspace is a smooth map. Subspaces of a smooth space are classified by their characteristic functions, which are smooth maps taking values in $\{0, 1\}$ equipped with its 'indiscrete' smooth structure. So, we say $\{0, 1\}$ with its indiscrete smooth structure is a 'weak subobject classifier' for the category of smooth spaces (see Definition 5.11).

- The quotient of a smooth space under any equivalence relation becomes a smooth space in a natural way, and the quotient map is smooth.
- The category of smooth spaces has all limits and colimits.
- Given smooth spaces X and Y, the set $\mathcal{C}^\infty(X,Y)$ of all smooth maps from X to Y can be made into a smooth space in such a way that the natural map
$$\mathcal{C}^\infty(X \times Y, Z) \to \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y,Z))$$
is a smooth map with a smooth inverse. So, the category of smooth spaces is Cartesian closed.
- More generally, given any smooth space B, the category of smooth spaces 'over B', that is, equipped with maps to B, is Cartesian closed. So, we say the category of smooth spaces is 'locally Cartesian closed' (see Definition 5.16 for details).

The goal of this paper is to present a unified approach to Chen spaces and diffeological spaces that explains why they share these convenient properties.

All this convenience comes with a price: both these categories contain many spaces whose local structure is far from that of Euclidean space. This should not be surprising. For example, the subset of a manifold defined by an equation between smooth maps,
$$Z = \{x \in M \colon f(x) = g(x)\},$$
is not usually a manifold in its own right. In fact, Z can easily be as bad as the Cantor set if $M = \mathbb{R}$. But it is a smooth space. It is nice having the solution set of an equation between smooth maps be a smooth space, but the price we pay is that a smooth space can be locally as bad as the Cantor set.

So, we should not expect the theory of smooth spaces to support the wealth of fine-grained results familiar from the theory of smooth manifolds. Instead, it serves as a large context for general ideas. For a taste of just how much can be done here, see Iglesias–Zemmour's book on diffeological spaces [18]. There is no real conflict, since smooth manifolds form a full subcategory of the category of smooth spaces. We can use the larger category for abstract constructions, and the smaller one for theorems that rely on good control over local structure.

Since we want differential geometers to embrace the notions we are describing, our treatment will be as self-contained as possible. This requires a little introduction to sheaves on sites, because the key fact underlying our main results is that both Chen spaces and diffeological spaces are examples of 'concrete sheaves on a concrete site'. For example, Chen spaces are sheaves on a site Chen: the category whose objects are convex subsets of $\mathbb{R}^n$ and whose morphisms are smooth maps, equipped with a certain Grothendieck topology. However, not *all* sheaves on this site count as Chen spaces, but only those satisfying a certain 'concreteness' property, which guarantees that any Chen space has a well-behaved underlying set. Formulating this property uses the fact that Chen itself is a 'concrete site'. Similarly, the category of diffeological spaces can be seen as the category of concrete sheaves on a concrete site $\mathsf{Diffeological}$.

The category of *all* sheaves on a site is extremely nice: it is a topos. Here, following ideas of Penon [34, 35] and Dubuc [11, 13], we show that the category of concrete sheaves on a concrete site is also nice, but slightly less so: it is a 'quasitopos'. This yields many of the good properties listed above.

Various other notions of 'smooth space' are currently being studied. Perhaps the most elegant approach is synthetic differential geometry [21], which drops the assumption that a smooth space be a set equipped with extra structure. This gives a topos of smooth spaces, and it allows a rigorous treatment of calculus using infinitesimals.

Most other approaches treat smooth spaces as sets equipped with a specified class of 'maps in', 'maps out', or 'maps in and out'. We recommend Stacey's work [40] for a detailed comparison of these approaches. Chen and Souriau take the 'maps in' approach, where a plot in a smooth space X is a map into X, and a function $f\colon X \to Y$ between smooth spaces is smooth when its composite with every plot in X is a plot in Y. Smith [38], Sikorski [22, 37] and Mostow [32] follow the 'maps out' approach instead, in which a smooth space X comes equipped with a collection of 'coplots' $\varphi\colon X \to C$ for certain spaces C, and a map $f\colon X \to Y$ between smooth spaces is smooth when its composite with every coplot on Y is a coplot on X. Frölicher takes the 'maps in and out' approach, in which a smooth space is equipped with both plots and coplots [15, 26]. This gives two ways to determine the smoothness of a map between smooth spaces, which are required to give the same answer. Our work covers a wide class of definitions that take the 'maps in' approach.

The structure of the paper is as follows. In Section 2, we define Chen spaces and diffeological spaces and give some examples. We also discuss the relation between these two formalisms, focusing on manifolds with corners and the work of Stacey [40]. In Section 3, we list many convenient properties shared by these categories. In Section 4, we recall the concept of a sheaf on a site and show that Chen spaces and diffeological spaces are 'concrete' sheaves on 'concrete' sites. Simplicial complexes give another interesting example. In Section 5, we show that any category of concrete sheaves on a concrete site is a quasitopos with all limits and colimits. Most of the properties described in Section 3 follow as a direct result.

2. Smooth spaces

Souriau's notion of a 'diffeological space' [39] is very simple:

Definition 2.1. An **open set** is an open subset of $\mathbb{R}^n$. A function $f\colon U \to U'$ between open sets is called **smooth** if it has continuous derivatives of all orders.

Definition 2.2. A **diffeological space** is a set X equipped with, for each open set U, a set of functions

$$\varphi\colon U \to X,$$

called **plots in** X, such that:

(1) If φ is a plot in X and $f\colon U' \to U$ is a smooth function between open sets, then φf is a plot in X.
(2) Suppose the open sets $U_j \subseteq U$ form an open cover of the open set U, with inclusions $i_j\colon U_j \to U$. If φi_j is a plot in X for every j, then φ is a plot in X.
(3) Every map from the one point of $\mathbb{R}^0$ to X is a plot in X.

Definition 2.3. Given diffeological spaces X and Y, a function $f\colon X \to Y$ is a **smooth map** if, for every plot φ in X, the composite $f\varphi$ is a plot in Y.

Chen actually considered several different definitions. Here we use his final, most refined approach [9], which closely resembles Souriau's:

Definition 2.4. A **convex set** is a convex subset of $\mathbb{R}^n$ with nonempty interior. A function $f\colon C \to C'$ between convex sets is called **smooth** if it has continuous derivatives of all orders.

Definition 2.5. A **Chen space** is a set X equipped with, for each convex set C, a set of functions

$$\varphi\colon C \to X,$$

called **plots in** X, satisfying these axioms:

(1) If φ is a plot in X and $f\colon C' \to C$ is a smooth function between convex sets, then φf is a plot in X.
(2) Suppose the convex sets $C_j \subseteq C$ form an open cover of the convex set C with its topology as a subspace of $\mathbb{R}^n$. Denote the inclusions as $i_j\colon C_j \to C$. If φi_j is a plot in X for every j, then φ is a plot in X.
(3) Every map from the one point of $\mathbb{R}^0$ to X is a plot in X.

Definition 2.6. Given Chen spaces X and Y, a function $f\colon X \to Y$ is a **smooth map** if, for every plot φ in X, the composite $f\varphi$ is a plot in Y.

It is instructive to see how Chen's definition evolved. Of course he did not speak of 'Chen spaces'; he called them 'differentiable spaces'. In 1973, he took a differentiable space to be a Hausdorff space X equipped with continuous plots $\varphi\colon C \to X$ satisfying axioms 1 and 3 above, where the domains C were closed convex subsets of Euclidean space [7]. In 1975, he added a preliminary version of axiom 2 and dropped the condition that X be Hausdorff [8].

Starting in 1977, Chen used a definition equivalent to the one above [9, 10]. In particular, he dropped the topology on X, the continuity of φ, and the condition that C be closed. This marks an important realization, emphasized by Stacey [40]: we can give a space a smooth structure without first giving it a topology. Indeed, we shall see that a smooth structure determines a topology!

The notion of a smooth function $f\colon C \to C'$ between convex sets is a bit subtle, particularly for points on the boundary of C. One tends to imagine C as either open or closed, but the generic situation is more messy. For example, C could be the closed unit disk D^2 minus the set Q of points on the unit circle with rational coordinates. Both Q and its complement are dense in the unit circle.

Situations like this, while far from our main topic of interest, deserve a little thought. So, suppose $C \subseteq \mathbb{R}^n$ and $C' \subseteq \mathbb{R}^m$ are convex subsets with nonempty interior. To define the kth derivative of a function from C to C', it suffices to define the first derivative of a function $F\colon C \to V$ for any finite-dimensional normed vector space V, since when this derivative exists it will be a function dF from C to the normed vector space of linear maps $\hom(\mathbb{R}^n, V)$. We can then define the derivative of this function, and so on. Therefore, we say that the derivative of F exists at the point $x \in C$ if there is a linear map $(dF)_x\colon \mathbb{R}^n \to V$ such that

$$\frac{\|F(y) - F(x) - dF_x(y-x)\|}{\|y - x\|} \to 0$$

as $y \to x$ for $y \in C - \{x\}$. Note that since C is convex with nonempty interior, dF_x is unique if it exists.

This is the usual definition going back to Fréchet, and scarcely worth remarking on, except for the obvious caveat that y must lie in C. In the case $C = [0,1]$, this means we are using one-sided derivatives at the endpoints. In the case of the convex set $D^2 - Q$, it means we are using a generalization of one-sided derivatives at all points on the boundary of this set, which is the unit circle minus Q.

Luckily, whenever C and C' are convex sets, we can characterize smooth functions $f\colon C \to C'$ in three equivalent ways:

(1) The function $f\colon C \to C'$ has continuous derivatives of all orders.
(2) The function $f\colon C \to C'$ has continuous derivatives of all orders in the interior of C, and these extend continuously to the boundary of C.
(3) If $\gamma\colon \mathbb{R} \to C$ is a smooth curve in C, then $f\gamma$ is a smooth curve in C'.

The equivalence of conditions 1 and 2 is not hard; the equivalence of 2 and 3 was proved by Kriegl [24], and appears as Theorem 24.5 in Kriegl and Michor's book [25].

Since most of our results apply both to Chen spaces and diffeological spaces, we lay down the following conventions:

Definition 2.7. We use **smooth space** to mean either a Chen space or a diffeological space, and use $\mathcal{C}^\infty$ to mean either the category of Chen spaces and smooth maps, or diffeological spaces and smooth maps. We use the term **domain** to mean either a convex set or an open set, depending on the context.

Henceforth, any statement about smooth spaces or the category $\mathcal{C}^\infty$ holds for *both* Chen spaces and diffeological spaces.

2.1. **Examples.** Next we give some examples. For these it is handy to call the set of plots in a smooth space its **smooth structure**. So, we may speak of taking a set and putting a smooth structure on it to obtain a smooth space.

(1) Any domain D becomes a smooth space, where the plots $\varphi\colon D' \to D$ are just the smooth functions.
(2) Any set X has a **discrete** smooth structure such that the plots $\varphi\colon D \to X$ are just the constant functions.
(3) Any set X has an **indiscrete** smooth structure where every function $\varphi\colon D \to X$ is a plot.
(4) Any smooth manifold X becomes a smooth space where $\varphi\colon D \to X$ is a plot if and only if φ has continuous derivatives of all orders. Moreover, if X and Y are smooth manifolds, then $f\colon X \to Y$ is a morphism in $\mathcal{C}^\infty$ if and only if it is smooth in the usual sense.
(5) Given any smooth space X, we can endow it with a new smooth structure, where we keep only the plots of X that factor through a chosen domain D_0. When $D_0 = \mathbb{R}$ this smooth structure is called the 'wire diffeology' in the theory of diffeological spaces [18]. While this construction gives many examples of smooth spaces, these seem to be useful mainly as counterexamples to naive conjectures.
(6) Any topological space X can be made into a smooth space where we take the plots to be all the continuous maps $\varphi\colon C \to X$. Since every smooth map is continuous this defines a smooth structure. Again, these examples mainly serve to disprove naive conjectures.

If Diff is the category of smooth finite-dimensional manifolds and smooth maps, our fourth example above gives a full and faithful functor

$$\mathrm{Diff} \to \mathcal{C}^\infty.$$

So, we can think of $\mathcal{C}^\infty$ as a kind of 'extension' or 'completion' of Diff with better formal properties.

Any smooth space X can be made into a topological space with the finest topology such that all plots $\varphi\colon D \to X$ are continuous. With this topology, smooth maps between smooth spaces are automatically continuous. This gives a faithful functor

$$\mathcal{C}^\infty \to \mathrm{Top}.$$

In particular, if we take a smooth manifold, regard it as a smooth space, and then turn it into a topological space this way, we recover its usual topology.

2.2. **Comparison.** We should also say a bit about how Chen spaces and diffeological spaces differ, and how they are related. To begin with, let us compare their treatment of manifolds with boundary, or more generally manifolds with corners [19, 27].

An n-dimensional manifold with corners M has charts of the form $\varphi\colon X_k \to M$, where

$$X_k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1, \ldots, x_k \geq 0\}$$

for $k = 0, \ldots, n$. The case $k = 1$ gives a half-space, familiar from manifolds with boundary. Since $X_k \subset \mathbb{R}^n$ is convex, any chart $\varphi\colon X_k \to M$ can be made into a plot in Chen's sense. So, if we make M into a Chen space where the plots $\varphi\colon C \to M$ are just maps that are smooth in the usual sense, it follows that any map between manifolds with corners $f\colon M \to N$ is smooth as a map of Chen spaces if and only if it is smooth in the usual sense.

