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CONVERGENCE ANALYSIS OF STRUCTURE-PRESERVING DOUBLING ALGORITHMS FOR RICCATI-TYPE MATRIX EQUATIONS*

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Abstract. In this paper, we introduce the doubling transformation, a structure-preserving transformation for symplectic pencils, and present its basic properties. Based on these properties, a unified convergence theory for the structure-preserving doubling algorithms for a class of Riccati-type matrix equations is established, using only elementary matrix theory.

Key words. matrix equation, structure-preserving doubling algorithm, convergence rate

AMS subject classifications. 15A24, 65H10, 93B50, 93D15

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1. Introduction.

In this paper, we investigate the convergence of the structure-preserving doubling algorithms (SDAs) for the symmetric positive (semi)definite solutions to the following Riccati-type matrix equations:

- Continuous-time algebraic Riccati equation (CARE) [22, 27]:

$$-XGX + A^T X + XA + H = 0, \tag{1.1}$$

  where $A, H, G \in \mathbb{R}^{n\times n}$ with G and H being symmetric positive semidefinite.
- Discrete-time algebraic Riccati equation (DARE) [22, 27]:

$$X = A^T X(I + GX)^{-1}A + H, \tag{1.2}$$

  where $A, H, G \in \mathbb{R}^{n\times n}$ with G and H being symmetric positive semidefinite.
- Nonlinear matrix equation with the plus sign (NME-P) [3]:

$$X + A^T X^{-1}A = Q, \tag{1.3}$$

  where $A, Q \in \mathbb{R}^{n\times n}$ with Q being symmetric positive definite.
- Nonlinear matrix equation with the minus sign (NME-M) [12]:

$$X - A^T X^{-1}A = Q, \tag{1.4}$$

  where $A, Q \in \mathbb{R}^{n\times n}$ with Q being symmetric positive definite.

The Riccati-type matrix equations occur in many important applications (see [3, 12, 22, 27] and references therein). The nonlinear matrix equations CARE and DARE have been studied extensively (see [1, 2, 4, 5, 6, 7, 19, 8, 14, 15, 18, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 34]). Recently, the nonlinear matrix equations NME-P and NME-M have been studied in [3, 10, 11, 12, 16, 17, 28, 32, 35].

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A class of methods, referred to as doubling algorithms, attracted much interest in the 1970s and '80s (see [2] and references therein). These methods originate from the fixed-point iteration derived from the DARE:

$$X_{k+1} = A^T X_k (I + GX_k)^{-1} A + H.$$

Instead of generating the sequence $\{X_k\}$, doubling algorithms generate $\{X_{2^k}\}$. Doubling algorithms were largely forgotten in the past decade. Recently, doubling algorithms have been revived for DAREs and CAREs because of their nice numerical behavior—a quadratic convergence rate, low computational cost, and high numerical reliability, despite the lack of a rigorous error analysis (see [19, 9, 8]). For the NME-Ps and NME-Ms, Meini [28] and Guo [17] proposed iterative methods with a numerical behavior similar to that of the SDAs for DAREs and CAREs.

In this paper, by employing techniques similar to those in [8], we derive two SDAs for solving NME-Ps and NME-Ms, similar to those proposed by Meini in [28]. In general, we discover that our SDAs can be viewed as repeated applications of some special structure-preserving transformations to the associated symplectic pencils. We first introduce these doubling transformations, then develop a unified convergence theory for the SDAs, based on the nice properties of the doubling transformations using only elementary matrix theory.

Throughout this paper, the symbols $\|\cdot\|_2$ denote the matrix spectral norm. For a given $n \times n$ matrix A we use $\rho(A)$ to denote the spectral radius of A. For real symmetric matrices X and Y we write $X > Y$ ($X \geq Y$) if $X - Y$ is symmetric positive definite (semidefinite).

The paper is organized as follows. In section 2, we introduce a structure-preserving transformation for symplectic pencils and show its basic properties. In section 3, we analyze the convergence of the SDAs for the DARE and the CARE. In section 4, we derive the SDAs for solving the NME-P and the NME-M by using the doubling transformations, and establish the convergence theory of SDAs. Concluding remarks are given in section 5.

2. Doubling transformation. In this section, we introduce a structure-preserving transformation for symplectic pencils and investigate its basic properties. Based on the swapping and collapsing techniques in [4, 7, 6, 5], we begin with the definition of the transformation.

For $M, L \in \mathbb{R}^{2n\times 2n}$, let $M - \lambda L$ be a symplectic pencil, i.e.,

$$MJM^T = LJL^T, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{2.1}$$

Define

$$\mathcal{N}(M, L) = \left\{ [M_*, L_*] \ : \ M_*, L_* \in \mathbb{R}^{2n\times 2n}, \ \text{rank}[M_*, L_*] = 2n, \ [M_*, L_*] \begin{bmatrix} L \\ -M \end{bmatrix} = 0 \right\}. \tag{2.2}$$

Since $\text{rank} \left[\begin{smallmatrix} L \\ -M \end{smallmatrix} \right] \leq 2n$, it follows that $\mathcal{N}(M, L) \neq \emptyset$. For any given $[M_*, L_*] \in \mathcal{N}(M, L)$, define

$$\widehat{M} = M_* M, \qquad \widehat{L} = L_* L. \tag{2.3}$$

The transformation

$$M - \lambda L \longrightarrow \widehat{M} - \lambda \widehat{L}$$

is called a *doubling transformation*.

