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Convergence and Divergence of Fejér Means of Fourier Series on One and Two-Dimensional Walsh and Vilenkin Groups

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Abstract: It is a highly celebrated issue in dyadic harmonic analysis the pointwise convergence of the Fejér (or $(C,1)$) means of functions on the Walsh and Vilenkin groups both in the point of view of one and two dimensional cases. We give a résumé of the very recent developments concerning this matter, propose unsolved problems and throw a glance at the investigation of Vilenkin-like systems too.

Keywords: Walsh, Vilenkin group, Vilenkin series, Fejér means, a.e. convergence, two-dimensional systems

1 Introduction

FIRST, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [1, 2]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0,1,\dots\}, \mathbb{P} := \mathbb{N} \setminus \{0\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0,1,...,m_k-1\}$, with the group operation addition $\mod m_k$. Since the group is discrete, then every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ $(j \in \{0,1,...,m_k-1\})$. Let

$$G_m := \underset{k=0}{\overset{\infty}{\times}} Z_{m_k}.$$

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Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}\,(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. If $m_j = 2$ for each j, then we call the Vilenkin group G_m as the Walsh group and denote by G_2. A Vilenkin group is metrizable in the following way:

$$d(x,y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x,y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m.

Furthermore, let $L^p(G_m)\,(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathscr{A}_n$ the σ-algebra generated by the sets $I_n(x)\,(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathscr{A}_n\,(n \in \mathbb{N})$ ($f \in L^1$).

Let a be a nonnegative real. We say that the function $f \in L^1(G_m)$ belongs to the logarithm space $L(\log^+ L)^a(G_m)$ if the integral

$$\|f\|_{L(\log^+ L)^a} := \int_{G_m} |f(x)| \left(\log^+(|f(x)|)\right)^a d\mu(x)$$

is finite.

Let X and Y be either $L(\log^+ L)^a(G_m)$ or $L^p(G_m)$ for some $1 \le p \le \infty$, and $a \ge 0$ with norms $\|.\|_X$ and $\|.\|_Y$. We say that operator T is of type (X,Y) if there exist an absolute constant $C > 0$ for which $\|Tf\|_Y \le C\|f\|_X$ for all $f \in X$. T is said to be of weak type (L^1,L^1) if there exist an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \le C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$.

Let $M_0 := 1, M_{n+1} := m_n M_n\,(n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0,1,...,m_i - 1\},\ i \in \mathbb{N}),$$

where only a finite number of n_i 's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp(2\pi\imath\frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & , \text{ if } x_n \neq 0, \\ m_n & , \text{ if } x_n = 0 \end{cases}$ $(x \in G_m,\, n \in \mathbb{N})$. The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\operatorname{mod} m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then , $\psi_{k\oplus n} = \psi_k \psi_n,\ \psi_n(x+y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1$ $(k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y,x) = D_n(y-x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f,$$

$$K_n(y,x) = K_n(y-x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(y-x),$$

$$(n \in \mathbb{P}, y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f,\ S_0 f = D_0 = K_0 = 0,\ f \in L^1(G_m)).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x),$$

$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y-x) d\mu(x) \quad (n \in \mathbb{P}, y \in G_m,\ f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0) \end{cases},$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

It is known that the operator which maps a function f to the maximal function $f^* := \sup |S_{M_n} f|$ is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \leq \infty$ (see e.g. [3]). Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstood.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\hat{f}(n_1, n_2) := \int_{G_m \times G_{\tilde{m}}} f(x^1, x^2) \bar{\psi}_{n_1}(x^1) \bar{\psi}_{n_2}(x^2) d\mu(x^1, x^2),$$

$$S_{n_1,n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(y^1) \psi_{k_2}(y^2),$$

$$D_{n_1,n_2}(y, x) = D_{n_1}(y^1 - x^1) D_{n_2}(y^2 - x^2)$$

$$:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \psi_{k_1}(y^1) \psi_{k_2}(y^2) \bar{\psi}_{k_1}(x^1) \bar{\psi}_{k_2}(x^2),$$

$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} S_{k_1,k_2} f,$$

$$K_{n_1,n_2}(y, x) = K_{n_1,n_2}(y - x) := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} D_{k_1,k_2}(y - x),$$

$$(y = (y^1, y^2), x = (x^1, x^2) \in G_m \times G_{\tilde{m}}).$$

It is also well-known that

$$\sigma_{n_1,n_2} f(y) = \int_{G_m \times G_{\tilde{m}}} f(x) K_{n_1,n_2}(y - x) d\mu(x),$$

$$S_{M_{n_1}, \tilde{M}_{n_2}} f(x) = M_{n_1} \tilde{M}_{n_2} \int_{I_{n_1}(x^1) \times I_{n_2}(x^2)} f = (E^1_{n_1} \otimes E^2_{n_2}) f(x).$$

The one and two-dimensional (C,α) means are defined as follows. Denote by K_n^α the kernel of the summability method (C,α), and call it the (C,α) kernel, or the Cesàro kernel for $\alpha \in \mathbb{R}$:

$$K_{n+1}^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} D_\nu, \quad A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k).$$

It is well-known [4, Ch. 3] that $A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}$, $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$, $A_n^\alpha \sim n^\alpha$. The (C,α) Cesàro means of the integrable function f is

$$\sigma_{n+1}^\alpha f(y) := \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f = \int_{G_m} f(x) K_n^\alpha (y-x)\, d\mu(x).$$

The two-dimensional version is

$$\sigma_{n_1+1,n_2+1}^\alpha f(y) := \frac{1}{A_{n_1}^\alpha}\frac{1}{A_{n_2}^\alpha}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} A_{n_1-k_1}^{\alpha-1} A_{n_2-k_2}^{\alpha-1} S_{k_1,k_2} f.$$

2 Some Known Results and Problems

One of the most celebrated issue in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on one and two-dimensional unbounded Vilenkin groups.

