Convergence Analysis of a Subdomain Iterative Method for the Finite Element Approximation of the Coupling of Stokes and Darcy Equations

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Summary

We consider a Galerkin Finite Element approximation of the Stokes-Darcy problem which models the coupling between surface and groundwater flows. Then we propose an iterative subdomain method for its solution, inspired to the domain decomposition theory. The convergence analysis that we develop is based on the properties of the discrete Steklov–Poincaré operators associated to the given coupled problem. An optimal preconditioner for Krylov methods is proposed and analyzed.

Key words: *Stokes and Darcy equations – Domain Decomposition Methods – Steklov-Poincaré operators – Finite Element Methods*

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1 Introduction

In a previous paper [5] we have introduced a differential system based on the coupling of (Navier) Stokes equations and Darcy equation for the modeling of the interaction between surface and subsurface flows.
These coupled models have interesting applications. They can be used to simulate the effect of flooding in dry areas. When further coupled with transport-diffusion equations they can be used to study the propagation and diffusion of pollutants dispersed in water.
Moreover similiar models can be used to describe the behaviour of water in a basin due to the motion of a body (a ship, or a boat) beneath the free surface (see [4]). In fact such a problem can be studied by decomposing the computational domain into two parts. We have an upper region where the Navier-Stokes equations are used to describe the motion of water near the moving body; then we consider a deeper region where the effects due to the motion of the body

can be omitted and simpler models, such as a Laplace equation for the velocity potential, can be adopted.
Similiar coupled models have been considered by other authors as well (see, e.g. [4, 8, 9, 10, 13]).
In another paper [6] the mathematical analysis of the coupled problem has been carried out. In particular, the Stokes-Darcy problem has been reformulated as an interface problem governed by a suitable Steklov–Poincaré operator. The properties of symmetry, continuity and positivity of the Steklov–Poincaré operator have been analyzed.
In this paper, after setting up our problem in Sect. 2, we provide a Galerkin finite element approximation of the Stokes-Darcy problem, and carry out its analysis (see Sect. 3). Then we reformulate the finite element problem as an interface problem which is governed by a discrete Steklov–Poincaré operator (DSP). We analyze the properties of DSP in Sect. 4, then we introduce and analyze preconditoned iterative methods for DSP. These methods can be regarded as domain decomposition methods for the finite element problem. Their algebraic formulation is provided in Sect. 6. The fact that the preconditioner of the DSP is optimal guarantees that Krylov methods converge with a rate independent of the finite element grid size. This is confirmed by our numeical experiments of Sect. 7.

2 Problem Setting

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain, which can be decomposed as the union of two non intersecting subdomains Ω_f and Ω_p separated by an hypersurface $\Gamma \subset \mathbb{R}^{d-1}$ called interface, i.e. $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\Omega_f \cap \Omega_p = \emptyset$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$.
From the physical point of view, Γ is a surface separating an upper domain Ω_f filled by a fluid, from a lower domain Ω_p formed by a porous medium. We assume that the fluid contained in Ω_f has an upper fixed surface (i.e. we do not consider the free surface fluid case) and can filtrate through the underlying porous medium.
In order to describe the motion of the fluid in Ω_f, we introduce the Stokes equations: $\forall t > 0$

$$\begin{array}{rcll} \dfrac{\partial \mathbf{u}_f}{\partial t} - \mathbf{div}\mathsf{T}(\mathbf{u}_f, p_f) & = & \mathbf{f} & \text{in}\,\Omega_f, \\ \mathrm{div}\mathbf{u}_f & = & 0 & \text{in}\,\Omega_f\ , \end{array} \tag{1}$$

which are the linear counterpart of the more general Navier-Stokes equations in which the momentum equation contains the convective term $(\mathbf{u}_f \cdot \nabla)\mathbf{u}_f$ as well.

In (1) $\mathsf{T}(\mathbf{u}_f, p_f) = \nu(\nabla\mathbf{u}_f + \nabla^t\mathbf{u}_f) - p_f\mathsf{I}$ is the stress tensor, $\nu > 0$ is the kinematic viscosity of the fluid, while $\mathbf{u}_f$ and p_f are the fluid velocity and pressure, respectively; ∇ is the gradient operator $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)^t$ with respect to the space coordinate $\mathbf{x} = (x_1, \dots, x_d)^t$.
In the lower domain Ω_p we define the piezometric head: $\varphi := z + \frac{p_p}{\rho_f g}$, where z is the elevation from a reference level, p_p is the pressure of the fluid in Ω_p, ρ_f its density and g is the gravity acceleration.

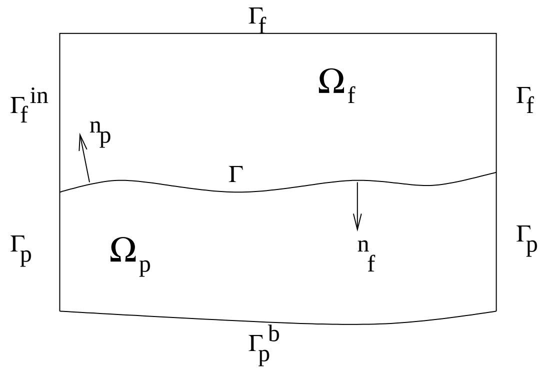

Figure 1: Schematic representation of a 2D vertical section of the computational domain

According to [1, 16], the fluid motion in Ω_p is described by the equations:

$$\begin{array}{rcll} S_0\dfrac{\partial\varphi}{\partial t} + n\mathrm{div}\mathbf{u}_p & = & 0 & \text{in } \Omega_p \\ \mathbf{u}_p & = & -\dfrac{\mathsf{K}}{n}\nabla\varphi & \text{in } \Omega_p \ , \end{array} \qquad (2)$$

where $\mathbf{u}_p$ is the fluid velocity, n is the volumetric porosity and K is the hydraulic conductivity tensor $\mathsf{K} = diag(K_1, \ldots, K_d)$ with $K_i \in L^\infty(\Omega_p)$, $i = 1, \ldots, d$. S_0 is the specific mass storativity coefficient. The second equation is the Darcy law.

2.1 Boundary and Interface Conditions

We consider Dirichlet boundary conditions for the Stokes problem: we assign an inflow $\mathbf{u}_f = \mathbf{u}_{in}$ on Γ_f^{in} and a no-slip condition $\mathbf{u}_f = \mathbf{0}$ on Γ_f, where $\partial\Omega_f = \Gamma \cup \Gamma_f \cup \Gamma_f^{in}$, $\partial\Omega_p = \Gamma \cup \Gamma_p^b \cup \Gamma_p$ (see Fig. 1).
For the Darcy problem, we set the piezometric head $\varphi = \varphi_p$ on Γ_p^b and we require the normal velocity to be null on Γ_p: $\mathbf{u}_p \cdot \mathbf{n}_p = 0$. $\mathbf{n}_p$ and $\mathbf{n}_f$ denote the unit outward normal vectors to the surfaces $\partial\Omega_p$ and $\partial\Omega_f$, respectively; in particular on Γ we have $\mathbf{n}_f = -\mathbf{n}_p$.
A description of other boundary conditions of physical relevance can be found in [6].
We supplement Stokes and Darcy problems with the following matching conditions on Γ:

$$\begin{array}{rcll} \mathbf{u}_p \cdot \mathbf{n}_f & = & \mathbf{u}_f \cdot \mathbf{n}_f \ , & \\ -[(\mathsf{T}(\mathbf{u}_f, p_f)) \cdot \mathbf{n}_f] \cdot \boldsymbol{\tau}_i & = & \dfrac{\alpha}{\sqrt{k_i}}(\mathbf{u}_f - \mathbf{u}_p) \cdot \boldsymbol{\tau}_i & i = 1, \ldots, d-1 \ , \\ -[(\mathsf{T}(\mathbf{u}_f, p_f)) \cdot \mathbf{n}_f] \cdot \mathbf{n}_f & = & g\varphi \ , & \end{array} \qquad (3)$$

where $k_i = \boldsymbol{\tau}_i \cdot \mathsf{K} \cdot \boldsymbol{\tau}_i$ ($\boldsymbol{\tau}_i$, $i = 1, \ldots, d-1$, being linear independent unit tangential vectors to the boundary), and α is a positive dimensionless parameter depending on the properties of the porous medium.
Conditions (3) impose the continuity of the normal velocity on Γ, as well as that of the normal component of the normal stress, but they allow pressure to be discontinuous across the interface (see [13, 8, 9]).

3 Galerkin Finite Element Approximation of the Stokes-Darcy Problem

From now on, we shall deal only with the stationary case for equations (1) and (2), and we take $\alpha = 0$ in condition (3), so that the latter becomes a natural boundary condition for Stokes problem in Ω_f.
Therefore, the differential formulation of the problem we are investigating reads:

$$\begin{aligned} -\mathbf{div}\mathsf{T}(\mathbf{u}_f, p_f) &= \mathbf{f} \quad \text{in } \Omega_f \\ \text{div}\mathbf{u}_f &= 0 \quad \text{in } \Omega_f \\ -\text{div}(\mathsf{K}\nabla\varphi) &= 0 \quad \text{in } \Omega_p \,, \end{aligned} \tag{4}$$

together with the boundary conditions described in the previous paragraph and the interface conditions (3) on Γ. The boundary Dirichlet datum $\mathbf{u}_{in}$ is supposed null in a neighborhood of the intersection $\overline{\Gamma} \cap \overline{\Gamma}_f^{in}$.
We shall introduce now the finite element Galerkin approximation of problem (4), and refer to [6] for the analysis in the continuous case.