An important feature of this kind of transformation is that it is structure-preserving, eigenspace-preserving, and eigenvalue-squaring, which has been shown in [4, 5, 33]. We quote the basic properties in the following theorem.

THEOREM 2.1. *Assume that the pencil $\widehat{M} - \lambda\widehat{L}$ is the result of a doubling transformation of the symplectic pencil $M - \lambda L$. Then we have the following:*

(a) *The pencil $\widehat{M} - \lambda\widehat{L}$ is symplectic.*

(b) *If $M\left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right] = L\left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right]S$, where $U, V \in \mathbb{R}^{n\times m}$ and $S \in \mathbb{R}^{m\times m}$, then $\widehat{M}\left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right] = \widehat{L}\left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right]S^2$.*

(c) *If the pencil $M - \lambda L$ has the Kronecker canonical form*

$$WMZ = \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \qquad WLZ = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix}, \tag{2.4}$$

where W, Z are nonsingular, J_r is a Jordan matrix, and N_{2n-r} is a nilpotent matrix, then there exists a nonsingular matrix $\widehat{W}$ such that

$$\widehat{W}\widehat{M}Z = \begin{bmatrix} J_r^2 & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \qquad \widehat{W}\widehat{L}Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r}^2 \end{bmatrix}. \tag{2.5}$$

Remark 2.1. (i) A subspace $\mathcal{W}$ of $\mathbb{R}^{2n}$ is called a *generalized eigenspace* of a pencil $M - \lambda L$ if $\mathcal{W}$ is spanned by the columns of $W = \left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right]$, where $U, V \in \mathbb{R}^{n\times m}$, and W has full column rank and satisfies $MW = LWS$ with $S \in \mathbb{R}^{m\times m}$. Therefore, part (b) of Theorem 2.1 tells us that if $\mathcal{W}$ is a generalized eigenspace of a symplectic pencil $M - \lambda L$, then it is still a generalized eigenspace after a doubling transformation. This is a cornerstone for the convergence theory of the SDAs for the Riccati-type matrix equations in the next two sections.

(ii) A pencil $M - \lambda L$ is called *regular* if $\det(M - \lambda L)$ does not vanish identically. It is well known that a pencil is regular if and only if it has a Kronecker canonical form as in (2.4). Thus, part (c) of Theorem 2.1 says that doubling transformations preserve regularity and that λ is a eigenvalue of $M - \lambda L$ if and only if λ^2 is an eigenvalue of $\widehat{M} - \lambda\widehat{L}$.

A symplectic pencil $M - \lambda L$ is said to be in *first standard symplectic form* (SSF-1) if it has the form

$$M = \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}, \qquad L = \begin{bmatrix} I & G \\ 0 & A^T \end{bmatrix}, \tag{2.6}$$

with $H, G \geq 0$; it is said to be in *second standard symplectic form* (SSF-2) if

$$M = \begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \qquad L = \begin{bmatrix} -P & I \\ A^T & 0 \end{bmatrix}, \tag{2.7}$$

with $P, Q \geq 0$.

Note that one standard symplective form cannot be transformed to another by left nonsingular and right symplectic equivalence transformations unless G in (2.6) or P in (2.7) is positive definite. The following theorem shows that the two standard symplectic forms are preserved by an appropriate choice of doubling transformations.

THEOREM 2.2. (a) *Let $M - \lambda L$ be in SSF-1. Then $[M_*, L_*] \in \mathcal{N}(M, L)$ can be constructed such that after the corresponding doubling transformation, $\widehat{M} - \lambda\widehat{L}$ is still in SSF-1.*

(b) *Let $M - \lambda L$ be in SSF-2. If $Q - P > 0$ and $Q - A^T(Q-P)^{-1}A \geq 0$, then $[M_*, L_*] \in \mathcal{N}(M, L)$ can be constructed such that after the corresponding doubling transformation, $\widehat{M} - \lambda \widehat{L}$ is still in SSF-2.*

