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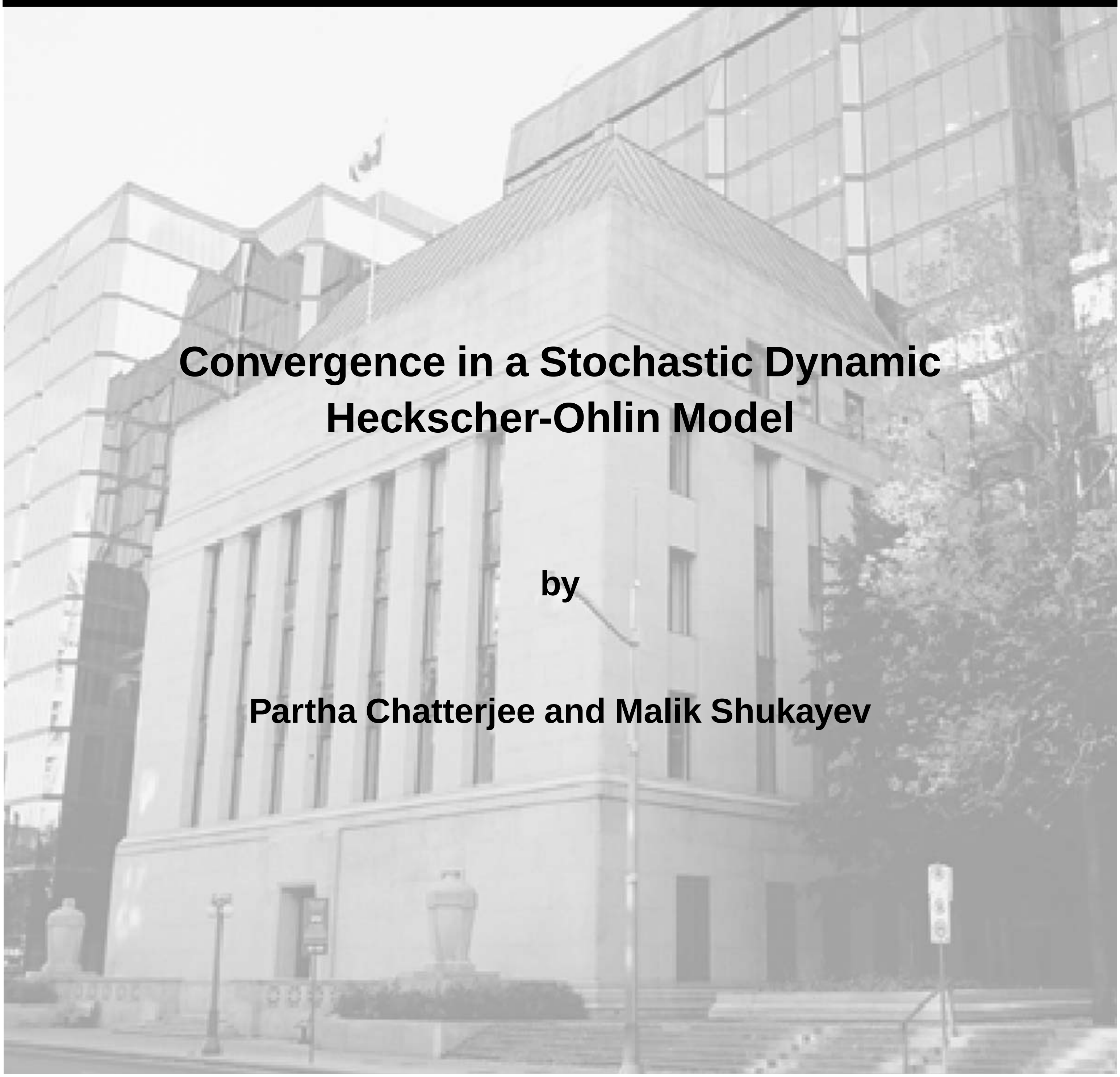

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The views expressed in this paper are those of the authors.
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Abstract

The authors characterize the equilibrium for a small economy in a dynamic Heckscher-Ohlin model with uncertainty. They show that, when trade is balanced period-by-period, the per capita output and consumption of a small open economy converge to an invariant distribution that is independent of the initial wealth. Further, at the invariant distribution, with probability one there are some periods in which the small economy diversifies. These results are in sharp contrast with those of deterministic dynamic Heckscher-Ohlin models, in which permanent specialization and non-convergence occur. One key feature of the authors' model is the presence of market incompleteness as a result of the period-by-period trade balance. The importance of market incompleteness, and not just uncertainty, in achieving the authors' results is illustrated through an analytical example. Further, numerical simulations show that the convergence occurs more quickly as the magnitude of the shocks increases. Thus, the results extend the predictions of income convergence, standard in one-sector neoclassical growth models, to the dynamic multi-country Heckscher-Ohlin environment.

JEL classification: F43, O41
Bank classification: Economic models

Résumé

Les auteurs caractérisent l'équilibre dans le cadre d'un modèle dynamique de Heckscher-Ohlin avec incertitude. Ils montrent que, lorsque les échanges extérieurs sont en équilibre à chaque période, la production et la consommation par tête dans une petite économie ouverte convergent vers une distribution invariante, indépendamment du niveau de richesse initial. En outre, une fois la distribution invariante établie, il existe à coup sûr certaines périodes où cette économie se diversifie. Ces résultats tranchent avec ceux que produisent les modèles de Heckscher-Ohlin dynamiques déterministes, où coexistent une spécialisation permanente et la non-convergence. Le modèle des auteurs se distingue en particulier par la présence de marchés incomplets, attribuable à la condition d'équilibre des échanges. Le rôle que jouent, non pas seulement l'incertitude, mais les marchés incomplets pour expliquer les résultats obtenus est illustré au moyen d'un exemple analytique. Par des simulations numériques, les auteurs montrent également que la convergence s'accélère avec l'intensification des chocs. Ainsi, les résultats étendent au modèle dynamique multipays de Heckscher-Ohlin les prédictions relatives à la convergence du revenu typiques des modèles de croissance monosectoriels néoclassiques.

Classification JEL : F43, O41
Classification de la Banque : Modèles économiques

1. Introduction

Will income levels in two countries, which start from different conditions, converge? Traditionally, deterministic closed-economy neoclassical growth models have been used to answer this question. These models predict that, as long as countries have the same preferences, technologies, and population dynamics, they will converge to the same level of per capita income from any positive initial wealth. Initial conditions therefore do not matter for the long-run income levels. Brock and Mirman (1972) extend this result to a stochastic environment by showing that countries will converge to the same invariant distribution of income irrespective of their positive initial wealth. More recently, Chen (1992), Ventura (1997), and Atkeson and Kehoe (2000) have shown that, in deterministic dynamic Heckscher-Ohlin models — that is, models with two or more tradable commodities produced using neoclassical production functions that differ in capital intensities — convergence may not occur, despite all countries being identical up to their initial conditions and all the production functions being strictly concave. Although the models vary in details, all of them rely on trade-induced factor-price equalization, which leads to the existence of multiple steady states. Initial conditions determine to which steady state a particular economy will converge. This result has led to a surge of interest in dynamic Heckscher-Ohlin models, with the view that such models can potentially account for the observed income differences across countries without resorting to non-convexities or structural differences between countries.

It is natural to wonder whether the results from a deterministic model will carry over to an uncertain world. We introduce technological uncertainty in a dynamic Heckscher-Ohlin model and, just as in the one-sector neoclassical growth model, we obtain income convergence across countries. We show that, when trade is balanced period-by-period, a standard assumption in deterministic Heckscher-Ohlin models, the per capita output and consumption of a small trading economy converge to an invariant distribution that is independent of the (positive) initial wealth. Furthermore, the introduction of uncertainty overturns another prediction of the deterministic model: that countries may permanently specialize, and therefore may never produce all tradable commodities. We find that, in an uncertain environment, when the income of an economy is within the invariant distribution, there will be some periods in which the small economy diversifies. It is important to note that, in our modelling strategy, we are following Atkeson and Kehoe (2000) in concentrating on the dynamics of a small trading economy that has no effect on world prices of tradable goods.

Why are the results in the stochastic version of the Heckscher-Ohlin model so different from those in the deterministic version? There are two ingredients in our model that are crucial for our results: uncertainty and market incompleteness arising from the period-by-period trade balance constraint. To understand the role played by these two factors, first consider the deterministic version. In such models, when countries diversify (i.e., when their aggregate capital-labour ratios are within the diversification cone), factor-price equalization means that the countries face the same rates of return to capital. Thus, when preferences are identical across countries, there is no incentive for agents in one country to accumulate capital if there is no incentive for agents in the other country to do so. This implies that if one country is in a steady state, so is the other. In particular, if the world economy is in a steady state, all capital-labour ratios within the diversification cone can be sustained as steady states. Consequently, any country that starts with a capital-labour ratio within the diversification cone will remain at that level of capital forever. Countries that start outside the diversification cone, with a low capital-labour ratio, will grow until they reach the lower boundary of the diversification cone. Such countries will never enter the diversification cone and will permanently specialize in the production of tradable goods that are less capital intensive. Therefore, in this case, the initial conditions determine the fate of the country in the long run.

In the presence of uncertainty and market incompleteness, however, the initial conditions are eventually irrelevant. In an uncertain world, two small economies starting from different initial conditions will find themselves in a similar situation in the future, in which they will surely diversify their production — produce all tradable goods — in at least some periods. Uncertainty by itself is, however, not enough for convergence. The period-by-period trade balance creates market incompleteness, and that is crucial for the convergence result. In the deterministic model, when all countries have capital-labour ratios within the diversification cone, the rental rates are equated across them, and there is no incentive to borrow and lend internationally. Thus, the absence of borrowing and lending as a result of the balanced trade in each period is irrelevant. If, however, different countries face different shocks, then the rate of return to capital is not the same across countries in each period: there can be mutual gains through risk-sharing if countries can borrow from, and lend to, each other. Period-by-period trade balance prevents that practice, forcing countries to self-insure by accumulating more capital when income is higher than expected, and de-accumulating capital when income is lower than expected. This pattern of capital accumulation in economies with uncertainties shifts the policy functions relative to those in economies without uncertainty, which implies that the capital-accumulation policy function in an economy with uncertainty no longer coincides with the 45-degree line, as it does in deterministic Heckscher-Ohlin models, and we no longer get a multiplicity of steady states.

The importance of the restriction on risk-sharing opportunities imposed by the balanced trade condition is illustrated through an analytical example. In this example, we retain the period-by-period balanced trade constraint, but assume away any risk-sharing opportunities. We assume that a small economy and the rest of the world economy both face the same realization of shocks each period. In this case, when there are no gains to be made from borrowing and lending from each other, there is no income convergence, and the small economy permanently specializes if it starts from outside the diversification cone. Thus, in this example, despite uncertainty, results are very similar to what we observe in deterministic models.

If, instead of setting borrowing and lending to zero, we allowed for limited borrowing, our results would remain unchanged as long as the limit on borrowing was sufficiently low.

We also simulate our model to determine how the speed of convergence depends on the size of the shocks that the economy faces. We find that the bigger the shocks are, the faster is the convergence. In the limit, when uncertainty vanishes, the convergence disappears, which suggests that, if uncertainty is small, initial conditions play an important role in the development of a country: it takes a long time for initially capital-poor countries to catch up with richer countries. For higher levels of uncertainty, however, initial conditions quickly cease to have an effect on per capita income levels across countries. The simulation also helps us to see the actual shape of investment policy functions and determine where the support for invariant distribution of capital is located vis-a-vis the diversification cone.

