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CONVERGENCE AND CONVOLUTIONS OF PROBABILITY MEASURES ON A TOPOLOGICAL GROUP

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A new technique is developed for studying the convergence of nets of probability measures on a topological group. It is applied to results concerned with the interplay between convergence and convolutions of measures like properties of the convolution mapping, divisibility of measures and convolution semigroups. Our method gives a unified and simple approach to these results.

There is a series of results on probability measures on a topological group whose common background is the interplay between convergence and convolutions of measures. Let us mention, for example, the continuity of the convolution mapping, the closedness of the set of infinitely divisible probability measures or the continuity of convolution semigroups. But although most of these results are well known, their proofs are quite complicated and often depend heavily on topological properties of the underlying group (like local compactness or metrizability). Furthermore in some cases one has studied those problems for uniformly tight nets of measures only. For nonmetrizable groups, however, this is a very restrictive hypothesis.

It is the purpose of this paper to present a unified approach to the results mentioned above, and to extend their domain of validity (if possible). Moreover we are able to give simpler and more direct proofs of the theorems. This is done via the concepts of tight and quasi-tight nets of measures. The important point of this idea is that a convergent net is always quasi-tight, and conversely a quasi-tight net always has a convergent subnet.

In the first two sections of this paper we study properties of quasi-tight and tight nets of probability measures on a topological group. In the following sections we give several applications of the techniques developed. At first we prove in Section 3 the continuity and a closure property of the convolution mapping. Then we derive two results on the divisibility of probability measures. In Section 4 we show that on a locally compact group the correspondence between probability measures and convolution operators is bicontinuous. Finally Section 5 is devoted to the problems of continuity and extension of convolution semigroups of probability measures. The results in the last two sections seem to be new.

Throughout this paper we have restricted ourselves to probability measures though many results will also hold for positive tight measures.

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0. Preliminaries. All topological spaces occurring in this paper are assumed to be Hausdorff spaces.

If T is a topological space we denote by $\mathfrak{K}(T)$ the system of compact subsets in T, by $\mathfrak{B}(T)$ the σ-algebra of Borel subsets in T, by $\mathfrak{C}(T)$ the space of *bounded* continuous functions on T with values in $\mathbb{R}$ and by $\mathfrak{C}^0(T)$ the subspace of functions vanishing at infinity.

Let X be a completely regular space. A τ-continuous measure on X is a σ-additive set function μ of $\mathfrak{B}(X)$ into the set $\mathbb{R}_+$ of positive real numbers such that $\mu(\bigcap_{i\in I} F_i) = \inf_{i\in I} \mu(F_i)$ for every family $(F_i)_{i\in I}$ of closed subsets in X filtering downwards. A *tight measure* on X is a τ-continuous measure μ on X such that $\sup\{\mu(K): K \in \mathfrak{K}(X)\} = \mu(X)$. For every $x \in X$ we denote by ε_x the unit mass in x (Dirac measure). By $\mathscr{M}_+(X)$ we denote the set of tight measures on X and by $\mathscr{W}(X)$ the subset of probability measures in $\mathscr{M}_+(X)$. A net $(\mu_i)_{i\in I}$ in $\mathscr{M}_+(X)$ is said to converge (weakly) to $\mu \in \mathscr{M}_+(X)$ if $\lim_{i\in I} \mu_i(f) = \mu(f)$ for every $f \in \mathfrak{C}(X)$. The corresponding topology on $\mathscr{M}_+(X)$ is called the weak topology. We consider on $\mathscr{M}_+(X)$ only the weak topology. A subset H in $\mathscr{M}_+(X)$ is called uniformly tight if $\sup\{\mu(1): \mu \in H\} < +\infty$ and if for every $\varepsilon > 0$ there exists a $K \in \mathfrak{K}(X)$ such that $\mu(\complement K) < \varepsilon$ for every $\mu \in H$. If H is uniformly tight H is relatively compact in $\mathscr{M}_+(X)$ by Prohorov's theorem ([4], page 63, Theorem 1).

By G we always denote a topological group, the composition in G being the multiplication, and e the identity in G. G being a Hausdorff space, it is well known that G is completely regular. By $\mathfrak{C}_u(G)$ we denote the space of bounded real functions on G uniformly continuous with respect to the left uniform structure on G. For $\mu, \nu \in \mathscr{M}_+(G)$ the product measure $\mu \otimes \nu$ is in $\mathscr{M}_+(G \times G)$. If m denotes multiplication in G (that is $m(x, y) = xy$ for all $x, y \in G$) the image of $\mu \otimes \nu$ under m is denoted by $\mu * \nu$ and called the convolution of μ and ν. It is $\mu * \nu \in \mathscr{M}_+(G)$.

1. Nets of probability measures on topological spaces. Let X be a completely regular topological space and let $(\mu_i)_{i\in I}$ be a net of measures in $\mathscr{W}(X)$. We call $(\mu_i)_{i\in I}$ a *compact net* if every subnet of $(\mu_i)_{i\in I}$ has a further subset that converges (in $\mathscr{W}(X)$).

We say that $(\mu_i)_{i\in I}$ is a *quasi-tight net* if for every $\varepsilon > 0$ there exists a compact set K in X such that $\liminf_{i\in I} \mu_i(0) > 1 - \varepsilon$ for every open set 0 in X containing K.

We say that $(\mu_i)_{i\in I}$ is a *tight net* if for every $\varepsilon > 0$ there exists a compact set K in X such that $\liminf_{i\in I} \mu_i(K) > 1 - \varepsilon$.

LEMMA 1.1. (i) *Each tight net is a quasi-tight net.*

(ii) *A net in $\mathscr{W}(X)$ is quasi-tight if and only if it is a compact net whose set of accumulation points is uniformly tight.*

(iii) *Each convergent net in $\mathscr{W}(X)$ is a quasi-tight net.*

PROOF. (i) is trivial.