Proof. (a) Applying block Gaussian elimination and row permutation to $\left[\begin{smallmatrix} L \\ -M \end{smallmatrix}\right]$, we get

$$M_* = \begin{bmatrix} A(I+GH)^{-1} & 0 \\ -A^T(I+HG)^{-1}H & I \end{bmatrix}, \qquad L_* = \begin{bmatrix} I & AG(I+HG)^{-1} \\ 0 & A^T(I+HG)^{-1} \end{bmatrix} \tag{2.8}$$

such that

$$M_* L = L_* M, \tag{2.9}$$

i.e., $[M_*, L_*] \in \mathcal{N}(M, L)$. Here the Sherman–Morrison–Woodbury formula (see, e.g., [13, p. 50]) is used. For more details, see [8]. We then compute $L_* L$ and $M_* M$ to produce

$$\widehat{M} = M_* M = \begin{bmatrix} \widehat{A} & 0 \\ -\widehat{H} & I \end{bmatrix}, \qquad \widehat{L} = L_* L = \begin{bmatrix} I & \widehat{G} \\ 0 & \widehat{A}^T \end{bmatrix}, \tag{2.10}$$

where

$$\widehat{A} = A(I+GH)^{-1}A, \tag{2.11}$$

$$\widehat{G} = G + AG(I+HG)^{-1}A^T, \tag{2.12}$$

$$\widehat{H} = H + A^T(I+HG)^{-1}HA. \tag{2.13}$$

It is clear that the resulting pencil is still in SSF-1.

(b) Similarly, under the condition $Q - P > 0$, we can compute $[M_*, L_*] \in \mathcal{N}(M, L)$ with

$$M_* = \begin{bmatrix} A(Q-P)^{-1} & 0 \\ -A^T(Q-P)^{-1} & I \end{bmatrix}, \qquad L_* = \begin{bmatrix} I & -A(Q-P)^{-1} \\ 0 & A^T(Q-P)^{-1} \end{bmatrix}. \tag{2.14}$$

Direct calculation gives rise to

$$\widehat{M} = M_* M = \begin{bmatrix} \widehat{A} & 0 \\ \widehat{Q} & -I \end{bmatrix}, \qquad \widehat{L} = L_* L = \begin{bmatrix} -\widehat{P} & I \\ \widehat{A}^T & 0 \end{bmatrix}, \tag{2.15}$$

where

$$\widehat{A} = A(Q-P)^{-1}A, \tag{2.16}$$

$$\widehat{Q} = Q - A^T(Q-P)^{-1}A, \tag{2.17}$$

$$\widehat{P} = P + A(Q-P)^{-1}A^T. \tag{2.18}$$

The assumption $Q - A^T(Q-P)^{-1}A \geq 0$ implies that the resulting pencil is still in SSF-2. ∎

Remark 2.2. The proof of Theorem 2.2 provided us with the well-defined computation formulae for the special structure-preserving doubling transformations, which is the basis for the SDAs for solving the Riccati-type matrix equations.

3. SDAs for preserving SSF-1. In this section, we first state the SDAs proposed in [8] and [9], respectively, for solving DAREs and CAREs. Then we use the technique established in the last section to develop the convergence theory of the SDAs.

3.1. SDA for solving DAREs. It is well known [27] that the DARE (1.2) has a symmetric positive semidefinite solution X (i.e., $X \geq 0$) if and only if X satisfies that

$$M \begin{bmatrix} I \\ X \end{bmatrix} = L \begin{bmatrix} I \\ X \end{bmatrix} S \tag{3.1}$$

for some stable matrix $S \in \mathbb{R}^{n \times n}$, where

$$M = \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}, \qquad L = \begin{bmatrix} I & G \\ 0 & A^T \end{bmatrix}. \tag{3.2}$$

Notice that the pencil $M - \lambda L$ is in SSF-1. Therefore, repeated applications of the special doubling transformation defined in (2.11)–(2.13) gives rise to the following structure-preserving doubling algorithm.

ALGORITHM SDA-1.

$$\begin{aligned} &A_0 = A, \quad G_0 = G, \quad H_0 = H, \\ &A_{k+1} = A_k (I + G_k H_k)^{-1} A_k, \\ &G_{k+1} = G_k + A_k G_k (I + H_k G_k)^{-1} A_k^T, \\ &H_{k+1} = H_k + A_k^T (I + H_k G_k)^{-1} H_k A_k. \end{aligned}$$

This is the SDA described in [8], in which extensive numerical experiments show that this algorithm is efficient and competitive.

3.2. SDA for solving CAREs. Assume that $X \geq 0$ solves the CARE (1.1). It is well known that the CARE (1.1) can be rewritten as

$$\mathcal{H} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} R, \tag{3.3}$$

where

$$\mathcal{H} = \begin{bmatrix} A & -G \\ -H & -A^T \end{bmatrix}, \qquad R = A - GX.$$

The matrix $\mathcal{H}$ is a Hamiltonian matrix, i.e., $(\mathcal{H}J)^T = \mathcal{H}J$. Using a Cayley transformation with some appropriate $\gamma > 0$, we can transform (3.3) into the form

$$M \begin{bmatrix} I \\ X \end{bmatrix} = L \begin{bmatrix} I \\ X \end{bmatrix} S, \tag{3.4}$$

where

$$M = \mathcal{H} + \gamma I, \qquad L = \mathcal{H} - \gamma I, \qquad S = (R - \gamma I)^{-1}(R + \gamma I).$$