We consider a regular triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra in the 3–dimensional case. We assume that the triangulations $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ induced on the subdomains Ω_f and Ω_p are compatible on Γ, that is they share the same edges (if $d = 2$) or faces (if $d = 3$) therein. Finally we suppose the triangulation $\mathcal{M}_h$ induced on Γ to be quasi-uniform (e.g. [15]).
Several choices of finite element spaces can be made.
If we indicate by H_{fh} and Q_h the finite element spaces which approximate the velocity and pressure fields respectively, there must exist a positive constant $\beta^* > 0$, independent of h, such that $\forall q_h \in Q_h$, $\exists \mathbf{v}_h \in H_{fh}$, $\mathbf{v}_h \neq 0$:

$$\int_{\Omega_f} q_h \,\text{div}\mathbf{v}_h \, d\Omega_f \geq \beta^* \|\mathbf{v}_h\|_{H^1(\Omega_f)} \|q_h\|_{L^2(\Omega_f)} \,. \tag{5}$$

Several families of finite element spaces satisfying the inf-sup condition (5) are provided in [3].
What matters for the analysis we are going to develop, is only to guarantee that the compatibility condition (5) holds. Therefore, in the following, for the sake of exposition, we will consider the special choice of piecewise quadratic elements for the velocity components and piecewise linear for the pressure.
More precisely we define the discrete spaces:

$$H_{fh} := (V_{fh})^d, \qquad d = 2, 3 \,, \tag{6}$$

where

$$V_{fh} := \{v_h \in X_{fh} |\, v_h = 0 \quad \text{on} \quad \Gamma_f^{in}\} \,, \tag{7}$$

$$\begin{aligned} X_{fh} := \{v_h \in C^0(\overline{\Omega}_f) |\, v_h = 0 \quad &\text{on} \quad \Gamma_f \\ \text{and} \quad v_{h|K} \in \mathbb{P}_2(K), \forall K \in \mathcal{T}_{fh}\} \,, \end{aligned} \tag{8}$$

and $\mathbb{P}_r$, $r \geq 0$, is the space of polynomials of degree less than or equal to r;

$$
\begin{aligned}
H^0_{fh} &:= \{\mathbf{v}_h \in H_{fh} |\, \mathbf{v}_h \cdot \mathbf{n}_f = 0 \quad \text{on } \Gamma\} \; ; && (9)\\
Q_h &:= \{q_h \in C^0(\overline{\Omega}_f) |\, q_{h|K} \in \mathbb{P}_1(K),\, \forall K \in \mathcal{T}_{fh}\} \; ; && (10)\\
X_{ph} &:= \{\psi_h \in C^0(\overline{\Omega}_p) |\, \psi_{h|K} \in \mathbb{P}_2(K),\, \forall K \in \mathcal{T}_{ph}\} \; ; && (11)\\
H_{ph} &:= \{\psi_h \in X_{ph} |\, \psi_h = 0 \quad \text{on} \quad \Gamma^b_p\} \; ; && (12)\\
H^0_{ph} &:= \{\psi_h \in H_{ph} | \psi_h = 0 \quad \text{on } \Gamma\} \; ; && (13)\\
W_h &:= H_{fh} \times H_{ph} \; . && (14)
\end{aligned}
$$

Finally, we consider the space $\Lambda_h := \{v_{h|_\Gamma} | v_h \in V_{fh}\}$ to approximate the trace space $H^{1/2}_{00}(\Gamma)$ on Γ (see [11]).

Let us consider the approximation of the boundary data. For the Darcy datum φ_p on Γ^b_p, supposing that $\varphi_p \in H^{1/2}(\Gamma^b_p) \cap C^0(\Gamma^b_p)$, we can consider the quadratic interpolant φ_{ph} of its nodal values on Γ^b_p, and then the extension $E_{ph}\varphi_{ph} \in X_{ph}$, such that $E_{ph}\varphi_{ph} = \varphi_{ph}$ at the nodes lying on Γ^b_p and $E_{ph}\varphi_{ph} = 0$ at the nodes of $\Omega_p \setminus \Gamma^b_p$.
We can proceed in the same way for the boundary datum $\mathbf{u}_{in}$ provided it belongs to $(H^{1/2}(\Gamma^{in}_f))^d \cap (C^0(\Gamma^{in}_f))^d$. We consider again its quadratic interpolant $\mathbf{u}_{inh}$ and then its extension $E_{fh}\mathbf{u}_{inh} \in Y_{fh}$ where $Y_{fh} := \{\mathbf{v}_h \in (C^0(\overline{\Omega}_f))^d |\ \mathbf{v}_h = \mathbf{0}$ on Γ_f, $\mathbf{v}_h \cdot \mathbf{n}_f = 0$ on Γ and $v^i_{h|_K} \in \mathbb{P}_2(K)\ \ \forall K \in \mathcal{T}_{fh},\ i = 1, \ldots, d\}$ (v^i_h being the i–th component of $\mathbf{v}_h$).
Finally, let us define $\varphi_{0h} := \varphi_h - E_{ph}\varphi_{ph} \in H_{ph}$, for all $\varphi_h \in X_{ph}$ such that $\varphi_h|_{\Gamma^b_p} = \varphi_{ph}$, and $\mathbf{u}^0_{fh} := \mathbf{u}_{fh} - E_{fh}\mathbf{u}_{inh} \in H_{fh}$, with $\mathbf{u}_{fh} \in (X_{fh})^d$ such that $\mathbf{u}_{fh}|_{\Gamma^{in}_f} = \mathbf{u}_{inh}$.

Remark 3.1 *Alternatively, we could consider a divergence free extension of* $\mathbf{u}_{in}$*. To this end, let us denote by* $\mathbf{w}_{inh}$ *the solution of the following problem: find* $\mathbf{w}_{inh} \in H_{fh}$ *s.t. for all* $q_h \in Q_h$

$$
-\int_{\Omega_f} q_h\, div\mathbf{w}_{inh} = \int_{\Omega_f} q_h\, div(E_{fh}\mathbf{u}_{inh}) \; . \tag{15}
$$

The solvability of (15) is guaranteed by the inf-sup condition (5).
Finally, we indicate by $\mathcal{E}_{fh}\mathbf{u}_{inh} = E_{fh}\mathbf{u}_{inh} + \mathbf{w}_{inh}$ *the discrete extension of* $\mathbf{u}_{in}$ *on* Γ^{in}_f*. We underline that* $\mathcal{E}_{fh}\mathbf{u}_{inh} = \mathbf{u}_{inh}$ *at the nodes on* Γ^{in}_f, $\mathcal{E}_{fh}\mathbf{u}_{in} = \mathbf{0}$ *at the nodes of* Γ_f *and that, thanks to (15),*

$$
\int_{\Omega_f} q_h\, div(\mathcal{E}_{fh}\mathbf{u}_{in}) = 0 \qquad \forall q_h \in Q_h \; .
$$

We observe that, in general we cannot impose also $\mathcal{E}_{fh}\mathbf{u}_{inh} \cdot \mathbf{n}_f = 0$ *on* Γ*, except for the special case when* $\mathbf{u}_{in}$ *satisfies* $\int_{\Gamma^{in}_f} \mathbf{u}_{in} \cdot \mathbf{n}_f = 0$.

We define the following bilinear forms: for all $\mathbf{v}, \mathbf{w} \in (H^1(\Omega_f))^d$,

$$
a_f(\mathbf{v}, \mathbf{w}) := \int_{\Omega_f} \frac{\nu}{2} \left(\nabla \mathbf{v} + \nabla^t \mathbf{v}\right) \cdot \left(\nabla \mathbf{w} + \nabla^t \mathbf{w}\right) \tag{16}
$$

$$\begin{aligned}\mathcal{A}(\underline{v},\underline{w}) := n\, a_f(\mathbf{v},\mathbf{w}) + \int_{\Omega_p} g\nabla\psi\cdot \mathsf{K}\nabla\varphi \\ + \int_\Gamma ng\,\varphi(\mathbf{w}\cdot\mathbf{n}_f) - \int_\Gamma ng\,\psi(\mathbf{v}\cdot\mathbf{n}_f)\end{aligned} \tag{17}$$

for all $\underline{v} = (\mathbf{v},\varphi)$, $\underline{w} = (\mathbf{w},\psi) \in (H^1(\Omega_f))^d \times H^1(\Omega_p)$;

$$\mathcal{B}(\underline{w},q) := -\int_{\Omega_f} n\,q\,\mathrm{div}\mathbf{w} \tag{18}$$

for all $\underline{w} = (\mathbf{w},\psi) \in (H^1(\Omega_f))^d \times H^1(\Omega_p)$, $q \in L^2(\Omega_f)$.
Finally, let us define the following linear functionals accounting for the discrete extensions of the boundary data:

$$\begin{aligned}< \mathcal{F}^*,\underline{w} >:= \int_{\Omega_f} n\mathbf{f}\cdot\mathbf{w} - na_f(E_{fh}\mathbf{u}_{inh},\mathbf{w}) \\ + \int_{\Omega_p} g\nabla\psi\cdot\mathsf{K}\nabla(E_{ph}\varphi_{ph}) + \int_\Gamma ng\,(E_{fh}\mathbf{u}_{inh}\cdot\mathbf{n}_f)\psi\ ,\end{aligned} \tag{19}$$

for all $\underline{w} = (\mathbf{w},\psi) \in (H^1(\Omega_f))^d \times H^1(\Omega_p)$;

$$< \mathcal{G}^*,\underline{w} >:= \int_{\Omega_f} n\,q\,\mathrm{div}(E_{fh}\mathbf{u}_{inh}) \tag{20}$$

for all $q \in L^2(\Omega_f)$.
The Galerkin approximation to the coupled Stokes/Darcy problem (4) reads: find $\underline{u}_h = (\mathbf{u}^0_{fh},\varphi_{0h}) \in W_h$ and $p_h \in Q_h$:

$$\begin{array}{ll}\mathcal{A}(\underline{u}_h,\underline{v}_h) + \mathcal{B}(\underline{v}_h,p_h) =< \mathcal{F}^*,\underline{v}_h > & \forall \underline{v}_h \in W_h \\ \mathcal{B}(\underline{u}_h,q_h) =< \mathcal{G}^*,q_h > & \forall q_h \in Q_h\ .\end{array} \tag{21}$$