Our paper encompasses various strands of the literature. First, it generalizes the dynamic Heckscher-Ohlin model in an important way by introducing uncertainty. It shows that deterministic dynamic Heckscher-Ohlin models studied by Chen (1992), Ventura (1997), and Atkeson and Kehoe (2000) are very special limiting cases of the stochastic environment. Second, the paper extends to an open-economy setting the convergence results of a closed-economy stochastic-growth model studied by many researchers, starting with Brock and Mirman (1972). Third, the paper contributes to the long literature on the income-fluctuations problem, which studies savings decisions under market incompleteness in environments that have many agents facing idiosyncratic shocks; Clarida (1987), Chamberlain and Wilson (2000), and Aiyagari (1994) are just a few examples of researchers in this category.

This paper is organized as follows. In section 2, we describe our model's environment. In section 3, we report the equilibrium results for the model, including those on convergence and diversification. In section 4, we simulate the model and discuss the speed of convergence. In section 5, we consider another version of the model with productivity shocks that are economy-wide, rather than sector-specific. We show that the convergence and diversification results still hold, but the range of capital-labour ratios observed in the invariant distribution is much bigger. In section 6, we provide an analytical example which shows that the constraint on risk-sharing opportunities is important to our results. We offer some conclusions in section 7. Appendix A provides the properties of the diversification cone boundaries. Appendix B provides the proofs.

2. The Environment

The economic environment consists of two economies: a *small economy* and the *rest-of-the-world economy*. The population is fixed in both countries. We assume that the population size in the small country is of measure zero relative to the rest of the world. Motivated by this assumption, and for brevity, we refer to the rest-of-the-world economy as simply the *world economy*.

The two economies are assumed to have identical preferences and technologies (up to the stochastic productivity factors), although the nature of the uncertainty faced by the economies could be different. In each economy there are two intermediate goods, a and m, and one final good, Y. The intermediate goods are produced using capital and labour in each intermediate-good sector. Technology for producing good a is less capital intensive than the technology for producing m. The intermediate goods are traded between the economies. The final good is produced by combining the two intermediate goods. The final good can be either invested or consumed domestically, but it cannot be traded across economies. Capital and labour are also immobile across borders.

2.1 Preferences

The agents in both the economies are assumed to have identical preferences. Representative agents in each economy supply labour inelastically and derive utility from consumption.

Assumption 1
The utility function, $u : \mathcal{R}_+ \rightarrow \mathcal{R}_+$*, has the following properties:*

(i) u *is continuous on* $\mathcal{R}_+$*, bounded below, and (without loss of generality)* $u(0) = 0$.

(ii) u *is twice continuously differentiable and strictly concave; i.e.,* $u'(c) > 0$, $u''(c) < 0 \;\; \forall \;\; c \in \mathcal{R}_{++}$.

(iii) $\lim_{c \rightarrow 0} u'(c) = \infty$.

2.2 Production

Each economy has access to three technologies: two intermediate-good technologies a and m, and one final-good technology, Y. All the production functions are assumed to be standard neoclassical production functions: homogeneous of degree one in all inputs, and twice continuously differentiable with positive and diminishing marginal products of each input.

The final good is produced by combining intermediate goods a and m:

$$Y = H(a, m), \tag{2.1}$$

$H(a, m)$ satisfies Assumption 2.[1]

[1]We use H_1 to represent the partial derivative of H with respect to its first argument. We do the same for all other first and second derivatives.

Assumption 2
$H(a, m)$ *exhibits constant returns to scale, and for all* $a \geq 0$ *and* $m \geq 0$,

(i) $H(0, m) = H(a, 0) = 0$.

(ii) $H_1(a, m) > 0$, $H_2(a, m) > 0$, $H_{11}(a, m) < 0$ *and* $H_{22}(a, m) < 0$.

There are two distinct production functions, which combine capital and labour to produce intermediate goods. The technology for producing intermediate good a is given by,

$$a = \lambda F(K_a, L_a), \tag{2.2}$$

where λ is the productivity factor and is potentially stochastic. K_a and L_a are capital and labour employed in sector a.

The technology for producing intermediate good m is given by,

$$m = \theta G(K_m, L_m), \tag{2.3}$$

where θ is the productivity factor and is also potentially stochastic. Similarly, K_m and L_m are capital and labour employed in sector m.

Assumptions for both production functions F and G are similar to that for H: they are constant returns to scale, and their marginal products of capital and labour are positive and strictly diminishing. In addition, the intermediate technologies satisfy the following boundary conditions.

Assumption 3
Inada conditions for intermediate technologies:

(i) *For all* $L > 0$, $\lim_{K \to 0} F_1(K, L) = \lim_{K \to 0} G_1(K, L) = \infty$.

(ii) For all $L > 0$, $\lim_{K \to \infty} F_1(K, L) = \lim_{K \to \infty} G_1(K, L) = 0$.

We also assume, as is standard in Heckscher-Ohlin models, that the good m technology is more capital intensive than the good a technology for all relevant factor-price ratios (i.e., there are no factor intensity reversals). More formally, we have Assumption 4.

Assumption 4
For all $K > 0$ *and* $L > 0$ $\quad \frac{F_2(K,L)}{F_1(K,L)} > \frac{G_2(K,L)}{G_1(K,L)}$.

2.3 International trade

As we stated above, final goods, capital, and labour are not tradable across the two countries. The only commodities that can be traded between the economies are the two intermediate goods. Thus, the quantities of intermediate goods utilized in a small economy for the production of final goods can differ from the quantities produced in the small economy. We assume, as is standard in deterministic dynamic Heckscher-Ohlin models, that trade is balanced in each period for each economy.

Assumption 5
In all periods t*, and for both countries* $(i = s, w)$,

$$p_{at}(a_t^{i,d} - a_t^i) + p_{mt}(m_t^{i,d} - m_t^i) = 0, \tag{2.4}$$

where variables with superscript d are quantities demanded in country i, variables without superscript d are quantities produced in country i, and p_{at}, p_{mt} are the world prices of intermediate goods. This assumption has no implication on the equilibrium outcomes in deterministic Heckscher-Ohlin models when each economy produces both intermediate goods. In that case, balanced trade is an equilibrium outcome. With country-specific productivity shocks, however, the period-by-period balanced trade constraint is binding and precludes risk-sharing opportunities through borrowing and lending. As we will show later, this constraint plays an important role in determining the equilibrium outcomes. Balanced trade implies that countries cannot borrow or lend. Thus, the balanced trade constraint will also be reflected in the budget constraint of a representative household.

2.4 Uncertainty

In this paper, except in section 5, we assume that the world economy faces no uncertainty. Further, productivity factors in the intermediate technologies used in the world economy, λ_t^w and θ_t^w, are both normalized to be equal to one for all t. The small economy, however, faces uncertainty: λ_t^s and θ_t^s are stochastic. We assume the following distributions for these productivity terms:

Assumption 6

(i) Both λ_t^s and θ_t^s are independently, identically distributed (i.i.d.) random variables drawn from their respective time-invariant distributions.

(ii) The support of λ_t^s is $\Lambda = [\underline{\lambda}, \overline{\lambda}]$, where $0 < \underline{\lambda} \leq \overline{\lambda} < \infty$, while the support of θ_t^i is $\Theta = [\underline{\theta}, \overline{\theta}]$, where $0 < \underline{\theta} \leq \overline{\theta} < \infty$.

(iii) $E[\lambda] = 1$ and $E[\theta] = 1$.

The last part of Assumption 6 ensures that the expected productivity of each sector in the small economy is equal to the productivity of the world economy's sectors.

Let η be the probability measure for the joint distribution of $z = (\lambda, \theta)$, defined on the Borel subsets of $Z = \Lambda \times \Theta$. The assumption that Λ and Θ are full supports implies that $\eta(A) > 0$ for any non-degenerate rectangle in the $Z = \Lambda \times \Theta$ space.

At this point, it is useful to state the timing of various events and decision processes in the economy. At the beginning of every period, the uncertainty about current productivity levels is resolved. The consumers, final-good producers, and intermediate-goods producers all take their decisions after that point. The consumers choose how much to consume and save. The savings decisions determine the next period's capital. The intermediate-goods producers decide how to allocate the capital and labour available in the economy between the two sectors. Note that the aggregate capital in the economy is decided before the uncertainty for the period is resolved (it is decided a period earlier), but the allocation of capital and labour across sectors takes place after the uncertainty is resolved. Also, the final-good producers decide the amount of each intermediate good to demand, which in turn determines the quantity of exports and imports of each intermediate good. With this timing, the subscript t signifies that a variable is measurable with respect to the information available up to period t, including period t productivity terms in both sectors.

3. Equilibrium in the World Economy and the Small Economy

In this section, we analyze the equilibrium of the world economy and the small economy. We begin with the world economy.

3.1 Equilibrium in the world economy

Our assumption that the small economy's population is of zero measure compared with the population of the world economy implies that the world economy behaves as a closed economy and that the prices of the intermediate goods are determined by the world economy's equilibrium alone.

In the absence of uncertainty, the world economy will converge to a unique steady state. In the steady state, the prices of the intermediate goods, p_a and p_m, and the interest rate in the world economy, will be constant across time.

In our analysis of the equilibrium of the small economy, we assume that the world is in the steady state. The world economy's equilibrium determines the intermediate-goods prices, p_a and p_m, prevailing universally in both the world and the small economy. Therefore, in our analysis, the prices of intermediate goods are given and constant across time. Also, since we are concentrating on the equilibrium of the small economy only, we drop the superscript i from all variables. We distinguish world variables with a superscript w whenever necessary.

3.2 Decision problems in the small economy

In the small economy, the representative household maximizes its lifetime expected utility subject to the period budget constraint, and taking prices of labour, w_t, and capital, r_t, as given. Thus, the representative household's decision problem is to choose consumption, c_t, investment, x_t, and capital, k_t, to solve:

$$\max_{\{c_t, x_t, k_t\}_{t=1}^{\infty}} \quad E_1[\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)] \quad s.t.$$

$$c_t + x_t \leq w_t + r_t k_{t-1}, \tag{3.1}$$

$$k_t = (1-\delta) k_{t-1} + x_t, \tag{3.2}$$

given the initial level of per capita capital, k_0.

Note that markets are incomplete; there are no contingent assets available to the households to insure themselves against risk. Moreover, the budget constraints noted above do not allow for borrowing or lending. Capital accumulation is the only available instrument to transfer resources across periods and states of nature. The lack of borrowing or lending is a reflection of the period-by-period balanced trade constraint described earlier.

The above maximization problem results in the following dynamic optimality conditions:

$$u'(c_t) = \beta E_t \left[u'(c_{t+1})(1 - \delta + r_{t+1}) \right], \tag{3.3}$$

$$c_t + k_t = w_t + r_t k_{t-1} + (1 - \delta) k_{t-1}. \tag{3.4}$$

These are the equations that determine the dynamics of per capita capital and per capita wealth in this model.

On the production side, there are two kinds of firms in the economy: final-good firms and intermediate-goods firms. We assume that each firm operates in a perfectly competitive environment. The representative final-good firm takes the prices of the intermediate goods as given and solves the following problem:

$$\begin{aligned} \min_{\{a_t^d, m_t^d\}} \quad & p_a a_t^d + p_m m_t^d, \\ s.t. \quad & \\ Y_t \leq \; & H(a_t^d, m_t^d). \end{aligned} \tag{3.5}$$

As noted earlier, variables with superscript d are the quantities demanded in the economy, while variables without the superscripts are the quantities produced in the economy. The first-order conditions for the final-good firm are,

$$p_a = H_1(a_t^d, m_t^d), \tag{3.6}$$

$$p_m = H_2(a_t^d, m_t^d). \tag{3.7}$$

Given world prices of intermediate goods, these equations determine the relative quantities of intermediate goods demanded in the small economy.