However, the subset $X_k \subset \mathbb{R}^n$ is typically not open. So, we cannot make a chart for a manifold with corners into a plot in the sense of diffeological spaces. Nonetheless, we can make any manifold with corners M into a diffeological space where the plots $\varphi\colon U \to M$ are the maps that are smooth in the usual sense, and then, in fact, a map between manifolds with corners is smooth as a map between diffeological spaces if and only if it is smooth in the usual sense!

The key to seeing this is the theorem of Kriegl mentioned above. Since the issues involved are local, it suffices to consider maps $f\colon X_k \to \mathbb{R}^m$. Suppose $f\colon X_k \to \mathbb{R}^m$ is smooth in the sense of diffeological spaces. Then the composite $f\gamma$ is smooth for any smooth curve $\gamma\colon \mathbb{R} \to X_k$. By Kriegl's theorem, this implies that f has continuous derivatives of all orders in the interior of X_k, extending continuously to the boundary. So, f is smooth in the usual sense for manifolds with corners. Conversely, any $f\colon X_k \to \mathbb{R}^m$ smooth in the usual sense is clearly smooth in the sense of diffeological spaces.

Stacey has given a more general comparison of Chen spaces versus diffeological spaces [40]. To briefly summarize this, let us write ChenSpace for the category of Chen spaces, and DiffeologicalSpace for the category of diffeological spaces. Stacey has shown that these categories are not equivalent. However, he has constructed some useful functors relating them. These take advantage of the fact that every open subset of $\mathbb{R}^n$ becomes a Chen space with its subspace smooth structure, and conversely, every convex subset of $\mathbb{R}^n$ becomes a diffeological space.

Using this, Stacey defines for any Chen space X a diffeological space $\mathrm{So}X$ with the same underlying set, where $\varphi\colon U \to \mathrm{So}X$ is a plot if and only if $\varphi\colon U \to X$ is a smooth map between Chen spaces. This extends to a functor

$$\mathrm{So}\colon \mathsf{Chen}\mathrm{Space} \to \mathsf{Diffeological}\mathrm{Space}$$

that is the identity on maps. He also defines for any diffeological space Y a Chen space $\mathrm{Ch}^\sharp Y$ with the same underlying set, where $\varphi\colon C \to \mathrm{Ch}^\sharp Y$ is a plot if and only if $\varphi\colon C \to Y$ is a smooth map between diffeological spaces. Again, this extends to a functor

$$\mathrm{Ch}^\sharp\colon \mathsf{Diffeological}\mathrm{Space} \to \mathsf{Chen}\mathrm{Space}$$

that is the identity on maps. Stacey shows that

$$\begin{gathered} f\colon X \to \mathrm{Ch}^\sharp Y \text{ is a smooth map between Chen space} \\ \Updownarrow \\ f\colon \mathrm{So}X \to Y \text{ is a smooth map between diffeological spaces.} \end{gathered}$$

In other words, $\mathrm{Ch}^\sharp$ is the right adjoint of So.

The functor So also has a left adjoint

$$\mathrm{Ch}^\flat\colon \mathsf{Diffeological}\mathrm{Space} \to \mathsf{Chen}\mathrm{Space}$$

which acts as the identity on maps. This time the adjointness means that

$$\begin{gathered} f\colon \mathrm{Ch}^\flat Y \to X \text{ is a smooth map between Chen spaces} \\ \Updownarrow \\ f\colon Y \to \mathrm{So}X \text{ is a smooth map between diffeological spaces.} \end{gathered}$$

Furthermore, Stacey shows that both these composites,

$$\mathsf{Diffeological}\mathrm{Space} \xrightarrow{\mathrm{Ch}^\sharp} \mathsf{Chen}\mathrm{Space} \xrightarrow{\mathrm{So}} \mathsf{Diffeological}\mathrm{Space},$$

$$\mathsf{Diffeological}\mathrm{Space} \xrightarrow{\mathrm{Ch}^\flat} \mathsf{Chen}\mathrm{Space} \xrightarrow{\mathrm{So}} \mathsf{Diffeological}\mathrm{Space},$$

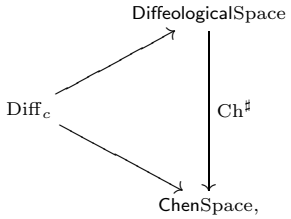
are equal to the identity. With a little work, it follows that both $\mathrm{Ch}^\sharp$ and $\mathrm{Ch}^\flat$ embed $\mathsf{Diffeological}\mathrm{Space}$ isomorphically as a full subcategory of $\mathsf{Chen}\mathrm{Space}$: a 'reflective' subcategory in the first case, and a 'coreflective' one in the second.

The embedding $\mathrm{Ch}^\flat$ is a bit strange: as shown by Stacey, even the ordinary closed interval fails to lie in its image! To see this, he takes I to be $[0,1] \subset \mathbb{R}$ made into a Chen space with its subspace smooth structure. If I were isomorphic to a Chen space in the image of $\mathrm{Ch}^\flat$, say $I \cong \mathrm{Ch}^\flat X$, we would then have $\mathrm{Ch}^\flat \mathrm{So}I = \mathrm{Ch}^\flat \mathrm{So}\mathrm{Ch}^\flat X = \mathrm{Ch}^\flat X \cong I$. However, he shows explicitly that $\mathrm{Ch}^\flat \mathrm{So}I$ is not isomorphic to I; it is the unit interval equipped with a nonstandard smooth structure.

The embedding $\mathrm{Ch}^\sharp$ lacks this defect, since $\mathrm{Ch}^\sharp \mathrm{So}I = I$. For an example of a Chen space not in the image of $\mathrm{Ch}^\sharp$, we can resort to $\mathrm{Ch}^\flat \mathrm{So}I$. Suppose there were a diffeological space X with $\mathrm{Ch}^\sharp X \cong \mathrm{Ch}^\flat \mathrm{So}I$. Then we would have $\mathrm{So}\mathrm{Ch}^\sharp X \cong \mathrm{So}\mathrm{Ch}^\flat \mathrm{So}I$; hence $X \cong \mathrm{So}I$. But this is a contradiction, since we know that $\mathrm{Ch}^\sharp$ applied to $\mathrm{So}I$ gives I, which is not isomorphic to $\mathrm{Ch}^\flat \mathrm{So}I$.

Luckily, the embedding $\mathrm{Ch}^\sharp$ works well for manifolds with corners. In particular, if Diff_c is the category of manifolds with corners and smooth maps, we have a

commutative triangle



where the diagonal arrows are the full and faithful functors described earlier.

3. Convenient properties of smooth spaces

Now we present some useful properties shared by Chen spaces and diffeological spaces. Following Definition 2.7, we call either kind of space a 'smooth space', and we use $\mathcal{C}^\infty$ to denote either the category of Chen spaces or the category of diffeological spaces. Most of the proofs are straightforward diagram chases, but we defer all proofs to Section 5.

- Subspaces

  Any subset $Y \subseteq X$ of a smooth space X becomes a smooth space if we define $\varphi\colon D \to Y$ to be a plot in Y if and only if its composite with the inclusion $i\colon Y \to X$ is a plot in X. We call this the **subspace** smooth structure.

  It is easy to check that, with this smooth structure, the inclusion $i\colon Y \to X$ is smooth. Moreover, it is a monomorphism in $\mathcal{C}^\infty$. Not every monomorphism is of this form. For example, the natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism. In Proposition 5.7, we show that a smooth map $i\colon Y \to X$ comes from the inclusion of a subspace precisely when i is a 'strong' monomorphism (see Definition 5.5).

  The 2-element set $\{0, 1\}$ with its indiscrete smooth structure is called the 'weak subobject classifier' for smooth spaces, and is denoted by Ω. The precise definition of a weak subobject classifier can be found in Definition 5.11, but the idea is simple: for any smooth space X, subspaces of X are in one-to-one correspondence with smooth maps from X to Ω. In particular, any subspace $Y \subseteq X$ corresponds to the characteristic function $\chi_Y : X \to \Omega$ given by

$$\chi_Y(x) = \begin{cases} 1 & x \in Y, \\ 0 & x \notin Y. \end{cases}$$

  In Proposition 5.13 we prove the existence of a weak subobject classifier in a more general context.

- Quotient spaces

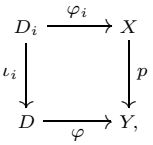
  If X is a smooth space and $\sim$ is any equivalence relation on X, the quotient space $Y = X/\sim$ becomes a smooth space if we define a plot in Y to be any function

$$\varphi\colon D \to Y$$

  for which there exists an open cover $\{D_i\}$ of D and a collection of plots in X,

$$\{\varphi_i\colon D_i \to X\}_{i \in I},$$

such that the following diagram commutes:



where $p\colon X \to Y$ is the function induced by the equivalence relation $\sim$ and $\iota_i\colon D_i \to D$ is the inclusion. We call this the **quotient space** smooth structure.

It is easy to check that with this smooth structure, the quotient map $p\colon X \to Y$ is smooth and an epimorphism in $\mathcal{C}^\infty$. Not every epimorphism is of this form: for example, the natural map from $\mathbb{R}$ with its standard smooth structure to $\mathbb{R}$ with its indiscrete smooth structure is also an epimorphism. In Proposition 5.10, we show that a smooth map $p\colon X \to Y$ comes from taking a quotient space precisely when p is a 'strong' epimorphism (see Definition 5.8).

- Terminal object

  The one element set 1 can be made into a smooth space in only one way, namely by declaring every function from every domain to 1 to be a plot. This smooth space is the terminal object of $\mathcal{C}^\infty$.

- Initial object

  The empty set $\emptyset$ can be made into a smooth space in only one way, namely by declaring every function from every domain to $\emptyset$ to be a plot. (Of course, such a function exists only for the empty domain.) This smooth space is the initial object of $\mathcal{C}^\infty$.

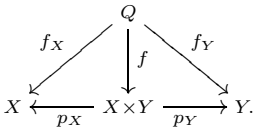
- Products

  Given smooth spaces X and Y, the product $X \times Y$ of their underlying sets becomes a smooth space, where $\varphi\colon D \to X \times Y$ is a plot if and only if its composites with the projections

$$p_X\colon X \times Y \to X, \qquad p_Y\colon X \times Y \to Y$$

  are plots in X and Y, respectively. We call this the **product smooth structure** on $X \times Y$.

  It is easy to check that with this smooth structure, p_X and p_Y are smooth. Moreover, for any other smooth space Q with smooth maps $f_X\colon Q \to X$ and $f_Y\colon Q \to Y$, there exists a unique smooth map $f\colon Q \to X \times Y$ such that the following diagram commutes:



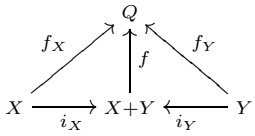
  So, $X \times Y$ is indeed the product of X and Y in the category $\mathcal{C}^\infty$.

- Coproducts

  Given smooth spaces X and Y, the disjoint union $X + Y$ of their underlying sets becomes a smooth space where $\varphi\colon D \to X + Y$ is a plot if and only if for each connected component U of D, $\varphi|_U$ is either the composite of a plot in X with the inclusion $i_X\colon X \to X + Y$, or the composite of a

plot in Y with the inclusion $i_Y\colon Y \to X + Y$. We call this the **coproduct smooth structure** on $X + Y$. Note that for Chen spaces the domains of the plots are convex and thus have only one connected component. So, in this case, φ is a plot in the disjoint union if and only if it factors through a plot in either X or Y.

It is easy to check that with this smooth structure, i_X and i_Y are smooth. Moreover, for any other smooth space Q with smooth maps $f_X\colon X \to Q$ and $f_Y\colon Y \to Q$, there exists a unique smooth map $f\colon X + Y \to Q$ such that



commutes. So, $X + Y$ is indeed the coproduct of X and Y in the category $\mathcal{C}^\infty$.

- Equalizers

  Given a pair $f, g\colon X \to Y$ of smooth maps between smooth spaces, the set

  $$Z = \{x \in X\colon f(x) = g(x)\} \subset X$$

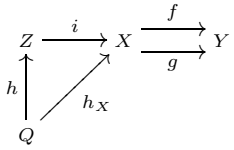
  becomes a smooth space with its subspace smooth structure, and the inclusion $i\colon Z \to X$ is the equalizer of f and g:

  $$Z \xrightarrow{\ i\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

  In other words, for any smooth space Q with a smooth map $h_X\colon Q \to X$ making the following diagram commute:

  $$Q \xrightarrow{\ h_X\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y,$$

  there exists a unique smooth map $h\colon Q \to Z$ such that

  commutes.

- Coequalizers

  Given a pair $f, g\colon X \to Y$ of smooth maps between smooth spaces, the quotient

  $$Z = Y/(f(x) \sim g(x))$$

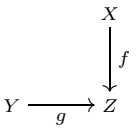
  becomes a smooth space with its quotient smooth structure, and the quotient map $p\colon Y \to Z$ is the coequalizer of f and g:

  $$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ p\ } Z.$$

The universal property here is dual to that of the equalizer: just turn all the arrows around.