(ii) We assume that X is embedded in its Stone–Čech compactification $\tilde{X}$ as a topological subspace. Let $(\mu_i)_{i\in I}$ be a quasi-tight net in $\mathscr{W}(X)$. By $\tilde{H}$ we denote the nonvoid set of accumulation points of $(\mu_i)_{i\in I}$ in the compact space $\mathscr{W}(\tilde{X})$ that contains $\mathscr{W}(X)$ by assumption. Given $\varepsilon > 0$ there exists a $K \in \mathfrak{K}(X)$ such that $\liminf_{i\in I} \mu_i(0) > 1 - \varepsilon$ for every set 0 in $[K] \equiv \{0 : 0 \text{ open set in } \tilde{X} \text{ and } 0 \supset K\}$. Since the mapping $\nu \to \nu(F)$ is upper semicontinuous on $\mathscr{W}(X)$ for every closed set F in $\tilde{X}$, the inequality $\mu(\bar{0}) \geqq 1 - \varepsilon$ holds for every $\mu \in \tilde{H}$ and $0 \in [K]$. It is $K = \bigcap_{0\in[K]} \bar{0}$. Hence the τ-continuity of μ implies $\mu(K) \geqq 1 - \varepsilon$ for every $\mu \in \tilde{H}$. Therefore $\tilde{H}$ is contained in $\mathscr{W}(X)$ ([4], page 32, Proposition 8) and is uniformly tight. Thus the quasi-tight net $(\mu_i)_{i\in I}$ has a convergent subnet. That (μ_i) even is a compact net follows from the observation that every subnet of it is also quasi-tight.

Let now $(\mu_i)_{i\in I}$ be a compact net in $\mathscr{W}(X)$ and assume that its set H of accumulation points is uniformly tight. Given $\varepsilon > 0$ there exists a $K \in \mathfrak{K}(X)$ such that $\mu(K) > 1 - \varepsilon$ for every $\mu \in H$. Let 0 be an open set in X containing K. Since the mapping $\nu \to \nu(0)$ is lower semicontinuous on $\mathscr{W}(X)$ the set $U \equiv \{\mu \in \mathscr{W}(X) : \mu(0) > 1 - \varepsilon\}$ is an open neighbourhood of H in $\mathscr{W}(X)$. We claim that there exists an $i_0 \in I$ such that $\mu_i \in U$ for every $i > i_0$. [Assume this would be wrong. Then there exists a subnet $(\mu_{i(j)})_{j\in J}$ such that $\mu_{i(j)} \in \complement U$ for every $j \in J$. However $(\mu_i)_{i\in I}$ being a compact net there would exist an accumulation point μ of $(\mu_i)_{i\in I}$ such that $\mu \in \complement U$. But this is a contradiction to $\mu \in H$.] Therefore it is $\liminf_{i\in I} \mu_i(0) \geqq 1 - \varepsilon$. Hence $(\mu_i)_{i\in I}$ is a quasi-tight net.

Finally (iii) is a simple consequence of (ii). □

REMARKS. 1. The concept of tight nets of measures has been introduced by Topsøe ([10], page 42) who also proved that tight nets are compact.

Quasi-tight nets of measures have been introduced by Štěpan ([9], page 133; he called them "nets satisfying condition (j)"). He proved in a different manner that quasi-tight nets are compact.

2. A net $(\mu_i)_{i\in I}$ in $\mathscr{W}(X)$ is tight if and only if $\sup_{K\in\mathfrak{K}(X)} \liminf_{i\in I} \mu_i(K) = 1$. On the other side $(\mu_i)_{i\in I}$ possesses a tight subnet if and only if $\sup_{K\in\mathfrak{K}(X)} \limsup_{i\in I} \mu_i(K) = 1$. Similar relations hold for quasi-tight nets.

3. There exist convergent nets of tight probability measures which are not tight nets.

[There exists a sequence of tight probability measures which converges but which is not uniformly tight ([1], page 137). It is immediate that this sequence cannot be a tight net.]

LEMMA 1.2. *The following assertions are equivalent*:

(i) *Each quasi-tight net in* $\mathscr{W}(X)$ *is a tight net.*
(ii) X *is locally compact.*

PROOF. Since "(ii) $\Rightarrow$ (i)" is trivial, we only have to show "(i) $\Rightarrow$ (ii)." Suppose that there exists a $x \in X$ such that the neighbourhood system $\mathfrak{B}(x)$ of x contains no relatively compact set. Thus for each $U \in \mathfrak{B}(x)$ and for each $K \in \mathfrak{K}(X)$

there is an element $x_{U,K} \in U \backslash K$. Obviously $(x_{U,K})_{U \in \mathfrak{B}(x), K \in \mathfrak{K}(X)}$ is a net in X converging to x. The corresponding net of Dirac measures converges to ε_x; it is therefore quasi-tight (Lemma 1) and hence tight by assumption. Thus there exist $U_0 \in \mathfrak{B}(x)$ and $C, K_0 \in \mathfrak{K}(X)$ such that $\varepsilon_{x_{U,K}}(C) > \frac{1}{2}$ and so $x_{U,K} \in C$ for all $U \in \mathfrak{B}(X)$ $U \subset U_0$ and for all $K \in \mathfrak{K}(X)$, $K \supset K_0$. But if $K_1 \equiv K_0 \cup C$ it is $x_{U_0,K_1} \in U_0 \backslash K_1$, a contradiction to $x_{U_0,K_1} \in C$. □

LEMMA 1.3. *For a nonvoid subset H in $\mathscr{W}(X)$ the following assertions are equivalent*:

(i) *H is uniformly tight.*
(ii) *Every net in H has a tight subnet.*

PROOF. We only have to show "(ii) $\Rightarrow$ (i)." Suppose that H is not uniformly tight. Then there exists an $\varepsilon > 0$ such that for every $C \in \mathfrak{K}(X)$ there is a $\mu_C \in H$ such that $\mu_C(C) \leqq 1 - \varepsilon$. $(\mu_C)_{C \in \mathfrak{K}(X)}$ is a net in H such that $\sup_{K \in \mathfrak{K}(X)} \limsup_{C \in \mathfrak{K}(X)} \mu_C(K) \leqq 1 - \varepsilon$. In view of Remark 2 it cannot have a tight subnet. □