The representative intermediate-goods firm in each economy chooses how to allocate the total capital and labour available in that economy across the two sectors. It takes world prices of intermediate goods and domestic factor prices as given and solves,

$$\min_{\{K_{at}, L_{at}, K_{mt}, L_{mt}\}} \quad r_t(K_{at} + K_{mt}) + w_t(L_{at} + L_{mt})$$

$$s.t.$$

$$p_a a_t + p_m m_t \leq p_a \lambda_t F(K_{at}, L_{at}) + p_m \theta_t G(K_{mt}, L_{mt}). \tag{3.8}$$

Let us define the intensive form of the intermediate production functions as:

$$f(k) = F(\frac{K}{L}, 1), \tag{3.9}$$

$$g(k) = G(\frac{K}{L}, 1). \tag{3.10}$$

The following equations give the first-order conditions in terms of the intensive production functions,

$$
\begin{aligned}
p_a \lambda_t f'(k_{at}) &\leq r_t, & (3.11)\\
p_a \lambda_t \left[f(k_{at}) - f'(k_{at})k_{at}\right] &\leq w_t, & (3.12)\\
p_m \theta_t g'(k^s_{mt}) &\leq r_t, & (3.13)\\
p_m \theta_t \left[g(k_{mt}) - g'(k_{mt})k^i_{mt}\right] &\leq w_t. & (3.14)
\end{aligned}
$$

Inequalities (3.11) and (3.12) hold with equality whenever sector a is operated with positive inputs, while inequalities (3.13) and (3.14) hold with equality whenever sector m is operated with positive inputs.

Thus, it is the intermediate firms that decide whether to produce both intermediate goods in positive quantities, or, in other words, whether the country will diversify. Their first-order conditions can be used to define the boundaries of the "cone of diversification," k^b_{at} and k^b_{mt}. Whenever the aggregate capital-labour ratio of a small economy belongs to the interior of this cone, $k_t \in (k^b_{at}, k^b_{mt})$, it is profitable to produce both intermediate goods in the small economy. The boundaries k^b_{at} and k^b_{mt} are defined as a solution to the following equations:

$$
\begin{aligned}
p_a \lambda_t f'(k^b_{at}) &= p_m \theta_t g'(k^b_{mt}), & (3.15)\\
p_a \lambda_t \left[f(k^b_{at}) - f'(k^b_{at})k^b_{at}\right] &= p_m \theta_t \left[g(k^b_{mt}) - g'(k^b_{mt})k^b_{mt}\right]. & (3.16)
\end{aligned}
$$

Equations (3.15) and (3.16) are the optimality conditions that equate marginal products of capital and labour in two intermediate sectors. They must be satisfied whenever both intermediate sectors are operated; i.e., when the economy's aggregate capital-labour ratio is within the cone of diversification. In this case, the optimal capital-labour ratios in intermediate sectors a and m are $k_{at} = k^b_{at}$

and $k_{mt} = k^b_{mt}$, respectively. This allows us to dispense with the superscript b; k_{at} and k_{mt} signify both the boundaries of the cone of diversification and the optimal capital-labour ratios in the two sectors of economies within the cone. The boundaries k_{at} and k_{mt} are stochastic, and are functions of λ_t and θ_t. We show in Appendix A that, whenever k_{at} and k_{mt} are positive, $k_{at} < k_{mt}$. Further, k_{at} and k_{mt} are increasing in $\rho = \frac{p_a \lambda}{p_m \theta}$.

A crucial point is that k_{at} and k_{mt} are independent of the domestic capital-labour ratio k_{t-1}. As long as the aggregate capital-labour ratios of two (or more) economies that face the same realizations of both productivity shocks λ_t and θ_t, fall within the cone of diversification $[k_{at}, k_{mt}]$, the economies will have the same capital-labour ratios in both sectors. Further, these economies will have the same factor prices, as is evident from the following equations:

$$r_t = p_a \lambda_t f'(k_{at}), \tag{3.17}$$

$$w_t = p_a \lambda_t [f(k_{at}) - f'(k_{at}) k_{at}]. \tag{3.18}$$

This is the essence of the factor-price equalization effect of international trade in goods. It plays a crucial role in creating multiple steady states and non-convergence in the environment without uncertainty, which is the focus of section 3.3.

The allocation of capital and labour between two intermediate sectors, however, depends on the domestic capital-labour ratio. Countries that have a higher capital-labour ratio devote a larger fraction of capital and labour to the capital-intensive sector m.

In any equilibrium the following market-clearing conditions must be satisfied:

$$
\begin{aligned}
a_t &= \lambda_t F(K_{at}, L_{at}), & (3.19)\\
m_t &= \theta_t G(K_{mt}, L_{mt}), & (3.20)\\
C_t + X_t &= H(a_t^d, m_t^d), & (3.21)\\
C_t &= c_t L, & (3.22)\\
X_t &= x_t L, & (3.23)\\
K_{at} + K_{mt} &= K_{t-1}, & (3.24)\\
L_{at} + L_{mt} &= L. & (3.25)
\end{aligned}
$$

The market-clearing conditions are standard. Observe that, in the market-clearing condition for capital, equation (3.24), aggregate capital is determined a period earlier than when it is allocated between the two intermediate sectors for production, a consequence of the aforementioned timing assumptions.

3.3 Equilibrium in the small economy without uncertainty

Before we discuss convergence in a stochastic environment, let us first understand why there are multiple steady-state equilibria, non-convergence, and specialization in the economies without uncertainty. Suppose a small economy faces no uncertainty and has $\lambda_t = 1(= \lambda_t^w)$ and $\theta_t = 1(= \theta_t^w)$ for all t and in all states of nature.

Since λ_t and θ_t are fixed, the boundaries of the diversification cone, k_{at} and k_{mt}, are constant over time. Further, since the technology is identical across the world economy and the small economy, the boundaries of the diversification cone in the two economies coincide: $k_a = k_a^w$ and $k_m = k_m^w$.

There are two possible scenarios for the small economy: it may start with a capital-labour ratio either within or outside of the diversification cone. First, suppose that the initial capital in the small economy, k_0, is within the diversification cone; i.e., $k_0 \in [k_a, k_m]$. Then, since $k_a = k_a^w$, we have

$$r_t = p_{at} f'(k_{at}) = r_t^w. \tag{3.26}$$

Similarly, $w_t = w_t^w$. Thus, factor-price equalization occurs across the economies. The fact that interest rates are equal across countries means that there is no incentive for cross-economy borrowing and lending, and trade is balanced period-by-period in the equilibrium. Further, identical rates of return in both economies mean that the incentives to accumulate capital are the same in both economies, and, since the world economy is in the steady state, the small economy will also be in the steady state at the initial capital-labour ratio. Thus, any capital-labour ratio within the diversification cone can be sustained as a steady state.

Second, consider the case where the small economy starts at a capital-labour ratio that is outside of the diversification cone. In particular, suppose that the economy starts with a very low capital-labour ratio, $k_0 < k_a$. In this case, as long as $k_{t-1} < k_a$, it is optimal to produce only the less capital-intensive good,

$$r_t = p_a f'(k_{t-1}) > r_t^w = p_a f'(k_a).$$

The interest rate in the small economy will be larger than the world interest rate, and the small economy will accumulate capital until it reaches (asymptotically) the lower boundary of the diversification cone; i.e., until it reaches the point where $k_{t-1} = k_a$. Once it reaches the boundary, again there is factor-price equalization, and the economy will stop accumulating capital. Hence, the small economy will be at a steady state at the lower boundary of the diversification cone and will produce only the less capital-intensive good. Therefore, the country that starts with a very low level of capital will permanently specialize in producing good a.

In the case where the economy starts at $k_0 > k_m$, the economy will de-accumulate capital until it reaches the upper boundary of the diversification cone, and it will remain there forever, producing only good m.

Hence, economies starting at any capital-labour ratio within the diversification cone will remain at that ratio, and those starting from outside the diversification cone will reach steady state at the boundaries of the diversification cone. This implies that there will be no convergence in per capita capital stock or income levels if various economies start with different initial capital-labour ratios.

There is one crucial difference between a one-sector closed economy and an open economy with two tradable sectors: although in the former the interest rate is a function of the aggregate capital in the economy, it is independent of the aggregate capital in the open economy within the diversification cone. As a result, there is a unique capital-labour ratio for a given interest rate in a closed economy. In the case of an open economy, several aggregate capital-labour ratios are sustainable for a given interest rate; all that differs is the share of the two intermediate goods. This is crucial in delivering multiple steady states in a deterministic dynamic Heckscher-Ohlin model.

3.4 Equilibrium in the stochastic small economy

We next describe the case when the small economy faces uncertainty; i.e., when λ_t and θ_t are stochastic.

We assume that, for all values of $z = (\lambda, \theta)$, the corresponding capital-labour ratios $k_a(\frac{p_a\lambda}{p_m\theta})$ and $k_m(\frac{p_a\lambda}{p_m\theta})$ are strictly positive and finite. This assumption is not crucial for our results, but it ensures that, at very low values of aggregate capital-labour ratio k, the small economy will produce only good a, while at very large values of capital the small economy will produce only good m.

Since p_a and p_m are constants and λ_t and θ_t are i.i.d., the variables $p_a\lambda_t$ and $p_m\theta_t$ are also i.i.d. Define the per capita income of the small economy as $y_t = w_t + r_t k_{t-1} + (1-\delta)k_{t-1}$. It is a function of the small economy's capital-labour ratio, k_{t-1}, and TFP shocks, $z = (\lambda, \theta)$: $y_t = y(k_{t-1}, z_t)$. The representative household's problem can be restated as

$$\max_{c_t,k_t} E\sum_{t=1}^{\infty}\beta^{t-1}u(c_t), \tag{3.27}$$

$$s.t.$$

$$c_t + k_t \leq y(k_{t-1}, z_t)$$

$$y(k_0, z_1) \text{ is given,}$$

where the expectation is defined over the Borel sigma-algebra of partial shock histories, $z^t = (z_0, z_1, ...z_t) \in Z^t$. This set-up of the household's problem makes it clear that, from the household's perspective, the problem is essentially the same as that faced by an agent in a one-sector stochastic growth model with i.i.d. shocks. Given this set-up, the optimal consumption and investment policy functions in any period t will be functions of current income, y_t, only. For our main result on convergence, we need to establish the continuity and monotonicity properties of our policy functions. To do that, we first need to understand the continuity and monotonicity properties of the income function, which is achieved in Lemma 1.

Lemma 1. ***Properties of the income function,*** y**.**

- *The income of the small economy, y, is continuous in k, λ, and θ. It is strictly increasing in k, non-decreasing in λ and θ, and strictly increasing in either λ or θ, or both.*
- *For every $z \in Z$, the function $y(\cdot, z) : R_+ \rightarrow R_+$ is concave, and continu-*

ously differentiable. For every $k > 0$, the derivative $\frac{\partial y(k,\cdot)}{\partial k}$ is continuous in λ and θ.

- *There exists the maximum sustainable level of capital, $\bar{k}$, such that $y(k,z) < \bar{k}$ for all $k > \bar{k}$ and for all $z \in Z$.*

Let $X = \left[0, \bar{k}\right]$. Define the value function $v(k_0, z_1)$ as the maximum lifetime expected utility attained in the program (3.27). It is a standard result that the value function is unique, bounded, strictly concave, continuously differentiable in k (for $k > 0$), and solves the following Bellman equation,

$$v(k,z) = \max_{k' \in [0, y(k,z)]} \left[u(y(k,z) - k') + \beta \int v(k', z')\eta(dz') \right]. \tag{3.28}$$

Further, for each $z \in Z$, $v(\cdot, z) : X \to R_+$ is strictly increasing and $v(0,z) = 0$.