Fine [5] proved every Walsh-Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C,α) summable for $\alpha > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [6]. Schipp [7] gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$. Define the maximal operator of the Fejér means of the integrable function f as $\sigma^* f := \sup_{n\in\mathbb{P}} |\sigma_n f|$. Schipp proved [7] that σ^* is of weak type (L^1,L^1). That σ^* is bounded from H^1 to L^1 was discovered by Fujii [8]. The Hardy space H^1 has several definitons. We give the most common one as follows. We say that a function $f \in L^1$ belongs to the Hardy space H^1 if its maximal function $f^* := \sup |S_{M_n} f|$ belongs to the Lebesgue space L^1.

The theorem of Schipp are generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [9], and later to bounded Vilenkin systems by Pál and Simon [10].

Now, what about the Vilenkin groups with unbounded generating sequences? The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m

is an unbounded one [11]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [11] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [11] that if $\frac{\log m_n}{M_n} \to \infty$, then there exists a function f continuous on G_m whose Fourier series are not $(C,1)$ summable on a set $S \subset G_m$ which is non-denumerable. On the other hand, Nurpeisov gave [12] a necessary and sufficient condition of the uniform convergence of the Fejér means $\sigma_{M_n} f$ of continuous functions on unbounded Vilenkin groups. Namely, define the uniform modulus of continuity as

$$\omega_n(f) := \sup_{h \in I_n(0), x \in G_m} |f(x+h) - f(x)|.$$

Nurpeisov proved [12]: A necessary and sufficient condition that the means $\sigma_{M_n} f$ of the Fourier series of the continuous function f converge uniformly to f on an unbounded Vilenkin group for all such an f is that

$$\omega_{n-1}(f) \log(m_n) = o(1).$$

Since the uniform modulus of continuity can be any nonincreasing real sequence which converges to zero (for the proof see [13, 14]), then as a consequence of this it is possible to give a sequence m increasing enough fast, and a function even in the Lipschitz class $\mathrm{Lip}(1)$, such that the M_nth Fejér means do not converge to the function uniformly.

So, it seems that it is impossible to give a (Hölder) function class such that the uniform convergence of the Fejér means would hold for all functions in this class if there is no condition on sequence m at all.

On the other hand, mean convergence of the full partial sums for $L^p, p > 1$, is known for the unbounded case. For the proof see [15]. This trivially implies the norm convergence $\sigma_n f \to f$ for all $f \in L^p$, where $1 < p < \infty$.

Concerning the a.e. convergence we can say a bit more. Namely, in 1999 the author [16] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. We could say that it is a trivial consequence of the a.e. convergence of the partial sums of the Fourier series of functions in $f \in L^p(G_m)$, where $p > 1$. The ,,only problem" with this that the a.e. convergence of the partial sums is the greatest open problem in the Vilenkin-Fourier analysis in the unbounded case. This is unknown even for the Lebesgue space $L^2(G_m)$.

In 2001 Simon proved [17] the following theorem with respect to the Fejér

means of L^1 functions. A sequence m is said to be strong quasi-bounded if

$$\frac{1}{M_{n+1}}\sum_{j=0}^{n-1} M_{j+1} < C\log m_n.$$

Then every bounded m is quasi-bounded, and there are also some unbounded ones. Let m be strong quasi-bounded. Then for all $f \in L^1(G_m)$

$$\sigma_{M_n} f(x) - f(x) = o(\max(\log m_0, \ldots, \log m_{n-1})).$$

Later, in 2003, the author of this paper improved [18] this result, and gave a partial answer for L^1 case. He discussed this partial sequence of the sequence of the Fejér means. Namely, if $f \in L^1(G_m)$, then he proved (see [18]) that $\sigma_{M_n} f \to f$ almost everywhere, where m is any sequence. This is also interesting in the point of view that if m is any unbounded sequence then there exists an integrable function f such that $\sigma_{M_n} f$ does not converge to f in the L^1 norm [11].

If there exists a constant C and $L \in \mathbb{P}$ such that for all $i, j \in \mathbb{P}$ we have

$$\frac{\min(m_i, m_{i+j})}{(m_{i+1} \cdot \cdots \cdot m_{i+j-1})^L} \leq C,$$

(the empty product is defined to be 1, and the constant C may depend on the sequence m - of course), then we call the Vilenkin group G_m a rarely unbounded Vilenkin group. Every bounded Vilenkin group is a rarely unbounded Vilenkin group. Unfortunately, not all unbounded ones are rarely unbounded, since for instance the rarely unboundedness implies the inequality $\min(m_i, m_{i+1}) \leq C$. So, e.g. if (m_n) tends to plus infinity, then G_m is not rarely unbounded. On the other hand, there are many unbounded Vilenkin groups, which are rarely unbounded ones.