Now assume that we have chosen a $\gamma > 0$ such that the matrices

$$A_\gamma = A - \gamma I \quad \text{and} \quad W_\gamma = A_\gamma^T + H A_\gamma^{-1} G \tag{3.5}$$

are nonsingular. Chu, Fan, and Lin [9] proposed a method for computing γ such that both A_γ and W_γ are well conditioned. Let

$$T_1 = \begin{bmatrix} A_\gamma^{-1} & 0 \\ HA_\gamma^{-1} & I \end{bmatrix}, \qquad T_2 = \begin{bmatrix} I & -A_\gamma^{-1}GW_\gamma^{-1} \\ 0 & -W_\gamma^{-1} \end{bmatrix}, \tag{3.6}$$

which are obtained by alternately applying block Gaussian elimination to the matrices L and M (see [9] for more details). Then, direct calculations give rise to

$$\widehat{M} = T_2T_1M = \begin{bmatrix} \widehat{A} & 0 \\ -\widehat{H} & I \end{bmatrix}, \qquad \widehat{L} = T_2T_1L = \begin{bmatrix} I & \widehat{G} \\ 0 & \widehat{A}^T \end{bmatrix},$$

where

$$\widehat{A} = I + 2\gamma W_\gamma^{-T}, \qquad \widehat{G} = 2\gamma A_\gamma^{-1}GW_\gamma^{-1}, \qquad \widehat{H} = 2\gamma W_\gamma^{-1}HA_\gamma^{-1}.$$

Here the Sherman–Morrison–Woodbury formula is used. Since $\gamma > 0$ and $H, G \geq 0$ implies that $\widehat{G}, \widehat{H} \geq 0$, it follows that the resulting pencil $\widehat{M} - \lambda\widehat{L}$ is in SSF-1. In addition, it follows from (3.4) that

$$\widehat{M} \begin{bmatrix} I \\ X \end{bmatrix} = \widehat{L} \begin{bmatrix} I \\ X \end{bmatrix} S. \tag{3.7}$$

Thus, beginning with (3.7), following the same lines as SDA-1 for solving the DARE, we can construct a matrix sequence to approximate the unique symmetric positive semidefinite solution X to the CARE (1.1). For more details, see [9].

3.3. Convergence analysis of SDA-1. Now we establish the convergence theory of SDA-1 using Theorem 2.1. The main results are listed in the following theorem.

THEOREM 3.1. *Assume that $X, Y \geq 0$ satisfies that*

$$X = A^TX(I + GX)^{-1}A + H, \tag{3.8}$$

$$Y = AY(I + HY)^{-1}A^T + G, \tag{3.9}$$

where $G, H \geq 0$, and let

$$S = (I + GX)^{-1}A, \qquad T = (I + HY)^{-1}A^T. \tag{3.10}$$

Then the matrix sequences $\{A_k\}$, $\{G_k\}$, and $\{H_k\}$ generated by SDA-1 satisfy

(a) $A_k = (I + G_kX)S^{2^k}$;

(b) $H \leq H_k \leq H_{k+1} \leq X$ *and*

$$X - H_k = (S^T)^{2^k}(X + XG_kX)S^{2^k} \leq (S^T)^{2^k}(X + XYX)S^{2^k}; \tag{3.11}$$

(c) $G \leq G_k \leq G_{k+1} \leq Y$ *and*

$$Y - G_k = (T^T)^{2^k}(Y + YH_kY)T^{2^k} \leq (T^T)^{2^k}(Y + YXY)T^{2^k}. \tag{3.12}$$

Proof. Notice that $U, V \geq 0$ implies that $I + UV$ is nonsingular and $V(I + UV)^{-1}, (I + UV)^{-1}U \geq 0$. It follows that SDA-1 is well defined and

$$H = H_0 \leq H_k \leq H_{k+1} \quad \text{and} \quad G = G_0 \leq G_k \leq G_{k+1}. \tag{3.13}$$

Define

$$M_k = \begin{bmatrix} A_k & 0 \\ -H_k & I \end{bmatrix}, \qquad L_k = \begin{bmatrix} I & G_k \\ 0 & A_k^T \end{bmatrix}.$$

Then the pencil $M_{k+1} - \lambda L_{k+1}$ is the result of doubling-transforming the pencil $M_k - \lambda L_k$. Since (3.8) implies

$$M_0 \begin{bmatrix} I \\ X \end{bmatrix} = L_0 \begin{bmatrix} I \\ X \end{bmatrix} S, \tag{3.14}$$

where S is defined by (3.10), repeated applications of part (b) of Theorem 2.1 produce

$$M_k \begin{bmatrix} I \\ X \end{bmatrix} = L_k \begin{bmatrix} I \\ X \end{bmatrix} S^{2^k}. \tag{3.15}$$

Equating the blocks of (3.15) then yields

$$A_k = (I + G_k X) S^{2^k}, \tag{3.16}$$

$$X - H_k = A_k^T X S^{2^k}. \tag{3.17}$$

Combining (3.16) with (3.17) gives rise to

$$X - H_k = (S^T)^{2^k} (X + X G_k X) S^{2^k}. \tag{3.18}$$

This, together with $(I + XG_k)X \geq 0$, implies that $X - H_k \geq 0$, i.e., $X \geq H_k$.