Remark 3.2 *Let us notice that considering the divergence null discrete extension $\mathcal{E}_{fh}\mathbf{u}_{inh}$ of Remark 3.1 instead of $E_{fh}\mathbf{u}_{inh}$, the linear functional $\mathcal{G}^*$ would be null.*

The existence, uniqueness and stability of the discrete solution of (21) can be proved following the same steps of the continuous case (see Sect. 3 of [6]), using in addition the theory developed by Brezzi for saddle-point problems in the finite dimensional case (see [2]).

Let $W = H_f \times H_p$, where $H_f := (H_{\Gamma_f\cup\Gamma_f^{in}})^d$ $(d = 2,3)$, being $H_{\Gamma_f\cup\Gamma_f^{in}} := \{v \in H^1(\Omega_f)|\, v = 0\,\mathrm{on}\,\Gamma_f \cup \Gamma_f^{in})\}$, and $H_p := \{\psi \in H^1(\Omega_p)|\psi = 0\,\mathrm{on}\,\Gamma_p^b\}$. Let $\underline{u} = (\mathbf{u}^0_f,\varphi_0) \in H_f \times H_p$, $p \in L^2(\Omega_f)$ be the solutions to the continuous counterpart of problem (21) (see Sect. 3 of [6]). The theory by Brezzi can be applied to obtain the following error estimates:

$$\|\underline{u} - \underline{u}_h\|_W \le \left(1 + \frac{\gamma}{\alpha}\right) \inf_{\underline{v}_h\in Z_h^0} \|\underline{u} - \underline{v}_h\|_W + \frac{1}{\alpha} \inf_{q_h\in Q_h} \|p - q_h\|_{L^2(\Omega_f)} \tag{22}$$

and

$$\begin{aligned}\|p - p_h\|_{L^2(\Omega_f)} \quad \le \quad & \frac{\gamma}{\beta^*}\left(1 + \frac{\gamma}{\alpha}\right) \inf_{\underline{v}_h\in Z_h^0} \|\underline{u} - \underline{v}_h\|_W \\ & + \left(1 + \frac{1}{\beta^*} + \frac{\gamma}{\alpha\beta^*}\right) \cdot \|p - q_h\|_{L^2(\Omega_f)}\ ,\end{aligned} \tag{23}$$

where β^* is the positive $h-$independent constant of the inf-sup condition (5); α and γ are respectively the coercivity and continuity constants of the bilinear form $\mathcal{A}(.,.)$ and they are independent of h too (for a detailed definition of these constants see Sect. 3 of [6]).
Finally Z_h^0 is the discrete space

$$Z_h^0 := \{\underline{v}_h \in W_h | \, \mathcal{B}(\underline{v}_h, q_h) = 0 \, \forall q_h \in Q_h\} \ ,$$

and $\| \cdot \|_W$ denotes the norm

$$\|\underline{w}\|_W = (\|\mathbf{w}\|^2_{H^1(\Omega_f)} + \|\psi\|^2_{H^1(\Omega_p)})^{\frac{1}{2}} \ ,$$

for all $\underline{w} = (\mathbf{w}, \psi) \in W$.
We remark that since constants α, γ and β^* are all independent of the discretization parameter h, (22) and (23) give optimal convergence results.

Remark 3.3 *Notice that no additional compatibility condition is required for the discrete spaces H_{fh} and H_{ph}. In fact, the mixed coupling terms on the interface appearing in the definition of the bilinear form $\mathcal{A}(.,.)$:*

$$\int_\Gamma ng\varphi_h(\mathbf{w}_h \cdot \mathbf{n}_f) - \int_\Gamma ng\psi_h(\mathbf{v}_h \cdot \mathbf{n}_f) \ ,$$

give null contribution when we consider $\mathbf{w}_h = \mathbf{v}_h$ and $\psi_h = \varphi_h$.

Finally, let us underline that the coupling condition $(3)_1$ imposing the continuity of normal velocity across the interface yields

$$-\frac{1}{n}\mathsf{K}\nabla\varphi \cdot \mathbf{n}_f = \mathbf{u}_f \cdot \mathbf{n}_f \ .$$

In the finite element approximation, this continuity equation has to be intended in the sense of the $L^2(\Gamma)$–projection on the finite element space H_{ph} on Γ. In fact, in (21) we are imposing

$$\int_\Gamma \left(-\frac{1}{n}\mathsf{K}\nabla\varphi_h \cdot \mathbf{n}_f - \mathbf{u}_{fh} \cdot \mathbf{n}_f \right) \psi_{h|_\Gamma} = 0, \tag{24}$$

for all $\psi_h \in H_{ph}$.
This is equivalent to require that $\Pi(\mathbf{u}_{fh} \cdot \mathbf{n}_f) = -(1/n) \cdot \mathsf{K}\nabla\varphi_h \cdot \mathbf{n}_f$, Π being the projection operator on $H_{ph|_\Gamma}$ with respect to the scalar product of $L^2(\Gamma)$.

Remark 3.4 *The coupling of Stokes and Darcy equations has been recently studied also by Layton, Schieweck and Yotov in [10]. In particular, considering a mixed formulation for Darcy problem and applying coupling conditions (3) with $\alpha \neq 0$, they analyse the Stokes-Darcy problem using a coupling strategy via Lagrange multipliers.*

4 Discrete Steklov–Poincaré Operators Associated to the Coupled Problem

The theory developed at the differential level for the Steklov–Poincaré operators associated to the Stokes-Darcy problem (see Sect. 4 of [6]) can be extended to the discrete operators associated with the Galerkin finite element approximation (21).

The following property is the discrete counterpart of Proposition 2 of [6], so we will not report its proof.

Proposition 4.1 *Problem (21) can be reformulated in an equivalent way as follows: find* $\mathbf{u}^0_{fh} \in H_{fh}$, $p_h \in Q_h$, $\varphi_{0h} \in H_{ph}$ *such that*

$$\begin{aligned}
& a_f(\mathbf{u}^0_{fh} + E_{fh}\mathbf{u}_{inh}, \mathbf{v}_h) - \int_{\Omega_f} p_h \, div\mathbf{v}_h = \int_{\Omega_f} \mathbf{f}\cdot\mathbf{v}_h \qquad \forall \mathbf{v}_h \in H^0_{fh} \ , \\
& \int_{\Omega_f} q_h \, div\mathbf{u}^0_{fh} = -\int_{\Omega_f} q_h \, div(E_{fh}\mathbf{u}_{inh}) \quad \forall q_h \in Q_h, \\
& \int_\Gamma (\mathbf{u}^0_{fh} + E_{fh}\mathbf{u}_{inh})\cdot\mathbf{n}_f \, \mu_h = -\frac{1}{n}\int_\Gamma (\mathsf{K}\nabla\varphi_{0h}\cdot\mathbf{n}_f)\mu_h \quad \forall \mu_h \in \Lambda_h, \\
& \int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla(\varphi_{0h} + E_{ph}\varphi_{ph}) = 0 \qquad \forall \psi_h \in H^0_{ph} \ , \\
& \int_\Gamma g\,\varphi_{0h}\mu_h = \int_{\Omega_f} \mathbf{f}\cdot(R_{1h}\mu_h) - a_f(\mathbf{u}^0_{fh} + E_{fh}\mathbf{u}_{inh}, R_{1h}\mu_h) \\
& \quad + \int_{\Omega_f} p_h \, div(R_{1h}\mu_h) \quad \forall \mu_h \in \Lambda_h \ ,
\end{aligned} \tag{25}$$

where R_{1h} *is any possible continuous extension operator from* Λ_h *to* H_{fh} *such that* $(R_{1h}\mu_h)\cdot\mathbf{n}_f = \mu_h$ *on* Γ*, for all* $\mu_h \in \Lambda_h$*.*

Now, we pose $\lambda_h = \mathbf{u}_{fh}\cdot\mathbf{n}_f$ on Γ; from (24) we obtain

$$\int_\Gamma \left(-\frac{1}{n}\mathsf{K}\nabla\varphi_h\cdot\mathbf{n}_f - \lambda_h \right) \psi_{h|\Gamma} = 0 \qquad \forall \psi_h \in H_{ph} \ ,$$

that is $\Pi\lambda_h = -(1/n)\mathsf{K}\nabla\varphi_h\cdot\mathbf{n}_f$, where Π is the projection operator introduced before.