The investment policy function $h(k,z)$ is defined so that

$$v(k,z) = u(y(k,z) - h(k,z)) + \beta \int v(h(k,z), z')\eta(dz'). \tag{3.29}$$

In the following proposition, we establish the existence, and the continuity and monotonicity properties, of both the investment policy function $h(k,z)$ and the consumption policy function $c(k,z)$.

Proposition 1. ***Existence, continuity, and monotonicity of the policy functions:***

- *There exist unique consumption and investment policy functions: $c_t = c(k_{t-1}, z_t)$, $k_t = h(k_{t-1}, z_t)$. They are both continuous with respect to k_t, λ_t, and θ_t, and measurable with respect to the Borel subsets of $Z = \Lambda \times \Theta$.*

- *Functions* $c(k_{t-1},(\lambda_t,\theta_t))$ *and* $h(k_{t-1},(\lambda_t,\theta_t))$ *are strictly increasing in* k_{t-1}*, non-decreasing in* λ_t *and* θ_t*, and strictly increasing in either* λ_t *or* θ_t*, or both. Also,* $c(0,z)=0$ *and* $h(0,z)=0$ *for all values of* z*.*

To explore the dynamic properties of the investment policy function, we need to be more specific about its shape. We start with fixed points of the function. For any realization z, we define k_z to be a fixed point for the investment policy function $h(k,z)$; i.e., k_z is such that $k_z = h(k_z,z)$. We can also define the maximum and minimum positive fixed points for any given realization, z, as follows:

$$k_z^{\max} = \max\{k>0|h(k,z)=k\}, \tag{3.30}$$
$$k_z^{\min} = \min\{k>0|h(k,z)=k\}, \tag{3.31}$$

We also define the *best*, $\overline{z}$, and the *worst*, $\underline{z}$, shocks in the sense of giving the most and the least amount of income, for any given level of capital available. Since the small economy's income is a non-decreasing function of both λ and θ, these are uniquely defined: $\overline{z} = \left(\overline{\lambda},\overline{\theta}\right)$ and $\underline{z} = (\underline{\lambda},\underline{\theta})$.

In the next proposition we show that the investment policy function possesses certain stability properties.

Proposition 2. ***Fixed points and stability properties of the investment policy function:***

- *The fixed point* $k_{\underline{z}}^{\max} > 0$ *exists and for all* $k > k_{\underline{z}}^{\max}$, $h(k,\underline{z}) < k$.
- *The fixed point* $k_{\overline{z}}^{\min} > 0$ *exists and for all* $k < k_{\overline{z}}^{\min}$, $h(k,\overline{z}) > k$.
- *The function* $h(k,z)$ *has a stable configuration; i.e.,* $k_{\underline{z}}^{\max} < k_{\overline{z}}^{\min}$.

Given that we have assumed the shocks to be i.i.d., the policy function $h(k, z)$ defines a Markov process on the set of capital-labour ratios, X. Let $\mathcal{B}$ be the Borel sigma field generated by X. For all $B \subset \mathcal{B}$, let $P(k_{t-1}, B) = \Pr(k_t \in B)$ be the transition probability function of the capital-labour ratio process in the small economy. Let $P^t(B) = \Pr(k_t \in B)$ be the probability measure for the small economy's capital-labour ratio in period t defined on Borel subsets B of X; it is generated by the transition probability function as

$$P^t(B) = \int_X P(k, B) P^{t-1}(dk),$$

starting from some initial distribution, P_0, defined on $(X, \mathcal{B})$. The invariant distribution over X, then, is any probability measure μ, such that

$$\mu(B) = \int_X P(k, B)\mu(dk).$$

The economy is generally assumed to start from a given value of the capital, which means that P^0 is a degenerate distribution concentrated on some positive value of the capital-labour ratio. Our objective is to prove that, no matter which positive value of capital we start from, the limit $\lim_{t\to\infty} P^t$ is the unique invariant distribution. More precisely, let δ_{k_0} be a degenerate distribution concentrated on k_0. Let $P^0(k_0, B) = \delta_{k_0},\quad P^1(k_0, B) = P(k_0, B)$, and $P^t(k_0, B) = \int_X P(k, B) P^{t-1}(k_0, dk)$ for any set $B \subset \mathcal{B}$. We need to show that $\lim_{t\to\infty} P^t(k_0, B) = \mu(B)$ for all positive k_0 and any Borel subset B in $\mathcal{B}$.

Theorem 1. ***Convergence***

There exists the unique invariant probability measure μ on $(X, \mathcal{B})$, such that $\lim_{t\to\infty} P^t(k_0, B) = \mu(B)$ for all $k_0 > 0$. The full support of μ is the unique non-degenerate compact interval on $\mathcal{R}_{++}$ given by $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$.

The above theorem means that, no matter where different small economies start from, their long-run average per capita capital stock will be the same. Thus, in the long run, there will be convergence in the per capita capital stock and, hence, convergence in the per capita income levels across countries. This result is in stark contrast to the result in the deterministic Heckscher-Ohlin model, where two countries with different initial conditions end up with different levels of steady-state variables.

One key assumption in our model is the balanced trade condition. As already noted, in the non-stochastic case the requirement that trade be balanced period-by-period does not constrain equilibrium when both tradable commodities are produced in the economy. In a case with uncertainty, however, there is an incentive for the small economy to borrow and lend from the world economy. Not being able to do that, the small economy will try to smooth consumption by saving more when income is higher than expected, and less when income is lower than expected. In addition, the rate of return in the diversification cone is determined by the realization of the shocks (see equations (3.11) and (3.13)). Thus, even within the diversification cone, the incentive to accumulate or de-accumulate capital in the small economy is different from that of the world economy. Thus, the small economy can grow (or have negative growth) inside the diversification cone. This is an important distinction between the stochastic and non-stochastic versions.

Theorem 1 states that support of the invariant distribution is the unique non-degenerate interval given by $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. The lower boundary of the interval $k_{\underline{z}}^{\max}$ is the maximum fixed point of the policy function for the *worst* possible shocks, while the upper boundary is the minimum fixed point of the policy function for the *best* possible shocks. As shown in Proposition 2, this interval is non-degenerate, and, since $k_{\underline{z}}^{\max} > 0$, its lower boundary is strictly positive.

Figure 1 illustrates the fact that the invariant distribution is unique. In the figure, we have drawn two policy functions: one for the worst shock, $\underline{z}$, and the other for the best shock, $\overline{z}$. The capital-labour ratio in the shaded region, marked as the invariant set, is the full support for the invariant distribution. Any economy that has a capital-labour ratio in that region will remain there—the worst that can happen is that the economy will face the worst shock each period, and then its capital-labour ratio will converge to the lower boundary. A country's capital-labour ratio goes to the upper boundary in the best possible case when the country faces the best shock every period. Since the policy function, $h(k, z)$, is continuous and non-decreasing in z, and the shocks come from a full compact support, every non-degenerate interval of capital-labour ratios within the invariant set is attainable with positive probability. Consider the case, however, when the initial capital-labour ratio is below the minimum point of the interval $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. A sequence of good shocks, which happens with positive probability, will eventually bring the ratio inside the interval. The case when the capital-labour ratio is above the interval is symmetric. Thus, $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ will be the unique full support for the invariant distribution.

This analysis of the support also helps us to determine whether the economy will diversify. To make this determination, we need to find out whether there is any intersection between the support of the invariant set and the diversification cone. Recall that k_{at} and k_{mt} are the capital labour ratios in sectors a and m in the small economy, whenever it produces positive amounts of both intermediate goods. In Appendix A we show that they are strictly increasing, continuous functions of $\rho = \frac{p_a \lambda}{p_m \theta}$. The minimum values of k_a, k_m are therefore the ones that correspond to $z^* = (\underline{\lambda}, \overline{\theta})$, and the maximum values of k_a, k_m are the ones that correspond to $z^{**} = (\overline{\lambda}, \underline{\theta})$.

Theorem 2. ***Diversification***
The fixed points of the optimal policy function satisfy $k_{\overline{z}}^{\min} > k_a(z^*)$ *and* $k_{\underline{z}}^{\max} < k_m(z^{**})$.[2]

[2]Given constant prices p_a and p_m, there is a unique single-valued map from $z = (\lambda, \theta)$ to $\rho = \frac{p_a \lambda}{p_m \theta}$. This allows us to write $k_a(z)$ and $k_m(z)$.

The above theorem implies that there is a positive measure of z such that $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}] \cap [k_a(z), k_m(z)]$ is a non-degenerate interval, which, in turn, implies, in an infinite-horizon setting, that the support of the invariant set will intersect with the diversification cone in at least some periods. Thus, the small economy will diversify in some periods. This result also differs from the finding in the non-stochastic version, where a small economy starting from outside the diversification cone will permanently specialize.

4. Simulation of the Small Economy

So far, we have shown that with uncertainty and borrowing constraints there will be convergence, but our results do not indicate how quickly that convergence will occur. We simulate our model to make that determination. In the simulations, we assume that there is no uncertainty in the production of the intermediate good m, and that uncertainty is present only in the production of good a.[3] We assume that there are two possible states, high and low, with equal probability,[4] and then simulate our model for different magnitudes of shocks. We fix the mean of the shocks in sector a, λ_t, to be 1 and take different symmetric deviations from that, λ^H being the good shock and λ^L the bad shock. We find that the bigger are the possible shocks in the small economy, the quicker is the convergence. This is illustrated in Figures 3 and 4, where we report two cases: (i) deviation from the mean is 1 per cent, and (ii) deviation from the mean is 10 per cent.

We also simulate the economy when there is no uncertainty. Figure 2 plots the path for the capital. This simulation replicates the results of the Heckscher-Ohlin models without uncertainty: a country that starts with a capital-labour ratio of less than k_a grows until it reaches the lower boundary of the diversification cone, and then its capital-labour ratio is fixed at that level. The case with countries that start with a capital-labour ratio of greater than k_m is symmetric.

[3]None of our qualitative results change because of this assumption.
[4]Simulations with continuous state space give similar results.

The fact that convergence occurs more quickly as the magnitude of shocks increases relates our convergence result with that for non-convergence in the deterministic version. The finding suggests that, for small degrees of uncertainty, it will take an extremely long time for economies to converge. In the limit, when uncertainty is driven to zero, convergence disappears altogether. Thus, the deterministic Heckscher-Ohlin model is a very special case of the stochastic model.

Our simulation is useful in another dimension as well: it allows us to see the actual shape of the investment policy function. A plot of the policy function in Figure 5 reveals the effect of uncertainty and market incompleteness in our model. Recall that, in the deterministic case, the investment policy function coincides with the 45-degree line everywhere within the diversification cone; at every point, there is a fixed point and a steady state. With uncertainty, the policy functions for high and low shocks shift apart from each other, as a consequence of the representative agent's self-insurance motives to accumulate more (less) capital when income is higher (lower) than expected owing to a high (low) productivity shock. This pattern of capital accumulation makes it much less likely for the small economy to find itself in a fixed point of the investment policy function. Without uncertainty or under complete markets, self-insurance would not be necessary and multiple stated states would become a possibility.

The above discussion suggests that there is nothing special about the uncertainty being introduced by sector-specific productivity shocks. Other kinds of uncertainty, like those introduced by endowment shocks, or by economy-wide productivity shocks, should deliver similar convergence results. This intuition is correct and in section 5 we consider a version of the model with economy-wide productivity shocks. There is, however, another interesting property of the model with sector-specific productivity shocks: Figure 5 shows that both policy functions—for the high shock and the low shock—tilt clockwise relative to the 45-degree line. The tilt shrinks the invariant set (full support of the invariant distribution) and, in this particular simulation, makes it fit completely within the diversification cone, owing to changes in comparative advantage induced by sector-specific shocks. Recall that changes in sectoral shocks affect the boundaries of the diversification cone; thus, the small economy's own diversification cone shifts as various values of sectoral shocks are realized. To take full advantage of uncertain changes in comparative advantage, the small economy is induced to have a capital-labour ratio closer to the intersection of its own possible diversification cones. This intersection is smaller than the world economy's diversification cone.