Similarly, (3.9) can be rewritten as

$$M_0 \begin{bmatrix} -Y \\ I \end{bmatrix} T = L_0 \begin{bmatrix} -Y \\ I \end{bmatrix}, \tag{3.19}$$

where T is defined by (3.10), and from (3.19) we can derive that

$$Y - G_k = (T^T)^{2^k} (Y + Y H_k Y) T^{2^k},$$

implying that $Y \geq G_k$. Thus, the theorem is proved. □

Let

$$W = \left[L \begin{bmatrix} I \\ X \end{bmatrix}, M \begin{bmatrix} -Y \\ I \end{bmatrix} \right], \qquad Z = \begin{bmatrix} I & -Y \\ X & I \end{bmatrix}.$$

Noting that $M_0 = M$, $L_0 = L$, and $X, Y \geq 0$, it follows from (3.14) and (3.19) that W and Z are nonsingular and satisfy

$$W^{-1} M Z = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}, \qquad W^{-1} L Z = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}.$$

Thus, by the spectral properties of symplectic pencils, it follows that if $\rho(S) < 1$, then we must have $\rho(T) = \rho(S) < 1$. In addition, it is well known that $0 \leq U \leq V$ implies that $\|U\|_2 \leq \|V\|_2$. Consequently from Theorem 3.1, we immediately get the following convergence result for SDA-1.

COROLLARY 3.2. *Under the hypothesis of Theorem* 3.1, *if* $\rho(S) < 1$, *then we have*

(a) $\|A_k\|_2 \le (1 + \|X\|_2\|Y\|_2)\|S^{2^k}\|_2 \longrightarrow 0$ *as* $k \to \infty$;
(b) $\|X - H_k\|_2 \le \|X + XYX\|_2\|S^{2^k}\|_2^2 \longrightarrow 0$ *as* $k \to \infty$;
(c) $\|Y - G_k\|_2 \le \|Y + YXY\|_2\|T^{2^k}\|_2^2 \longrightarrow 0$ *as* $k \to \infty$.

Remark 3.1. (i) Convergence results similar to those in Corollary 3.2 were obtained in [8]. In contrast to the work of [8], however, our analysis is simpler and our convergence results are stronger. In Theorem 3.1, we show explicit expressions of A_k, $X - H_k$, and $Y - G_k$, respectively. Furthermore, Corollary 3.2 contains simple upper bounds of $\|A_k\|_2$, $\|X - H_k\|_2$, and $\|Y - G_k\|_2$ in terms of *only* S, X, and Y.

(ii) Again from parts (b) and (c) of Theorem 3.1, the matrix sequences $\{H_k\}$ and $\{G_k\}$ are monotonically increasing and bounded above, and hence there exist symmetric positive semidefinite matrices $\bar{H}$ and $\bar{G}$ such that

$$\lim_{k\to\infty} H_k = \bar{H}, \qquad \lim_{k\to\infty} G_k = \bar{G}.$$

Corollary 3.2 tells us that if $\rho(S) < 1$, then $X = \bar{H}$ and $Y = \bar{G}$.

Remark 3.2. Let $G = BR^{-1}B^T \ge 0$, with $R > 0$, let $H = C^TC \ge 0$ in the DARE (3.8), and assume that (A, B) is stabilizable and (A, C) is detectable. Then it is well known that the DARE (3.8) and its dual (3.9), respectively, have symmetric positive semidefinite solutions X and Y, and that $\rho(S) < 1$ (see, e.g., [25, 29] for details). Thus the conditions in Corollary 3.2 are satisfied. In fact, it is easy to verify that if the DARE (3.8) and its dual (3.9), respectively, have symmetric positive semidefinite solutions X and Y, with $S = (I + GX)^{-1}A$ and $\rho(S) < 1$, then (A, B) is stabilizable and (A, C) is detectable. A similar argument also holds for the CARE (1.1) (see [9] for details).

4. SDAs for preserving SSF-2. In this section, we shall use the doubling transformations defined in the last section to derive two SDAs for solving the NME-Ps and NME-Ms. Then, we use the technique established in the last section to develop the convergence theory of these SDAs.

4.1. SDA for solving NME-Ps. It is easy to verify that the NME-P (1.3) has a symmetric positive definite X (i.e., $X > 0$) if and only if X satisfies

$$M \begin{bmatrix} I \\ X \end{bmatrix} = L \begin{bmatrix} I \\ X \end{bmatrix} S \tag{4.1}$$

for some matrix $S \in \mathbb{R}^{n\times n}$, where

$$M = \begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & I \\ A^T & 0 \end{bmatrix}. \tag{4.2}$$

Notice that the pencil $M - \lambda L$ is in SSF-2. Therefore, applying the special doubling transformation defined in (2.16)–(2.18) repeatedly gives rise to the following SDA.