Now, if $\int_{\Gamma_f^{in}} \mathbf{u}_{inh}\cdot\mathbf{n}_f \neq 0$, we introduce a function $\lambda_{*h} \in \Lambda_h$, $\lambda_{*h} := \tilde{c}_*\gamma_h$ where γ_h is a piecewise linear function on Γ such that $\gamma_h(\mathbf{x}) = 0$ if $\mathbf{x}$ is a node on $\partial\Gamma$ and $\gamma_h(\mathbf{x}) = 1$ if $\mathbf{x}$ is a node on $\Gamma \setminus \partial\Gamma$, while $\tilde{c}_* \in \mathbb{R}$ is defined as

$$\tilde{c}_* := -\frac{\int_{\Gamma_f^{in}} \mathbf{u}_{inh}\cdot\mathbf{n}_f}{\int_\Gamma \gamma_h} \ .$$

Therefore

$$\int_\Gamma \lambda_{*h} = -\int_{\Gamma_f^{in}} \mathbf{u}_{inh}\cdot\mathbf{n}_f \ . \tag{26}$$

Should the normal component of the datum $\mathbf{u}_{inh}$ have zero mean over Γ_f^{in}, the analysis we are going to develop would still be valid by setting $\lambda_{*h} = 0$ and considering the whole trace space Λ_h instead of the trace space Λ_{0h} defined below.

We split λ_h as the sum of two components: $\lambda_h = \lambda_{0h} + \lambda_{*h}$, where λ_{*h} is the function introduced in (26), and $\lambda_{0h} \in \Lambda_{0h}$ with

$$\Lambda_{0h} := \left\{ \mu_h \in \Lambda_h \,\middle|\, \int_\Gamma \mu_h = 0 \right\} \ . \tag{27}$$

We introduce the two auxiliary problems:

i) find $\boldsymbol{\omega}^*_{0h} \in H^0_{fh}, \pi^*_h \in Q_{0h}$ s.t. $\forall \mathbf{v}_h \in H^0_{fh}$, $\forall q_h \in Q_{0h}$

$$\begin{aligned} & a_f(\boldsymbol{\omega}^*_{0h}, \mathbf{v}_h) - \int_{\Omega_f} \pi^*_h \, \mathrm{div}\mathbf{v}_h = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}_h - a_f(E_{fh}\mathbf{u}_{inh} + E_{\Gamma h}\lambda_{*h}, \mathbf{v}_h) \\ & \int_{\Omega_f} q_h \, \mathrm{div}\boldsymbol{\omega}^*_{0h} = - \int_{\Omega_f} q_h \, \mathrm{div}(E_{fh}\mathbf{u}_{inh} + E_{\Gamma h}\lambda_{*h}) \ , \end{aligned} \tag{28}$$

where we have set $Q_{0h} := \{q_h \in Q_h | \int_{\Omega_f} q_h = 0\}$ and $E_{\Gamma h}\lambda_{*h} \in H_{fh}$ denotes a suitable discrete extension of λ_{*h}, such that $E_{\Gamma h}\lambda_{*h} \cdot \mathbf{n}_f = \lambda_{*h}$ on Γ;

ii) find $\varphi^*_{0h} \in H_{ph}$ s.t. $\forall \psi_h \in H_{ph}$

$$\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla\varphi^*_{0h} = - \int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla(E_{ph}\varphi_{ph}) + \int_\Gamma n\lambda_{*h}\psi_h \ . \tag{29}$$

Moreover, let us define the following extension operators:

$$\begin{aligned} & R_{fh} \colon \Lambda_{0h} \to H_{fh} \times Q_{0h}, \\ & \eta_h \to R_{fh}\eta_h := (R^1_{fh}\eta_h, R^2_{fh}\eta_h) \end{aligned}$$

such that $(R^1_{fh}\eta_h) \cdot \mathbf{n}_f = \eta_h$ on Γ and

$$\begin{aligned} & a_f(R^1_{fh}\eta_h, \mathbf{w}_h) - \int_{\Omega_f} (R^2_{fh}\eta_h)\mathrm{div}\mathbf{w}_h \ = 0 \\ & \int_{\Omega_f} q_h \, \mathrm{div}(R^1_{fh}\eta_h) \ = 0 \end{aligned} \tag{30}$$

$\forall \mathbf{w}_h \in H^0_{fh}$, $\forall q_h \in Q_{0h}$;

$$R_{ph} \colon \Lambda_{0h} \to H_{ph}, \qquad \eta_h \to R_{ph}\eta_h$$

such that

$$\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla(R_{ph}\eta_h) \ = \int_\Gamma n\eta_h\psi_h \quad \forall \psi_h \in H_{ph}. \tag{31}$$

Now we can define the *discrete Steklov–Poincaré* operator $S_h : \Lambda_{0h} \to \Lambda'_h$ as follows:

$$\begin{aligned} < S_h\eta_h, \mu_h > \ := & \ a_f(R^1_{fh}\eta_h, R_{1h}\mu_h) - \int_{\Omega_f} (R^2_{fh}\eta_h) \, \mathrm{div}(R_{1h}\mu_h) \\ & + \int_\Gamma g(R_{ph}\eta_h)\mu_h \end{aligned} \tag{32}$$

$\forall \eta_h \in \Lambda_{0h}$, $\forall \mu_h \in \Lambda_h$.
It can be split as sum of two sub-operators $S_h = S_{fh} + S_{ph}$, which are associated with the Stokes and Darcy problems, respectively, and are defined as follows:

$$< S_{fh}\eta_h, \mu_h > := a_f(R^1_{fh}\eta_h, R_{1h}\mu_h) - \int_{\Omega_f} (R^2_{fh}\eta_h) \, \mathrm{div}(R_{1h}\mu_h) \ , \tag{33}$$

$$< S_{ph}\eta_h, \mu_h >:= \int_\Gamma g\,(R_{ph}\eta_h)\mu_h \ , \tag{34}$$

for all $\eta_h \in \Lambda_{0h},\ \mu_h \in \Lambda_h$.
Finally, let χ_h be the linear functional:

$$\begin{aligned} < \chi_h, \mu_h > \ &:= \int_{\Omega_f} \mathbf{f} \cdot (R_{1h}\mu_h) - a_f(\boldsymbol{\omega}^*_{0h} + E_{fh}\mathbf{u}_{in} + E_{\Gamma h}\lambda_{*h}, R_{1h}\mu_h) \\ &\quad + \int_{\Omega_f} \pi^*_h \,\mathrm{div}(R_{1h}\mu_h) - \int_\Gamma g\,\varphi^*_{0h}\mu_h \end{aligned} \tag{35}$$

for all $\mu_h \in \Lambda_h$.

A characterization of the solution of problem (25) in terms of the solution of the Steklov–Poincaré discrete interface problem is given in the following result, which is the discrete counterpart of Theorem 1 of [6]:

Theorem 4.1 *The solution to (25) can be characterized as follows:*

$$\begin{aligned} \mathbf{u}^0_{fh} &= \boldsymbol{\omega}^*_{0h} + R^1_{fh}\lambda_{0h} + E_{\Gamma h}\lambda_{*h}, \\ p_h &= \pi^*_h + R^2_{fh}\lambda_{0h} + \hat{p}_{fh}, \\ \varphi_{0h} &= \varphi^*_{0h} + R_{ph}\lambda_{0h} \ , \end{aligned} \tag{36}$$

where $\hat{p}_{fh} = (meas(\Omega_f))^{-1}\int_{\Omega_f} p_h$ *, and* $\lambda_{0h} \in \Lambda_{0h}$ *is the solution of the discrete Steklov–Poincaré interface problem:*

$$< S_h\lambda_{0h}, \mu_{0h} >=< \chi_h, \mu_{0h} > \quad \forall \mu_{0h} \in \Lambda_{0h} \ . \tag{37}$$

Moreover, $\hat{p}_{fh}$ *can be obtained from* λ_{0h} *by solving the algebraic equation*

$$\hat{p}_{fh} = \frac{1}{meas(\Gamma)} < S_h\lambda_{0h} - \chi_h, \varepsilon_h > \ , \tag{38}$$

where $\varepsilon_h \in \Lambda_h$ *is a given function that satisfies*

$$\frac{1}{meas(\Gamma)}\int_\Gamma \varepsilon_h = 1 \ . \tag{39}$$

Proof. We refer to the one of Theorem 1 of [6] since it follows the same ideas. □

4.1 Analysis of the Discrete Steklov–Poincaré Operators

Let us investigate some properties of the discrete Steklov–Poincaré operators S_{fh}, S_{ph} and S_h that will allow us to prove existence and uniqueness for problem (37). Since their proofs are similar to those of the continuous case, we shall only sketch them, referring to [6] for more details.