- Pullbacks

  Since $\mathcal{C}^\infty$ has products and equalizers, it also has pullbacks, also known as 'fibered products'. Given a diagram of smooth maps



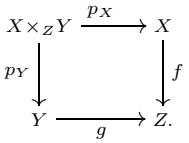
  we equip the set

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

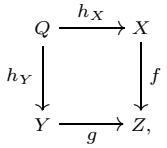
  with its smooth structure as a subspace of the product $X \times Y$. The natural functions

$$p_X \colon X \times_Z Y \to X, \qquad p_Y \colon X \times_Z Y \to Y$$

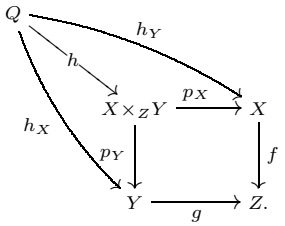
  are then smooth, and it is easy to check that this diagram is a pullback square:



  In other words, given any commutative square of smooth maps like this:



  there exists a unique smooth map $h \colon Q \to X \times_Z Y$ making the following diagram commute:



  More generally, we can compute any limit of smooth spaces by taking the limit of the underlying sets and endowing the result with a suitable smooth structure. This follows from Proposition 5.12, where we show that $\mathcal{C}^\infty$ has all small limits, together with the fact that the forgetful functor from $\mathcal{C}^\infty$ to Set preserves limits, since it is the right adjoint of the functor equipping any set with its discrete smooth structure.

- Pushouts

  Since $\mathcal{C}^\infty$ has coproducts and coequalizers, it also has pushouts. Given a diagram of smooth maps

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \\ Y, & & \end{array}$$

  we equip the set

$$X +_Z Y = (X + Y)/(f(z) \sim g(z))$$

  with its smooth structure as a quotient space of the coproduct $X + Y$. The natural functions

$$i_X \colon X \to X +_Z Y, \qquad i_Y \colon Y \to X +_Z Y$$

  are then smooth, and in fact this diagram is a pushout square:

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_X} \\ Y & \xrightarrow[i_Y]{} & X+_Z Y. \end{array}$$

  The universal property here is dual to that of the pullback and can also be easily checked.

  More generally, we can compute any limit of smooth spaces by taking the limit of the underlying sets and endowing the result with a suitable smooth structure. This follows from Proposition 5.23, where we show that $\mathcal{C}^\infty$ has all small colimits, together with the fact that the forgetful functor from $\mathcal{C}^\infty$ to Set preserves colimits, since it is the left adjoint of the functor equipping any set with its indiscrete smooth structure.

- Mapping spaces

  Given smooth spaces X and Y, the set

$$\mathcal{C}^\infty(X, Y) = \{f \colon X \to Y \colon f \text{ is smooth}\}$$

  becomes a smooth space, where a function $\tilde{\varphi} \colon D \to \mathcal{C}^\infty(X, Y)$ is a plot if and only if the corresponding function $\varphi \colon D \times X \to Y$ given by

$$\varphi(x, y) = \tilde{\varphi}(x)(y)$$

  is smooth. With this smooth structure one can show that the natural map

$$\begin{array}{ccc} \mathcal{C}^\infty(X \times Y, Z) & \to & \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z)) \\ f & \mapsto & \tilde{f} \end{array}$$

$$\tilde{f}(x)(y) = f(x, y)$$

  is smooth, with a smooth inverse. So, we say that the category $\mathcal{C}^\infty$ is Cartesian closed (see Definition 5.15).

- Parametrized mapping spaces

  Mapping spaces are a special case of parametrized mapping spaces. Fix a smooth space B as our parameter space, or 'base space'. Define a smooth space **over** B to be a smooth space Y equipped with a smooth map $p \colon Y \to$

B called the **projection**. For each point $b \in B$, define the **fiber** of Y over b to be the set

$$Y_b = \{y \in Y : p(y) = b\},$$

made into a smooth space with its subspace smooth structure. We can think of a smooth space over B as a primitive sort of 'bundle', without any requirement of local triviality. Note that given smooth spaces X and Y over B, the pullback or 'fibered product' $X \times_B Y$ is again a smooth space over B. In fact this is the product in a certain category of smooth spaces over B.

If Y and Z are smooth spaces over B, let

$$\mathcal{C}^\infty_B(Y, Z) = \bigsqcup_{b \in B} \mathcal{C}^\infty(Y_b, Z_b).$$

We make this into a smooth space, the **parametrized mapping space**, as follows. First define a function

$$p : \mathcal{C}^\infty_B(Y, Z) \to B$$

sending each element of $\mathcal{C}^\infty(Y_b, Z_b)$ to $b \in B$. This will be the projection for the parametrized mapping space. Then, note that given any smooth space X and any function

$$\tilde{f} : X \to C^\infty_B(Y, Z),$$

we get a function from X to B, namely $p\tilde{f}$. If this is smooth we can define the pullback smooth space $X \times_B Y$. Then we can define a function

$$f : X \times_B Y \to Z$$

by

$$f(x, y) = \tilde{f}(x)(y).$$

This allows us to define the smooth structure on $\mathcal{C}^\infty_B(Y, Z)$: for any domain D, a function

$$\tilde{\varphi} : D \to C^\infty_B(Y, Z)$$

is a plot if and only if $p\tilde{\varphi}$ is smooth and the corresponding function

$$\varphi : D \times_B Y \to Z$$

is smooth. With this smooth structure, one can check that $p : \mathcal{C}^\infty_B(Y, Z) \to B$ is smooth. So, the parametrized mapping space is again a smooth space over B.

The point of the parametrized mapping space is that given smooth spaces X, Y, Z over B, there is a natural isomorphism of smooth spaces

$$\mathcal{C}^\infty_B(X \times_B Y, Z) \cong \mathcal{C}^\infty_B(X, \mathcal{C}^\infty_B(Y, Z)).$$

We summarize this by saying that $\mathcal{C}^\infty$ is 'locally' Cartesian closed (see Definition 5.16). In the case where B is a point, this reduces to the fact that $\mathcal{C}^\infty$ is Cartesian closed.

The following theorem subsumes most of the above remarks:

Definition 3.1. A **quasitopos** is a locally Cartesian closed category with finite colimits and a weak subobject classifier.

Theorem 3.2. *The category of smooth spaces, $\mathcal{C}^\infty$, is a quasitopos with all (small) limits and colimits.*

Proof. In Theorem 5.25 we show that this holds for any category of 'generalized spaces', that is, any category of concrete sheaves on a concrete site. In Proposition 4.13 we prove that $\mathsf{ChenSpace}$ is equivalent to a category of this kind, and in Proposition 4.15 we show the same for $\mathsf{DiffeologicalSpace}$. □

4. Smooth spaces as generalized spaces

The concept of a 'generalized space' was developed in the context of quasitopos theory by Antoine [1], Penon [34, 35] and Dubuc [11, 13]. Generalized spaces form a natural framework for studying Chen spaces, diffeological spaces, and even simplicial complexes. For us, a category of generalized spaces will be a category of 'concrete sheaves' over a 'concrete site'. For a self-contained treatment, we start by explaining some basic notions concerning sheaves and sites. We motivate all these notions with the example of Chen spaces, and in Proposition 4.13, we prove that Chen spaces are concrete sheaves on a concrete site. We also prove similar results for diffeological spaces and simplicial complexes.

We can define sheaves on a category as soon as we have a good notion of when a family of morphisms $f\colon D_i \to D$ 'covers' an object D. For this, our category should be what is called a 'site'. Usually a site is defined to be a category equipped with a 'Grothendieck topology'. However, as emphasized by Johnstone [20], we can get away with less: it is enough to use a 'Grothendieck pretopology', or 'coverage'. The difference is not very great, since every coverage on a category determines a Grothendieck topology with the same sheaves. Coverages are simpler to define, and for our limited purposes they are easier to work with. So, we shall take a site to be a category equipped with a coverage. Two different coverages may determine the same Grothendieck topology, but knowledgeable readers can check that everything we do depends only on the Grothendieck topology.

Definition 4.1. A **family** is a collection of morphisms with common codomain.

Definition 4.2. A **coverage** on a category D is a function assigning to each object $D \in \mathsf{D}$ a collection $\mathcal{J}(D)$ of families $(f_i\colon D_i \to D | i \in I)$ called **covering families**, with the following property:

- Given a covering family $(f_i\colon D_i \to D | i \in I)$ and a morphism $g\colon C \to D$, there exists a covering family $(h_j\colon C_j \to C | j \in J)$ such that each morphism gh_j factors through some f_i.

Definition 4.3. A **site** is a category equipped with a coverage. We call the objects of a site **domains**.

In Lemma 4.12 we describe a coverage on the category Chen, whose objects are convex sets and whose morphisms are smooth functions. For this coverage, a covering family is just an open cover in the usual sense. This makes Chen into a site, and Chen spaces will be 'concrete sheaves' on this site. To understand how this works, let us quickly review sheaves and then explain the concept of 'concreteness'.

Definition 4.4. A **presheaf** X on a category D is a functor $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$. For any object $D \in \mathsf{D}$, we call the elements of $X(D)$ **plots in X with domain D**.

Usually the elements of $X(D)$ are called 'sections of X over D'. However, given a Chen space X there is a presheaf on Chen assigning to any convex set D the set $X(D)$ of all plots $\varphi\colon D \to X$. So, it will guide our intuition to quite generally call an object $D \in \mathsf{D}$ a 'domain' and elements of $X(D)$ 'plots'.

Axiom 1 in the definition of a Chen space is what gives us a contravariant functor from Chen to Set: it says that given any morphism $f\colon C \to D$ in Chen, we get a function

$$X(f)\colon X(D) \to X(C)$$

sending any plot $\varphi\colon D \to X$ to the plot $\varphi f\colon C \to X$. Axiom 2 says that the resulting presheaf on Chen is actually a sheaf:

Definition 4.5. Given a covering family $(f_i\colon D_i \to D | i \in I)$ in D and a presheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$, a collection of plots $\{\varphi_i \in X(D_i) | i \in I\}$ is called **compatible** if whenever $g\colon C \to D_i$ and $h\colon C \to D_j$ make the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{h} & D_j \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_j} \\ D_i & \xrightarrow[f_i]{} & D, \end{array}$$

then $X(g)(\varphi_i) = X(h)(\varphi_j)$.

Definition 4.6. Given a site D, a presheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ is a **sheaf** if it satisfies the following condition:

- Given a covering family $(f_i\colon D_i \to D | i \in I)$ and a compatible collection of plots $\{\varphi_i \in X(D_i) | i \in I\}$, then there exists a unique plot $\varphi \in X(D)$ such that $X(f_i)(\varphi) = \varphi_i$ for each $i \in I$.

On any category, there is a special class of presheaves called the 'representable' ones:

Definition 4.7. A presheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ is called **representable** if it is naturally isomorphic to $\hom(-, D)\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ for some $D \in \mathsf{D}$.

The site Chen is 'subcanonical':

Definition 4.8. A site is **subcanonical** if every representable presheaf on this site is a sheaf.

We shall include this property in the definition of a 'concrete site'. But there is a much more important property that we shall also require. A Chen space X gives a special kind of sheaf on the site Chen: a 'concrete' sheaf, roughly meaning that for any $D \in \mathsf{Chen}$, elements of $X(D)$ are certain functions from the underlying set of D to some fixed set. Of course, this notion relies on the fact that D has an underlying set! The following definition ensures that this is the case for any object D in a concrete site.

Definition 4.9. A **concrete site** D is a subcanonical site with a terminal object 1 satisfying the following conditions:

- The functor $\hom(1, -)\colon \mathsf{D} \to \mathrm{Set}$ is faithful.
- For each covering family $(f_i\colon D_i \to D | i \in I)$, the family of functions $(\hom(1, f_i)\colon \hom(1, D_i) \to \hom(1, D) | i \in I)$ is **jointly surjective**, meaning that the union of their images is all of $\hom(1, D)$.

Quite generally, any object D in a category D with a terminal object has an underlying set $\hom(1, D)$, often called its set of 'points'. The requirement that $\hom(1, -)$ be faithful says that two morphisms $f, g\colon C \to D$ in D are equal when they induce

the same functions from points of C to points of D. In other words: objects have enough points to distinguish morphisms. In this situation we can think of objects of D as sets equipped with extra structure. The second condition above then says that the underlying family of functions of a covering family is itself a 'covering', in the sense of being jointly surjective.

Henceforth, we let D stand for a concrete site. Now we turn to the notion of 'concrete sheaf'. There is a way to extract a set from a sheaf on a concrete site. Namely, a sheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ gives a set $X(1)$. In the case of a sheaf coming from a Chen space, this is the set of one-point plots $\varphi\colon 1 \to X$. Axiom 3 implies that it is the underlying set of the Chen space. Furthermore, for any sheaf X on a concrete site, there is a way to turn a plot $\varphi \in X(D)$ into a function $\underline{\varphi}$ from $\hom(1, D)$ to $X(1)$. To do this, set

$$\underline{\varphi}(d) = X(d)(\varphi).$$

A simple computation shows that for the sheaf coming from a Chen space, this process turns any plot into its underlying function. (See Proposition 4.13 for details.) In this example, we lose no information when passing from φ to the function $\underline{\varphi}$: distinct plots have distinct underlying functions. The notion of 'concrete sheaf' makes this idea precise quite generally:

Definition 4.10. Given a concrete site D, we say that a sheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ is **concrete** if for every object $D \in \mathsf{D}$, the function sending plots $\varphi \in X(D)$ to functions $\underline{\varphi}\colon \hom(1, D) \to X(1)$ is one-to-one.

We can think of concrete sheaves as 'generalized spaces', since they generalize Chen spaces and diffeological spaces. Every concrete site gives a category of generalized spaces:

Definition 4.11. Given a concrete site D, a **generalized space** or D **space** is a concrete sheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$. A **map** between D spaces $X, Y\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ is a natural transformation $F\colon X \Rightarrow Y$. We define $\mathsf{D}\mathrm{Space}$ to be the category of D spaces and the maps between them.