ALGORITHM SDA-2.

$$\begin{aligned} &A_0 = A, \quad Q_0 = Q, \quad P_0 = 0, \\ &A_{k+1} = A_k(Q_k - P_k)^{-1}A_k, \\ &Q_{k+1} = Q_k - A_k^T(Q_k - P_k)^{-1}A_k, \\ &P_{k+1} = P_k + A_k(Q_k - P_k)^{-1}A_k^T. \end{aligned}$$

Remark 4.1. To ensure that the iterations in SDA-2 are well defined, the matrix $Q_k - P_k$ must be symmetric positive definite for all k. This can be guaranteed if the

NME-P (1.3) has a symmetric positive solution (see Theorem 4.1), as we shall prove below.

Remark 4.2. It is interesting to note that SDA-2 is essentially the same as Algorithm 3.1 proposed in [28] with $Q_k - P_k$ and Q_k in SDA-2 replaced by Q_k and X_k, respectively. In other words, Algorithm 3.1 in [28] is an SDA. It was pointed out that this algorithm has very nice numerical behavior, with quadratical convergence rate, low computational cost, and good numerical stability. For more details, see [28, 17].

4.2. SDA for solving NEM-Ms. It is proved in [12] that there always exists a unique positive definite solution X to the NME-M

$$X - A^T X^{-1} A = Q \tag{4.3}$$

and, moreover, that the spectral radius of $X^{-1}A$ is strictly less than 1. The solution X is closely related to the generalized eigenspace of the pencil

$$M - \lambda L = \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} - \lambda \begin{bmatrix} 0 & I \\ A^T & 0 \end{bmatrix}. \tag{4.4}$$

In fact, it is easy to verify that a symmetric positive definite matrix X is a solution to the NME-M (4.3) if and only if X satisfies that

$$M \begin{bmatrix} I \\ X \end{bmatrix} = L \begin{bmatrix} I \\ X \end{bmatrix} S \tag{4.5}$$

for some matrix $S \in \mathbb{R}^{n\times n}$.

Although the pencil (4.4) is not symplectic, we can use the same technique as described in section 2 to transform it into a symplectic pencil. Take

$$M_* = \begin{bmatrix} AQ^{-1} & 0 \\ A^T Q^{-1} & -I \end{bmatrix}, \qquad L_* = \begin{bmatrix} I & AQ^{-1} \\ 0 & A^T Q^{-1} \end{bmatrix};$$

then we have

$$M_* L = L_* M. \tag{4.6}$$

Direct calculations lead to

$$\widehat{M}_0 = M_* M = \begin{bmatrix} \widehat{A} & 0 \\ \widehat{Q} & -I \end{bmatrix}, \qquad \widehat{L}_0 = L_* L = \begin{bmatrix} \widehat{P} & I \\ \widehat{A}^T & 0 \end{bmatrix},$$

where

$$\widehat{A} = AQ^{-1}A, \quad \widehat{Q} = Q + A^T Q^{-1} A, \quad \widehat{P} = AQ^{-1}A^T. \tag{4.7}$$

The pencil $\widehat{M}_0 - \lambda \widehat{L}_0$ is symplectic but neither an SSF-1 nor an SSF-2.

Assume that $X > 0$ is the unique symmetric positive solution to the NME-M (4.3). Then it satisfies (4.5) with $S = X^{-1}A$. From part (b) of Theorem 2.1, we have

$$\widehat{M}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \widehat{L}_0 \begin{bmatrix} I \\ X \end{bmatrix} S^2. \tag{4.8}$$

Now let

$$\begin{bmatrix} I \\ \widehat{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \widehat{P} & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix}, \qquad \widehat{M} = \begin{bmatrix} \widehat{A} & 0 \\ \widehat{Q} + \widehat{P} & -I \end{bmatrix}, \qquad \widehat{L} = \begin{bmatrix} 0 & I \\ \widehat{A}^T & 0 \end{bmatrix}.$$

Then it follows from (4.8) that

$$\widehat{M} \begin{bmatrix} I \\ \widehat{X} \end{bmatrix} = \widehat{L} \begin{bmatrix} I \\ \widehat{X} \end{bmatrix} S^2. \tag{4.9}$$

Clearly, the pencil $\widehat{M} - \lambda \widehat{L}$ is in SSF-2. Thus, beginning with (4.9), following the same lines as SDA-2 for solving the NME-P (1.3), we can construct an approximating matrix sequence with limit $\widehat{X}$. Then the unique symmetric positive definite solution X to the NME-M (4.3) can be obtained by computing $X = \widehat{X} - \widehat{P}$.