Lemma 4.1 *The discrete Steklov–Poincaré operators enjoy the following properties:*

1. *S_{fh} and S_{ph} are linear continuous operators on Λ_{0h},* i.e. $S_{fh}\eta_h \in \Lambda_0'$, $S_{ph}\eta_h \in \Lambda_0'$, $\forall \eta_h \in \Lambda_{0h}$ *;*

2. *S_{fh} is symmetric and coercive;*

3. *S_{ph} is symmetric and positive.*

Proof.
i) Making the special choice $R_{1h} = R^1_{fh}$, the operator S_{fh} can be represented as follows

$$< S_{fh}\eta_h, \mu_h >= a_f(R^1_{fh}\eta_h, R^1_{fh}\mu_h) \ , \tag{40}$$

for all $\eta_h, \mu_h \in \Lambda_{0h}$.
Now, proceeding as in Lemma 2 of [6], we can define the function $\mathbf{z}_h(\mu_h) := R^1_{fh}\mu_h - \mathcal{H}_h\mu_h \in H^0_{fh}$, $\mathcal{H}_h$ being the Galerkin approximation of the harmonic extension operator defined in (44) of [6].
Using the inf-sup condition (5.3.43) of [14] (p. 173), we have for all $\mu_h \in \Lambda_{0h}$

$$\|R^2_{fh}\mu_h\|_{L^2(\Omega_f)} \leq \frac{2\nu}{\beta^*}\|R^1_{fh}\mu_h\|_{H^1(\Omega_f)} \ ,$$

and therefore

$$\|R^1_{fh}\mu_h\|_{H^1(\Omega_f)} \quad \leq \quad \frac{4}{n\kappa_f}\left(1+\frac{1}{\beta^*}\right)\|\mathcal{H}_h\mu_h\|_{H^1(\Omega_f)} \, . \tag{41}$$

Now, thanks to the Uniform Extension Theorem (see [14], Theorem 4.1.3), there exists a positive constant $C_{\Omega_f} > 0$, depending on the measure of the subdomain Ω_f, but independent of the parameter h, such that

$$\|\mathcal{H}_h\mu_h\|_{H^1(\Omega_f)} \leq C_{\Omega_f}\|\mu_h\|_\Lambda \ , \qquad \forall \mu_h \in \Lambda_h \ .$$

Therefore, (41) gives $\forall \mu_h \in \Lambda_{0h}$

$$\|R^1_{fh}\mu_h\|_{H^1(\Omega_f)} \leq \frac{4C_{\Omega_f}}{n\kappa_f}\left(1+\frac{1}{\beta^*}\right)\|\mu_h\|_\Lambda \ , \tag{42}$$

where κ_f is a positive constant arising from the following Korn inequality: $\forall \mathbf{v}_h = (v_{1h}, \ldots, v_{dh}) \in H_{fh}$, $\exists \kappa_f > 0$:

$$\int_{\Omega_f}\sum_{j,l=1}^{d}\left(\frac{\partial v_{jh}}{\partial x_l}+\frac{\partial v_{lh}}{\partial x_j}\right)^2 \geq \kappa_f\|\mathbf{v}_h\|^2_{H^1(\Omega_f)} \ . \tag{43}$$

From (42) we deduce the continuity of S_{fh}:

$$| < S_{fh}\mu_h, \eta_h > | \leq \tilde{\beta}_f\|\mu_h\|_\Lambda\|\eta_h\|_\Lambda \ , \tag{44}$$

where $\tilde{\beta}_f$ is the positive constant, independent of h,

$$\tilde{\beta}_f := \frac{8\nu}{n^2}\left[\frac{C_{\Omega_f}}{n\kappa_f}\left(1+\frac{1}{\beta^*}\right)\right]^2 \ . \tag{45}$$

Proceeding as for the continuous case, we can prove that S_{ph} is continuous with constant $\beta_p := [gC_p^2(1+C_{\Omega_p})]m_K^{-1}$, independent of h. C_p, C_{Ω_p} and m_K are positive constants introduced in [6].

ii) S_{fh} is symmetric thanks to (40) and the proof of its coercivity follows the one in the continuous case, the coercivity constant $\alpha_f := (n\nu\kappa_f)/(2C_f)$ being the same (see (50) in [6]).

iii) This property follows from point 3. of the proof of Lemma 2 of [6].
□

As a consequence of Lemma 4.1, we have the following result:

Corollary 4.1 *The discrete Steklov–Poincaré operator S_h is symmetric, continuous and coercive, uniformly with respect to h. Moreover S_h and S_{fh} are uniformly spectrally equivalent,* i.e. *there exist two constants k_1 and k_2 independent of h,* s.t. $\forall \eta_h \in \Lambda_h$,

$$k_1 < S_{fh}\eta_h, \eta_h > \leq < S_h\eta_h, \eta_h > \leq \; k_2 < S_{fh}\eta_h, \eta_h > .$$

Remark 4.1 *Thanks to Lax–Milgram Lemma, Corollary 4.1 guarantees that the discrete Steklov–Poincaré equation (37) has a solution, and that this solution is unique.*

5 An Iterative Method for the Numerical Solution of the Coupled Problem

The iterative method we propose to compute the solution of the Stokes–Darcy problem reads as follows:
given $\mathbf{u}_{inh}$, construct λ_{*h} as indicated in Sect. 4;
then let $\lambda_h^0 \in \Lambda_{0h}$ be the initial guess; for $k \geq 0$:
find $\varphi_{0h}^{k+1} \in H_{ph}$:

$$\begin{aligned} &\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla\varphi_{0h}^{k+1} - \int_\Gamma n\,\psi_h\,\lambda_{0h}^k \\ &\quad = -\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla(E_{ph}\varphi_{ph}) + \int_\Gamma n\,\psi_h\lambda_{*h} \qquad \forall\psi \in H_{ph}\;; \end{aligned} \tag{46}$$

find $(\mathbf{u}_{fh}^0)^{k+1} \in H_{fh},\ p_h^{k+1} \in Q_h$:

$$\begin{aligned} &a_f((\mathbf{u}_{fh}^0)^{k+1}, \mathbf{w}_h) - \int_{\Omega_f} p_h^{k+1}\,\mathrm{div}\mathbf{w}_h + \int_\Gamma g\varphi_h^{k+1}\mathbf{w}_h \cdot \mathbf{n}_f \\ &= \int_{\Omega_f} \mathbf{f}\cdot\mathbf{w}_h - a_f(E_{fh}\mathbf{u}_{in}, \mathbf{w}_h) \qquad \forall\mathbf{w}_h \in H_{fh}, \\ &\int_{\Omega_f} q_h\,\mathrm{div}(\mathbf{u}_{fh}^0)^{k+1} = -\int_{\Omega_f} q_h\,\mathrm{div}(E_{fh}\mathbf{u}_{inh}) \quad \forall q_h \in Q_h, \end{aligned} \tag{47}$$

with $\varphi_h^{k+1} = \varphi_{0h}^{k+1} + E_{ph}\varphi_{ph}$;

$$\lambda_{0h}^{k+1} := \theta(\mathbf{u}_{fh}^{k+1}\cdot\mathbf{n}_f - \lambda_{*h})_{|\Gamma} + (1-\theta)\lambda_{0h}^k\,, \tag{48}$$

being θ a positive relaxation parameter and $\mathbf{u}_{fh}^{k+1} = (\mathbf{u}_{fh}^0)^{k+1} + E_{fh}\mathbf{u}_{inh}$.

Remark 5.1 *Note that* $\lambda_{0h}^k \in \Lambda_{0h}$ *for all* $k \geq 0$*. In fact,* $\lambda_{0h} \in \Lambda_{0h}$ *given, suppose* $\lambda_{0h}^k \in \Lambda_0$*. Then* $\int_\Gamma \lambda_{0h}^{k+1} = \theta \int_\Gamma (\mathbf{u}_{fh}^{k+1} \cdot \mathbf{n}_f|_\Gamma - \lambda_{*h})$*. Now, since* $\int_{\Omega_f} div \mathbf{u}_{fh}^{k+1} = 0$*, thanks to the divergence theorem we have* $\int_\Gamma \mathbf{u}_{fh}^{k+1} \cdot \mathbf{n}_f = -\int_{\Gamma_f^{in}} \mathbf{u}_{inh} \cdot \mathbf{n}_f$*, and recalling (26) the thesis follows.*

Following the general theory developed in [14], the above iterative method can be reinterpreted as a preconditioned Richardson method for the Steklov–Poincaré problem (37).