It would be interesting to see what happens in the model where the changes in comparative advantage are absent and only self-insurance motives are at work. In section 5 we consider a model with economy-wide productivity shocks, and show that, for it, the entire world economy's diversification cone is a strict subset of the invariant set.

5. A Model with Economy-Wide Shocks

In this section, we assume that both intermediate sectors are affected by the same productivity shock.[5] That is, we impose a restriction that $\Lambda = \Theta$ and that λ and θ are perfectly correlated. It is immediately apparent that this model is a special case of the one with sector-specific shocks. The convergence result of Theorem 1 then immediately applies.

With an economy-wide productivity shock, the boundaries of the diversification cone of the small economy are fixed and coincide with the ones of the world economy. This becomes immediately apparent from equations (3.15) and (3.16) once we substitute λ for θ. Thus, in this case, for all z we have $k_a(z) = k_a^w$ and $k_m(z) = k_m^w$. We are ready to prove the following proposition.

Proposition 3. *If both intermediate sectors are affected by the same productivity*

[5] It is trivial to show that this set-up is equivalent to the —perhaps more intuitive— environment with productivity shocks on the final-good technology only. We stick with this interpretation for ease of comparison with the previous sections.

shock (e.g., λ_t), then the fixed points of the optimal policy function satisfy $k_{\underline{z}}^{\max} < k_a^w$ and $k_{\overline{z}}^{\min} > k_m^w$.

This proposition proves that, with economy-wide shocks, the entire diversification cone is a proper subset of the invariant set. Therefore, in the invariant distribution, a small economy may visit not only the entire diversification cone, but also some areas outside of it.

We simulate the model with economy-wide shocks to determine which kind of dynamics and policy functions the model will generate. Figure 6 shows the optimal policy functions for both high- and low-productivity shocks. Without sector-specific shocks, the entire diversification cone is in the invariant set. Figure 7 shows the dynamics of the capital-labour ratio for two small economies — one initially poor, and the other, initially rich — that face the same sequence of realized economy-wide shocks. They do converge, in accordance with our theoretical results, but it takes a lot of time. Further, in the invariant distribution the two small economies visit not only the entire diversification cone, but also areas outside of it, again in accordance with theoretical predictions.

The example with economy-wide shocks highlights again the role of sector-specific shocks: they make the invariant set smaller than the diversification cone, but they are not necessary for convergence. The self-insurance motive, owing to income uncertainty and market incompleteness, determines the convergence result.

Are country-specific shocks with market incompleteness necessary for convergence? Do countries have to be different? Could we prove similar convergence results with "global" technological shocks that affect all countries? Perhaps it is just uncertainty in income that matters, and not how it is introduced. The next section develops a model in which all countries are affected by the same productivity shocks. We show analytically that, in this particular example, countries do not converge and may permanently specialize in producing only one tradable good.

6. Analytical Example with No Convergence

In this example, we assume a different structure for uncertainty. We assume that both the world economy and the small economy face identical shocks; i.e., $\lambda_t^w = \lambda_t^s$ and $\theta_t^w = \theta_t^s$ for all t.

We use specific functional forms for the utility and the production functions. The utility function is logarithmic, $u(c) = \ln(c)$, and all production functions are Cobb-Douglas. The individual production functions are given by,

- final-good technology $H(a, m) = a^{\mu} m^{1-\mu}$,
- intermediate-good a technology $\lambda F(K, L) = \lambda K^{\alpha} L^{1-\alpha}$,
- intermediate-good m technology $\theta G(K, L) = \theta K^{\gamma} L^{1-\gamma}$,

where $1 > \gamma > \alpha > 0$. Further, we assume full depreciation (i.e., $\delta = 1$). Under these assumptions, we can find an analytical solution to the dynamic problems of both the world economy and the small economy. Since in this example we provide the dynamics of both the world and the small economy, we distinguish them using superscript w for the world economy and s for the small economy.

Given the specific functional forms, it is easy to show that, in the world economy, the allocation of labour across intermediate sectors is fixed:

$$L_{at}^w = \frac{(1-\alpha)\mu}{(1-\alpha)\mu + (1-\gamma)(1-\mu)}, \tag{6.1}$$

$$L_{mt}^w = \frac{(1-\gamma)(1-\mu)}{(1-\alpha)\mu + (1-\gamma)(1-\mu)}. \tag{6.2}$$

Further, optimal capital-labour ratios in both intermediate sectors of the world economy are proportional to the aggregate capital-labour ratio:

$$k_{at}^{w} = \frac{1}{L_a(1-\Omega)+\Omega} k_{t-1}^{w}, \tag{6.3}$$

$$k_{mt}^{w} = \frac{\Omega}{L_a(1-\Omega)+\Omega} k_{t-1}^{w}, \tag{6.4}$$

where $\Omega = \frac{\gamma(1-\alpha)}{\alpha(1-\gamma)} > 1$. Denote $\phi_a = \frac{1}{L_a(1-\Omega)+\Omega}$, and $\phi_m = \frac{\Omega}{L_a(1-\Omega)+\Omega}$. Thus, the capital-labour ratio in each sector is a constant fraction of the aggregate capital-labour ratio. Note that $\phi_a \in (0,1)$, while $\phi_m > 1$, a consequence of technology m being more capital intensive than technology a.

The optimal capital-labour ratio in the world economy evolves according to the following law of motion:

$$k_t^w = A_t \left(k_{t-1}^w\right)^q, \tag{6.5}$$

where $A_t = Q\lambda_t^{\mu}\theta_t^{1-\mu}$ is an aggregate productivity (Q is a fixed positive number) and $q = \alpha\mu + \gamma(1-\mu)$. Since $q < 1$, the world capital-labour ratio converges to a unique invariant distribution. The above law of motion for the world capital-labour ratio determines a Markov process for intermediate-good prices p_{at}, p_{mt}. Note that the prices of the intermediate goods are no longer constant across time.

Suppose that, at the beginning of period t, the small economy's capital-labour ratio is $k_{t-1}^s > 0$. Let $\tau_t = \frac{k_{t-1}^s}{k_{t-1}^w}$ be the capital-labour ratio in the small economy relative to that in the world economy. There are three possible cases to consider:

- If $\tau_t \leq \phi_a$, then $k_{t-1}^s \leq k_{at}^w = \phi_a k_{t-1}^w$. In this case, the small economy will produce only good a in period t. The optimal level of investment in this case will be $k_t^s = \alpha\beta p_{at}\lambda_t \left(k_{t-1}^s\right)^{\alpha} = \alpha\beta\tau_t^{\alpha} p_{at}\lambda_t \left(k_{t-1}^w\right)^{\alpha}$.

- If $\phi_a < \tau_t < \phi_m$, then $k^s_{t-1} \in (k^w_{at}, k^w_{mt})$ and the small economy will produce both goods, a and m, in period t. The optimal level of investment in this case will be $k^s_t = \alpha\beta\tau_t\phi_a^{\alpha-1}p_{at}\lambda_t\left(k^w_{t-1}\right)^{\alpha}$, or, equivalently, $k^s_t = \gamma\beta\tau_t\phi_m^{\gamma-1}p_{mt}\theta_t\left(k^w_{t-1}\right)^{\gamma}$.

- Finally, if $\tau_t \geq \phi_m$, the small economy will produce only good m in period t, and will invest $k^s_t = \gamma\beta\tau_t^{\gamma}p_{mt}\theta_t\left(k^w_{t-1}\right)^{\gamma}$.

This investment rule implies that, in the next period, $t+1$, the capital-labour ratio in the small economy relative to that in the world economy will depend on whether the small economy is inside the diversification cone. Thus, in period $t+1$,

- if $\tau_t < \phi_a$, then $\frac{k^s_t}{k^w_t} = \frac{\alpha\beta\tau_t^{\alpha}p_{at}\lambda_t\left(k^w_{t-1}\right)^{\alpha}}{\alpha\beta\phi_a^{\alpha-1}p_{at}\lambda_t\left(k^w_{t-1}\right)^{\alpha}} = \tau_t\left(\frac{\tau_t}{\phi_a}\right)^{\alpha-1} > \tau_t$,

- if $\tau_t \in [\phi_a, \phi_m]$, then $\frac{k^s_t}{k^w_t} = \frac{\alpha\beta\tau_t\phi_a^{\alpha-1}p_{at}\lambda_t\left(k^w_{t-1}\right)^{\alpha}}{\alpha\beta\phi_a^{\alpha-1}p_{at}\lambda_t\left(k^w_{t-1}\right)^{\alpha}} = \tau_t$,

- if $\tau_t > \phi_m$, then $\frac{k^s_t}{k^w_t} = \frac{\gamma\beta\tau_t^{\gamma}p_{mt}\theta_t\left(k^w_{t-1}\right)^{\gamma}}{\gamma\beta\phi_m^{\gamma-1}p_{mt}\theta_t\left(k^w_{t-1}\right)^{\gamma}} = \tau_t\left(\frac{\tau_t}{\phi_m}\right)^{\gamma-1} < \tau_t$.

Therefore, whenever the small economy has an aggregate capital-labour ratio outside the diversification cone $[k_{at}, k_{mt}]$, the optimal investment policy will push it closer to the diversification cone. If, on the other hand, the small economy starts within the diversification cone, it will maintain a constant ratio between the domestic aggregate capital-labour ratio, k^s_t, and the world aggregate capital-labour ratio, k^w_t. Thus, if two small economies start within the diversification cone, but with different capital-labour ratios relative to that of the world economy, they will maintain those relative positions. Hence, there is no convergence in capital or income. Also, if any small economy starts with capital-labour ratios outside the diversification cone, they will always specialize in the production of only one commodity.

The results are therefore the same as in the non-stochastic version of the dynamic Heckscher-Ohlin model:

- multiplicity of invariant distributions of capital,
- no income convergence, and
- permanent specialization in production.

The only difference between this example and the stochastic version considered earlier is that in this example both the small and the world economies face identical shocks; i.e., $z_t^w = z_t^s$ for all *t*. As a result of global shocks, there are no risk-sharing opportunities. Inside the diversification cone, the small economy and the world economy have the same return to capital. There is no incentive for borrowing or lending between the economies, and the trade balance constraint does not bind. Balanced trade is an equilibrium outcome in this case, as in the non-stochastic version.

The fact that the trade balance constraint binds, as countries realize different productivity shocks, is crucial for our convergence and diversification results. It is not just uncertainty that is important. Uncertainty occurs in this example, and yet the results are very similar to those for deterministic models.

On a more technical level, this example also shows the importance of i.i.d. shocks: it is the only assumption that is violated. As a result of the world economy being disturbed by shocks, intermediate-good prices p_{at} and p_{mt} follow a Markov process induced by the optimal capital accumulation of the world economy. It follows that the small economy faces autocorrelated productivity shocks $\lambda_t p_{at}$ and $\theta_t p_{mt}$, rather than the i.i.d. shocks in the previous sections. The example shows that, with suitably correlated shocks, there could be a multiplicity of invariant distributions of capital.

7. Conclusion

This paper shows that, in an uncertain world, when markets are not complete, different economies will have the same average long-run income irrespective of where they start from. This reverses the predictions of the deterministic dynamic Heckscher-Ohlin model. Our results extend the predictions of income convergence, standard in one-sector neoclassical growth models, to the dynamic multi-country Heckscher-Ohlin environment. In another departure from the deterministic dynamic Heckscher-Ohlin model, our results show there will be some periods in which a small open economy diversifies, even if it starts with a very low capital stock. In the deterministic model, countries can permanently specialize in producing a subset of tradable goods. We find that the restriction on risk-sharing opportunities imposed by the period-by-period balanced trade requirement is crucial for our results. Our results remain unchanged, however, if we allow for limited borrowing and lending.