COROLLARY (Prohorov). *Let X be locally compact. Then each compact subset H in $\mathscr{W}(X)$ is uniformly tight.*

PROOF. H being compact each net in H has a convergent subnet which is tight by Lemma 1.1 and Lemma 1.2. The assertion follows by Lemma 1.3. □

2. Nets of probability measures on topological groups. Let G be a topological group. Let $(\mu_i)_{i \in I}$, $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$ be three nets in $\mathscr{W}(G)$ (possessing the same index set I) such that $\mu_i = \alpha_i * \beta_i$ for every $i \in I$. Let us see how some well-known results on uniformly tight nets of measures can be recast to results on quasi-tight nets and tight nets.

LEMMA 2.1. *If (α_i) and (β_i) are (quasi-) tight nets then $(\alpha_i \otimes \beta_i)$ and (μ_i) are (quasi-) tight nets (in $\mathscr{W}(G \times G)$* resp. *$\mathscr{W}(G)$).*

[For $\alpha, \beta \in \mathscr{W}(G)$ and $A, B \in \mathfrak{B}(G)$ we always have $\alpha \otimes \beta(A \times B) = \alpha(A)\beta(B)$ and $\alpha * \beta(AB) \geqq \alpha(A)\beta(B)$. Moreover for $K, K' \in \mathfrak{K}(G)$ and every open set 0 in $G \times G$ (resp. in G) such that $K \times K' \subset 0$ (resp. $KK' \subset 0$) there always exist open sets U, U' in G such that $K \subset U$, $K' \subset U'$ and $U \times U' \subset 0$ (resp. $UU' \subset 0$).]

LEMMA 2.2. *If (α_i) and (μ_i) are (quasi-) tight nets then (β_i) is a (quasi-) tight net.*

[Let $A, B \in \mathfrak{B}(G)$ such that $\liminf_{i \in I} \mu_i(A) > 1 - \varepsilon$, $\liminf_{i \in I} \alpha_i(B) > 1 - \varepsilon$ for some $\varepsilon > 0$. It follows:

$$\begin{aligned}\mu_i(A) &= \int \beta_i(x^{-1}A)\alpha_i(dx) = \int_B \beta_i(x^{-1}A)\alpha_i(dx) + \int_{\complement B} \beta_i(x^{-1}A)\alpha_i(dx) \\ &\leqq \beta_i(B^{-1}A) + (1 - \alpha_i(B))\end{aligned}$$

and therefore $\liminf_{i \in I} \beta_i(B^{-1}A) \geqq 1 - 2\varepsilon$.

If (α_i) and (μ_i) are tight nets then we can choose for A and B compact sets. $B^{-1}A$ then being compact (β_i) is also a tight net.

If (α_i) and (μ_i) are quasi-tight nets, then we can choose $C, K \in \mathfrak{K}(G)$ and for A and B arbitrary open sets such that $C \subset A$, $K \subset B$. If 0 is an open set containing $K^{-1}C$ then A and B moreover can be chosen in such a way that $B^{-1}A \subset 0$. Hence (β_i) is a quasi-tight net.]

LEMMA 2.3. (i) *If (μ_i) is a tight net then there exist $x_i \in G$ (for every $i \in I$), such that $(\alpha_i * \varepsilon_{x_i})$ and $(\varepsilon_{x_i^{-1}} * \alpha_i)$ are tight nets.*

(ii) *If in addition there exist a $K \in \mathfrak{K}(G)$, an $\varepsilon \in]0, 1[$ and an $i_0 \in I$ such that $\alpha_i(K) \geqq \varepsilon$ for every $i > i_0$, then (α_i) and (β_i) themselves are tight net.*

PROOF. (i) (Compare with [7], page 59, Theorem 2.2.) Let $(\varepsilon_n)_{n\geqq 1}$ be a sequence of positive real numbers such that $\sum_{n\geqq 1} \varepsilon_n < +\infty$. Then there exist $K_n \in \mathfrak{K}(G)$ and $i(n) \in I$ such that $\mu_i(\complement K_n) < \varepsilon_n$ for every $i > i(n)$. Let $I_n \equiv \{i \in I : i > i(n)\}$, $R_n \equiv I_n \backslash I_{n+1}$ (for every $n \in \mathbb{N}$) and $I_\infty \equiv \bigcap_{n\geqq 1} I_n$. W.l.o.g. we can assume $i(n) < i(n+1)$ and $I_1 = I$. Then it is $I_{n+1} \subset I_n$ and $\bigcup_{n\geqq 1} R_n = I \backslash I_\infty$.

Let $(\delta_n)_{n\geqq 1}$ be a sequence of positive real numbers descending to zero and such that $\sum_{n\geqq 1} \varepsilon_n \delta_n^{-1} < \frac{1}{2}$. Let $A_n \equiv \complement K_n$, $E_{in} \equiv \{x \in G : \alpha_i(A_n x^{-1}) < \delta_n\}$, $F_i \equiv \bigcap_{m=1}^{n} E_{im}$ for $i \in R_n$ and $F_i \equiv \bigcap_{n\geqq 1} E_{in}$ for $i \in I_\infty$. It is $F_i \neq \varnothing$ for every $i \in I$ ([7], page 60).

For each $i \in I$ let x_i be any element of F_i. Then the net $(\alpha_i * \varepsilon_{x_i})$ is obviously tight. Finally $(\varepsilon_{x_i^{-1}} * \beta_i)$ is a tight net by Lemma 2.2.