Now let us give some examples:

Lemma 4.12. *Let* Chen *be the category whose objects are convex sets and whose morphisms are smooth functions. The category* Chen *has a subcanonical coverage where* $(i_j\colon C_j \to C | j \in J)$ *is a covering family if and only if the convex sets* $C_j \subseteq C$ *form an open covering of the convex set* $C \subseteq \mathbb{R}^n$ *with its usual subspace topology, and* $i_j\colon C_j \to C$ *are the inclusions.*

Proof. Given such a covering family $(i_j\colon C_j \to C | j \in J)$ and $g\colon D \to C$ in Chen, then $\{g^{-1}(i_j(C_j))\}$ is an open cover of D which factors through the family i_j as functions on sets. We can refine this cover by convex open balls to obtain a covering family of D which factors through the family i_j in Chen. Since the covers are open covers in the usual sense, it is clear that the site is subcanonical. □

We henceforth consider Chen as a site with the above coverage. Since any 1-point convex set is a terminal object, Chen is a concrete site. This allows us to define a kind of generalized space called a 'Chen space' following Definition 4.11.

Proposition 4.13. *A Chen space is the same as a* Chen *space. More precisely, the category of Chen spaces and smooth maps is equivalent to the category* $\mathsf{Chen}\mathrm{Space}$.

Proof. Let $\mathcal{C}^\infty$ stand for the category of Chen spaces and smooth maps. We begin by constructing functors from $\mathcal{C}^\infty$ to $\mathsf{ChenSpace}$ and back. To reduce confusion, just for now we use italics for objects and morphisms in $\mathcal{C}^\infty$, and boldface for those in $\mathsf{ChenSpace}$.

First, given $X \in \mathcal{C}^\infty$, we construct a concrete sheaf $\mathbf{X}$ on Chen. For each convex set C, we define $\mathbf{X}(C)$ to be the set of all plots $\varphi\colon C \to X$, and given a smooth function $f\colon C' \to C$ between convex sets, we define $\mathbf{X}(f)\colon \mathbf{X}(C) \to \mathbf{X}(C')$ as follows:

$$\mathbf{X}(f)\varphi = \varphi f.$$

Axiom 1 in Chen's definition guarantees that φf lies in $\mathbf{X}(C')$, and it is easy to check that $\mathbf{X}$ is a presheaf. Axiom 2 ensures that this presheaf is a sheaf.

To check that $\mathbf{X}$ is concrete, first note that axiom 3 gives a bijection between the underlying set of X and the set $\mathbf{X}(1)$, sending any point $x \in X$ to the one-point plot whose image is x. Then, let $\varphi \in \mathbf{X}(C)$ and compute $\underline{\varphi}\colon \hom(1, C) \to X(1) \cong X$:

$$\underline{\varphi}(c) = X(c)(\varphi) = \varphi(c),$$

where at the last step we identify the smooth function $c \in \hom(1, C)$ with the one point in its image. So, $\underline{\varphi}$ is the underlying function of the plot φ. It follows that the map sending φ to $\underline{\varphi}$ is one-to-one, so $\mathbf{X}$ is concrete.

Next, given a smooth map $f\colon X \to Y$ between Chen spaces, we construct a natural transformation $\mathbf{f}\colon X \to \mathbf{Y}$ between the corresponding sheaves. For this, we define

$$\mathbf{f}_C\colon \mathbf{X}(C) \to Y(C)$$

by

$$\mathbf{f}_C(\varphi) = f\varphi.$$

To show that $\mathbf{f}$ is natural, we need the following square to commute for any smooth function $g\colon C' \to C$:

$$\begin{array}{ccc} \mathbf{X}(C) & \xrightarrow{\mathbf{f}_C} & \mathbf{Y}(C) \\ {\scriptstyle \mathbf{X}(g)}\downarrow & & \downarrow{\scriptstyle \mathbf{Y}(g)} \\ \mathbf{X}(C') & \xrightarrow[\mathbf{f}_{C'}]{} & \mathbf{Y}(C'). \end{array}$$

This just says that $(f\varphi)g = f(\varphi g)$.

We leave it to the reader to verify that this construction defines a functor from $\mathcal{C}^\infty$ to $\mathsf{ChenSpace}$.

To construct a functor in the other direction, we must first construct a Chen space X from any concrete sheaf $\mathbf{X}$ on Chen. For this we take $X = \mathbf{X}(1)$ as the underlying set of the Chen space, and we take as plots in X with domain C all functions of the form $\underline{\varphi}$, where $\varphi \in \mathbf{X}(C)$. Axiom 1 in the definition of a Chen space follows from the fact that $\mathbf{X}$ is a presheaf. Axiom 2 follows from the fact that $\mathbf{X}$ is a sheaf. Axiom 3 follows from the fact that $X = \mathbf{X}(1)$. Next, we must construct a function $f\colon X \to Y$ from a natural transformation $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ between concrete sheaves. For this we set

$$f = \mathbf{f}_1\colon \mathbf{X}(1) \to \mathbf{Y}(1).$$

Again, we leave it to the reader to check that this construction defines a functor.

Finally, we must check that the composite of these functors in either order is naturally isomorphic to the identity. This is straightforward in the case where we turn a Chen space $X \in \mathcal{C}^\infty$ into a concrete sheaf $\mathbf{X}$ and back into a Chen space. When we turn a concrete sheaf $\mathbf{X}$ into a Chen space X and back into a concrete sheaf $\mathbf{X}'$, we have

$$\mathbf{X}'(C) = \{\underline{\varphi} \colon C \to X(1)\},$$

but the latter is naturally isomorphic to $X(C)$ via the function

$$\begin{array}{ccc} X(C) & \to & X'(C) \\ \varphi & \mapsto & \underline{\varphi} \end{array}$$

thanks to the fact that $\mathbf{X}$ is concrete. □

Diffeological spaces work similarly:

Lemma 4.14. *Let* Diffeological *be the category whose objects are open subsets of* $\mathbb{R}^n$ *and whose morphisms are smooth maps. The category* Diffeological *has a subcanonical coverage where* $(i_j \colon U_j \to U | j \in J)$ *is a covering family if and only if the open sets* $U_j \subseteq U$ *form an open covering of the open set* $U \subseteq \mathbb{R}^n$*, and* $i_j \colon U_j \to U$ *are the inclusions.*

Proof. The proof is a simpler version of the proof for Chen, since we are considering open but not necessarily convex sets. □

We henceforth treat Diffeological as a site with this coverage. The one-point open subset of $\mathbb{R}^0$ is a terminal object for Diffeological, so this is a concrete site. As before, we have:

Proposition 4.15. *A diffeological space is the same as a* Diffeological *space. More precisely, the category of diffeological spaces is equivalent to the category* DiffeologicalSpace.

Proof. The proof of the corresponding statement for Chen spaces applies here as well. □

An example of a very different flavor is the category of simplicial complexes:

Definition 4.16. An **(abstract) simplicial complex** is a set X together with a family K of nonempty finite subsets of X such that:

(1) Every singleton lies in K.
(2) If $S \in K$ and $T \subseteq S$, then $T \in K$.

A **map** of simplicial complexes $f \colon (X, K) \to (Y, L)$ is a function $f \colon X \to Y$ such that $S \in K$ implies $f(S) \in L$.

We can geometrically realize any simplicial complex (X, K) by turning each n-element set $S \in K$ into a geometrical $(n-1)$-simplex. Then axiom 1 above says that any point of X corresponds to a 0-simplex, while axiom 2 says that any face of a simplex is again a simplex.

To view the category of simplicial complexes as a category of generalized spaces, we use the following site.

Lemma 4.17. *Let* F *be the category with nonempty finite sets as objects and functions as morphisms. There is a subcanonical coverage on* F *where for each object* D *in* F *there is exactly one covering family, consisting of all inclusions* $D' \hookrightarrow D$.

Proof. Given a covering family $(f_i \colon D_i \hookrightarrow D | i \in I)$ and a function $g \colon C \to D$, each function in a covering family having C as codomain composed with g clearly factors through some f_i. For instance, take f_i to be the identity function on D. The coverage is clearly subcanonical since each covering includes the identity morphism. □

Henceforth we make F into a concrete site with the above coverage. Since every covering family contains the identity, this coverage is 'vacuous': every presheaf is a sheaf. Presheaves on F have been studied by Grandis under the name **symmetric simplicial sets**, since they resemble simplicial sets whose simplices have unordered vertices [17]. It turns out that *concrete* sheaves on F are simplicial complexes:

Proposition 4.18. *The category of* F *spaces is equivalent to the category of simplical complexes.*

Proof. We define a functor from the category of F spaces to the category of simplicial complexes. We use n to stand for an n-element set. Since the underlying set $\hom(1, n)$ of $n \in \mathsf{F}$ is naturally isomorphic to n, we shall not bother to distinguish between the two.

Given an F space, that is, a concrete sheaf $\mathbf{X} \colon \mathsf{F}^{op} \to \mathrm{Set}$, we define a simplicial complex (X, S) with $X = \mathbf{X}(1)$ and $K = \{\mathrm{im}\underline{\varphi} | \varphi \in \mathbf{X}(n), n \in \mathsf{F}\}$. To check axiom 1, we note that a point $x \in X$ is a plot $\varphi \in \mathbf{X}(1)$, and $\{x\} = \mathrm{im}\underline{\varphi} \in K$. To check axiom 2, we fix an object n, a plot $\varphi \in \mathbf{X}(n)$ and a subset $Y \subseteq \mathrm{im}\underline{\varphi} \in K$. We consider $\underline{\varphi}^{-1}(Y) \subseteq n$ and let m be the object in F representing the finite set of cardinality $|\underline{\varphi}^{-1}(Y)|$. There is an inclusion $m \hookrightarrow \underline{\varphi}^{-1}(Y) \subseteq n$, and the commutativity of

$$\begin{array}{ccc} S(n) & \longrightarrow & S(m) \\ \downarrow & & \downarrow \\ \hom(n, S(1)) & \longrightarrow & \hom(m, S(1)) \end{array}$$

shows that Y is an element of K and that the structure defined is, in fact, a simplicial complex.

Given a natural transformation $\mathbf{f} \colon \mathbf{X} \Rightarrow \mathbf{Y}$ between F spaces we obtain a map $f = \mathbf{f}_1 \colon \mathbf{X}(1) \to \mathbf{Y}(1)$. By the commutativity of

$$\begin{array}{ccc} \mathbf{X}(n) & \longrightarrow & \mathbf{X}(n) \\ \downarrow & & \downarrow \\ \hom(n, \mathbf{X}(1)) & \longrightarrow & \hom(n, \mathbf{Y}(1)) \end{array}$$

we see that this defines a map of simplicial complexes and this process clearly preserves identities and composition. Since a map $\mathbf{f} \colon \mathbf{X} \Rightarrow \mathbf{Y}$ of F spaces is completely determined by the function $\mathbf{f}_1 \colon \mathbf{X}(1) \to \mathbf{Y}(1)$ it is clear that this functor is faithful. We see that the functor is full since given a map of simplicial complexes $f \colon (X, K) \to (Y, L)$ and a morphism between finite sets $j \colon m \to n$, then the

naturality square

$$\begin{array}{ccc} \mathbf{X}(n) & \xrightarrow{\mathbf{X}(j)} & \mathbf{X}(m) \\ {\scriptstyle \mathbf{f}(n)}\downarrow & & \downarrow{\scriptstyle \mathbf{f}(m)} \\ \mathbf{Y}(n) & \xrightarrow[\mathbf{Y}(j)]{} & \mathbf{Y}(m) \end{array}$$

commutes, thus defining a natural transformation between F spaces.