The results of the deterministic version and the stochastic version seem to fit into two different extremes, but our simulation results give a sense of continuity between the two cases: the smaller the shocks are, the slower is the convergence, and in the limit when there is no uncertainty there is no convergence. The speed with which countries will catch up with each other depends on how much uncertainty exists in the world.

Further, we show that it is not necessary to assume that shocks are sector-specific in order to prove convergence, but that shocks play a role in contracting the set of possible capital-labour ratios observed in the long run.

Finally, we construct an example that shows the importance of country-specific shocks, and of the balanced trade constraint being binding. We show that, if global shocks affect all countries, the income convergence may disappear.

References

Aiyagari, S.R. 1994. "Uninsured Idiosyncratic Risk and Aggregate Saving." *Quarterly Journal of Economics* 109(3): 659–84.

Atkeson, A. and P.J. Kehoe. 2000. *Paths of Development for Early- and Late-Bloomers in a Dynamic Heckscher-Ohlin Model.* Federal Reserve Bank of Minneapolis, Research Department Staff Report No. 256 .

Brock, W.A. and L.J. Mirman. 1972. "Optimal Economic Growth And Uncertainty: The Discounted Case." *Journal of Economic Theory* 4: 479–13.

Chamberlain, G. and C. Wilson. 2000. "Optimal Intertemporal Consumption under Uncertainty." *Review of Economic Dynamics* 3: 365–95.

Chatterjee, P. and M.D. Shukayev. 2006. "Note on Positive Lower Bound of Capital in the Stochastic Growth Model." Available at: <http://www.bschool.nus.edu.sg/staff/bizpc/research/research.htm>.

Chen, Z. 1992. "Long-Run Equilibria in a Dynamic Heckscher-Ohlin Model." *Canadian Journal of Economics* 25(4): 923–43.

Clarida, R. 1987. "Consumption, Liquidity Constraints and Asset Accumulation in the Presence of Random Income Fluctuations." *International Economic Review* 28: 339–51.

Hopenhayn, H.A. and E.C. Prescott. 1992. "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies." *Econometrica* 60(6): 1387–1406.

Stokey, N., R. Lucas, and E. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Cambridge: Harvard University Press.

Ventura, J. 1997. "Growth and Interdependence." *Quarterly Journal of Economics* 112: 57–84.