In 2007 we proved [19] the following two theorems Let G_m be a rarely unbounded Vilenkin group. Then the operator σ^* is of weak type (L^1, L^1). A straightforward consequence of this theorem is: Let G_m be a rarely unbounded Vilenkin group, and $f \in L^1(G_m)$. Then we have the a.e. relation $\sigma_n f \to f$.

In my opinion, it is highly likely that the methods of the papers [16, 18, 19] can be applied and improved in order to prove the a.e. relation $\sigma_n f \to f$ for all $f \in L\log^+ L$ and m - at least. Anyway, it is not an easy task...

Besides, I think that the original Fejér-Lebesgue theorem holds on all (bounded or not bounded - not only rarely unbounded) Vilenkin groups. However, to prove it seems to be much more difficult.

What can be said in the case of two-dimensional functions? This is "another story". For double trigonometric Fourier series Marcinkiewicz and Zygmund [20] proved that $\sigma_{m,n} f \to f$ a.e. as $m, n \to \infty$ provided the integral lattice points (m, n)

remain in some positive cone, that is provided $\beta^{-1} \leq m/n \leq \beta$ for some fixed parameter $\beta \geq 1$. It is known that the classical Fejér means are dominated by decreasing functions whose integrals are bounded but this fails to hold for the one-dimensional Walsh-Fejér kernels. This growth difference is exacerbated in higher dimensions so that the trigonometric techniques are not powerful enough for the Walsh case.

In 1992 Móricz, Schipp and Wade [21] proved that $\sigma_{2^{n_1},2^{n_2}} f \to f$ a.e. for each $f \in L^1(G_2^2)$, when $n_1, n_2 \to \infty$, $|n_1 - n_2| \leq \alpha$ for some fixed α. Later, Gát and Weisz proved (independently, in the same year) this for the whole sequence, that is, the theorem of Marcinkiewicz and Zygmund with respect to the Walsh-Paley system (see [22] and [23]). For the bounded Vilenkin case see the paper of Weisz [24], and the paper of Blahota and the author [25]. In the paper [25] we generalize this theorem with respect to two-dimensional bounded Vilenkin-like systems.

If we do not provide a "cone restriction" for the indices in $\sigma_{n,k} f$ that is, we discuss the convergence of this two-dimensional Fejér means in the Pringsheim sense, then the situation changes. In 1992 Móricz, Schipp and Wade [21] proved with respect to the two-parameter Walsh-Paley system that $\sigma_{n,k} f \to f$ a.e. for each $f \in L \log^+ L$, when $\min\{n,k\} \to \infty$. Later, in 2002 Weisz generalized [26] this with respect to two-dimensional bounded Vilenkin systems.

In 2000 Gát proved [27] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved the existence of a function $f \in L^1$ such that $f \in L \log^+ L \delta(L)$, and $\sigma_{n,k} f$ does not converge to f a.e. as $\min\{n,k\} \to \infty$. This result with respect to the bounded two-dimensional Vilenkin case is also due to the author [28].

It is an interesting question that is it possible to weaken somehow the ,,cone restriction" in a way that a.e. convergence remains for each function in L^1. Maybe for some ,,interim space" if not for space L^1. The answer is negative both in the point of view of space and in the point of view of restriction. Namely, in 2001 Gát proved [29] the theorem below:

Let $\delta : [0,+\infty) \to [0,+\infty)$ measurable, $\delta(+\infty) = 0$ and let $w : \mathbb{N} \to [1,+\infty)$ be an arbitrary increasing function such that

$$\sup_{x\in\mathbb{N}} w(x) = +\infty.$$

Moreover, $\vee n := \max\{n_1, n_2\}$, $\wedge n := \min\{n_1, n_2\}$. The, there exists a function f in the space $L \log^+ L \delta(L)$ such that

$$\sigma_{n_1,n_2} f \not\to f$$

a.e. as $\wedge n \to \infty$ such that the restriction condition $\frac{\vee n}{\wedge n} \leq w(\wedge n)$ is also fulfilled. That is there is no ,,interim" space. Either we have space $L \log^+ L$ and ,,no restriction at

all", or the „cone restriction" and then the maximal convergence space is L^1. As a consequence of this we have that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for each $f \in L^1(G_2^2)$ as $\min\{n_1,n_2\} \to \infty$, provided that

$$\frac{\vee n}{\wedge n} \leq w(\wedge n)$$

if and only if

$$\sup w(x) < \infty.$$

What can be said in the two-dimensional case with respect to unbounded Vilenkin systems? In 1997 Wade proved [30] the following. Let $\beta_{k,j} := \max\left\{m_0,\dots,m_{k-1},\tilde{m}_0,\dots,\tilde{m}_{j-1}\right\}$. The sequence m is called δ-quasi bounded, $0 \leq \delta < 1$, if the sums

$$\sum_{j=0}^{n-1} m_j/(m_{j+1}\dots m_n)^{\delta}$$

are (uniformly) bounded. Let the generating sequences $m,\tilde{m}$ be δ-quasi bounded. Then for all $f \in L^1(G_m \times G_{\tilde{m}})$ we have

$$\sigma_{M_n,\tilde{M}_k} f(x) - f(x) = o(\beta_{n,k}\beta^{2r}_{n+r,k+r}),$$

as $n,k \to \infty$, provided that $|n-k| < \alpha$, where $\alpha, r \in \mathbb{N}$ are some constants for almost every $x \in G_m \times G_{\tilde{m}}$.