(ii) Since $(\alpha_i * \varepsilon_{x_i})$ is a tight net there exist a $C \in \mathfrak{K}(G)$ and an $i_1 \in I$ such that $\alpha_i(Cx_i^{-1}) = \alpha_i * \varepsilon_{x_i}(C) > 1 - \varepsilon$ for every $i > i_1$. W.l.o.g. we may assume $i_1 > i_0$. By assumption we have $\alpha_i(K) \geqq \varepsilon$ for every $i > i_1$. Thus $Cx_i^{-1} \cap K \neq \varnothing$ and therefore $x_i \in K^{-1}C$ for every $i > i_1$. By Lemma 2.2 (α_i) is a tight net and so is (β_i). □

In a special case Lemma 2.3 can be strengthened. Following Böge a topological group G is called *root compact* if for every $C \in \mathfrak{K}(G)$ and for every $n \in \mathbb{N}$ there exists a $C_n \in \mathfrak{K}(G)$ such that the following property holds: Every sequence $x_1, \cdots, x_{n-1} \in G$, $x_n = e$ such that $Cx_iCx_j \cap Cx_{i+j} \neq \varnothing$ for all $i, j \in \{1, \cdots, n\}$, $i + j \leqq n$ is contained in C_n ([8], page 231).

LEMMA 2.4. *Let G be a root compact group and $(\mu_i)_{i\in I}$ a tight net in $\mathscr{W}(G)$. If for every $i \in I$ there exists an $\alpha_i \in \mathscr{W}(G)$ such that $\alpha_i^n = \mu_i$ (for some fixed $n \in \mathbb{N}$) then $(\alpha_i)_{i\in I}$ itself is a tight net.*

PROOF. Let $\varepsilon \in]0, \frac{1}{3}[$. There exist $i_\varepsilon \in I$ and $C \in \mathfrak{K}(G)$ such that $\mu_i(C) > 1 - \varepsilon$ for all $i > i_\varepsilon$. Since $\alpha_i^p * \alpha_i^{n-p} = \mu_i$ for $p = 1, \cdots, n$ there exist $x_{i,p} \in G$ (all $i > i_\varepsilon$ and $p = 1, \cdots, n$) such that $\alpha_i^p(Cx_{i,p}) \geqq 1 - \varepsilon$. Then we have $\alpha_i^{p+q}(Cx_{i,p+q}) \geqq 1 - \varepsilon > \frac{2}{3}$ and $\alpha_i^{p+q}(Cx_{i,p}Cx_{i,q}) \geqq \alpha_i^p(Cx_{i,p})\alpha_i^q(Cx_{i,q}) \geqq (1 - \varepsilon)^2 > 1 - 2\varepsilon > \frac{1}{3}$ and therefore $Cx_{i,p}Cx_{i,q} \cap Cx_{i,p+q} \neq \varnothing$ for every $i > i_\varepsilon$ and for $p, q \in \{1, \cdots, n\}$ such that $p + q \leqq n$. W.l.o.g. $x_{i,n} = e$ for every $i > i_\varepsilon$. By definition of root compactness there exists a $C_n \in \mathfrak{K}(G)$ such that $x_{i,p} \in C_n$ for every $i > i_\varepsilon$ and for $p = 1, \cdots, n$. Thus it is $\alpha_i(CC_n) \geqq 1 - \varepsilon$ for every $i > i_\varepsilon$. This completes the proof of the lemma. □

3. Applications to the convolution mapping and to divisibility. We are now going to show how the concepts of quasi-tight nets and tight nets do work.

PROPOSITION 3.1. *Let G be a topological group. Then the convolution mapping ψ from $\mathscr{W}(G) \times \mathscr{W}(G)$ into $\mathscr{W}(G)$ (defined by $\psi(\mu, \nu) = \mu * \nu$) is continuous.*

PROOF. Let $(\mu_i)_{i \in I}$ and $(\nu_i)_{i \in I}$ be nets in $\mathscr{W}(G)$ converging to $\mu \in \mathscr{W}(G)$ and $\nu \in \mathscr{W}(G)$ resp. By Lemma 1.1 and Lemma 2.1 then $(\mu_i \otimes \nu_i)_{i \in I}$ is a quasi-tight net in $\mathscr{W}(G \times G)$. Let $\lambda \in \mathscr{W}(G \times G)$ be an accumulation point of $(\mu_i \otimes \nu_i)_{i \in I}$ and let $(\mu_{i(j)} \otimes \nu_{i(j)})_{j \in J}$ be a subnet converging to λ (Lemma 1.1). If for $f, g \in \mathfrak{C}(G)$ the function $f \times g$ is defined by $(f \times g)(x, y) = f(x)g(y)$, it follows:

$$\begin{aligned}\lambda(f \times g) &= \lim_{j \in J} \mu_{i(j)} \otimes \nu_{i(j)}(f \times g) = \lim_{j \in J} \mu_{i(j)}(f)\nu_{i(j)}(g) \\ &= \mu(f)\nu(g) = \mu \otimes \nu(f \times g) .\end{aligned}$$

Hence by definition of product measure we must have $\lambda = \mu \otimes \nu$. This shows us $\lim_{i \in I} \mu_i \otimes \nu_i = \mu \otimes \nu$. Since the mapping $\alpha \otimes \beta \rightarrow \alpha * \beta$ of $\mathscr{W}(G \times G)$ into $\mathscr{W}(G)$ is continuous (by definition of convolution) we finally have $\lim_{i \in I} \mu_i * \nu_i = \mu * \nu$. $\square$

REMARKS. 1. This result has already been proved by Csiszár ([5], page 32; even for τ-continuous measures) using a different method.