Figure 1: Invariant Set

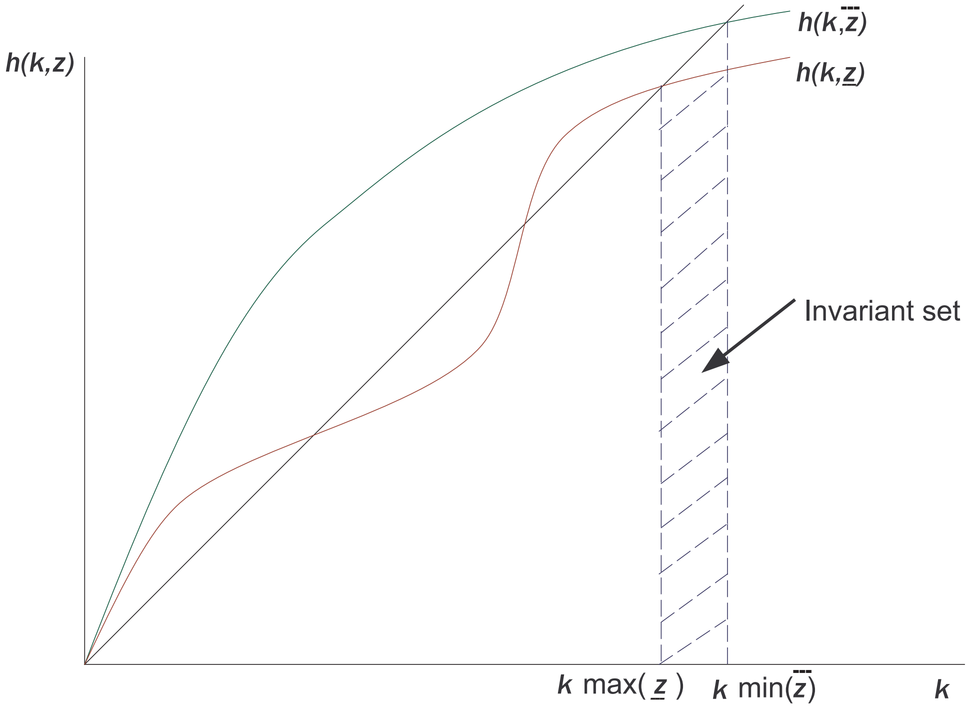


Figure 2: Path of Capital: No Uncertainty

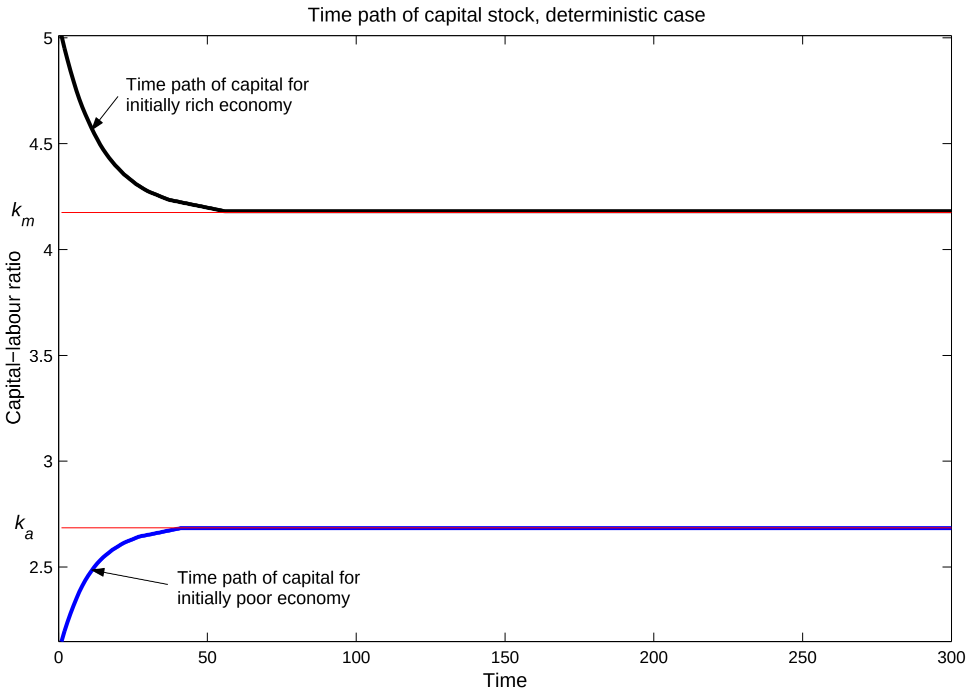


Figure 3: Path of Capital: 1 per cent Shock

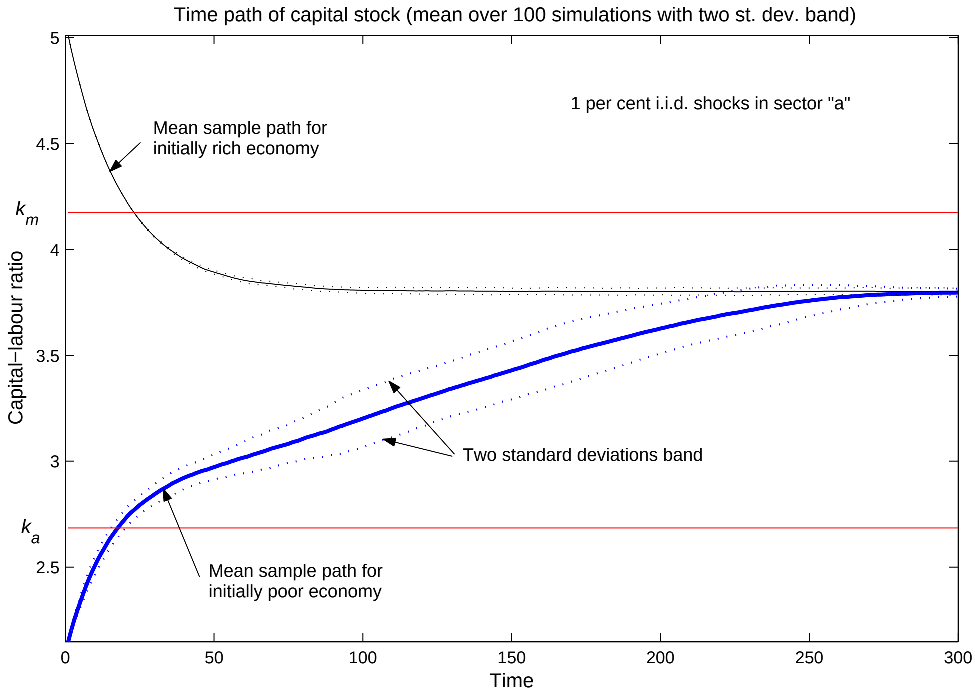


Figure 4: Path of Capital: 10 per cent Shock

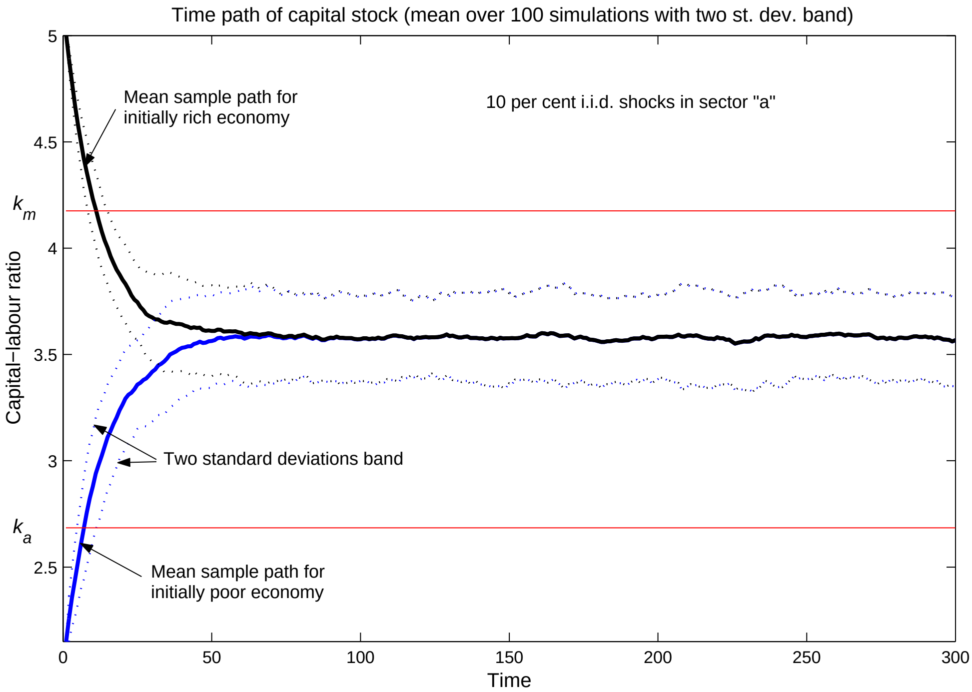


Figure 5: Policy Functions for "High" and "Low" Shocks

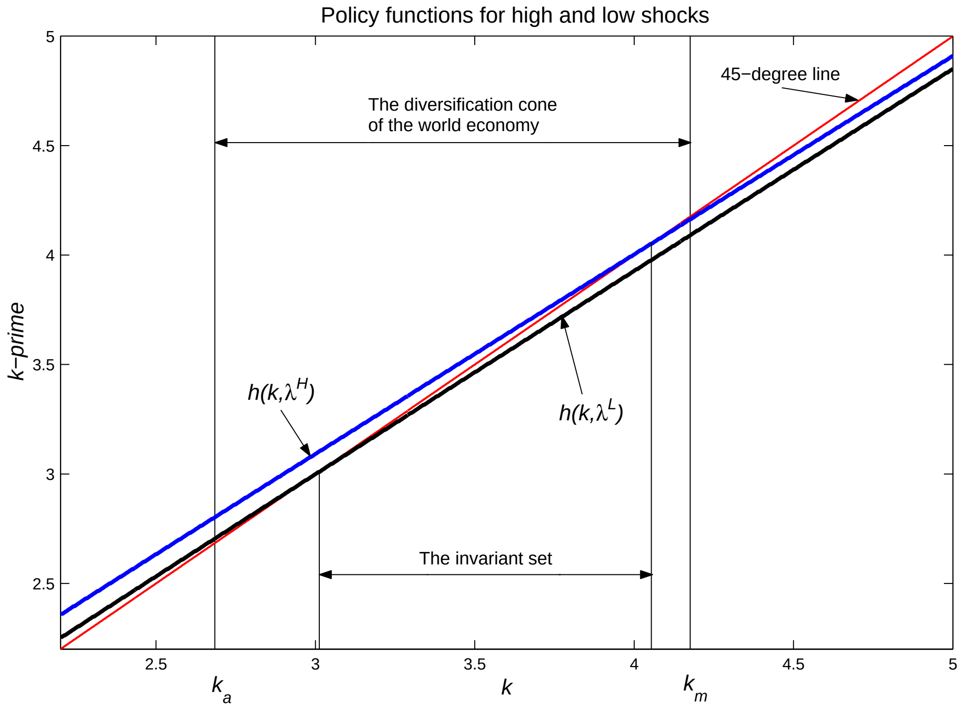


Figure 6: Policy Functions for “High” and “Low” Economy-Wide Shocks

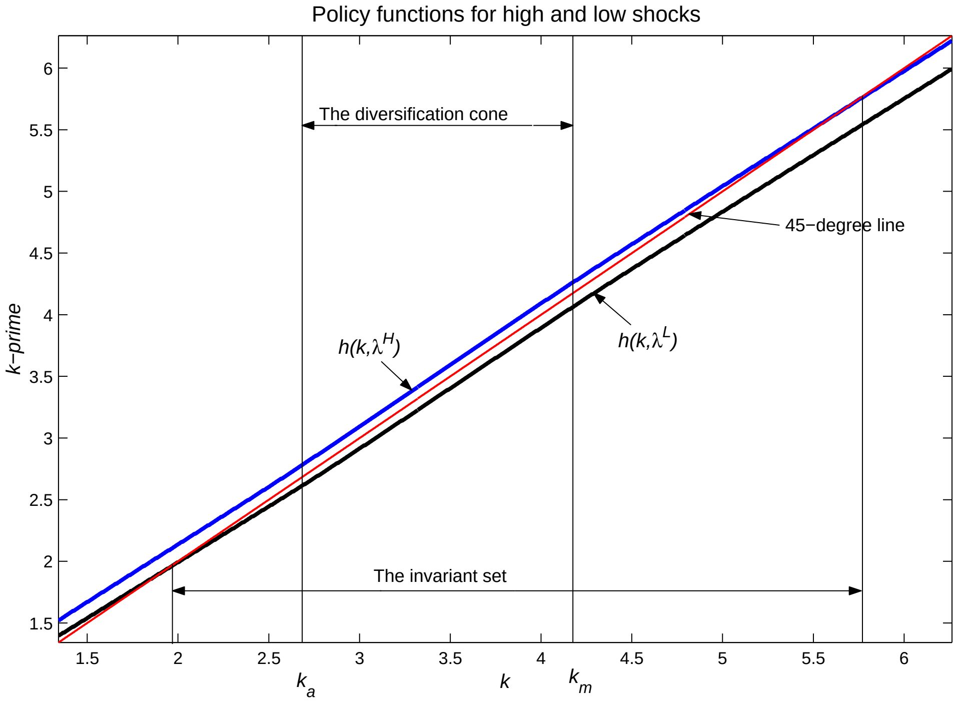
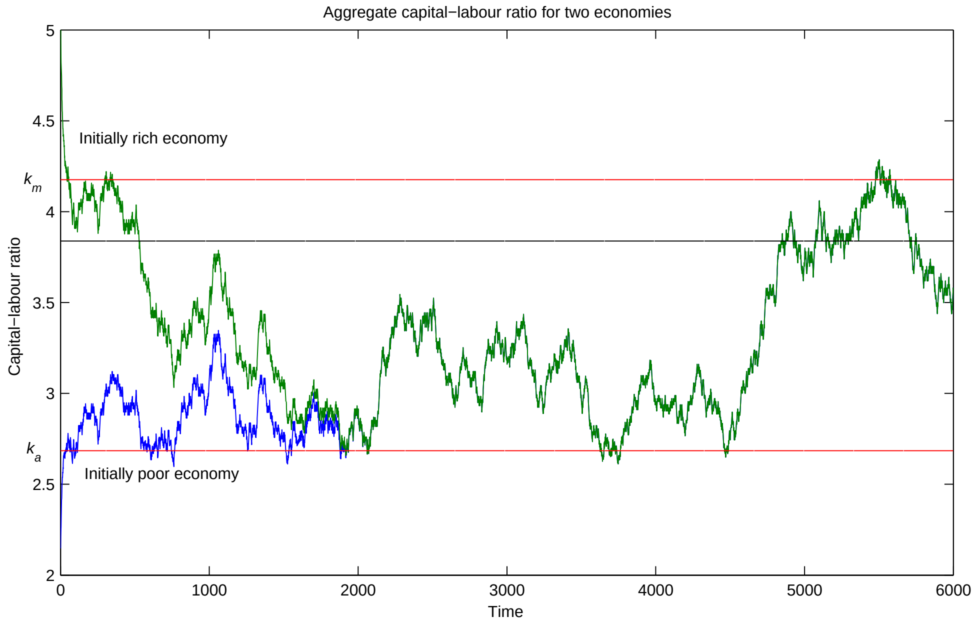


Figure 7: Capital Dynamics with Economy-Wide Shocks



Appendix A: Properties of the Diversification Cone Boundaries

In this appendix, we ignore time subscripts and the country superscripts, since, if the small economy produces both intermediate goods, the analysis applies to both economies.

To simplify the notation we denote:

$$\begin{aligned} x &= k_a, \\ y &= k_m, \\ \sigma &= p_a \lambda, \\ \chi &= p_m \theta, \\ \rho &= \frac{\sigma}{\chi}. \end{aligned}$$

We establish that when both intermediate sectors produce, the following equalities must hold:

$$r = \sigma f'(x) = \chi g'(y), \tag{A1}$$

$$w = \sigma \left[f(x) - f'(x)x \right] = \chi \left[g(y) - g'(y)y \right]. \tag{A2}$$

Dividing the second set of equalities by the first, we obtain

$$\frac{f(x) - f'(x)x}{f'(x)} = \frac{g(y) - g'(y)y}{g'(y)}.$$

This is an implicitly defined function, $y = \Omega(x)$, the derivative of which can be found using the implicit function theorem. For that define

$$\Phi(x,y)=\frac{f(x)}{f'(x)}-x-\frac{g(y)}{g'(y)}+y.$$

We then have:

$$\begin{aligned}\Omega'(x) &= -\frac{\Phi_1(x,y)}{\Phi_2(x,y)}=-\frac{\frac{[f'(x)]^2-f''(x)f(x)}{[f'(x)]^2}-1}{-\frac{[g'(y)]^2-g''(y)g(y)}{[g'(y)]^2}+1}\\ &= \frac{f''(x)f(x)}{[f'(x)]^2}\frac{[g'(y)]^2}{g''(y)g(y)}>0.\end{aligned}$$

Thus, k_m is a strictly increasing and continuous function of k_a. Also from the above equations, it is clear that, whenever $k_a=0$, so is $k_m=\Omega(k_a)$.

From (A1) and (A2), we can derive another expression that we will need later:

$$\rho f(x)-g(y)=\rho f'(x)x-g'(y)y=\rho f'(x)(x-y),$$

or, alternatively,

$$\rho f'(x)=\frac{g(y)-\rho f(x)}{y-x}. \tag{A3}$$

Once we have $\Omega'(x)$, we can use the equality $\sigma f'(x)=\chi g'(\Omega(x))$ to find x as a function of sectoral shocks. It is clear that x depends only on the ratio of sectoral shocks $\rho=\frac{\sigma}{\chi}$. Define a function of x and ρ :

$$\widetilde{\Phi}(x,\rho)=\rho f'(x)-g'(\Omega(x)).$$

By the implicit function theorem, we again can find the derivative of $x = \pi(\rho)$:

$$\begin{aligned}
\pi'(\rho) &= -\frac{\widetilde{\Phi}_2(x,\rho)}{\widetilde{\Phi}_1(x,\rho)} = -\frac{f'(x)}{\rho f''(x) - g''(\Omega(x))\Omega'(x)} \\
&= -\frac{f'(x)}{\rho f''(x) - g''(y)\frac{f''(x)f(x)}{[f'(x)]^2}\frac{[g'(y)]^2}{g''(y)g(y)}} \\
&= -\frac{f'(x)}{\rho f''(x) - \frac{f''(x)f(x)}{[f'(x)]^2}\frac{[g'(y)]^2}{g(y)}} \\
&= -\frac{f'(x)}{\rho f''(x)(1 - \frac{\rho f(x)}{g(y)})} \\
&= -\frac{g(y)}{\rho^2 f''(x)(y-x),}
\end{aligned}$$

where in the last two formulas we used the fact that $\frac{[g'(y)]^2}{[f'(x)]^2} = \rho^2$, and $g(y) - \rho f(x) = \rho f'(x)(y-x)$. The assumption of g technology being more capital intensive than f technology implies that $(y-x) > 0$. We therefore obtain $\pi'(\rho) > 0$. Hence, k_a and k_m are strictly increasing functions of ρ whenever both sectors are operated with positive inputs.

Appendix B: Proofs

B.1 Proof of Lemma 1

Fix some $z = (\lambda, \theta)$ and positive k.

If $k \in (0, k_a(z))$, then $y = p_a \lambda f(k) + (1-\delta)k$. Hence, $\frac{\partial y}{\partial k} = p_a \lambda f'(k) + 1 - \delta > 0$. Also, $\frac{\partial y}{\partial \lambda} = p_a f(k) > 0$ and $\frac{\partial y}{\partial \theta} = 0$. Thus, in $k < k_a(z)$, case monotonicity and continuity of y are established. For concavity of $y(\cdot, z)$ and continuity of $\frac{\partial y}{\partial k}$, observe that its $\frac{\partial^2 y}{\partial k^2} = p_a \lambda f''(k) < 0$, $\frac{\partial^2 y}{\partial k \partial \lambda} = p_a f'(k)$, $\frac{\partial^2 y}{\partial k \partial \theta} = 0$.

If $k > k_m(z)$, then $y = p_m \theta g(k) + (1-\delta)k$. Hence, $\frac{\partial y}{\partial k} = p_m \theta g'(k) + 1 - \delta > 0$. Also, $\frac{\partial y}{\partial \lambda} = 0$ and $\frac{\partial y}{\partial \theta} = p_m g(k) > 0$. Thus, in $k > k_m(z)$, case monotonicity and continuity of y are established. For concavity of $y(\cdot, z)$ and continuity of $\frac{\partial y}{\partial k}$, observe that $\frac{\partial^2 y}{\partial k^2} = p_m \theta g''(k) < 0$, $\frac{\partial^2 y}{\partial k \partial \lambda} = 0$, and $\frac{\partial^2 y}{\partial k \partial \theta} = p_m g'(k)$.