On the other hand, there was nothing concerning the pointwise convergence before the following manuscript of the author.
In [31] we proved the following theorem. Let $f \in (L\log^+ L)(G_m \times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1},\tilde{M}_{n_2}} f \to f$ almost everywhere, where $\min\{n_1,n_2\} \to \infty$ provided that the distance of the indices is bounded, that is, $|n_1 - n_2| < \alpha$ for some fixed constant $\alpha > 0$. Here it is necessary to emphasize that in this paper $m,\tilde{m}$ can be any sequences.

Another question. What is the situation with the (C,α) summation of 2-dimensional Walsh-Fourier series? What is this?

$$\sigma^{\alpha}_{n_1+1,n_2+1} f = \frac{1}{A^{\alpha}_{n_1} A^{\alpha}_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} A^{\alpha-1}_{n_1-k_1} A^{\alpha-1}_{n_2-k_2} S_{k_1,k_2} f.$$

In 1999 Weisz proved [32], that

$$\sigma^{\alpha}_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1,n_2\} \to \infty$ for each $f \in L\log^+ L(G_2^2)$ and $\alpha > 0$.

The question is that is it possible to give a ,,larger" convergence space for the (C,α) summability method ($\alpha > 0$)? Is there such an α? If $\alpha \leq 1$, then not. Because for the $(C,1)$ method one can not give such a ,,larger" space.

Problem.

- What is the situation with the (C,α) methods, for $\alpha > 1$? We mean the Walsh and bounded Vilenkin case.

- The is no divergence result with respect to two-dimensional unbounded (any of the two generating sequence is unbounded) Vilenkin groups at all. May be this is surprising, since it is very usual that to construct divergence examples on unbounded Vilenkin groups is easier. But, I think not in this issue. The construction of our example of divergence in [28] does not work in this situation.

What can be said in the case of the Walsh-Kaczmarz system? What is this Walsh-Kaczmarz system? This is nothing else, but a rearrangement of the Walsh-Paley system. Introduce it as follows.

If $n > 0$, then let $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$. The n-th Walsh-Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

as if $n > 0$, $\kappa_0(x) := 1, x \in G_2$. Then the elements of the a Walsh-Kaczmarz system and the Walsh-Paley system is a dyadic blockwise rearrangements of each other. This means as follows:

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}.$$

In 1998 Gát proved [33] the Fejér-Lebesgue theorem for the Walsh-Kaczmarz system. That is, $\sigma_n f \to f$ a.e. for each $f \in L^1(G_2)$. In 2004 Simon [34] generalized the result of Gát above for (C,α) summation methods. The Fejér-Lebesgue theorem with respect to the character system of the p-series fields (Vilenkin groups with a constant m) in the Kaczmarz rearrangement is verified by the author and Nagy [35].

What is the situation with the Cesàro summation of 2-dimensional Walsh-Kaczmarz series? In 2001 Simon proved [36], that $\sigma_{n_1,n_2} f \to f$ a.e. as if $\min\{n_1,n_2\} \to \infty$ (in the Pringsheim sense) for every $f \in L\log^+ L(G_2^2)$. He also proved the restricted ,,cone" convergence for functions belonging to $L^1(G_2^2)$. With respect to this I propose the following unsolved problems.

Problem.

- What is the maximal convergence space of the two-dimensional $(C,1)$ summability method taken in the Pringsheim sense? Is it $L\log^+ L(G_2^2)$ again, as in the case of the two-dimensional Walsh-Paley system?

- Does not exist an ,,interim" space like in the Walsh-Paley case?
- What can be said in the case of (C,α) summation?
- The whole 2-dimensional story with respect to the character system of the p-series fields in the Kaczmarz rearrangement is open.

It seems also to be interesting to discuss the almost everywhere convergence of Marcinkiewicz means $\frac{1}{n}\sum_{j=0}^{n-1} S_{j,j} f$ of integrable functions on two-dimensional unbounded Vilenkin groups. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. It seems in a certain point of view that the one dimensional Fejér means. For the trigonometric, Walsh-Paley, and bounded Vilenkin case see the papers of Zhizhiasvili, Weisz and Gát [37, 38, 39]. With respect to the Walsh case see also the papers of Goginava [40, 41]. The a.e. convergence of Marczinkiewicz means of two-dimensional integrable functions with respect to the two-dimensional Walsh-Kaczmarz system is due to Nagy [42]. Some of the results summarized in this paper (including the proofs) can also be found in [43]. For the time being there is no result known with respect to this issue on unbounded case. I think it possible to discuss the a.e. convergence

$$\frac{1}{M_n}\sum_{j=0}^{M_n-1} S_{j,j} f$$

on unbounded Vilenkin groups for integrable functions f. I think it can be done with the methods written in the paper [18]. At this point we wrote about the Vilenkin systems and one of its special cases, the Walsh-Paley system. Now, we are going to have a look at a class of generalization of the Vilenkin systems.