2. This proposition holds true (with the same proof) if G is a completely regular topological semigroup.

PROPOSITION 3.2. *Let G be a locally compact or a metrizable group. For some $K \in \mathfrak{K}(G)$ and some $\varepsilon \in]0, 1[$ we define $A \equiv \{\mu \in \mathscr{W}(G) : \mu(K) \geqq \varepsilon\}$. Then the convolution mapping ψ transforms closed sets in $A \times A$ into closed sets in $\mathscr{W}(G)$.*

PROOF. Let F be a closed set in $A \times A$ and $((\mu_i, \nu_i))_{i \in I}$ a net in F such that $(\mu_i * \nu_i)_{i \in I}$ converges to some $\lambda \in \mathscr{W}(G)$. We have to show $\lambda \in \psi(F)$.

W.l.o.g. we may assume that $(\mu_i * \nu_i)_{i \in I}$ is a tight net.

[If G is locally compact $(\mu_i * \nu_i)_{i \in I}$ itself is a tight net (Lemma 1.1 and Lemma 1.2). If G is metrizable so is $\mathscr{W}(G)$ ([2], page 238). Hence there exists a sequence in $\{\mu_i * \nu_i : i \in I\}$ converging to λ. G being metrizable this sequence is uniformly tight ([2], page 241). Therefore we can substitute $(\mu_i * \nu_i)_{i \in I}$ by this sequence.]

By Lemma 2.3 (μ_i) and (ν_i) are themselves tight nets. Let $\mu, \nu \in \mathscr{W}(G)$ be accumulation points of (μ_i) resp. (ν_i). Since $A \times A$ and therefore F is closed in $\mathscr{W}(G) \times \mathscr{W}(G)$ we have $(\mu, \nu) \in F$, and thus $\lambda = \mu * \nu = \psi(\mu, \nu)$ by Proposition 3.1. $\square$

COROLLARY. *Let G be a locally compact or metrizable group and suppose that G is σ-compact. Then the convolution mapping ψ transforms closed sets in $\mathscr{W}(G) \times \mathscr{W}(G)$ into F_σ-sets in $\mathscr{W}(G)$.*

PROOF. Let $(K_n)_{n \geqq 1}$ be an increasing sequence in $\mathfrak{K}(G)$ such that $G = \bigcup_{n \geqq 1} K_n$. For each $n \in \mathbb{N}$ we define $A_n = \{\mu \in \mathscr{W}(G) : \mu(K_n) \geqq \frac{1}{2}\}$. Then A_n is closed in $\mathscr{W}(G)$, $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\mathscr{W}(G) \times \mathscr{W}(G) = \bigcup_{n \geq 1} A_n \times A_n$. If now F is a closed set in $\mathscr{W}(G) \times \mathscr{W}(G)$ we have $\psi(F) = \psi(\bigcup_{n\geq 1}(F \cap (A_n \times A_n))) = \bigcup_{n \geq 1} \psi(F \cap (A_n \times A_n))$. But by Proposition 3.2 each $\psi(F \cap (A_n \times A_n))$ is closed in $\mathscr{W}(G)$. □

REMARK. This corollary has been proved in [6] for metrizable σ-compact groups.

PROPOSITION 3.3. *Let G be a topological group and $\mu, \nu \in \mathscr{W}(G)$. Then the following assertions are equivalent:*

(i) *There exists a $\lambda \in \mathscr{W}(G)$ such that $\nu = \mu * \lambda$.*
(ii) *$\nu(f) \leqq \sup_{y \in G} \mu * \varepsilon_y(f)$ for every $f \in \mathfrak{C}(G)$.*

PROOF. "(i) ⟹ (ii)" It is $\nu(f) = \mu * \lambda(f) = \int\int f(xy)\mu(dx)\lambda(dy) = \int \mu * \varepsilon_y(f)\lambda(dy) \leqq \sup_{y \in G} \mu * \varepsilon_y(f)$ for every $f \in \mathfrak{C}(G)$.

"(ii) ⟹ (i)" Let $C \equiv \{\mu * \alpha : \alpha = \sum_{j=1}^n a_j \varepsilon_{x_j}, a_j > 0, \sum_{j=1}^n a_j = 1, x_j \in G$ for $j = 1, \cdots, n; n \in \mathbb{N}\}$. The closure D of C in $\mathscr{W}(G)$ is the closed convex hull of $\{\mu * \varepsilon_y : y \in G\}$. According to the bipolar theorem ([3], page 52, Proposition 3) it follows from (ii) that $\nu \in D$. Hence there exists a net $(\mu * \alpha_i)_{i \in I}$ in C such that $\nu = \lim_{i \in I} \mu * \alpha_i$. By Lemma 1.1 $(\mu * \alpha_i)_{i \in I}$ is a quasi-tight net. Therefore $(\alpha_i)_{i \in I}$ is a quasi-tight net (Lemma 2.2). Hence by Lemma 1.1 again there exists a subnet $(\alpha_{i(j)})_{j \in J}$ converging to some $\lambda \in \mathscr{W}(G)$. Finally by Proposition 3.1 we get $\nu = \lim_{j \in J} \mu * \alpha_{i(j)} = \mu * \lim_{j \in J} \alpha_{i(j)} = \mu * \lambda$. □

REMARK. This proposition has been proved in [6] for metrizable and for locally compact groups. But the proofs for these two cases are completely different.

PROPOSITION 3.4. *Let G be a root compact topological group and let G be locally compact or metrizable. Then the set $\mathfrak{U}(G)$ of infinitely divisible measures in $\mathscr{W}(G)$ is closed.*

PROOF. Let $(\mu_i)_{i \in I}$ be a net in $\mathfrak{U}(G)$ converging to some $\mu \in \mathscr{W}(G)$. W.l.o.g. we may assume that (μ_i) is a tight net (see the proof of Proposition 3.2). For every $i \in I$ let $\alpha_i \in \mathscr{W}(G)$ be a root of μ_i of order n ($n \in \mathbb{N}$). By Lemma 2.4 (α_i) is a tight net and has therefore a subnet $(\alpha_{i(j)})_{j \in J}$ converging to some $\alpha \in \mathscr{W}(G)$ (Lemma 1.1). By Proposition 3.1 we conclude $\alpha^n = \mu$. Consequently $\mu \in \mathfrak{U}(G)$. □

REMARK. This proposition has been proved by Hazod and Siebert ([8], Section 3), but the proof given above is considerably more easy and direct.