Lemma 5.1 *The iterative substructuring scheme (46)–(48) to compute the solution of the finite element approximation of the coupled problem Stokes–Darcy (21) is equivalent to a preconditioned Richardson method for the discrete Steklov–Poincaré equation (37), the preconditioner being the operator* S_{fh} *introduced in (33).*

Proof. Since $E_{fh}\mathbf{u}_{inh} \cdot \mathbf{n}_f = 0$ on Γ, (48) reduces to:

$$\lambda_{0h}^{k+1} = \theta[((\mathbf{u}_{fh}^0)^{k+1} - \lambda_{*h}) \cdot \mathbf{n}_f]_{|\Gamma} + (1-\theta)\lambda_{0h}^k \, . \tag{49}$$

Let $R_{1h} : \Lambda_h \to H_{fh}$ be the extension operator introduced in Proposition 4.1. For all $\mu_h \in \Lambda_h$, we can rewrite $(47)_1$ as:

$$\begin{aligned} &a_f((\mathbf{u}_{fh}^0)^{k+1}, R_{1h}\mu_h) - \int_{\Omega_f} p_h^{k+1} \operatorname{div}(R_{1h}\mu_h) + \int_\Gamma g\varphi_h^{k+1}\mu_h \\ &= \int_{\Omega_f} \mathbf{f} \cdot (R_{1h}\mu_h) - a_f(E_{fh}\mathbf{u}_{in}, R_{1h}\mu_h) \end{aligned} \tag{50}$$

for all $\mu_h \in \Lambda_h$.
Let us define $\hat{p}_{fh}^{k+1} := (meas(\Omega_f))^{-1} \int_{\Omega_f} p_h^{k+1}$; then we set

$$p_{0h}^{k+1} := p_h^{k+1} - \hat{p}_{fh}^{k+1} \, , \tag{51}$$

and we note that $p_{0h}^{k+1} \in L_0^2(\Omega_f)$. Then (50) gives:

$$\begin{aligned} &a_f((\mathbf{u}_{fh}^0)^{k+1}, R_{1h}\mu_h) - \int_{\Omega_f} p_{0h}^{k+1} \operatorname{div}(R_{1h}\mu_h) + \int_\Gamma g\varphi_h^{k+1}\mu_h \\ &= \int_{\Omega_f} \mathbf{f} \cdot (R_{1h}\mu_h) + \int_{\Omega_f} \hat{p}_{fh}^{k+1} \operatorname{div}(R_{1h}\mu_h) - a_f(E_{fh}\mathbf{u}_{in}, R_{1h}\mu_h) \end{aligned} \tag{52}$$

for all $\mu_h \in \Lambda_h$.
Let $\boldsymbol{\omega}_{0h}^*$, π_h^* and φ_{0h}^* be the solutions to problems (28) and (29), respectively. Subtracting from both members in (52) the following terms:

$$a_f(\boldsymbol{\omega}_{0h}^* + E_{\Gamma h}\lambda_{h*}, R_{1h}\mu_h) - \int_{\Omega_f} \pi_h^* \operatorname{div}(R_{1h}\mu_h) + \int_\Gamma g\,\varphi_{0h}^*\mu_h \, ,$$

we have

$$\begin{aligned} &a_f((\mathbf{u}_{fh}^0)^{k+1} - \boldsymbol{\omega}_{0h}^* - E_{\Gamma h}\lambda_{h*}, R_{1h}\mu_h) \\ &\quad - \int_{\Omega_f} (p_{0h}^{k+1} - \pi_h^*) \operatorname{div}(R_{1h}\mu_h) + \int_\Gamma g(\varphi_h^{k+1} - \varphi_{0h}^*)\mu_h \\ &= \int_{\Omega_f} \mathbf{f} \cdot (R_{1h}\mu_h) + \int_{\Omega_f} \pi_h^* \operatorname{div}(R_{1h}\mu_h) \\ &\quad - a_f(\boldsymbol{\omega}_{0h}^* + E_{\Gamma h}\lambda_{h*} + E_{fh}\mathbf{u}_{in}, R_{1h}\mu_h) \\ &\quad - \int_\Gamma g\,\varphi_{0h}^*\mu_h + \int_{\Omega_f} \hat{p}_{fh}^{k+1} \operatorname{div}(R_{1h}\mu_h) \end{aligned} \tag{53}$$

for all $\mu_h \in \Lambda_h$.
Since $\int_\Omega \mathrm{div}(\boldsymbol{\omega}_0^* + E_{\Gamma h}\lambda_{*h} + E_{fh}\mathbf{u}_{inh}) = 0$ and $\int_\Omega \mathrm{div}((\mathbf{u}_{fh}^0)^{k+1} + E_{fh}\mathbf{u}_{inh}) = 0$, we obtain $\int_\Omega \mathrm{div}((\mathbf{u}_{fh}^0)^{k+1} - \boldsymbol{\omega}_0^* - E_{\Gamma h}\lambda_{*h}) = 0$. Now, if we apply the divergence theorem and recall that $(\mathbf{u}_{fh}^0)^{k+1} \in H_{fh}$, $\boldsymbol{\omega}_{0h}^* \in H_{fh}^0$ and $E_{\Gamma h}\lambda_{*h} \in H_{fh}$, we can see that $[(\mathbf{u}_{fh}^0)^{k+1} - E_{\Gamma h}\lambda_{*h}] \cdot \mathbf{n}_{f_{|\Gamma}} \in \Lambda_{0h}$. Therefore

$$
\begin{aligned}
&a_f((\mathbf{u}_{fh}^0)^{k+1} - \boldsymbol{\omega}_{0h}^* - E_{\Gamma h}\lambda_{*h}, R_{1h}\mu_h) - \int_{\Omega_f} (p_{0h}^{k+1} - \pi_h^*)\,\mathrm{div}(R_{1h}\mu_h) \\
&=< S_{fh}(((\mathbf{u}_{fh}^0)^{k+1} - E_{\Gamma h}\lambda_{*h}) \cdot \mathbf{n}_f)_{|\Gamma}, \mu_h >
\end{aligned}
\tag{54}
$$

for all $\mu_h \in \Lambda_h$.
Moreover, if we subtract (29) from (46), we obtain

$$
\int_{\Omega_p} \nabla\psi_h \cdot \mathsf{K}\nabla(\varphi_{0h}^{k+1} - \varphi_{0h}^*) = \int_\Gamma n\lambda_{0h}^k\psi_h \quad \forall\psi_h \in H_{ph}\ ,
$$

that is, thanks to (31), $\varphi_{0h}^{k+1} - \varphi_{0h}^* = R_{ph}\lambda_{0h}^k$. Therefore

$$
\int_\Gamma g(\varphi_h^{k+1} - \varphi_{0h}^*)\mu_h =< S_{ph}\lambda_{0h}^k, \mu_h > \quad \forall\mu_h \in \Lambda_h.
$$

Finally, if we apply the divergence theorem to the last right hand side term in (53) and we recall definition (35), we can rewrite the right hand side of (53) as

$$
< \chi_h, \mu_h > + \hat{p}_{fh}^{k+1} \int_\Gamma \mu_h \qquad \forall\mu_h \in \Lambda_h\ . \tag{55}
$$

Now, for all $\mu_h \in \Lambda_{0h}$, it follows:

$$
\begin{aligned}
&< S_{fh}(((\mathbf{u}_{fh}^0)^{k+1} - E_{\Gamma h}\lambda_{*h}) \cdot \mathbf{n}_f)_{|\Gamma}, \mu_h > \\
&+ < S_{ph}\lambda_{0h}^k, \mu_h > = < \chi_h, \mu_h > \ .
\end{aligned}
\tag{56}
$$

Therefore we can conclude that the Dirichlet-Neumann iterative method (46)-(48) is equivalent to: $\forall k \geq 0$, find $\lambda_{0h}^{k+1} \in \Lambda_{0h}$ s.t.

$$
\text{given } \lambda_{0h}^0 \in \Lambda_{0h}, \qquad \lambda_{0h}^{k+1} = \lambda_{0h}^k + \theta_h S_{fh}^{-1}(\chi_h - S_h\lambda_{0h}^k)\ , \tag{57}
$$

that is to a preconditioned Richardson iterative scheme for the discrete Steklov–Poincaré equation (37).
□

This reinterpretation is useful for carrying out the convergence analysis of scheme (46)-(48), as illustrated in Sect. 5.1.

5.1 Convergence Analysis of the Iterative Method

Our aim is now to prove the convergence of the sequence $\{((\mathbf{u}_{fh}^0)^k, p_h^k, \varphi_{0h}^k)\}_k$ generated by the iterative method (46)-(48) to the exact solution $(\mathbf{u}_{fh}^0, p_h, \varphi_{0h})$ of the coupled problem (21).
To this end, we shall apply the following abstract convergence result (see [14], Theorem 4.2.2 and Remark 4.2.4):

Theorem 5.1 *Let X be a (real) Hilbert space, and X' its dual space. We consider a linear invertible continuous operator $\mathcal{Q}\colon X \to X'$, which can be split as $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$, where both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are linear operators. Taken $\mathcal{Z} \in X'$, let $\xi \in X$ be the unknown solution to the equation*

$$\mathcal{Q}\xi = \mathcal{Z} \ ,$$

and consider for its solution the preconditioned Richardson method

$$\mathcal{Q}_2(\xi^{k+1} - \xi^k) = \theta(\mathcal{Z} - \mathcal{Q}\xi^k), \quad k \geq 0.$$

Suppose that the following conditions are satisfied:
a) $\mathcal{Q}_2$ is symmetric, continuous and coercive with constants β_2 and α_2, respectively;
b) $\mathcal{Q}_1$ is continuous with constant β_1;
c) $\mathcal{Q}$ is coercive with constant κ.
Then for any given $\xi^0 \in X$ and for any $0 < \theta < \theta_{\max}$, with

$$\theta_{\max} := \frac{2\kappa\alpha_2^2}{\beta_2(\beta_1 + \beta_2)^2} \ ,$$

the sequence

$$\xi^{k+1} = \xi^k + \theta\mathcal{Q}_2^{-1}(\mathcal{Z} - \mathcal{Q}\xi^k)$$

converges in X to the solution of problem $\mathcal{Q}\xi = \mathcal{Z}$.

We can now prove the main result of this Section.