4.3. Convergence analysis of SDA-2. Now we establish the convergence theory of SDA-2 based on Theorem 2.1. The main results are listed in the following theorem.

THEOREM 4.1. *Assume that $X > 0$ satisfies that*

$$X + A^T X^{-1} A = Q, \tag{4.10}$$

where $Q > 0$, and let $S = X^{-1}A$. Then the matrix sequences $\{A_k\}$, $\{Q_k\}$, and $\{P_k\}$ generated by SDA-2 satisfy

(a) $A_k = (X - P_k)S^{2^k}$;

(b) $0 \le P_k \le P_{k+1} < X$ *and*

$$Q_k - P_k = (X - P_k) + A_k^T (X - P_k)^{-1} A_k > 0; \tag{4.11}$$

(c) $X \le Q_{k+1} \le Q_k \le Q$ *and*

$$Q_k - X = (S^T)^{2^k} (X - P_k) S^{2^k} \le (S^T)^{2^k} X S^{2^k}. \tag{4.12}$$

Proof. Using mathematical induction, denote

$$M_k = \begin{bmatrix} A_k & 0 \\ Q_k & -I \end{bmatrix}, \qquad L_k = \begin{bmatrix} -P_k & I \\ A_k^T & 0 \end{bmatrix},$$

where $P_0 = 0$.

For $k = 1$, since $Q_0 - P_0 = Q > 0$, it follows that A_1, Q_1, P_1 are all well defined. Using (4.10), we have

$$\begin{bmatrix} X & A \\ A^T & Q \end{bmatrix} = \begin{bmatrix} I & 0 \\ A^T X^{-1} & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I & X^{-1}A \\ 0 & I \end{bmatrix} > 0. \tag{4.13}$$

Further computations yield

$$\begin{bmatrix} I & -AQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X & A \\ A^T & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q^{-1}A^T & I \end{bmatrix} = \begin{bmatrix} X - AQ^{-1}A^T & 0 \\ 0 & Q \end{bmatrix}. \tag{4.14}$$

Combining (4.14) and (4.13), we obtain

$$X - P_1 = X - AQ^{-1}A^T > 0. \tag{4.15}$$

From (4.10), it is easy to verify that X satisfies

$$M_0 \begin{bmatrix} I \\ X \end{bmatrix} = L_0 \begin{bmatrix} I \\ X \end{bmatrix} S$$

with $S = X^{-1}A$. Since $M_1 - \lambda L_1$ is the result of doubling-transforming $M_0 - \lambda L_0$, part (b) of Theorem 2.1 leads to

$$M_1 \begin{bmatrix} I \\ X \end{bmatrix} = L_1 \begin{bmatrix} I \\ X \end{bmatrix} S^2. \tag{4.16}$$

Equating the blocks of (4.16) gives rise to

$$A_1 = (X - P_1)S^2, \qquad Q_1 - X = A_1^T S^2.$$

This, together with (4.15), implies that

$$Q_1 - P_1 = (X - P_1) + A_1^T (X - P_1)^{-1} A_1 > 0, \tag{4.17}$$

$$Q_1 - X = (S^T)^2 (X - P_1) S^2 \geq 0. \tag{4.18}$$

Obviously, the inequalities $Q = Q_0 \geq Q_1$ and $0 = P_0 \leq P_1$ hold. Thus, we have proved that the theorem is true for $k = 1$.

Next, considering the $k+1$ case, we assume that the theorem is true for all positive integers less than or equal to k. Since $Q_k - P_k > 0$, it follows that A_{k+1}, Q_{k+1}, P_{k+1} are all well defined. Similar to the proof of (4.15), (4.11) implies

$$X - P_{k+1} = (X - P_k) - A_k (Q_k - P_k)^{-1} A_k^T > 0.$$

Recall that $M_{j+1} - \lambda L_{j+1}$ is the result of doubling-transforming $M_j - \lambda L_j$ for $j = 0, 1, \ldots, k$. Applying part (b) of Theorem 2.1 $k+1$ times, we get

$$M_{k+1} \begin{bmatrix} I \\ X \end{bmatrix} = L_{k+1} \begin{bmatrix} I \\ X \end{bmatrix} S^{2^{k+1}}. \tag{4.19}$$

From (4.19), following the same lines as the proof of (4.17) and (4.18), it can be proved that

$$Q_{k+1} - P_{k+1} = (X - P_{k+1}) + A_{k+1}^T (X - P_{k+1})^{-1} A_{k+1} > 0,$$
$$Q_{k+1} - X = (S^T)^{2^{k+1}} (X - P_{k+1}) S^{2^{k+1}} \geq 0.$$

Clearly, $P_k \leq P_{k+1}$ and $Q_k \geq Q_{k+1}$. This shows that the theorem is also true for integers $k+1$. By induction principle, the theorem is true for all positive integers k. ◻

Remark 4.3. Similar results were obtained in [28] by using properties of cyclic reduction and spectral properties of block Toeplitz matrices with nonnegative definite matrix-valued generating functions. In contrast, our analysis is simpler and the results are stronger. In Theorem 4.1, we show the explicit expressions of A_k and $Q_k - X$, as well as the monotonicity properties of $\{P_k\}$ and $\{Q_k\}$. Furthermore, in part (b) we prove that $Q_k - P_k$ is symmetric positive definite for all k, which guarantees that SDA-2 is well defined.