4. Applications to convolution operators. Let X be a completely regular space, T a topological space and $f \in \mathfrak{C}(T \times X)$. For each $\mu \in \mathscr{W}(X)$ the function F_μ, defined by $F_\mu(t) = \int f(t, x)\mu(dx)$ for all $t \in T$, is continuous and bounded ([4], page 66, corollary). The space $\mathfrak{C}(T)$ equipped with the topology of compact convergence will be denoted by $\mathfrak{C}_c(T)$.

PROPOSITION 4.1. *Let $(\mu_i)_{i \in I}$ be a tight net in $\mathscr{W}(X)$ that converges to some $\mu \in \mathscr{W}(X)$. Then the net $(F_{\mu_i})_{i \in I}$ converges to F_μ in the topology of $\mathfrak{C}_c(T)$.*

PROOF. Obviously it is $\lim_{i\in I} F_{\mu_i}(t) = F_\mu(t)$ for every $t \in T$. Let $\varepsilon > 0$. There exist a $C \in \mathfrak{K}(X)$ and an $i_0 \in I$ such that $\mu_i(C) > 1 - \varepsilon$ for every $i > i_0$ and $\mu(C) > 1 - \varepsilon$. Let $K \in \mathfrak{K}(T)$ and $t \in K$. By compactness of C there exists a neighbourhood U of t (in K) such that $|f(t, x) - f(t', x)| < \varepsilon$ for every $t' \in U$ and for all $x \in C$.

Since K is compact there exist $t_1, \cdots, t_m \in K$ and for each t_j a neighbourhood U_j (in K) such that $K = U_1 \cup \cdots \cup U_m$ and $|f(t, x) - f(t_j, x)| < \varepsilon$ for every $t \in U_j$ and for all $x \in C$ $(j = 1, \cdots, m)$. Furthermore there exists an $i_1 \in I$, $i_1 > i_0$, such that $|F_{\mu_i}(t_j) - F_\mu(t_j)| < \varepsilon$ for $j = 1, \cdots, m$ and for every $i > i_1$. Let $t \in K$ and $i > i_1$. Then there is a $j \in \{1, \cdots, m\}$ such that $t \in U_j$, and we obtain:

$$|F_{\mu_i}(t) - F_\mu(t)| \leqq |F_{\mu_i}(t) - F_{\mu_i}(t_j)| + |F_{\mu_i}(t_j) - F_\mu(t_j)| + |F_\mu(t_j) - F_\mu(t)| .$$

But for $\nu = \mu_i$ or $\nu = \mu$ we have:

$$\begin{aligned} |F_\nu(t) - F_\nu(t_j)| &\leqq \int |f(t, x) - f(t_j, x)|\nu(dx) \\ &= \int_C |f(t, x) - f(t_j, x)|\nu(dx) + \int_{X\setminus C} |f(t, x) - f(t_j, x)|\nu(dx) \\ &\leqq \varepsilon + 2\|f\|\varepsilon . \end{aligned}$$

From these estimations the assertion follows easily. □

We give an application of this result. Let G be a topological group. For each $\mu \in \mathscr{W}(G)$ we define

$$T_\mu f(x) = \int f(xy)\mu(dy) \qquad (\text{all } f \in \mathfrak{C}(G), \text{ all } x \in G) .$$

Then we have $T_\mu f \in \mathfrak{C}(G)$ ([4], page 66, corollary), and T_μ is a bounded linear operator on $\mathfrak{C}(G)$, $\|T_\mu\| = \|\mu\| = 1$. T_μ is called the *convolution operator* to μ. For all $\mu, \nu \in \mathscr{W}(G)$ it is $T_{\mu*\nu} = T_\mu T_\nu$, and $\mathfrak{C}_u(G)$ as well as $\mathfrak{C}^0(G)$ are invariant subspaces of $\mathfrak{C}(G)$ with respect to T_μ. We have the following result:

PROPOSITION 4.2. *Let G be a locally compact group and $(\mu_i)_{i\in I}$ a net in $\mathscr{W}(G)$. Then the following assertions are equivalent*:

(i) *The net $(\mu_i)_{i\in I}$ converges weakly to a measure $\mu \in \mathscr{W}(G)$.*

(ii) *The net $(T_{\mu_i})_{i\in I}$ converges in the strong operator topology on $\mathfrak{C}^0(G)$ to an operator T on $\mathfrak{C}^0(G)$ such that $\|T\| = 1$.*

PROOF. "(i) ⟹ (ii)" Let T be the restriction of T_μ to $\mathfrak{C}^0(G)$. If $G' = G \cup \{\omega\}$ is the one point compactification of G, we define $\omega x = \omega$ for every $x \in G$. Then the mapping $(x, y) \to xy$ from $G' \times G$ into G' is continuous. We extend each $g \in \mathfrak{C}^0(G)$ to a continuous function g' on G' by $g'(\omega) \equiv 0$. Consequently the mapping $(t, x) \to f(t, x) \equiv g'(tx)$ from $G' \times G$ into $\mathbb{R}$ is continuous and bounded.

Since $(\mu_i)_{i\in I}$ is a tight net (Lemma 1.1 and Lemma 1.2), by Proposition 4.1 the net $(F_{\mu_i})_{i\in I}$ converges to F_μ in $\mathfrak{C}(G')$ (equipped with the topology of uniform convergence). From $F_{\mu_i}(t) = (T_{\mu_i} g)(t)$ for all $t \in G$ and $F_{\mu_i}(\omega) = \int f(\omega, x)\mu_i(dx) = \int g'(\omega)\mu_i(dx) = 0$ the assertion follows.