Corollary 5.1 *The iterative method (46)-(48) converges to the solution $(\mathbf{u}^0_{fh}, p_h, \varphi_{0h}) \in H_{fh} \times Q_h \times H_{ph}$ of the coupled problem (21), for any choice of the initial guess $\lambda^0_{0h} \in \Lambda_{0h}$, and for suitable values of the relaxation parameter θ.*

Proof. Upon setting $X = \Lambda_{0h}$, $\mathcal{Q} = S_h$, $\mathcal{Q}_1 = S_{ph}$, $\mathcal{Q}_2 = S_{fh}$ and $\mathcal{Z} = \chi_h$, the proof follows from Theorem 5.1, whose hypotheses are satisfied thanks to Lemma 4.1 and to Corollary 4.1. In fact, for an initial guess $\lambda^0_{0h} \in \Lambda_{0h}$, and any $\theta \in (0, \theta_{\max})$, with $\theta_{\max}$ defined as

$$\theta_{\max} := \frac{2\alpha_f^3}{\tilde{\beta}_f(\tilde{\beta}_f + \beta_p)^2} \ ,$$

the sequence defined in (57) converges to the solution of the Steklov–Poincaré equation (37). Taking the limit $k \to \infty$ in the iterative procedure (46)-(48), it follows that $\{((\mathbf{u}^0_{fh})^k, p^k_h, \varphi^k_{0h})\}_k \xrightarrow[k\to\infty]{} (\mathbf{u}^0_{fh}, p_h, \varphi_{0h})$.
□

The constants α_f, $\tilde{\beta}_f$ and β_p are those introduced in Lemma 4.1. Since they are all independent of h, $\theta_{\max}$ is also independent of h.

6 Algebraic Formulation of the Coupled Problem. The Schur Complement Matrix.

After having introduced suitable bases for the discrete spaces H_{fh}, Q_h and H_{ph}, we indicate by u_{int} the vector of the values of the unknown $\mathbf{u}^0_{fh}$ at the nodes of $\Omega_f \setminus \Gamma$ plus those of $(\mathbf{u}^0_{fh} \cdot \boldsymbol{\tau}_i)$ $(i = 1, \ldots, d-1)$ at the nodes lying on the interface Γ.
Moreover u_Γ indicates the vector of the values of $(\mathbf{u}^0_{fh} \cdot \mathbf{n}_f)$ at the nodes of Γ, and p the one of the values of the unknown pressure p_h at the nodes of Ω_f.
Finally $\boldsymbol{\phi}_{int}$ indicates the vector of the values of the piezometric head φ_{0h} at the nodes on $\Omega_p \setminus \Gamma$, and $\boldsymbol{\phi}_\Gamma$ those at the nodes on Γ.
Let us denote by N_Γ the number of nodes lying on the interface Γ, so that $dim(\mathsf{u}_\Gamma) = dim(\boldsymbol{\phi}_\Gamma) = N_\Gamma$.
The matrix form of problem (21) reads:

$$\begin{pmatrix} A & B^t & A_\Gamma & 0 & 0 \\ B_1 & 0 & B_\Gamma & 0 & 0 \\ A_{f\Gamma} & B^t_\Gamma & A_{\Gamma\Gamma} & M_\Gamma & 0 \\ 0 & 0 & -M^t_\Gamma & \tilde{A}_\Gamma & A^t_{p\Gamma} \\ 0 & 0 & 0 & A_{p\Gamma} & A_{pp} \end{pmatrix} \begin{pmatrix} \mathsf{u}_{int} \\ \mathsf{p} \\ \mathsf{u}_\Gamma \\ \boldsymbol{\phi}_\Gamma \\ \boldsymbol{\phi}_{int} \end{pmatrix} = \begin{pmatrix} \mathsf{f}_f \\ \mathsf{f}_{in} \\ \mathsf{f}_\Gamma \\ \mathsf{f}_{p\Gamma} \\ \mathsf{f}_p \end{pmatrix} \tag{58}$$

6.1 Matrix Interpretation of the Substructuring Iterative Method

The iterative scheme (46)–(48) corresponds to the following steps.
Let $\boldsymbol{\lambda}^k_0 \in \mathbb{R}^{N_\Gamma}$ be the vector of the values of λ_{0h} at the k–th step at the nodes of Γ.
The following algebraic system corresponds to (46):

$$\begin{pmatrix} \tilde{A}_\Gamma & A^t_{p\Gamma} \\ A_{p\Gamma} & A_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}^{k+1}_\Gamma \\ \boldsymbol{\phi}^{k+1}_{int} \end{pmatrix} = \begin{pmatrix} \mathsf{f}_{p\Gamma} + M^t_\Gamma \boldsymbol{\lambda}^k_0 + M^t_\Gamma \boldsymbol{\lambda}_* \\ \mathsf{f}_p \end{pmatrix}, \tag{59}$$

where $\boldsymbol{\lambda}_*$ is the vector whose components are the (known) values of λ_{*h} at the nodes on Γ.
By eliminating $\boldsymbol{\phi}^{k+1}_{int}$ from (59), we obtain

$$\left(\tilde{A}_\Gamma - A^t_{p\Gamma} A^{-1}_{pp} A_{p\Gamma}\right) \boldsymbol{\phi}^{k+1}_\Gamma = \mathsf{f}_{p\Gamma} - A^t_{p\Gamma} A^{-1}_{pp} \mathsf{f}_p + M^t_\Gamma \boldsymbol{\lambda}^k + M^t_\Gamma \boldsymbol{\lambda}_* \,. \tag{60}$$

Now use $\boldsymbol{\phi}^{k+1}_\Gamma$ to compute the unknown vector u^{k+1}_Γ by solving the following system (which corresponds to the Stokes problem (47)):

$$\begin{pmatrix} A & B^t & A_\Gamma \\ B_1 & 0 & B_\Gamma \\ A_{f\Gamma} & B^t_\Gamma & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathsf{u}^{k+1}_{int} \\ \mathsf{p}^{k+1} \\ \mathsf{u}^{k+1}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathsf{f}_f \\ \mathsf{f}_{in} \\ \mathsf{f}_\Gamma - M_\Gamma \boldsymbol{\phi}^{k+1}_\Gamma \end{pmatrix} \tag{61}$$

Finally, according to (48), we set

$$\boldsymbol{\lambda}^{k+1}_0 := \theta(\mathsf{u}^{k+1}_\Gamma - \boldsymbol{\lambda}_*) + (1-\theta)\boldsymbol{\lambda}^k_0 \,, \tag{62}$$

and we iterate restarting from (59) until the convergence test

$$\frac{\|\boldsymbol{\lambda}_0^{k+1} - \boldsymbol{\lambda}_0^k\|_{\mathbb{R}^{N_\Gamma}}}{\|\boldsymbol{\lambda}_0^{k+1}\|_{\mathbb{R}^{N_\Gamma}}} \leq \epsilon$$

is satisfied for a prescribed tolerance ϵ; $\|\cdot\|_{\mathbb{R}^{N_\Gamma}}$ denotes the Euclidean norm in $\mathbb{R}^{N_\Gamma}$.

6.2 Algebraic Formulation of the Discrete Steklov–Poincaré Operator S_h: the Schur Complement Matrix

With obvious choice of notation, system (58) has the following block form:

$$\begin{pmatrix} C & C_\Gamma^1 & 0 \\ C_\Gamma^2 & A_{\Gamma\Gamma} & E \\ 0 & F & D \end{pmatrix} \begin{pmatrix} \mathsf{U} \\ \mathsf{u}_\Gamma \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \mathsf{f}_1 \\ \mathsf{f}_\Gamma \\ \mathsf{f}_2 \end{pmatrix} . \tag{63}$$

By writing $\mathsf{u}_\Gamma = \mathsf{u}_\Gamma^0 + \boldsymbol{\lambda}_*$, system (63) reduces to:

$$\begin{pmatrix} C & C_\Gamma^1 & 0 \\ C_\Gamma^2 & A_{\Gamma\Gamma} & E \\ 0 & F & D \end{pmatrix} \begin{pmatrix} \mathsf{U} \\ \mathsf{u}_\Gamma^0 \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \tilde{\mathsf{f}}_1 \\ \tilde{\mathsf{f}}_\Gamma \\ \tilde{\mathsf{f}}_2 \end{pmatrix} , \tag{64}$$

where $\tilde{\mathsf{f}}_1 = \mathsf{f}_1 - C_\Gamma^1 \boldsymbol{\lambda}_*$, $\tilde{\mathsf{f}}_\Gamma = \mathsf{f}_\Gamma - A_{\Gamma\Gamma}\boldsymbol{\lambda}_*$ and $\tilde{\mathsf{f}}_2 = \mathsf{f}_2 - F\boldsymbol{\lambda}_*$.
Upon eliminating the unknowns U and $\boldsymbol{\phi}$, we obtain the reduced system:

$$\Sigma_h \mathsf{u}_\Gamma^0 = \boldsymbol{\chi}_h \tag{65}$$

where we have defined

$$\Sigma_h := \left(A_{\Gamma\Gamma} - C_\Gamma^2 C^{-1} C_\Gamma^1\right) + \left(-ED^{-1}F\right) \tag{66}$$

and

$$\boldsymbol{\chi}_h := \tilde{\mathsf{f}}_\Gamma - C_\Gamma^2 C^{-1}\tilde{\mathsf{f}}_1 - ED^{-1}\tilde{\mathsf{f}}_2 . \tag{67}$$