3 Vilenkin-like Systems

Denote G_{m_k} a set of cardinality m_k. Suppose that each (coordinate) set has the discrete topology and measure μ_k which maps every singleton of G_{m_k} to $\frac{1}{m_k}$ ($\mu_k(G_{m_k}) = 1$), $k \in \mathbb{N}$. Let G_m be the compact set formed by the complete direct product of G_{m_k} with the product of the topologies and measures (μ). Thus each $x \in G_m$ is a sequence $x := (x_0, x_1, ...)$, where $x_k \in G_{m_k}$, $k \in \mathbb{N}$. G_m is called a Vilenkin space. That is, in this situation we do not have any algebraical operation of the set G_m. This is the main difference between a Vilenkin group and a Vilenkin space.

The complex valued functions which we call the generalized Rademacher functions $r_k^n : G_m \to \mathbb{C}$ have these properties:

i. r_k^n is $\mathscr{A}_{k+1}$ measurable (i.e. $r_k^n(x)$ depends only on $x_0,...,x_k$ $(x \in G_m)$), $r_k^0 = 1$ for all $k,n \in \mathbb{N}$.

ii. If M_k is a divisor of n and l and if $n^{(k+1)} = l^{(k+1)}$ $(k,l,n \in \mathbb{N})$, then

$$E_k(r_k^n \overline{r_k^l}) = \begin{cases} 1 \text{ if } n_k = l_k, \\ 0 \text{ if } n_k \neq l_k \end{cases}$$

($\overline{z}$ is the complex conjugate of z).

iii. If M_{k+1} is a divisor of n (that is, $n = n_{k+1}M_{k+1} + n_{k+2}M_{k+2} + ... + n_{|n|}M_{|n|}$), then

$$\sum_{j=0}^{m_k-1} |r_k^{jM_k+n}(x)|^2 = m_k$$

for all $x \in G_m$.

iv. There exists a $\delta > 1$ for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$.

Define the Vilenkin-like system $\psi = (\psi_n : n \in \mathbb{N})$ as follows.

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_{(k)}}, \quad n \in \mathbb{N}.$$

We would like to mention some examples for Vilenkin-like systems.

Example A, the Vilenkin and the Walsh system

Let $G_{m_k} := Z_{m_k}$ be the m_k-th $(2 \leq m_k \in \mathbb{N})$ discrete cyclic group $(k \in \mathbb{N})$. In this case let $r_k^n(x) := (\exp(2\pi\imath x_k/m_k))^{n_k}$, where $\imath := \sqrt{-1}, x \in G_m$.

Example B, the group of m-adic integers

Let $G_{m_k} := \{0,1,...,m_k-1\}$ for all $k \in \mathbb{N}$. Define on G_m the following (commutative) addition: Let $x,y \in G_m$. Then $x+y = z \in G_m$ is defined in a recursive way. $x_0 + y_0 = t_0 m_0 + z_0$, where (of course) $z_0 \in \{0,1,...,m_0-1\}$ and $t_0 \in \mathbb{N}$. Suppose that $z_0,...,z_k$ and $t_0,...,t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1}m_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0,1,...,m_{k+1}-1\}$ and $t_{k+1} \in \mathbb{N}$. Then G_m is called the group of m-adic integers (if $m_k = 2$ for all $k \in \mathbb{N}$, then 2-adic integers). In this case let

$$r_k^n(x) := \left(\exp\left(2\pi\imath\left(\frac{x_k}{m_k} + \frac{x_{k-1}}{m_k m_{k-1}} + ... + \frac{x_0}{m_k m_{k-1}...m_0}\right)\right)\right)^{n_k}.$$

Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n_{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$. Then the system $\psi := (\psi_n : n \in \mathbb{N})$ is the character system of the group of m-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic)

integers. Since $|r_k^n| = 1$, *i*, *iii* and *iv* are trivial. *ii* is also easy to see and well-known [44, p. 91]. For more on the group of *m*-adic (if $m_k = 2$ for each $k \in \mathbb{N}$ then 2-adic) integers see e.g. [45, 46, 9].

Example C, noncommutative Vilenkin groups

Let σ be an equivalence class of continuous irreducible unitary representations of a compact group G. Denote by Σ the set of all such σ. Σ is called the dual object of G. The dimension of a representation $U^{(\sigma)}$, $\sigma \in \Sigma$, is denoted by d_σ and let

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle \qquad i, j \in \{1, ..., d_\sigma\}$$

be the coordinate functions for $U^{(\sigma)}$, where $\xi_1, ..., \xi_{d_\sigma}$ is an orthonormal basis in the representation space of $U^{(\sigma)}$. (For the notations see [45, vol 2, p. 3].) According to the Weyl-Peter's theorem (see e.g. [45, vol 2, p. 24]), the system of functions $\sqrt{d_\sigma} u_{i,j}^{(\sigma)}$, $\sigma \in \Sigma$, $i, j \in \{1, ..., d_\sigma\}$ is an orthonormal basis for $L^2(G)$. If G is a finite group, then Σ is finite too. If $\Sigma := \{\sigma_1, ..., \sigma_s\}$, then $|G| = d_{\sigma_1}^2 + ... + d_{\sigma_s}^2$.