We can also reverse the process described, taking a simplicial complex (X, K) and defining an F space $\mathbf{X}$ whose image is isomorphic to (X, K). For each $n \in \mathsf{F}$, we let $\mathbf{X}(n)$ be the set of n-element sets $S \in K$. The downward closure property of simplicial complexes guarantees that this is an F space, and it is easy to see that one can construct an isomorphism from the image of this F space under our functor to (X, K). Thus, we have obtained an equivalence of categories. □

5. Convenient properties of generalized spaces

In this section we establish convenient properties of any category of generalized spaces. We begin with some handy notation. In Section 4 we introduced three closely linked notions of 'underlying set' or 'underlying function' in the context of a concrete site D. It will now be convenient, and we hope not confusing, to denote all three of these by an underline:

- The underlying set of a domain: $\underline{D} = \hom(1, D)$

  Any concrete site D has an 'underlying set' functor $\hom(1, -) \colon \mathsf{D} \to \mathrm{Set}$. Henceforth we denote this functor by an underline:

  $$\_ \colon \mathsf{D} \to \mathrm{Set}.$$

  So, any domain $D \in \mathsf{D}$ has an underlying set $\underline{D}$, and any morphism $f \colon C \to D$ in D has an underlying function $\underline{f} \colon \underline{C} \to \underline{D}$. The concreteness condition on D says that this underlying set functor is faithful.
- The underlying set of a generalized space: $\underline{X} = X(1)$

  Any generalized space $X \colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ has an underlying set $X(1)$. Henceforth we denote this set as $\underline{X}$. Similarly, any map of generalized spaces $f \colon X \to Y$ has an underlying function $f_1 \colon X(1) \to Y(1)$, which we henceforth write as $\underline{f} \colon \underline{X} \to \underline{Y}$. It is easy to check that these combine to give an 'underlying set' functor

  $$\_ \colon \mathsf{DSpace} \to \mathrm{Set}.$$

  In Proposition 5.1 we show that this underlying set functor is also faithful.
- The underlying function of a plot: $\underline{\varphi}(d) = X(d)(\varphi)$

  For any generalized space $X \colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$, any plot $\varphi \in X(D)$ has an underlying function $\underline{\varphi} \colon \underline{D} \to \underline{X}$ defined as above. The concreteness condition in the definition of 'generalized space' says that the map from plots to their underlying functions is one-to-one. One can check that this map defines a natural transformation

  $$\_ \colon X(D) \to \underline{X}^{\underline{D}}.$$

Proposition 5.1. *The underlying set functor* $\_ \colon \mathsf{DSpace} \to \mathrm{Set}$ *is faithful.*

Proof. Given D spaces X and Y, suppose $f, g\colon X \to Y$ have $\underline{f} = \underline{g}$. We need to show that $f = g$. Recall that f and g are natural transformations between the functors $X, Y\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$, so given $D \in \mathsf{D}$ the following squares commute for each $d \in \underline{D}$:

$$\begin{array}{ccc} X(D) & \xrightarrow{f_D} & Y(D) \\ {\scriptstyle X(d)}\downarrow & & \downarrow{\scriptstyle Y(d)} \\ X(1)=\underline{X} & \xrightarrow[\underline{f}]{} & \underline{Y}=Y(1), \end{array} \qquad \begin{array}{ccc} X(D) & \xrightarrow{g_D} & Y(D) \\ {\scriptstyle X(d)}\downarrow & & \downarrow{\scriptstyle Y(d)} \\ X(1)=\underline{X} & \xrightarrow[\underline{g}]{} & \underline{Y}=Y(1). \end{array}$$

We need to show that for any $\varphi \in X(D)$, $f_D(\varphi) = g_D(\varphi)$ in $Y(D)$. Since the natural transformation

$$\_\colon Y(D) \to \underline{Y}^{\underline{D}}$$

is one-to-one, it suffices to show that

$$\underline{f_D(\varphi)}(d) = \underline{g_D(\varphi)}(d)$$

for all $d \in D$, or in other words,

$$Y(d)f_D(\varphi) = Y(d)g_D(\varphi).$$

By the above commuting squares, this amounts to showing that

$$\underline{f}X(d)(\varphi) = \underline{g}X(d)(\varphi),$$

but this follows from $\underline{f} = \underline{g}$. □

There is a further relation between the two 'underlying set' functors mentioned above. In the case of Chen spaces, every convex set naturally becomes a Chen space with the same underlying set. This happens quite generally:

Proposition 5.2. *Every representable presheaf on* D *is a* D *space. The underlying set of the* D *space* $\hom(-, D)\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ *is equal to* $\underline{D}$.

Proof. Since a concrete site is subcanonical by definition, every representable presheaf on D is a sheaf. So, to show that the representable presheaves are D spaces, we just need to show that they are *concrete* sheaves. Suppose $X \cong \hom(-, D)$ is a representable presheaf. Then given $C \in \mathsf{D}$, the map $\_\colon \hom(C, D) \to \underline{C}^{\underline{D}}$ takes a morphism $f\colon C \to D$ to its underlying function $\underline{f}\colon \underline{C} \to \underline{D}$ and thus is one-to-one. It follows that $\hom(-, D)$ is concrete. The underlying set of this D space is $\hom(1, D)$, which is just $\underline{D}$. □

5.1. **Subspaces, quotient spaces, and limits.** With these preliminaries in hand, we now study subspaces and quotient spaces of D spaces and show that the category of D spaces has a weak subobject classifier, Ω. In the process we will show that DSpace has limits.

For Chen spaces or diffeological spaces, Ω is just the 2-element set $2 = \{0, 1\}$ equipped with its indiscrete smooth structure. In general, Ω will have the 2-element set as its underlying set, and for any $D \in \mathsf{D}$, every function $\varphi\colon \underline{D} \to 2$ will count as a plot. So, $\Omega(D)$ will be the power set of $\underline{D}$:

Proposition 5.3. *There is a* D *space* Ω *such that for any object* $D \in \mathsf{D}$, $\Omega(D) = 2^{\underline{D}}$, *and for any morphism* $f\colon C \to D$ *in* D, $\Omega(f)\colon 2^{\underline{D}} \to 2^{\underline{C}}$ *sends any plot* $\varphi\colon \underline{D} \to 2$ *to the plot* $\varphi\underline{f}\colon \underline{C} \to 2$.

Proof. Ω is clearly a presheaf. To show that it is a sheaf, we suppose that $(f_i\colon D_i \to D | i \in I)$ is a covering family and $\{\varphi_i \in \Omega(D_i) | i \in I\}$ is a compatible family of plots, and show that there exists a unique plot $\varphi \in \Omega(D)$ with $\Omega(f_i)\varphi = \varphi_i$. The compatible family of plots consists of functions $\varphi_i\colon \underline{D_i} \to 2$, and we need to show that there exists a unique function $\varphi\colon \underline{D} \to 2$ with $\varphi f_i = \varphi_i$.

For the existence of φ, suppose $d \in \underline{D}$. Then since the family $(\underline{f_i}\colon \underline{D_i} \to \underline{D})$ is jointly surjective we can find an i such that there exists $d' \in \underline{D_i}$ with $\underline{f_i}(d') = d$. We define $\varphi(d) = \varphi_i(d')$. To show that $\varphi(d)$ is independent of the choice of i, suppose that $d' \in \underline{D_i} \cap \underline{D_j}$ and consider morphisms $g\colon 1 \to D_i$ and $h\colon 1 \to D_j$ such that $\underline{g}(1) = d' = \underline{h}(1)$. Since the plots were chosen to be compatible with the family, we have $\Omega(g)(\varphi_i) = \Omega(h)(\varphi_j)$. In other words, $\varphi_i(d') = \varphi_j(d')$. Uniqueness follows from the family being jointly surjective. Finally, since plots $\varphi \in \Omega(D)$ are in one-to-one correspondence with functions $\underline{\varphi}\colon \underline{D} \to 2$, the sheaf Ω is concrete. □

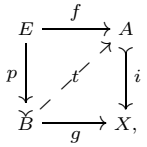
Proposition 5.4. *A monomorphism (resp. epimorphism) in* DSpace *is a map $f\colon X \to Y$ for which the underlying function $\underline{f}$ is injective (resp. surjective).*

Proof. For one direction, recall that $\_\colon \mathsf{DSpace} \to \mathsf{Set}$ is faithful by Proposition 5.1, so a morphism $f\colon X \to Y$ in DSpace is monic (resp. epic) if its image under this functor has the same property.

Conversely, suppose f is monic. Then the map from $\hom(1, X)$ to $\hom(1, Y)$ given by composing with f is injective, but this says precisely that $\underline{f}$ is injective.

Next, suppose f is epic. Then the map from $\hom(Y, \Omega)$ to $\hom(X, \Omega)$ given by composing with f is injective, but this says that the map from $2^{\underline{Y}}$ to $2^{\underline{X}}$ sending $\chi\colon \underline{Y} \to 2$ to $\chi f\colon \underline{X} \to 2$ is injective, which implies that f is surjective. □

Definition 5.5. In any category, a monomorphism $i\colon A \to X$ is **strong** if given any epimorphism $p\colon E \to B$ and morphisms f, g making the outer square here commute:

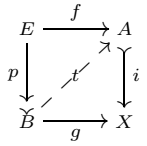


then there exists a unique $t\colon B \to A$ making the whole diagram commute.

Definition 5.6. We say that a morphism of D spaces $i\colon A \to X$ makes A a **subspace** of X if for any plot $\varphi \in X(D)$ with $\underline{\varphi}(\underline{D}) \subseteq \underline{i}(\underline{A})$, there exists a unique plot $\psi \in A(D)$ with $i_D(\psi) = \varphi$.

Proposition 5.7. *A morphism of* D *spaces $i\colon A \to X$ is a strong monomorphism if and only if i makes A a subspace of X.*

Proof. Suppose $i\colon A \to X$ is a subspace of X. Given an epimorphism $p\colon E \to B$ and morphisms f, g such that the outer square here:



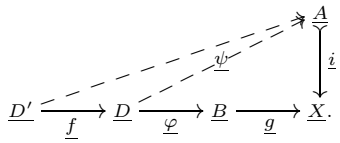
commutes, we need to prove that there exists a unique $t\colon E \to B$ making the whole diagram commute. Define functions $t_D\colon B(D) \to A(D)$ as follows. Note that for any plot $\varphi \in B(D)$, the plot $g_D(\varphi) \in X(D)$ has

$$\underline{g_D(\varphi)}(\underline{D}) = \underline{g\varphi}(\underline{D}) \subseteq \underline{i}(\underline{A}),$$

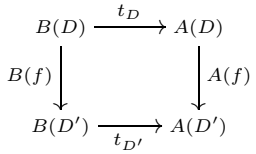
where in the first step we use the naturality of the map sending a plot to its underlying function, and in the second we use the commutative diagram of underlying functions. By Definition 5.6 it follows that there exists a unique plot $\psi \in A(D)$ with $i_D(\psi) = g_D(\varphi)$. We set

$$t_D(\varphi) = \psi.$$

We can check that t is a natural transformation by considering a morphism $f\colon D' \to D$ in D and the following diagram:

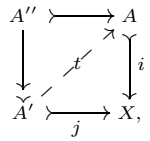


By the description of t_D above, we see that $\underline{g\varphi}$ is uniquely lifted to a plot $\underline{\psi}\colon \underline{D} \to \underline{A}$. Then $\underline{g\varphi f}$ also has a unique lift, which must be $\underline{\psi f}\colon \underline{D'} \to \underline{A}$. We have seen that the naturality square



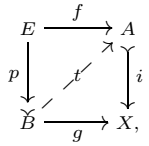
commutes, and thus that t is a map of D spaces. The lower triangle commutes by construction. The upper triangle commutes since $f = i^{-1}gp = tp$. We can check that t is unique at the level of the underlying functions, where it follows from the commutativity of the diagram and that i is a monomorphism. Now we have shown that i is a strong monomorphism.

Conversely, suppose $i\colon A \to X$ is a strong monomorphism and consider a plot $\varphi \in X(D)$ for some $D \in \mathsf{D}$ with $\underline{\varphi}(\underline{D}) \subseteq \underline{i}(\underline{A})$. We give the set $\underline{A'} := \underline{\varphi}(\underline{D}) \subseteq \underline{X}$ the subspace structure from X and we give $\underline{A''} := \underline{i}^{-1}\underline{\varphi}(\underline{D})$ the subspace structure of A. Then a DSpace epimorphism from A'' to A' is induced by restricting $\underline{i}$ and we have the following commutative diagram:



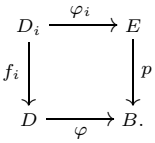
where t exists and is unique since i is a strong monomorphism. Since A' is a subspace of X and $\underline{\varphi}(\underline{D}) = \underline{A'}$, there exists a unique plot $\psi \in A'(D)$ such that $j_D(\psi) = \varphi$. Thus we have $t_D(\psi) \in A(D)$ and, by commutativity of the diagram, $i_D t_D(\psi) = j_D(\psi) = \varphi$. For any other $\psi' \in A(D)$ with $i_D(\psi') = \varphi$, we have $\psi' = t_D(\psi)$ since i is a monomorphism, and thus $t_D(\psi)$ is unique as desired. □

Definition 5.8. In any category, an epimorphism $p\colon E \to B$ is **strong** if given any monomorphism $i\colon A \to X$ and morphisms f, g making the outer square below commute:



then there exists a unique $t\colon B \to A$ making the whole diagram commute.

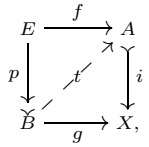
Definition 5.9. We say that a morphism of D spaces $p\colon E \to B$ makes B a **quotient space** of E if for every plot $\varphi \in B(D)$, there exists a covering family $(f_i\colon D_i \to D | i \in I)$ in D and a collection of plots $\{\varphi_i \in E(D_i) | i \in I\}$ such that the following diagram commutes:



With this definition, the underlying map $\underline{p}\colon \underline{E} \to \underline{B}$ is a surjection and thus defines an equivalence relation, $e_1 \sim e_2$ if and only if $\underline{p}(e_1) = \underline{p}(e_2)$, such that $\underline{B}$ is the quotient of $\underline{E}$ by this equivalence relation. The extra condition that every plot in B comes locally from a plot in E gives the following theorem:

Proposition 5.10. *A morphism of* D *spaces* $p\colon E \to B$ *is a strong epimorphism if and only if* p *makes* B *a quotient space of* E.