"(ii) ⟹ (i)" T is a positive bounded linear operator on $\mathfrak{C}^0(G)$ such that $\|T\| = 1$. Since every convolution operator commutates with the left translation on $\mathfrak{C}^0(G)$

the same is true for T by assumption. Hence there is a $\mu \in \mathscr{W}(G)$ such that T is the restriction of T_μ to $\mathfrak{C}^0(G)$. It is $\lim_{i \in I} \mu_i(f) = \lim_{i \in I} T_{\mu_i} f(e) = T_\mu f(e) = \mu(f)$ for every $f \in \mathfrak{C}^0(G)$. Now the assertion follows from [4], page 61, Proposition 9. □

5. Applications to convolution semigroups. In [8], Section 1, we have announced two results on convolution semigroups which can now be derived from our results above. For this purpose let D be a dense subsemigroup of the additive semigroup $\mathbb{R}_+^*$ (= positive real numbers) with the property "$r, s \in D$ and $r < s \Rightarrow s - r \in D$." By D_0 we denote the extended semigroup $D \cup \{0\}$.

Let G be a topological group and f a homomorphism from D into $\mathscr{W}(G)$. Let $\mu_r \equiv f(r)$ for all $r \in D$. Then $S = (\mu_r)_{r \in D}$ is called a convolution semigroup in $\mathscr{W}(G)$. S is said to be 0-*continuous* if the limit $\mu_0 = \lim_{r \downarrow 0, r \in D} \mu_r$ in $\mathscr{W}(G)$ exists. In this case μ_0 will be an idempotent. S is said to be continuous if the mapping f is continuous (here D is equipped with the relative topology with respect to $\mathbb{R}$).

At first we examine the connection between these two concepts of continuity.

PROPOSITION 5.1. *Let $S = (\mu_r)_{r \in D_0}$ be a 0-continuous convolution semigroup and suppose that μ_0 is the identity in S (that means $\mu_0 * \mu_r * \mu_0 = \mu_r$ for all $r \in D$). Then the semigroup S is continuous.*

PROOF. For each $r \in D_0$ we denote by S_r the restriction of the convolution operator T_{μ_r} to $\mathfrak{C}_u(G)$. In view of $\mu_r(f) = S_r f(e)$ it suffices to show $\lim_{r \to r_0} ||S_r f - S_{r_0} f|| = 0$ for all $f \in \mathfrak{C}_u(G)$ and $r, r_0 \in D_0$ ([10], page 40, Theorem 8.1). According to $||S_r|| = ||\mu_r|| = 1$ and $S_r S_0 = S_0 S_r = S_r$ (all $r \in D_0$) we have for all $r \in D_0$, $h \in D$ and $f \in \mathfrak{C}_u(G)$:

$$||S_{r+h} f - S_r f|| = ||S_r S_h f - S_r S_0 f|| \leqq ||S_h f - S_0 f||$$

and

$$||S_{r-h} f - S_r f|| = ||S_{r-h} S_0 f - S_{r-h} S_h f|| \leqq ||S_0 f - S_h f|| .$$

Hence it suffices to show $\lim_{r \downarrow 0} ||S_0 f - S_r f|| = 0$.

μ_0 being an idempotent in $\mathscr{W}(G)$ there exists a compact subgroup H in G such that μ_0 is the normed Haar measure on H ([11], page 228, corollary). Obviously S_0 is a projector on $\mathfrak{C}_u(G)$.

In the first part let be given $f \in \mathfrak{C}_u(G)$ such that $S_0 f = f$ and $\varepsilon > 0$. There exists a neighbourhood U of the identity in G such that $|f(x) - f(xy)| \leqq \varepsilon$ for all $x \in G$ and for all $y \in U$. In view of $f(xz) = f(x)$ for all $z \in H$, w.l.o.g. we may assume $U = UH$. Then there exists a $r_0 \in D$ such that $\mu_r(\complement U) \leqq \varepsilon$ for all $r \in D$, $r \leqq r_0$ (S is 0-continuous). For all $x \in G$ and $r \in D$, $r \leqq r_0$ the following estimation holds:

$$\begin{aligned} |f(x) - S_r f(x)| &\leqq \int |f(x) - f(xy)| \mu_r(dy) \\ &= \int_U |f(x) - f(xy)| \mu_r(dy) + \int_{\complement U} |f(x) - f(xy)| \mu_r(dy) \\ &\leqq \varepsilon \mu_r(U) + 2||f|| \mu_r(\complement U) \leqq \varepsilon(1 + 2||f||) . \end{aligned}$$

This shows $\lim_{r \downarrow 0} ||f - S_r f|| = 0$.

Now let $f \in \mathfrak{C}_u(G)$ be arbitrary. Then it is $S_0 S_0 f = S_0 f$ and $S_r S_0 f = S_r f$, and so we have in this case $\lim_{r \downarrow 0} \|S_0 f - S_r f\| = 0$. □

PROPOSITION 5.2. *Let G be a topological group and $S = (\mu_r)_{r \in D}$ a convolution semigroup in $\mathscr{W}(G)$. Then S is 0-continuous if and only if S is continuous, and in this case $\mu_0 = \lim_{r \downarrow 0, r \in D} \mu_r$ is the identity in S.*

PROOF. 1. Let S be 0-continuous and $\mu_0 = \lim_{r \downarrow 0, r \in D} \mu_r$. If $(r_i)_{i \in I}$ is a net in D such that $\lim_{i \in I} r_i = 0$ we have $\lim_{i \in I} \mu_{r_i} = \mu_0$. Therefore $(\mu_{r_i})_{i \in I}$ is a quasi-tight net (Lemma 1.1). Let $r \in D$ and (w.l.o.g.) $r_i < r$ for all $i \in I$. From $\mu_r = \mu_{r_i} * \mu_{r-r_i}$ for all $i \in I$ it follows that $(\mu_{r-r_i})_{i \in I}$ is a quasi-tight net (Lemma 2.2). Hence there exists a subnet $(\mu_{r-r_j})_{j \in J}$ converging to some $\mu \in \mathscr{W}(G)$ (Lemma 1.1). From $\mu_r = \mu_{r_j} * \mu_{r-r_j}$ and $\lim_{j \in J} \mu_{r_j} = \mu_0$ we conclude $\mu_r = \mu_0 * \mu = \mu * \mu_0$. This shows that $\mu_0 * \mu_r * \mu_0 = \mu_r$ (all $r \in D$). Therefore μ_0 is the identity in S. By Proposition 1 the semigroup S is continuous.