Let G_{m_k} be a finite group with order m_k, $k \in \mathbb{N}$. Let $\{r_k^{sM_k} : 0 \le s < m_k\}$ be the set of all normalized coordinate functions of the group G_{m_k} and suppose that $r_k^0 \equiv 1$. Thus for every $0 \le j < m_k$ there exist a $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$ such that

$$r_k^{sM_k} = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_{m_k}),$$

$r_k^n := r_k^{n_k M_k}$. Let ψ be the product system of r_k^j, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n^{(k)}}(x_k) \qquad (x \in G_m),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. We should remark that if G_{m_k} is the discrete cyclic group of order $m_k, k \in \mathbb{N}$ then G_m coincides with the Vilenkin group and ψ with respect to the corresponding order, is the Vilenkin system [47, 44, 1, 2]. In [47] it is proved that the system ψ satisfies the properties i, ii, iii. *iv* is satisfied because: $m_k = |G_{m_k}| = d_{\sigma_{k,1}}^2 + ... + d_{\sigma_{k,s_k}}^2$, where $\{\sigma_{k,i} : i = 1, ..., k_s\} = \Sigma_k$ (the dual object of G_{m_k}) and $d_{\sigma_{k,i}}$ is the dimension of $\sigma_{k,i}$. $\|r_k^j\|_\infty \le \sqrt{d}$, where d is one of $d_{\sigma_{k,i}}$ and since d is a divisor of m_k [45, vol 2 p. 44], [47] and since at least one of $d_{\sigma_{k,i}}$ is 1, then $d < \sqrt{m_k}$. From this we have that there exists a $\delta > 1$ (may depend on the sequence m) such that *iv* holds for all $n, k \in \mathbb{N}$. For more on this system and noncommutative Vilenkin groups see [47, 48].

Example D, a system in the field of number theory

Let

$$r_k^n(x) := \exp\left(2\pi\imath \sum_{j=k}^{\infty} \frac{n_j}{M_{j+1}} \sum_{i=0}^{k} x_i M_i\right)$$

for $k,n \in \mathbb{N}$ and $x \in G_m$. Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}$, $n \in \mathbb{N}$.

Then, $\psi := (\psi_n : n \in \mathbb{N})$ is a Vilenkin-like system (introduced in [49]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [49]. *i* is trivial and since $|r_k^n| = 1$, then so are *iii* and *iv*. It is easy to prove *ii* (see [49]). This system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. For the definition of these arithmetical functions see also the book of Mauclaire [50, p. 25].

Example E, the UDMD product system

The notion of the UDMD product system was introduced by F. Schipp [51] on the Walsh-Paley group. Let functions $\alpha_k : G_m \to \mathbb{C}$ satisfy: $|\alpha_k| = 1$ and α_k is $\mathscr{A}_k$ measurable. Let $r_k^n(x) := (-1)^{x_k n_k} \alpha_k(x)$. *i* is trivial and since $|r_k^n| = 1$, so are *iii* and *iv*. The proof of *ii* is simple. Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$ $(n \in \mathbb{N})$. The system $\psi := (\psi_n : n \in \mathbb{N})$ is called an UDMD product system. For more on UDMD product systems see [51, 46].

Example F, The Vilenkin-like diaphony

The special system called Vilenkin-like diaphony is defined by Grozdanov [52] on the group of *m*-adic integers.

We mention some results and problems with respect to Vilenkin-like systems and Cèsaro summability. The Fejér-Lebesgue theorem for the group of 2-adic integers was proved by Gát [53] and for the general system also by Gát [54]. The only result with respect to the general (C,α) is summation is available for the group of 2-adic integers proved also by Gát [55]. The general case is not discussed yet. I also feel it highly likely that the method of the papers [55, 54] make it easy to prove the a.e. (C,α) summability of Vilenkin-like systems of integrable functions. The two (more) dimensional situations have many unsolved problems. The only result available is that the author of this paper with Blahota proved [25] the a.e. convergence of cone restricted two-dimensional Fejér means of integrable functions with respect to Vilenkin systems, but only in the case when $|r_k^n| = 1$ for all k,n. (However, this obviously contains the case of UDMD product systems, the character system of *m*-adic integers.) Besides, there is no divergence result known for the two-dimensional Fejér means at all. Finally, we mention the work of Volosivets [56] in which - among others - he proved Efimov type inequalities with respect to the best approximation with Vilenkin-like systems. Therefore, we think it possible to investigate the relationship of the best approximation and the Fejér means.

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References

[1] N. J. Vilenkin, "On a class of complete orthonormal systems," *Izv. Akad. Nauk. SSSR, Ser. Math.*, vol. 11, pp. 363–400, 1947, russian.

[2] G. H. Agaev, N. J. Vilenkin, G. M. Dzhafarli, and A. Rubinstein, *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups*. Baku: "ELM", 1981, in Russian.

[3] D. L. Burkholder, "Distribution function inequalities for martingales," *Ann. Probability*, vol. 1, pp. 19–42, 1973.

[4] A. Zygmund, *Trigonometric Series*. Cambridge, UK: University Press, 1959.