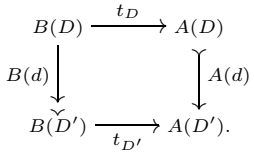
Proof. Suppose that $p\colon E \to B$ makes B a quotient space of E. By Proposition 5.4, p is an epimorphism since $\underline{p}$ is surjective. To show that p is a strong epimorphism, given any monomorphism $i\colon A \to X$ and morphisms f, g making the outer square commute:



we need to prove that there exists a unique $t\colon B \to A$ making the whole diagram commute. We do this by first constructing the underlying function

$$\begin{array}{rccc} \underline{t}\colon & \underline{B} & \to & \underline{A} \\ & x & \mapsto & \underline{f}(y), \end{array}$$

where $\underline{p}(y) = x$. The respective surjectivity and injectivity of $\underline{p}$ and $\underline{i}$ guarantee that $\underline{t}$ as just defined is the unique function making the diagram of underlying sets commute. To show that $\underline{t}$ induces a map of D spaces, we need to check that the following naturality square commutes for every map $d\colon D' \to D$ in D:



Since B and A are concrete sheaves, we can check that this diagram commutes at the level of underlying functions of plots. Given a plot $\varphi \in B(D)$, we examine its two images in $A(D')$. First, we have

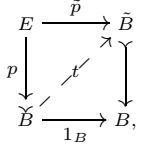
$$\begin{array}{rccc} \underline{A(d)(t_D(\varphi))}\colon & \underline{D'} & \to & \underline{A} \\ & c & \mapsto & \underline{f}(y), \end{array}$$

where $y \in \underline{E}$ such that $\underline{p}(y) = \underline{\varphi}\underline{d}(c)$. The second image has an underlying function defined as follows:

$$\begin{array}{rccc} \underline{t_{D'}(B(d)(\varphi))}\colon & \underline{D'} & \to & \underline{A} \\ & c & \mapsto & \underline{f}(y'), \end{array}$$

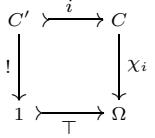
where $y' \in \underline{E}$ such that $\underline{p}(y') = \underline{\varphi}\underline{d}(c)$. We are just left to check that given $y, y' \in \underline{p}^{-1}(\underline{\varphi}\underline{d}(c))$, then $\underline{f}(y) = \underline{f}(y')$. This follows from the commutativity $if = gp$ and that $\underline{i}$ is injective.

Conversely, let $p\colon E \to B$ be a strong epimorphism. Consider the concrete presheaf with plots $p_D(E(D))$ for every $D \in \mathsf{D}$. By the process of sheafification described in Section 5.3 we obtain a DSpace, which we will denote by $\tilde{B}$. We consider the following commutative diagram:



where $\tilde{p}$ has the same underlying function as p and the unlabeled arrow is the DSpace map induced by the identity function on $\underline{B}$. Since p is a strong epimorphism, there exists a unique t making the diagram commute. It follows that $\underline{t} = 1_{\underline{B}}$ and that $B(D) \subseteq \tilde{B}(D)$ for every $D \in \mathsf{D}$. Every plot $\varphi \in B(D)$ can then be considered as a plot in $\tilde{B}(D)$, which arises in two ways. Either φ came from a plot in $E(D)$, in which case we consider the covering family with just the identity map $(1\colon D \to D)$, and there exists a plot $\hat{\varphi} \in E(D)$ which maps to φ, or φ arose from sheafification. In the latter case, there exists a family $(f_i\colon D_i \to D | i \in I)$ in D and a compatible collection of plots $\{\underline{p\varphi_i} \in p_{D_i}(E(D_i)) | i \in I\}$ each of which is the restriction of $\underline{\varphi}$. We have shown that p makes B a quotient space of E. □

Definition 5.11. In a category $\mathcal{C}$ with finite limits, a **weak subobject classifier** is an object Ω equipped with a morphism $\top\colon 1 \to \Omega$ such that, given any strong monomorphism $i\colon C' \to C$ in $\mathcal{C}$, there is a unique morphism $\chi_i\colon C \to \Omega$ making



a pullback.

To show that DSpace has a weak subobject classifier for any D, we need to first show that it has finite limits. It is almost as easy to show that it has arbitrary limits. For this, we use the category $\mathrm{Sh}(\mathsf{D})$ with sheaves on D as objects and natural transformations between these as morphisms. It is well known [33] that this category has all limits, which can be computed 'pointwise': if $F\colon C \to \mathrm{Sh}(\mathsf{D})$

is any diagram of sheaves indexed by a small category C, then its limit exists and is given by

$$(\lim F)(D) = \lim F(D).$$

Proposition 5.12. DSpace *has all (small) limits, which may be computed pointwise.*

Proof. Let $F\colon C \to \mathsf{DSpace}$ be a diagram. The limit of the underlying diagram of sheaves can be computed pointwise; we will show that this limit has the concreteness property. It will follow that this limit is also the limit in DSpace, since a morphism in $\mathrm{Sh}(D)$ between sheaves that happen to be objects of DSpace is automatically a morphism in DSpace.

For any domain D, the diagram $F\colon C \to \mathsf{DSpace}$ gives two diagrams of sets, namely its composites with the functors

$$\begin{array}{ccc} \mathsf{DSpace} & \to & \mathrm{Set} \\ X & \mapsto & X(D) \end{array}$$

and

$$\begin{array}{ccc} \mathsf{DSpace} & \to & \mathrm{Set} \\ X & \mapsto & \underline{X}^{\underline{D}}. \end{array}$$

There is a natural transformation between these two diagrams of sets, namely

$$\_\colon F(\alpha)(D) \to \underline{F(\alpha)}^{\underline{D}}\,.$$

Because each $F(\alpha)$ is a concrete sheaf, each component of this natural transformation is one-to-one. So, taking the limits of both diagrams, we get a one-to-one function

$$\lim_{\alpha\in C} F(\alpha)(D) \to \lim_{\alpha\in C} \underline{F(\alpha)}^{\underline{D}},$$

which by the properties of limits can be reinterpreted as a one-to-one function

$$\lim_{\alpha\in C} F(\alpha)(D) \to (\lim_{\alpha\in C} \underline{F(\alpha)})^{\underline{D}}\,.$$

Next, since $\underline{F(\alpha)} = F(\alpha)(1)$ and limits of sheaves are computed pointwise, we can reinterpret this as a one-to-one function

$$(\lim_{\alpha\in C} F(\alpha))(D) \to \underline{(\lim_{\alpha\in C} F(\alpha))}^{\underline{D}}\,.$$

One can check that this function is none other than

$$\_\colon (\lim_{\alpha\in C} F(\alpha))(D) \to \underline{(\lim_{\alpha\in C} F(\alpha))}^{\underline{D}}\,.$$

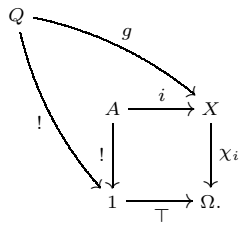
Since this is one-to-one, the limit of F is a concrete sheaf. □

It follows that the terminal object 1 is the unique sheaf with exactly one plot for each object $D \in \mathsf{D}$.

Proposition 5.13. *For any concrete site* D*, the category of* D *spaces has a weak subobject classifier* Ω.

Proof. We define Ω as in Proposition 5.3, and we define the map $\top\colon 1 \to \Omega$ as the constant map to $1 \in \{0,1\} = \underline{\Omega}$. Given a strong monomorphism $i\colon A \to X$ of D spaces we define its characteristic map $\chi_i\colon X \to \Omega$ to have the underlying function $\underline{\chi}_i$ given by $\underline{\chi}_i(x) = 1$ if x is in the image of $\underline{i}$ and $\underline{\chi}_i(x) = 0$ otherwise. We need

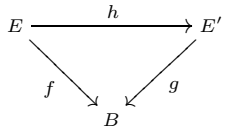
to check that this definition makes A a pullback. Consider a D space Q with maps making the following diagram commute:



We need to show that there exists a unique function $\underline{h}\colon \underline{Q} \to \underline{A}$ defining a map of D spaces $h\colon Q \to A$. Since the outer edges of the diagram commute we can define $\underline{h}$ such that $\underline{mh} = \underline{g}$. Since A is a subspace of X it is clear that h is a map of D spaces. □

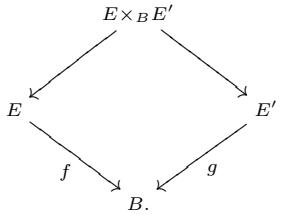
5.2. **Parametrized mapping spaces.** We next turn to the existence of parametrized mapping spaces between D spaces over a fixed base B.

Definition 5.14. Given an object B in a category $\mathcal{C}$, the category of **objects over** B (sometimes called the **slice category** of B) has morphisms $f\colon E \to B$ in $\mathcal{C}$ as objects and commuting triangles



as morphisms. We denote this category by $\mathcal{C}_B$.

We can think of these as 'bundles' over B, in a very general sense, not assuming any sort of local triviality. The product in the category of objects over B is given by the pullback:



Definition 5.15. A category $\mathcal{C}$ is **Cartesian closed** if it has finite products and for every object $Y \in \mathcal{C}$, the functor

$$- \times Y\colon \mathcal{C} \to \mathcal{C}$$

has a right adjoint, called the **internal hom** and denoted by

$$\mathcal{C}(Y, -)\colon \mathcal{C} \to \mathcal{C}.$$

The fact that $\mathcal{C}(X, -)$ is right adjoint to $- \times X$ means that we have a natural bijection of sets

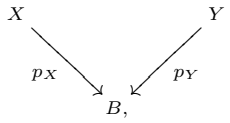
$$\hom(X \times Y, Z) \cong \hom(X, \mathcal{C}(Y, Z)),$$

but a standard argument shows that we also have a natural isomorphism in $\mathcal{C}$:

$$\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

Definition 5.16. A category $\mathcal{C}$ is called **locally Cartesian closed** if for every $B \in \mathcal{C}$, the category $\mathcal{C}_B$ of objects over B is Cartesian closed. Given objects X, Y over B, we call the internal hom $\mathcal{C}_B(X, Y)$ a **parametrized mapping space**.

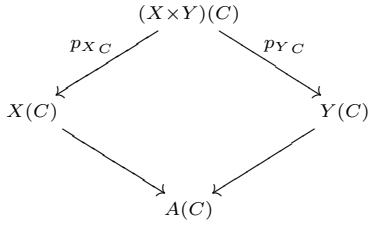
We want to show that the category of D spaces is locally Cartesian closed. To do this we need to determine the product and internal hom in the category of D spaces over some D space B. Given two spaces over B,



the product is given by the pullback $X \times_B Y$ in the category of D spaces. It is easily checked that the universal property holds by considering the universal property of the pullback. Alternatively, the D space structure on $X \times_B Y$ can be quickly obtained by the following lemma:

Lemma 5.17. *The monomorphism* $m\colon X \times_A Y \to X \times Y$ *in* DSpace *given by the inclusion of sets* $\underline{X \times_A Y} \hookrightarrow \underline{X \times Y}$ *is a strong monomorphism.*

Proof. Given $C \in \mathsf{D}$, then for any $\varphi \in (X \times Y)(C)$ such that $\underline{\varphi}(\underline{C}) \subseteq \underline{X \times_A Y}$, the following diagram commutes:



and thus $\varphi \in (X \times_A Y)(C) \subseteq (X \times Y)(C)$. □

Proposition 5.18. *For any concrete site* D*, the category of* D *spaces is locally Cartesian closed.*

Proof. To keep our notation terse, we write $\mathcal{D}$ for DSpace in what follows. So, given a D space B, $\mathcal{D}_B$ stands for the category of D spaces over B, and given $X, Y \in \mathcal{D}_B$ we will denote their parametrized mapping space by $\mathcal{D}_B(X, Y)$.

Of course, we need to prove that this internal hom exists. To describe it, we start by describing its underlying set, which we denote by $\underline{\mathcal{D}_B}(X, Y)$. First, suppose $p\colon X \to B$ is any D space over B. Given a point $b \in \underline{B}$, let the **fiber of** X **over** b, denoted by X_b, be the set

$$\underline{p}^{-1}(b) \subseteq \underline{X}$$

with the unique D space structure for which the inclusion $i\colon X_b \to X$ makes X_b into a subspace of X. Then, we have

$$\underline{\mathcal{D}_B}(X, Y) = \coprod_{b \in \underline{B}} \underline{\mathcal{D}}(X_b, Y_b),$$

where $\underline{\mathcal{D}}(X_b, Y_b)$ is the underlying set of $\mathcal{D}(X_b, Y_b)$. This set sits over the set $\underline{B}$ in an obvious way, which we denote as

$$q\colon \underline{\mathcal{D}_B}(X, Y) \to \underline{B}.$$

Next we describe the plots for $\mathcal{D}_B(X,Y)$. Given $C \in \mathsf{D}$ and a function $\underline{\varphi}\colon \underline{C} \to \underline{\mathcal{D}}_B(X,Y)$, we need to say when this is the function underlying a plot with domain C. We can consider C as a representable D space (see Proposition 5.2). By composing $\underline{\varphi}$ with q we see that $\underline{C}$ is a set over $\underline{B}$. We say that $\underline{\varphi}$ determines a plot for $\mathcal{D}_B(X,Y)$ if this composite function $q\underline{\varphi}$ underlies a map of D spaces and

$$\underline{C} \times_{\underline{B}} \underline{X} \xrightarrow{\varphi \times_B 1} \coprod_{b \in \underline{B}} \underline{\mathcal{D}}(X_b, Y_b) \times_{\underline{B}} \underline{X} \xrightarrow{\mathrm{ev}} \underline{Y}$$

underlies a map of D spaces, where $\underline{\mathrm{ev}}$ is the evaluation map.

We need to check that these plots for $\mathcal{D}_B(X,Y)$ indeed describe a sheaf. Let $(f_i\colon C_i \to C | i \in I)$ be a covering family with a compatible collection of plots $\{\underline{\varphi_i}\colon \underline{C_i} \to \underline{C}\}$. From this collection we can define a function $\underline{\varphi}\colon \underline{C} \to \underline{\mathcal{D}}_B(X,Y)$. It is clear, since B is a sheaf, that φ satisfies the first condition of being a plot for $\mathcal{D}_B(X,Y)$. We then just need to check that

$$\underline{C} \times_{\underline{B}} \underline{X} \xrightarrow{\varphi \times_B 1} \underline{\mathcal{D}}_B(X,Y) \times_{\underline{B}} \underline{X} \xrightarrow{\mathrm{ev}} \underline{Y}$$

underlies a map of smooth spaces.