It was proved in [11] that if the NME-P (1.3) has a symmetric positive definite solution, then all symmetric solutions are positive definite with the maximal and minimal solutions X_+ and X_-. Since Theorem 4.1 is true for any symmetric positive definite solution X, the following result follows immediately.

COROLLARY 4.2. *Under the hypothesis of Theorem* 4.1*, we have*

$$Q_k - P_k > X_+ - X_- \geq 0$$

for all k, where X_+ and X_- are the maximal and minimal solutions of (1.3)*, respectively.*

In addition, from Theorem 4.1, we obtain the following corollary.

COROLLARY 4.3. *Under the hypothesis of Theorem* 4.1*, if $\rho(S) < 1$, then we have*

(a) $\|A_k\|_2 \leq \|X\|_2 \|S^{2^k}\|_2 \longrightarrow 0$ *as* $k \to \infty$*;*

(b) $\|X - Q_k\|_2 \leq \|X\|_2 \|S^{2^k}\|_2^2 \longrightarrow 0$ *as* $k \to \infty$*.*

Remark 4.4. (i) Here we see that the upper bounds of $\|A_k\|_2$ and $\|X - Q_k\|_2$ are in terms of *only* X and $S \equiv X^{-1}A$.

(ii) By Theorem 4.1, the matrix sequence $\{Q_k\}$ is monotonically decreasing and bounded below by $X > 0$. Hence, there exists $\bar{Q} > 0$ such that $\lim_{k\to\infty} Q_k = \bar{Q}$. Corollary 4.3 tells us that if $\rho(S) < 1$, then $X = \bar{Q}$. In fact, X will then be the maximal solution of (1.3). Moreover, it has been proved that X is the maximal solution of (1.3) if and only if $\rho(S) \leq 1$ (see [17]). Now assuming that $X = X_+$ is the maximal solution of (1.3), it is natural to ask whether $\bar{Q} = X_+$ if $\rho(S) = 1$. In [17], Guo proved that if $\rho(S) = 1$ and all eigenvalues of S on the unit circle are semisimple, then $\bar{Q} = X_+$ is still true. In this case, the convergence is at least linear with rate 1/2. When S has nonsemisimple unimodular eigenvalues, it is unclear whether $\bar{Q} = X_+$.

Remark 4.5. It is proved that the NME-P (1.3) has a symmetric positive definite solution X if and only if the nonlinear matrix equation

$$Y + AY^{-1}A^T = Q \tag{4.20}$$

has a symmetric positive solution Y (see, for e.g., [28]). Assume that the maximal solution of (4.20) is Y_+. Then it follows from (4.20) that

$$\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix} \begin{bmatrix} I \\ Q - Y_+ \end{bmatrix} T = \begin{bmatrix} 0 & I \\ A^T & 0 \end{bmatrix} \begin{bmatrix} I \\ Q - Y_+ \end{bmatrix}, \tag{4.21}$$

where $T = Y_+^{-1}A^T$. Similar to the proof of (4.12), we can show from (4.21) that

$$0 \leq Q - Y_+ - P_k = (T^T)^{2^k}(Q_k - Q + Y_+)T^{2^k} \leq (T^T)^{2^k} Y_+ T^{2^k},$$

where P_k and Q_k are generated by SDA-2. Since $\rho(T) = \rho(Y_+^{-1}A^T) = \rho(X_+^{-1}A)$ (see, e.g., [28]), where X_+ is the maximal solution of the NME-P (1.3), it follows that $\lim_{k\to\infty} P_k = Q - Y_+$ under the conditions of Corollary 4.3. If A is nonsingular, then $X_- = Q - Y_+$ (see [28]), where X_- is the minimal solution of the NME-P (1.3), and thus in this case we have $\lim_{k\to\infty} P_k = X_-$.

Remark 4.6. Since $\lim_{k\to\infty}(Q_k - P_k) = X_+ - X_-$ if A is nonsingular and $\rho(S) < 1$, the lower bound $X_+ - X_-$ in Corollary 4.2 is the best one. However, $X_+ - X_-$ may be singular and, indeed, it can be the zero matrix. For example, the NME-P with $Q = I$ and $A = \frac{1}{2}I$ has $X_+ = X_- = \frac{1}{2}I$.

5. Conclusions. In this paper, we have introduced a structure-preserving transformation for symplectic pencils, referred to as the doubling transformation, and investigated its basic properties. Based on these nice properties, a unified convergence theory for the SDAs for solving a class of Riccati-type matrix equations has been established, using only elementary matrix theory.

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