[5] N. J. Fine, "Cesáro summability of Walsh-Fourier series," *Proc. Nat. Acad. Sci. U.S.A.*, vol. 41, pp. 558–591, 1955.

[6] J. Marcinkiewicz, "Quelques théorémes sur les séries orthogonales," *Ann Soc. Polon. Math.*, vol. 16, pp. 85–96, 1937.

[7] F. Schipp, "Über gewiessen Maximaloperatoren," *Annales Univ. Sci. Budapestiensis , Sectio Math.*, vol. 18, pp. 189–195, 1975.

[8] N. Fujii, "A maximal inequality for H^1 functions on the generalized Walsh-Paley group," *Proc. Amer. Math. Soc.*, vol. 77, pp. 111–116, 1979.

[9] M. H. Taibleson, "Fourier series on the ring of integers in a p-series field," *Bull. Amer. Math. Soc.*, vol. 73, pp. 623–629, 1967.

[10] J. Pál and P. Simon, "On a generalization of the concept of derivative," *Acta Math. Acad. Sci. Hungar.*, vol. 29, pp. 155–164, 1977.

[11] J. Price, "Certain groups of orthonormal step functions," *Canadian J. Math.*, vol. 9, pp. 413–425, 1957.

[12] R. Ž. Nurpeisov, "On the uniform $(C,1)$ summability of Fourier series and integrability of series with respect to multiplicative orthonormal systems," *Analysis Math.*, vol. 15, pp. 127–143, 1989.

[13] A. I. Rubinstein, "On the modulus of continuity of functions defined on 0-dimesnional groups," *Mat. Zametki*, vol. 23, pp. 379–388, 1978.

[14] S. Fridli, "On the modulus of continuity with respect to functions defined on Vilenkin groups," *Acta Mat. Hungar.*, vol. 45, no. 3-4, pp. 393–396, 1985.

[15] F. Schipp, "On L^p-norm convergence of series with respect to product systems," *Analysis Math.*, vol. 2, pp. 49–63, 1976.

[16] G. Gát, "Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups," *Journal of Approximation Theory*, vol. 101, no. 1, pp. 1–36, 1999.

[17] P. Simon, "Cesáro means of Vilenkin-Fourier series," *Publ. Math.*, vol. 59, no. 1-2.

[18] G. Gát, "Cesàro means of integrable functions with respect to unbounded Vilenkin systems," *Journal of Approximation Theory*, vol. 124, no. 1, pp. 25–43, 2003.

[19] ——, "Almost everywhere convergence of Fejér means of L^1 functions on rarely unbounded Vilenkin groups," *Acta Math. Sinica (English Series)*, vol. 23, no. 12, pp. 2269–2294, 2007.

[20] J. Marcinkiewicz and J. A. Zygmund, "On the summability of double Fourier series," *Fund. Math.*, vol. 32, pp. 122–132, 1939.

[21] F. Móricz, F. Schipp, and W. Wade, "Cesàro summability of double Walsh-Fourier series," *Trans Amer. Math. Soc.*, vol. 329, pp. 131–140, 1992.

[22] G. Gát, "Pointwise convergence of the Cesàro means of double Walsh series," *Ann. Univ. Sci. Budap. Rolando Eoetvoes, Sect. Comput.*, vol. 16, pp. 173–184, 1996.

[23] F. Weisz, "Cesáro summability of two-dimensional Walsh-Fourier series," *Trans. Amer. Math. Soc.*, vol. 348, pp. 2169–2181, 1996.

[24] ——, "Hardy spaces and Cesàro means of two-dimensional Fourier series." in *Proceedings of the international conference dedicated to Károly Tandori on his 70th birthday*, P. Vértesi *et al.*, Eds., Budapest, Hungary, Aug. 21–25, 1995, vol. Approximation theory and function series, budapest: János Bolyai Mathematical Society. Bolyai Soc. Math. Stud. 5, pp. 353-367, (1996).

[25] I. Blahota and G. Gát, "Pointwise convergence of double Vilenkin-Fejér means," *Stud. Sci. Math. Hungar*, vol. 36, 2000.

[26] F. Weisz, "Summability results of Walsh- and Vilenkin-Fourier series," in *Functions, Series, Operators – Alexits Memorial Conference*, Budapest (Hungary) 1999, 2002, pp. 443–464.

[27] G. Gát, "On the divergence of the $(C,1)$ means of double Walsh-Fourier series," *Proc. Am. Math. Soc.*, vol. 128, no. 6, pp. 1711–1720, 2000.

[28] ——, "On the divergence of the Fejér means of integrable functions on two-dimensional Vilenkin groups," *Acta Math. Hung.*, vol. 107, no. 1-2, pp. 17–33, 2005.

[29] ——, "Divergence of the $(C,1)$ means of d-dimensional Walsh-Fourier series," *Anal. Math.*, vol. 27 (3), no. 1, pp. 157–171, 2001.

[30] W. Wade, "Growth of Cesáro means of double Vilenkin-Fourier series of unbounded type, analysis of divergence. Control and management of divergent processes," in *Proc. of the 7th Int. Workshop in Analysis and Its Applications, IWAA*, Orono, ME, USA, June 1–6, 1997, boston, MA: Birkhauser. Applied and Numerical Harmonic Analysis, (1999), pp. 41-50.