Let $\psi\colon C' \to C \times_B X$ be a plot. Projecting out of the first coordinate, we obtain a map $\pi_1\psi\colon C' \to C$ in D. Recalling the property in the definition of a covering family, corresponding to the family covering C and the map $\pi_1\psi\colon C' \to C$, there exists a covering family $(g_j\colon D_j \to C' | j \in J)$ with the following property: for each $j \in J$, there exists a map $h_{ij}\colon D_j \to C_i$ for some $i \in I$, such that the following diagram commutes:

$$\begin{array}{ccccc} C' & \xrightarrow{\psi} & C \times_B X & \xrightarrow{\pi_1} & C \\ {\scriptstyle g_j}\uparrow & & & & \uparrow{\scriptstyle f_i} \\ D_j & & \xrightarrow[h_{ij}]{} & & C_i. \end{array}$$

We define for each $j \in J$, a function $\underline{\tau_{ij}}\colon \underline{D_j} \to \underline{C_i} \times_{\underline{B}} \underline{X}$ by $\underline{\tau_{ij}}(d) = (\underline{h_{ij}}(d), \underline{\pi_2\psi g_j}(d))$. These functions $\underline{\tau_{ij}}$ will be plots if the projection to each component is a plot. This is clear since $\underline{h_{ij}}$ is a map in D and ψg_j is a plot in $C \times_B X$. We consider the following diagram for each $j \in J$:

$$\begin{array}{ccccccccc} D_j & \xrightarrow{g_j} & C' & \xrightarrow{\psi} & C \times_B X & \xrightarrow{\varphi \times_B 1} & \mathcal{D}_B(X,Y) \times_B X & \xrightarrow{\mathrm{ev}} & Y \\ & {\scriptstyle \tau_{ij}}\searrow & & & & & \nearrow{\scriptstyle \varphi_i \times_B 1} & & \\ & & & & C_i \times_B X. & & & & \end{array}$$

It is easy to check that this diagram commutes since for $d \in \underline{D}$,

$$(\underline{\varphi_i h_{ij}}(d), \underline{\pi_2\psi g_j}(d)) = (\underline{\varphi\pi_1\psi g_j}(d), \underline{\pi_2\psi g_j}(d)),$$

which follows from $\varphi\pi_1\psi g_j = \varphi f_i h_{ij} = \varphi_i h_{ij}$. It follows that for each $j \in J$, that

$$D_j \xrightarrow{g_j} C' \xrightarrow{\psi} C \times_B X \xrightarrow{\varphi \times_B 1} \mathcal{D}_B(X,Y) \times_B X \xrightarrow{\mathrm{ev}} Y$$

is a plot.

We now check that $\{\mathrm{ev}(\varphi \times_B 1_X)\psi g_j \in Y(D_j)\}_{j \in J}$ is a compatible collection of plots corresponding to the covering family $(g_j\colon D_j \to C' | j \in J)$. Let $d \in$

$\underline{g_j}(\underline{D_j}) \cap \underline{g_k}(\underline{D_k})$ with $e_j \in \underline{D_j}$, $e_k \in \underline{D_k}$ such that $\underline{g_j}(e_j) = d = \underline{g_k}(e_k)$. We have

$$\underline{\mathrm{ev}(\varphi \times_B 1_X)\psi g_j}(e_j) = \underline{(\varphi_i \times_B 1_X)\tau_{ij}}(e_j) = (\underline{\varphi_i h_{ij}}(e_j), \underline{\pi_2 \psi g_j}(e_j))$$

and

$$\underline{\mathrm{ev}(\varphi \times_B 1_X)\psi g_k}(e_k) = \underline{(\varphi_l \times_B 1_X)\tau_{lk}}(e_k) = (\underline{\varphi_l h_{lk}}(e_k), \underline{\pi_2 \psi g_k}(e_k)).$$

We need to show equality of the rightmost terms. The second components are clearly equal. The equality of the first components follows from $\underline{f_i h_{ij}}(e_j) = \underline{f_l h_{lk}}(e_k)$, and that the family of plots $\{\varphi_i \in \mathcal{D}_B(X,Y)(C_i) | i \in I\}$ is compatible. Since Y is a smooth space, we have by the sheaf condition that $\mathrm{ev}(\varphi \times_B 1_X)\psi$ is a plot. We have now shown that

$$C \times_B X \xrightarrow{\varphi \times_B 1} \mathcal{D}_B(X,Y) \times_B X \xrightarrow{\mathrm{ev}} Y$$

is a smooth map. It follows that $\mathcal{D}_B(X,Y)$ is a sheaf.

Given D spaces X,Y,Z over B with projections p_X, p_Y, and p_Z respectively, we define a bijective correspondence between functions of the form

$$f\colon \underline{Z \times_B X} \to \underline{Y}$$

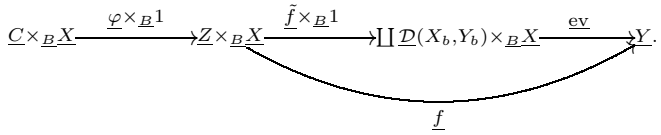
and

$$\tilde{f}\colon \underline{Z} \to \coprod_{b \in \underline{B}} \underline{\mathcal{D}}(X_b, Y_b).$$

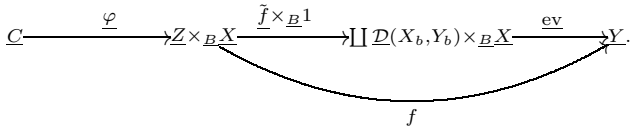
Given f we define $\tilde{f}$ in the following way: given $z \in \underline{Z}$ with $\underline{p_Z}(z) = b \in \underline{B}$, if X_b is empty, then $\tilde{f}(z)$ is defined to be the unique map $!\colon \emptyset \to Y_b$. Otherwise, $\tilde{f}(z) := f(z,-)\colon X_b \to Y_b$. Starting with a map $\tilde{f}\colon \underline{Z} \to \coprod_{b \in \underline{B}} \underline{\mathcal{D}}(X_b, Y_b)$, the map $f\colon \underline{Z} \times_{\underline{B}} \underline{X} \to \underline{Y}$ is defined by $f(z,x) := \tilde{f}(z)(x)$.

Next we must show that f defines a map of D spaces if and only if $\tilde{f}$ defines a map of D spaces, and that this correspondence is natural.

First, let us show that if we have a map of D spaces $f\colon Z \times_B X \to Y$, then the function $\underline{\tilde{f}}\colon \underline{Z} \to \coprod_{b \in \underline{B}} \underline{\mathcal{D}}(X_b, Y_b)$ constructed above determines a map of D spaces. Given an object $C \in \mathsf{D}$ and a plot of $Z(C)$, $\underline{\varphi}\colon \underline{C} \to \underline{Z}$, we treat C as a D space over B and obtain a function $\underline{\varphi} \times_{\underline{B}} 1\colon \underline{C} \times_{\underline{B}} \underline{X} \to \underline{Z} \times_{\underline{B}} \underline{X}$. Since $\underline{\varphi}$ is a plot, this determines a map of D spaces. One can check that the following diagram commutes and it follows that $\tilde{f}$ determines a DSpace map:



Conversely, given a DSpace map $\tilde{f}\colon Z \to \coprod_{b \in \underline{B}} \mathcal{D}(X_b, Y_b)$ and a plot $\underline{\varphi}\colon \underline{C} \to \underline{Z} \times_{\underline{B}} \underline{X}$, the composite along the top of the following diagram is a plot of Y and the commutativity of the diagram implies that the function $f\colon \underline{Z} \times_{\underline{B}} \underline{X} \to \underline{Y}$ defines a DSpace map:



To check naturality in the Z variable, we consider a DSpace map $g\colon Z' \to Z$ and ask that the following diagram commutes. We note that we only need to check the commutativity of the functions of the underlying sets:

$$\begin{array}{ccc} \mathcal{D}(Z, \coprod \mathcal{D}(X_b, Y_b)) & \longrightarrow & \mathcal{D}(Z \times_B X, Y) \\ \downarrow & & \downarrow \\ \mathcal{D}(Z', \coprod \mathcal{D}(X_b, Y_b)) & \longrightarrow & \mathcal{D}(Z' \times_B X, Y). \end{array}$$

Let $f\colon Z \to \coprod_{b \in \underline{B}} \mathcal{D}(X_b, Y_b)$ be an element in the top left corner. Following the diagram down and right, we first obtain a map $fg\colon Z' \to \coprod_{b \in \underline{B}} \mathcal{D}(X_b, Y_b)$ and then a map $\widetilde{fg}\colon Z' \times_B X \to Y$. Given $(z', x) \in Z' \times_B X$, we see that $\widetilde{fg}(z', x) = \tilde{f}(g(z'), x)$.

Following the diagram the other way, f is first taken to $\tilde{f}$, and then $\tilde{f}$ is taken to a map from $Z' \times_B X$ to Y by the pullback of Z' and X. This induces a map $Z' \times_B X$ to $Z \times_B X$ which we compose with $\tilde{f}$ to obtain the desired map. It follows that $(z', x) \in Z' \times_B X \mapsto (g(z'), x) \in Z \times_B X \mapsto \tilde{f}(g(z'), x) \in Y$. Thus the diagram commutes. Given a DSpace map $h\colon Y \to Y'$, the commutativity of the following diagram is given by composing at each step with h in the appropriate manner:

$$\begin{array}{ccc} \mathcal{D}(Z, \coprod \mathcal{D}(X_b, Y_b)) & \longrightarrow & \mathcal{D}(Z \times_B X, Y) \\ \downarrow & & \downarrow \\ \mathcal{D}(Z, \coprod \mathcal{D}(X_b, {Y'}_b)) & \longrightarrow & \mathcal{D}(Z \times_B X, Y'). \end{array}$$

It follows that the correspondence is natural in Y, and thus that DSpace is locally Cartesian closed. □

5.3. **Colimits.** In Proposition 5.12 we showed that the category of D spaces has limits, which can be computed pointwise. To compute colimits in DSpace, we need some facts about 'sheafification' and also 'concretization'.

Given any site D, sheafification is a functor that takes presheaves on D to sheaves on D, but does not affect presheaves that are already sheaves [33]. We denote this functor by

$$S\colon \mathrm{Set}^{\mathsf{D}^{\mathrm{op}}} \to \mathrm{Sh}(\mathsf{D}),$$

where $\mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}$ is the category of presheaves on D and $\mathrm{Sh}(\mathsf{D})$ is the category of sheaves on D. (In both of these categories, the morphisms are just natural transformations.) The functor S is left adjoint to the inclusion

$$I\colon \mathrm{Sh}(\mathsf{D}) \to \mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}.$$

Since S is a left adjoint, it preserves colimits. So, to compute a colimit of sheaves we can compute the colimit of their underlying presheaves as objects in $\mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}$ and then sheafify the result.

Grothendieck's 'plus construction' [33] gives an explicit recipe for sheafification. Given a presheaf X, the plus construction gives a new presheaf X^+ by taking the colimit over covering families of compatible collections for those families:

$$X^+(C) = \mathrm{colim}_{R \in \mathcal{J}(C)} \mathcal{F}(R, X),$$

where $\mathcal{F}(R, X)$ is the set of compatible collections for the covering family R of $C \in \mathsf{D}$. Applying the plus construction to any presheaf gives a **separated** presheaf, which is like a sheaf except that the existence property in Definition 4.6 is dropped, and only uniqueness is required. Applying the plus construction to any separated

presheaf gives a sheaf. So, if X is a presheaf, X^{++} is a sheaf, and in fact it is the sheafification of X.

Next we turn to concretization, which makes presheaves 'concrete':

Definition 5.19. Given a concrete site D, we say that a presheaf $X\colon \mathsf{D}^{\mathrm{op}} \to \mathrm{Set}$ is **concrete** if for every object $D \in \mathsf{D}$, the function sending plots $\varphi \in X(D)$ to functions $\underline{\varphi}\colon \hom(1, D) \to X(1)$ is one-to-one. We denote the category of concrete presheaves on D and natural transformations between these by $\mathrm{Conc}(\mathrm{Set}^{\mathsf{D}^{\mathrm{op}}})$.