2. Now we assume that S is continuous. If $(r_i)_{i \in I}$ is a net in D such that $\lim_{i \in I} r_i = 0$ and if $r \in D$ we have $\mu_{r+r_i} = \mu_r * \mu_{r_i}$ and $\lim_{i \in I} \mu_{r+r_i} = \mu_r$. Therefore $(\mu_{r_i})_{i \in I}$ is a quasi-tight net and possesses a subnet converging to some $\mu \in \mathscr{W}(G)$ (Lemma 2.2 and Lemma 1.1). W.l.o.g. we may assume $\lim_{i \in I} \mu_{r_i} = \mu$. If $(s_j)_{j \in J}$ is another net in D converging to 0 such that $\lim_{j \in J} \mu_{s_j} = \mu' \in \mathscr{W}(G)$ exists we obtain by the continuity of convolution and the continuity of S:

$$\begin{aligned}\mu = \lim_{i \in I} \mu_{r_i} &= \lim_{i \in I} (\lim_{j \in J} \mu_{r_i + s_j}) \\ &= \lim_{i \in I} (\mu_{r_i} * \lim_{j \in J} \mu_{s_j}) = \lim_{i \in I} \mu_{r_i} * \mu' = \mu * \mu' .\end{aligned}$$

Similarly it follows that $\mu' = \mu * \mu'$ and therefore $\mu' = \mu$. Hence $\mu_0 = \lim_{r \downarrow 0, r \in D} \mu_r$ exists. Consequently S is 0-continuous. □

Finally we prove a result on the extension of continuous convolution semigroups.

PROPOSITION 5.3. *Let G be a topological group and $S = (\mu_r)_{r \in D_0}$ a continuous convolution semigroup in $\mathscr{W}(G)$. In each of the following situations there exists a (unique) continuous convolution semigroup $(\nu_t)_{t \in \mathbf{R}_+}$ in $\mathscr{W}(G)$ such that $\nu_r = \mu_r$ for every $r \in D_0$.*

(i) *There exists a $r_0 \in D$ such that $\{\mu_r : r \in D \cap \,]0, r_0]\}$ is relatively compact in $\mathscr{W}(G)$.*

(ii) *There exist a $r_0 \in D$, an $\varepsilon \in \,]0, 1[$ and a $K \in \mathfrak{K}(G)$ such that $\mu_r(K) \geqq \varepsilon$ for every $r \in D \cap [0, r_0]$.*

(iii) *G is locally compact.*

PROOF. (i) W.l.o.g. $r_0 = 1$. Let $I \equiv [0, 1]$. We choose $t \in I$. Let $(r_\alpha)_{\alpha \in A}$ and $(s_\alpha)_{\alpha \in A}$ be two nets in $B \equiv D_0 \cap I$ converging to t, such that $\mu = \lim \mu_{r_\alpha}$ and $\nu = \lim \mu_{s_\alpha}$ exist in $\mathscr{W}(G)$. Such nets exist by assumption. W.l.o.g. $r_\alpha \leqq s_\alpha$ for every $\alpha \in A$. Since S is a continuous convolution semigroup with identity μ_0 (Proposition 5.2) and since $\lim (s_\alpha - r_\alpha) = 0$ it follows that $\nu = \lim \mu_{s_\alpha} = \lim \mu_{s_\alpha - r_\alpha} * \mu_{r_\alpha} = \mu$. Therefore $\nu_t \equiv \lim_{r \to t, r \in B} \mu_r$ exists. Obviously $\nu_s * \nu_t = \nu_{s+t}$

for $s, t \in I$ such that $s + t \in I$. Thus $\{\nu_t : t \in I\}$ can be extended to a convolution semigroup $(\nu_t)_{t \in \mathbf{R}_+}$ in $\mathscr{W}(G)$ such that $\nu_r = \mu_r$ for every $r \in D_0$. But the semigroup $(\nu_t)_{t \in \mathbf{R}_+}$ is also continuous. [Given $f \in \mathfrak{C}(G)$ and $\varepsilon > 0$ there exists a $r_1 \in D$ such that $|\mu_0(f) - \mu_r(f)| \leqq \varepsilon$ for every $r \in D_0 \cap [0, r_1]$ (since S is continuous). Now by definition of ν_t we immediately get $|\nu_0(f) - \nu_t(f)| \leqq \varepsilon$ for every $t \in [0, r_1]$.] The uniqueness of $(\nu_t)_{t \in \mathbf{R}_+}$ is clear.

(ii) W.l.o.g. $r_0 = 1$. Since $\mu_r * \mu_{1-r} = \mu_1$ for every $r \in [0, 1] \cap D_0$ we deduce from the proof of Lemma 2.3 that $\{\mu_i : i \in [0, 1] \cap D_0\}$ is uniformly tight and therefore relatively compact (according to Prohorov's theorem). Now apply (i).

(iii) Given $\varepsilon \in]0, 1[$ there exists a $f \in \mathfrak{C}(G)$, $0 \leqq f \leqq 1$, having a compact support K such that $\mu_0(f) > \varepsilon$. Since the semigroup S is continuous it is $\lim_{r \downarrow 0, r \in D} \mu_r(f) = \mu_0(f)$. Therefore there exists a $r_0 \in D$ such that $\mu_r(K) \geqq \mu_r(f) > \varepsilon$ for every $r \in D_0 \cap [0, r_0]$. Now apply (ii). □

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