[31] G. Gát, "On the pointwise convergence of Cesáro means of two-variable functions with respect to unbounded Vilenkin systems," *Journal of Approximation Theory*, vol. 128, no. 1, pp. 69–99, 2004.

[32] F. Weisz, "Maximal estimates for the (C,α) means of d-dimensional Walsh-Fourier series," *Proc. Am. Math. Soc.*, vol. 128 (8), pp. 2337–2345, 2000.

[33] G. Gát, "On $(C,1)$ summability of integrable functions with respect to the Walsh-Kaczmarz system," *Stud. Math.*, vol. 130, no. 2, pp. 135–148, 1998.

[34] P. Simon, "(C,α) summability of Walsh–kaczmarz–Fourier series," *J. Approximation Theory*, vol. 127 (1), pp. 39–60, 2000.

[35] G. Gát and K. Nagy, "Cesàro summability of the character system of the p-series field in the Kaczmarz rearrangement." *Anal. Math.*, vol. 28, no. 1, pp. 1–23, 2002.

[36] P. Simon, "Cesáro summability with respect to two-parameter Walsh systems," *Monatsh. Math.*, vol. 131 (4), pp. 321–334, 2001.

[37] L. V. Zhizhiasvili, "Generalization of a theorem of Marcinkiewicz," *Izv. Akad. nauk USSR Ser Mat.*, vol. 32, pp. 1112–1122, 1968, in Russian.

[38] F. Weisz, "Convergence of double Walsh-Fourier series and hardy spaces," *Approximation Theory Appl.*, vol. 17, no. 2, pp. 32–44, 2001.

[39] G. Gát, "Convergence of Marcinkiewicz means of integrable functions with respect to two-dimensional Vilenkin systems," *Georgian Math. Journal*, vol. 10 (3), pp. 467–478, 2004.

[40] U. Goginava, "On the convergence and summability of N-dimensional Fourier series with respect to the Walsh-Paley systems in the spaces $L^p([0,1]^N), p \in [1,\infty]$," *Georgian Math. J.*, vol. 7, pp. 53–72, 2000.

[41] ——, "Almost everywhere summability of multiple Fourier series," *Math. Anal. and Appl.*, vol. 287, no. 1, pp. 90–100, 2003.

[42] K. Nagy, "On the two-dimensional Marcinkiewicz means with respect to Walsh-Kaczmarz system," *J. Approximation Theory*, vol. 142, no. 2, pp. 138–165, 2006.

[43] F. Weisz, *Summability of multi-dimensional Fourier series and Hardy spaces- Mathematics and Its Applications*. Dordrecht, Boston, London: Kluwer Academic Publishers, 2002.

[44] F. Schipp, W. Wade, P. Simon, and J. Pál, *Walsh series: an introduction to dyadic harmonic analysis*. Bristol and New York: Adam Hilger, 1990.

[45] E. Hewitt and K. Ross, "Abstract harmonic analysis. volume i, ii." in *Grundlehren der Mathematischen Wissenschaften. 115*. Berlin: Springer-Verlag, 1994, pp. viii, 519–771.

[46] F. Schipp and W. Wade, "Fast Fourier transforms on binary fields," *Approximation Theory Appl.*, vol. 14, no. 1, pp. 91–100, 1998.

[47] G. Gát and R. Toledo, "l^p-norm convergence of series in compact, totally disconnected groups," *Anal. Math.*, vol. 22, no. 1, pp. 13–24, 1996.

[48] G. Gát, "Pointwise convergence of Fejér means on compact totally disconnected groups," *Acta Sci. Math. (Szeged)*, vol. 60, no. 1-2, pp. 311–319, 1995.

[49] ——, "On almost even arithmetical functions via orthogonal systems on Vilenkin groups," *Acta Arith.*, vol. 60, no. 2, pp. 105–123, 1991.

[50] J. L. Mauclaire, *Intégration et théorie des nombres. (Integration and number theory)*. Travaux en Cours. Paris: Hermann, 1986, in French.

[51] F. Schipp and W. Wade, "Norm convergence and summability of Fourier series with respect to certain product systems," in *Proc. 6th Southeast. Approximation Theory Conf.*, Memphis/TN (USA), 1991, lect. Notes Pure Appl. Math. 138, pp. 437-452, (1992).

[52] V. Grozdanov, "The weighted *b*-adic diaphony." *J. Complexity*, vol. 22, no. 4, pp. 490–513, 2006, in English.

[53] G. Gát, "On the almost everywhere convergence of Fejér means of functions on the group of 2-adic integers," *Journal of Approximation Theory*, vol. 90, no. 1, pp. 88–96, 1997, in English.

[54] ——, "On $(C,1)$ summability for Vilenkin-like systems," *Stud. Math.*, vol. 144, no. 2, pp. 101–120, 2001.

[55] ——, "Almost everywhere convergence of Cesàro means of Fourier series on the group of 2-adic integers," *Acta Math. Hung.*, vol. 116, no. 3, pp. 209–211, 2007.

[56] S. S. Volosivets, "Convergence of Fourier series with respect to multiplicative systems and the *p*-fluctuation continuity modulus," *Sib. Mat. Zh.*, vol. 47, no. 2, pp. 241–258, 2006, in Russian and English.