For any presheaf X on D there is a concrete presheaf $L(X)$ for which $L(X)(D)$ consists of equivalence classes of plots $\varphi \in X(D)$, where $\varphi \sim \varphi'$ if and only if $\underline{\varphi} = \underline{\varphi}'$. Since these equivalence classes can be identified with functions $\underline{D} \to \underline{X}$, the image of L on a morphism $f\colon X \to Y$ is completely determined by the function $\underline{f}\colon \underline{X} \to \underline{Y}$. It follows that L preserves identities and composition. So, we obtain a functor called **concretization**:

$$L\colon \mathrm{Set}^{\mathsf{D}^{\mathrm{op}}} \to \mathrm{Conc}(\mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}).$$

On the other hand, there is an obvious inclusion functor

$$R\colon \mathrm{Conc}(\mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}) \to \mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}.$$

Lemma 5.20. *L is left adjoint to R.*

Proof. Given a presheaf X, a concrete presheaf Y and a natural transformation $f\colon L(X) \to Y$, we obtain a natural transformation $\tilde{f}\colon X \to R(Y)$ pointwise as $\tilde{f}_D\colon X(D) \to R(Y)(D)$ defined by

$$\varphi \mapsto f_D([\varphi]),$$

where we think of $f_D([\varphi])$ in $Y(D)$ as a plot of $R(Y)(D)$ under the inclusion. Conversely, given a natural transformation $f\colon X \to R(Y)$ we define $\tilde{f}_D\colon L(X)(D) \to Y(D)$ pointwise by

$$[\varphi] \mapsto f_D(\varphi),$$

which is well defined since the equivalence relation is defined by underlying functions. This defines a bijective correspondence. To check naturality in the first argument, it is sufficient to show that given $h\colon X \to X'$ and $g\colon L(X') \to Y$ that the following square commutes for every $D \in \mathsf{D}$:

$$\begin{array}{ccc} \hom(L(X')(D),Y(D)) & \longrightarrow & \hom(X'(D),R(Y)(D)) \\ \downarrow & & \downarrow \\ \hom(L(X)(D),Y(D)) & \longrightarrow & \hom(X(D),R(Y)(D)). \end{array}$$

Along the top and right, g_D gets sent to a map $\varphi \mapsto g_D([h_D(\varphi)])$ and along the left and bottom to a map $\varphi \mapsto g_D(L(h)_D([\varphi]))$. These are equal since maps between presheaves preserve the equivalence class. Naturality in the second argument follows similarly. □

Lemma 5.21. *Any concrete presheaf on a concrete site is a separated presheaf.*

Proof. Clear. □

It follows that for any concrete presheaf, sheafification is the same as *one* application of Grothendieck's plus construction. This brings us to the following lemma:

Lemma 5.22. *Given a concrete presheaf X, X^+ is a concrete sheaf.*

Proof. In the interest of presenting a simple argument, we now replace the coverage on our concrete site D by the Grothendieck topology $\mathcal{J}$ which has the same sheaves.

$X^+(D)$ is the colimit of a diagram (indexed by the sieves in $\mathcal{J}(D)$) of sets of compatible families of plots in $X(D)$. Given sieves R and R' indexed by the sets I and J, respectively, there is a function in the colimit diagram from the set of compatible families for a sieve R to the set of compatible families for the sieve R' exactly when $R' \subseteq R$. This function takes a compatible family $\{\varphi_i\}$ to the compatible family $\{\varphi_j\} \subseteq \{\varphi_i\}$ for R'.

Since the sheafification process is pointwise a colimit of sets, given plots $\varphi, \varphi' \in X^+(D)$ with the same underlying functions $\underline{\varphi} = \underline{\varphi'}$, there must be compatible families in the diagram which are mapped to each φ and φ'. Since given a morphism $f_i \colon C \to D$ in R we have $X^+(f)(\varphi) = \varphi_i$ and $X^+(f)(\varphi') = {\varphi'}_i$, it follows that if the two plots are in the image of compatible families $\{\varphi_i\}_{i \in I}$ and $\{{\varphi'}_i\}_{i \in I}$ for the same sieve $R \in \mathcal{J}(D)$ with indexing set I, then we have $\{\varphi_i\}_{i \in I} = \{{\varphi'}_i\}_{i \in I}$. Hence $\varphi = \varphi'$.

If φ and φ' are in the image of families for sieves R and R', respectively, then we can show that each of these functions factors through a set of compatible families for the common refinement $R \cap R'$. Since the sieve $R \cap R'$ is a subset of each R and R', the existence of the functions factoring through this set is guaranteed as long as $R \cap R'$ is in $\mathcal{J}(D)$. That the intersection of two covering sieves is a covering sieve follows directly from the axioms of a Grothendieck topology. Since we have focused almost entirely on coverages, we refer the reader to Mac Lane and Moerdijk [33] for an explanation and proof. Now the preimages of φ and φ' will be sent to the same compatible family for $R \cap R'$ and thus $\varphi = \varphi'$. □

Proposition 5.23. *The category* DSpace *has all (small) colimits.*

Proof. Given a diagram of D spaces $F \colon A \to \mathsf{DSpace}$, let $\tilde{F} \colon A \to \mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}$ be the underlying diagram of presheaves. We can compute the colimit P of $\tilde{F}$ pointwise. Given any presheaf we can concretize and then sheafify to obtain a D space. Since each of these functors is a left adjoint, this entire process preserves colimits. Also, if the presheaf is already a D space, then the process will have no effect. So we can apply this process to P and $\tilde{F}$, and it follows that the D space obtained from the presheaf P is the colimit of F in DSpace. □

It is interesting to note that in two of his papers, Chen called spaces satisfying axioms 1 and 3 but not necessarily axiom 2 'predifferentiable' spaces [8, 9]. These are the same as concrete presheaves on Chen. Chen described a systematic process for improving any predifferentiable space to a Chen space. This process is just the plus construction! The point is that by Lemma 5.22, we can turn a concrete presheaf into a concrete sheaf using the plus construction.

The following result is an easy spinoff of what we have done:

Proposition 5.24. *Every* D *space is a colimit of representable* D *spaces.*

Proof. It is well known that any presheaf is the colimit of representable presheaves [33]. So, given a D space X, there is a diagram of representables having the underlying presheaf of X as its colimit in $\mathrm{Set}^{\mathsf{D}^{\mathrm{op}}}$. As in Proposition 5.23, applying the concretization functor L and then the sheafification functor S, we can send this diagram into DSpace while preserving colimits. Since each representable is a D space by Proposition 5.2 and X was chosen to be a D space, we obtain a diagram exhibiting X as a colimit of representables in DSpace. □

Most of our results on generalized spaces can be summarized in this theorem:

Theorem 5.25. *For any concrete site* D*, the category of* D *spaces is a quasitopos with all (small) limits and colimits.*

Proof. Recall that a 'quasitopos' is a locally Cartesian closed category with finite colimits and a weak subobject classifier. We showed that DSpace has all limits in Proposition 5.12, that it has a weak subobject classifier in Proposition 5.13, that it is locally Cartesian closed in Proposition 5.18, and that it has all colimits in Proposition 5.23. □

Acknowledgements

This work had its origin in collaborations between the first author, Urs Schreiber [3, 4] and Toby Bartels [5]. We thank James Dolan for invaluable help, such as explaining the notion of 'concrete sheaf' that we use here, and pointing out the example of simplicial complexes. We also thank the n-Category Café regulars, including Bruce Bartlett, Todd Trimble and especially Andrew Stacey for online discussions, and Dan Christensen and Chris Rogers for many helpful conversations.

References

[1] P. Antoine, Etude élémentaire des catégories d'ensembles structurés, I, II, *Bull. Soc. Math. Belgique*, **18** (1966), 142–164, 387–414. MR0200321 (34:220); MR0213271 (35:4135)

[2] J. Baez, Quantum Gravity Seminar notes, U. C. Riverside, ⟨http://math.ucr.edu/home/baez/qg-spring2007/index.html#quantization⟩, Spring 2007.

[3] J. Baez and U. Schreiber, Higher gauge theory II: 2-connections. Available at arXiv:hep-th/0412325.

[4] J. Baez and U. Schreiber, Higher gauge theory. Contemp. Math. 431, Amer. Math. Soc., Providence, RI, 2007. MR2342821 (2008k:53099)

[5] T. Bartels, Higher gauge theory: 2-bundles. Available at arXiv:math/0410328. MR2709030

[6] R. Brown, Ten topologies for $X \times Y$, *Quart. J. Math.* **14** (1963), 303–319. MR0159299 (28:2516)

[7] K.-T. Chen, Iterated integrals of differential forms and loop space homology, *Ann. of Math.* (2) **97** (1973), 217–246. MR0380859 (52:1756)

[8] K.-T. Chen, Iterated integrals, fundamental groups and covering spaces, *Trans. Amer. Math. Soc.* **206** (1975), 83–98. MR0377960 (51:14129)

[9] K.-T. Chen, Iterated path integrals, *Bull. Amer. Math. Soc.* **83** (1977), 831–879. MR0454968 (56:13210)

[10] K.-T. Chen, On differentiable spaces, *Categories in Continuum Physics*, *Lecture Notes in Math.* **1174**, Springer, Berlin, 1986, 38-42. MR842915 (87i:58003)

[11] E. J. Dubuc, Concrete quasitopoi, *Lecture Notes in Math.* **753** (1979), 239–254. MR555548 (81e:18010)

[12] E. J. Dubuc and L. Español, Topological functors as familiarly-fibrations. Available at arXiv:math/0611701.

[13] E. J. Dubuc and L. Español, Quasitopoi over a base category. Available at arXiv: math/0612727.
[14] C. Ehresmann, Catégories topologiques et catégories differentiables, *Coll. Géom. Diff. Globale (Bruxelles, 1958)*, Centre Belge Rech. Math. Louvain, 1959, 137-150. MR0116360 (22:7148)
[15] A. Frölicher, Smooth structures, *Category Theory (Gummersbach, 1981) Lecture Notes in Math.* **962**, Springer, Berlin, 1982, 69–81. MR682945 (84g:58008)
[16] R. Goldblatt, *Topoi: the Categorial Analysis of Logic*, North–Holland, New York, (1984). Also available at ⟨http://cdl.library.cornell.edu/cgi-bin/cul.math/docviewer?did=Gold010⟩. MR766560 (85m:03002)
[17] M. Grandis, Finite sets and symmetric simplicial sets, *Th. Appl. Cat.* **8** (2001), 244–252. MR1825431 (2002c:18010)
[18] P. Iglesias-Zemmour, *Diffeology*. Draft available at ⟨http://math.huji.ac.il/∼piz/Site/The%20Book/The%20Book.html⟩.
[19] K. Jänich, On the classification of $O(n)$-manifolds, *Math. Ann.* **176** (1968), 53–76. MR0226674 (37:2261)
[20] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, Vol. 2, Chap. C2, Oxford U. Press, Oxford, 2002. MR2063092 (2005g:18007)
[21] A. Kock, *Synthetic Differential Geometry*, London Math. Soc. Lecture Notes **333**, Cambridge U. Press, Cambridge, 2006. Also available at ⟨http://home.imf.au.dk/kock/⟩. MR2244115 (2007b:51013)
[22] M. Kreck, *Differential Algebraic Topology*, draft available at ⟨http://www.him.uni-bonn.de/kreck-stratifolds⟩.
[23] A. Kriegl, A Cartesian closed extension of the category of smooth Banach manifolds, in *Categorical Topology (Toledo, Ohio, 1983)*, eds. H. L. Bentley *et al.*, Heldermann, Berlin, 1984, pp. 323–336. MR785022 (86i:58005)
[24] A. Kriegl, Remarks on germs in infinite dimensions, *Acta Math. Univ. Comenianae* **66** (1997), 1–18. MR1474553 (98i:46040)
[25] A. Kriegl and P. Michor, *The Convenient Setting of Global Analysis*, AMS, Providence, Rhode Island, 1997. Also available at ⟨http://www.ams.org/online_bks/surv53/⟩. MR1471480 (98i:58015)
[26] M. Laubinger, A Lie algebra for Frölicher groups, available at arXiv:0906.4486.
[27] G. Laures, On cobordism of manifolds with corners, *Trans. Amer. Math. Soc.* **252** (2000), 5667–5668. MR1781277 (2001i:55007)
[28] K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge U. Press, Cambridge, 2005. MR2157566 (2006k:58035)
[29] S. Mac Lane, *Categories for the Working Mathematician*, Springer, Berlin, 1998. MR1712872 (2001j:18001)
[30] P. May, *A Concise Course in Algebraic Topology*, Chap. 5, U. Chicago Press, Chicago, 1999. Also available at ⟨http://www.math.uchicago.edu/∼may/CONCISE/ConciseRevised.pdf⟩. MR1702278 (2000h:55002)
[31] P. Michor, A convenient setting for differential geometry and global analysis, *Cahiers Top. Géom. Diff.* **25** (1984), 63–109 and 113–178. MR764972 (86g:58014a)
[32] M. Mostow, The differentiable space structures of Milnor classifying spaces, simplicial complexes, and geometric realizations, *J. Diff. Geom.* **14** (1979), 255–293. MR587553 (84e:57030)
[33] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, Berlin, 1992. MR1300636 (96c:03119)
[34] J. Penon, Quasi-topos, *C.R. Acad. Sci. Paris, Sér. A* **276** (1973), 237–240. MR0332921 (48:11246)
[35] J. Penon, Sur les quasi-topos, *Cahiers Top. Géom. Diff.* **18** (1977), 181–218. MR0480693 (58:848)
[36] U. Schreiber and K. Waldorf, Parallel transport and functors, *J. Homotopy Relat. Struct.* **4** (2009), no. 1, 187–244. MR2520993 (2010k:53039)
[37] R. Sikorski, Differential modules, *Colloq. Math.* **24** (1971-72), 45–79. MR0482794 (58:2845)
[38] J. W. Smith, The de Rham theorem for general spaces, *Tôhoku Math. J.* **18** (1966), 115–137. MR0202154 (34:2027)
[39] J. M. Souriau, Groupes differentiels, in *Differential Geometrical Methods in Mathematical Physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979)*, *Lecture Notes in Math.* **836**, Springer, Berlin, 1980, pp. 91–128. MR607688 (84b:22038)

[40] A. Stacey, Comparative smootheology, available at arXiv:0802.2225.
[41] N. E. Steenrod, A convenient category of topological spaces, *Michigan Math. J.* **14** (1967), 133–152. Also available at ⟨http://projecteuclid.org/⟩. MR0210075 (35:970)

Department of Mathematics, University of California, Riverside, California 92521
E-mail address: baez@math.ucr.edu

Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, Ontario, Canada K1N 6N5
E-mail address: hoffnung@uottawa.ca