In (66) the first term $\Sigma_{fh} := A_{\Gamma\Gamma} - C_\Gamma^2 C^{-1} C_\Gamma^1$ arises from domain Ω_f, whereas $\Sigma_{ph} := -ED^{-1}F$ from Ω_p.
The matrices Σ_{fh} and Σ_{ph} are the algebraic counterparts of the operators S_{fh} and S_{ph}, respectively.
Thanks to Lemma 4.1 and Corollary 4.1, the matrices Σ_{fh}, Σ_{ph} and Σ_h are symmetric and positive definite.
Now, we observe that from Corollary 4.1 the following inequality arises:

$$[\Sigma_{fh}\boldsymbol{\mu}, \boldsymbol{\mu}] \leq [\Sigma_h \boldsymbol{\mu}, \boldsymbol{\mu}] \leq \left(1 + \frac{\beta_p}{\alpha_f}\right) [\Sigma_{fh}\boldsymbol{\mu}, \boldsymbol{\mu}] , \tag{68}$$

for all $\boldsymbol{\mu} \in \mathbb{R}^{N_\Gamma}$, having denoted by $[.,.]$ the Euclidean scalar product in $\mathbb{R}^{N_\Gamma}$; α_f and β_p are the constants introduced in the proof of Lemma 4.1.
From (68), it follows that the spectral condition number χ_{sp} of the matrix $\Sigma_{fh}^{-1}\Sigma_h$ is bounded independently of h; precisely:

$$\chi_{sp}\left(\Sigma_{fh}^{-1}\Sigma_h\right) \leq 1 + \frac{\beta_p}{\alpha_f} .$$

Σ_{fh} is therefore an optimal preconditioner for Σ_h; hence, the preconditioned Richardson scheme (57) (and henceforth the iterative method (46)–(48)) converges with a rate independent of h. Moreover, since we have found an optimal preconditioner for the interface problem, we can apply other Krylov methods (e.g. Conjugate Gradient, GMRES...) with the same preconditioner in order to have more effective methods.

7 Numerical Results

We present two test cases in 2D. Let $\Omega = (0,1) \times (0,2)$ with $\Omega_p = (0,1)^2$, $\Omega_f = (0,1) \times (1,2)$ and $\Gamma = \{(x,y) \in \overline{\Omega} | \, y = 1, 0 \leq x \leq 1\}$.
For Stokes problem we have adopted $\mathbb{P}_2 - \mathbb{P}_1$ finite elements, while $\mathbb{P}_2$ finite elements have been used for Darcy equation.
The solution of the coupled problem is computed using both the iterative Dirichlet-Neumann (DN) method (46)-(48) and by solving iteratively the Steklov–Poincaré interface problem (65) by the Conjugate Gradient (as we have pointed out in Sect. 4, this is a symmetric problem).
We remark that the DN method demands at each iteration the solution of the Darcy equation (46) and of the Stokes problem (47) (or, equivalently, (59) and (61) respectively).
In the CG method we have to compute at each step the matrix-vector product $\Sigma_h \mathbf{x}$, or $(I + \Sigma_{fh}^{-1} \Sigma_{ph})\mathbf{x}$ if we address directly the preconditioned system, where the matrices Σ_h, Σ_{fh}, Σ_{ph} have been defined in Sect. 6.2 and I is the identity matrix. These products correspond to the solution of the homogeneous problems (30) and (31). Moreover, the solution of the Stokes problem (28) and that of the Darcy equation (29) can be carried out off-line at the beginning of the procedure in order to compute the right hand side $\boldsymbol{\chi}_h$, which has been defined in (67).

We consider first of all the following exact solution for the coupled problem (4):

$$\begin{aligned} (\mathbf{u}_f)_1 &= -\cos\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}y\right) + 1 \\ (\mathbf{u}_f)_2 &= \sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right) - 3 \\ p &= -\left(\frac{2}{\pi} + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}y\right) + 3y \\ \varphi &= -\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}y\right) + 3y \end{aligned} \tag{69}$$

Remark 7.1 *In Darcy's equation a non null forcing term has been considered. This implies the presence of an additional term in (19), but it does not affect the theory that we have developed.*

In our computation, four different unstructured meshes have been considered, whose number of elements in Ω and of nodes on Γ are reported in Table 7.1, together with the number of iterations to reach convergence. A tolerance of 10^{-10} has been prescribed for the convergence test on the interface variable; the relaxation parameter θ in (48) has been set equal to 0.8.
Figures 2 and 3 show the computed errors at each step for the adopted iterative methods when using the finest mesh (logarithmic scale has been considered

Number of elements	Number of nodes on Γ	CG Iterations	DN Iterations
172	13	5	15
688	27	5	15
2752	55	5	15
11008	111	5	15

Table 1: Computational meshes and number of iterations (problem 69)

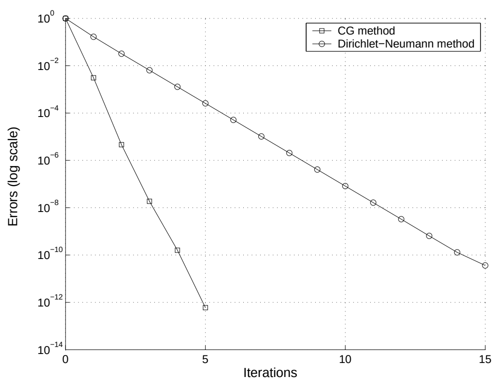

Figure 2: Computed errors (problem (69))

on the y–axis), and the computed interface variable with respect to the exact solution, using the four different meshes.

For the second test, we consider the analytic solution:

$$\begin{aligned} (\mathbf{u}_f)_1 &= 0 \\ (\mathbf{u}_f)_2 &= 0.1\,x(x-1) \\ p &= \frac{-y+2}{3} \\ \varphi &= -x(x-1)(y-1) + \frac{y^3}{3} - y^2 + y \end{aligned} \tag{70}$$

with the following forcing term for Stokes problem:

$$\mathbf{f} = \left(0, -2 \cdot 10^{-2} - \frac{1}{3}\right)^t .$$

We have solved this problem using the same finite elements and the same iterative methods as for the previous test case, considering three different computational meshes. For this test we have fixed the tolerance equal to 10^{-5} and the relaxation parameter $\theta = 0.5$ for the DN method (46)-(48). The convergence results are reported in Table 7.2, while Fig. 7.3 shows the computed errors versus the iterations. Finally, in Fig. 5 we have plot the computed solution.

The results that we have obtained show that the number of iterations needed to reach the fixed tolerance is invariant with respect to h in accordance with

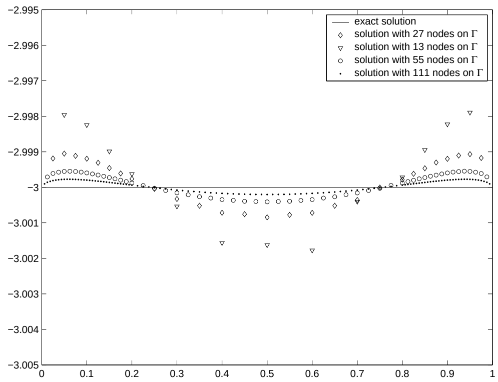

Figure 3: Computed normal velocities on the interface Γ (problem (69))

Number of elements	Number of nodes on Γ	CG Iterations	DN Iterations
688	27	9	16
2454	41	9	16
2752	55	9	16

Table 2: Computational meshes and number of iterations (problem (70))

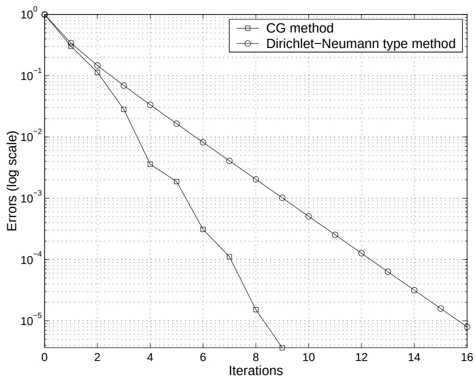

Figure 4: Computed errors (problem 70)

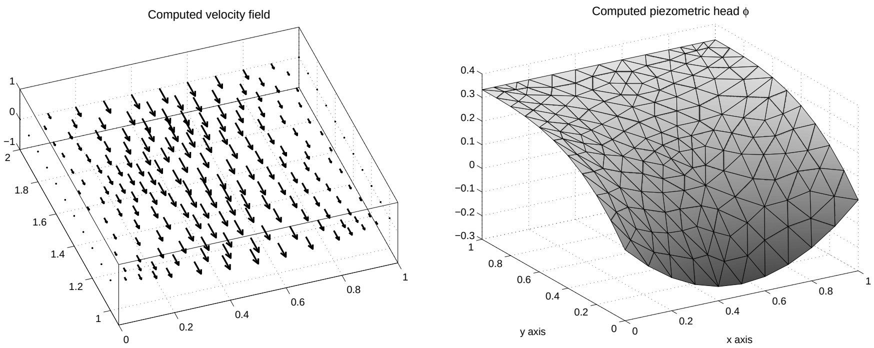

Figure 5: Computed velocity field in Ω_f (above) and piezometric head in Ω_p (below) for problem (70)

our theory (this fact indirectly proves that Σ_{fh} is an optimal preconditioner for solving the interface Steklov–Poincaré problem (37)).

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