If $k \in (k_a(z), k_m(z))$, then $y = p_a \lambda f(k_a(z)) L_a + p_m \theta g(k_m(z)) L_m + (1-\delta)k$, where $L_a + L_m = 1$ and $k_a(z) L_a + k_m(z) L_m = k$. By the assumption of perfect competition, we know that optimal k_a, k_m, L_a, L_m maximize y. Therefore, by the envelope theorem, we have $\frac{\partial y}{\partial \lambda} = p_a f(k_a(z)) L_a > 0$. Similarly, $\frac{\partial y}{\partial \theta} = p_m g(k_m) L_m > 0$. It is also easy to see that $\frac{\partial y}{\partial k} = p_a \lambda f'(k_a(z)) + 1 - \delta = p_m \theta g'(k_m(z)) + 1 - \delta > 0$. This establishes monotonicity and continuity results for y. From Appendix A it follows that, for all $k_a(z) < k < k_m(z)$, $\frac{\partial r}{\partial (p_a \lambda)}$ and $\frac{\partial r}{\partial (p_m \theta)}$ are well-defined finite numbers. This, along with the fact that $r = p_a \lambda f'(k_a(z))$, implies that the derivative $\frac{\partial y}{\partial k}$ is continuous in λ and θ. To establish concavity of $y(\cdot, z)$ and continuity of $\frac{\partial y}{\partial k}$ in k, observe that $\frac{\partial^2 y}{\partial k^2} = \frac{\partial r}{\partial k} = 0$ by the fact that interest rate r is independent of k in the diversification cone.

We need to check that when $k = k_a(z)$ or $k = k_m(z)$, the left and right limits of y and its three partial derivatives are equal. They are equal, since $L_a = 1$ when $k = k_a(z)$, and $L_m = 1$ when $k = k_m(z)$.

Since k and z are arbitrary, we establish monotonicity and continuity of y and $\frac{\partial y}{\partial k}$ everywhere in the domain.

Finally, the existence of the upper bound $\overline{k}$ on the set of sustainable capital-labour ratios is implied by the Inada conditions: $\lim_{k\to\infty} p_a\overline{\lambda}f'(k) = \lim_{k\to\infty} p_m\overline{\theta}g'(k) = 0$.

B.2 Proof of Proposition 1

The assumptions on the utility function $u(c)$ place this problem into the domain of "Bounded Return Problems," as defined in section 9.2 of Stokey, Lucas, and Prescott (1989). It is straightforward to verify that their assumptions 9.4 to 9.12 are satisfied by our model. The results of the first part of Proposition 1 then follow from theorems 9.6, 9.7, 9.8, and 9.10 in Stokey, Lucas, and Prescott (1989).

It is obvious that $c(0, z) = 0$ and $h(0, z) = 0$. It is easy to show that both policy functions are strictly increasing, continuous functions of y.[1] Therefore, these policy functions inherit all the continuity and monotonicity properties of y.

B.3 Proof of Proposition 2

The proof of the main theorem in Chatterjee and Shukayev (2006) can be applied to show that the policy function $h(k, \underline{z}) = 0$ has at least one positive and stable fixed point for the worst possible shock. Once this is established, the first two results of the proposition follow trivially from monotonicity and boundedness of the investment policy function.

To prove the last result, we will show that $k_{\underline{z}} < k_{\overline{z}}$ for any fixed points of $h(k, \underline{z})$ and $h(k, \overline{z})$ correspondingly. To show this, we first prove the following two claims:

[1] For example, see proofs of lemmas 1.1 and 1.2 in Brock and Mirman (1972).

Claim 1: For any fixed point $k_{\overline{z}}$ of $h(k, \overline{z})$, we have $1 > \beta \int_Z y'(k_{\overline{z}}, z)\eta(dz)$.

Proof: From the Euler equation, we have

$$u'(c(k_{\overline{z}}, \overline{z})) = \beta \int_Z u'(c(k_{\overline{z}}, z))y'(k_{\overline{z}}, z)\eta(dz).$$

Since $u'(c(k_{\overline{z}}, z)) \geq u'(c(k_{\overline{z}}, \overline{z}))$, with strict inequality for some $z \in Z$,

$$\begin{aligned} u'(c(k_{\overline{z}}, \overline{z})) &> \beta u'(c(k_{\overline{z}}, \overline{z})) \int_Z y'(k_{\overline{z}}, z)\eta(dz) \\ 1 &> \beta \int_Z y'(k_{\overline{z}}, z)\eta(dz). \end{aligned}$$

Claim 2: For any fixed point $k_{\underline{z}}$ of $h(k, \underline{z})$, we have $1 < \beta \int_Z y'(k_{\underline{z}}, z)\eta(dz)$.

Proof: From the Euler equation, we have

$$u'(c(k_{\underline{z}}, \underline{z})) = \beta \int_Z u'(c(k_{\underline{z}}, z))y'(k_{\underline{z}}, z)\eta(dz).$$

Since $u'(c(k_{\underline{z}}, z)) \leq u'(c(k_{\underline{z}}, \underline{z}))$, with strict inequality for some $z \in Z$,

$$\begin{aligned} u'(c(k_{\underline{z}}, \underline{z})) &< \beta u'(c(k_{\underline{z}}, \underline{z})) \int_Z y'(k_{\underline{z}}, z)\eta(dz) \\ 1 &< \beta \int_Z y'(k_{\underline{z}}, z)\eta(dz). \end{aligned}$$

The above two claims, along with the fact that $y'(k, z)$ is decreasing in k for every value of z, establish $k_{\underline{z}} < k_{\overline{z}}$.

B.4 Proof of Theorem 1

We prove Theorem 1 by showing that Theorem 2 of Hopenhayn and Prescott (1992) can be applied to our model. We first need to check that the optimal capital sequence $\{k_t = h(k_{t-1}, z_t)\}_{t=1}^{\infty}$ is bounded away from zero. The proof of the main theorem in Chatterjee and Shukayev (2006) can be applied to show that, for any $k_0 > 0$, there exists $\underline{k} \in (0, k_0)$ such that for all $t = 1, 2, 3, ..., k_t \geq \underline{k}$. Thus, without loss of generality, we can take $X = [\underline{k}, \overline{k}]$. We show next that the three assumptions of the Theorem 2 of Hopenhayn and Prescott (1992) are satisfied:

- The domain set X contains its lower and upper bounds.
  Since $X = \left[\underline{k}, \overline{k}\right]$ is a compact set, it satisfies this assumption.

- The transition probability $P(k, B)$ is increasing in k in the sense of first-order stochastic dominance.
  Since $h(k, z)$ is increasing in k for every z, $P(k, B)$ is indeed increasing.

- Monotone Mixing Condition: there exist some $\tilde{k} \in X$ and an integer M such that $P^M(\overline{k}, [\underline{k}, \tilde{k}]) > 0$ and $P^M(\underline{k}, [\tilde{k}, \overline{k}]) > 0$.

For brevity, let us define $y' = \frac{\partial y}{\partial k}$.

Consider the set $\tilde{K} = \{k \in X | \ \beta \int_Z y'(k, z)\eta(dz) = 1\}$. Continuity and monotonicity of $y'(\cdot, z)$ for every z guarantee that $\tilde{K}$ is non-empty, although, in general, it may contain more than one point. Let $\tilde{k}$ be any point in $\tilde{K}$. Let the sequence $\{k_n\}_{n=0}^{\infty}$ be generated as $k_n = h(k_{n-1}, \underline{z})$ with $k_0 = \overline{k}$. By the monotonicity of optimal policy rule, $\{k_n\}$ is decreasing, and we know from Proposition 2 that $k_n \to k_{\underline{z}}^{\max}$. For any $\varepsilon > 0$, the rectangle $[(\underline{\lambda}, \underline{\theta}), (\underline{\lambda} + \varepsilon, \underline{\theta} + \varepsilon)]$ has a positive measure under η. This, together with the continuity of $h(k, \cdot)$, implies that the probability of entering into any neighbourhood of $k_{\underline{z}}^{\max}$ in a finite number of steps is positive.

From Claim 2 in the proof of Proposition 2, we have $1 < \beta \int_Z y'(k_{\underline{z}}^{\max}, z)\eta(dz)$. Hence, $k_{\underline{z}}^{\max} < \tilde{k}$. An exactly symmetric line of argument establishes that $k_{\overline{z}}^{\min} > \tilde{k}$ and that the sequence $\{k_n\}_{n=0}^{\infty}$ started from $k_0 = \underline{k}$ enters with positive probability into any neighbourhood of $k_{\overline{z}}^{\min}$ in a finite number of steps. The above results prove that there exists some integer M such that $P^M(\overline{k}, [\underline{k}, \tilde{k}]) > 0$ and $P^M(\underline{k}, [\tilde{k}, \overline{k}]) > 0$. Thus, all the assumptions of Theorem 2 in Hopenhayn and Prescott (1992) are satisfied, which establishes the desired convergence result.

The full support for this invariant distribution is $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. To see this fact observe that the sequence $\{k_n\}_{n=0}^{\infty}$ generated as $k_n = h(k_{n-1}, z)$, started from any $k_0 > k_{\underline{z}}^{\max}$, enters with positive probability into any neighbourhood of $k_{\underline{z}}^{\max}$. Similarly, $\{k_n\}_{n=0}^{\infty}$ generated as $k_n = h(k_{n-1}, z)$, started from any $k_0 < k_{\overline{z}}^{\min}$, enters with positive probability into any neighbourhood of $k_{\overline{z}}^{\min}$. It is also clear that, once in $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$, the Markov process $P^t(k_0, \cdot)$ cannot leave this set. Thus, $k_{\underline{z}}^{\max}$ and $k_{\overline{z}}^{\min}$ must be the boundaries of the ergodic set. To show that the whole interval $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ is an ergodic set, choose any open interval $(k^1, k^2) \in [k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$ of a certain length $l > 0$, and any point $k_0 \in [k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$. Without loss of generality, assume that $k_0 < k^1$. Observe that, for any $k \in (k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min})$, the image $h(k, Z)$ is a non-degenerate interval $[h(k, \underline{z}), h(k, \overline{z})]$, such that k belongs to the interior of this interval. We can then construct an increasing sequence, $k_n = h(k_{n-1}, z_{n-1})$, such that $0 < \frac{\varepsilon}{2} < |k_n - k_{n-1}| < \varepsilon < \frac{l}{2}$. Clearly, this sequence will enter (k^1, k^2) in a finite number of steps, say in N steps. By continuity of $h(\cdot, \cdot)$, this sequence can be constructed with a positive measure of shock histories: $z^N = (z_0, z_1, ..., z_N) \in Z \times Z \times ... \times Z$ (N times). Obviously, for $k_0 > k^2$, we can construct a decreasing sequence. We have therefore proved that $P^N(k_0, (k^1, k^2)) > 0$ for some finite N. This establishes irreducibility, and hence the ergodicity of $[k_{\underline{z}}^{\max}, k_{\overline{z}}^{\min}]$.

B.5 Proof of Theorem 2

We will prove the following two claims, which together with Claims 1 and 2 in the proof of Proposition 2 establish that $k_{\overline{z}}^{\min} > k_a(z^*)$ and $k_{\underline{z}}^{\max} < k_m(z^{**})$.

Claim 3: If $k \leq k_a(z^*)$, then $1 \leq \beta \int_Z y'(k,z)\eta(dz)$.

Proof: For all $z \in Z$, we have $k \leq k_a(z^*) \leq k_a(z)$ and $k \leq k_a(z^*) \leq k_a^w$. Hence,

$$\begin{aligned}\beta \int_Z y'(k,z)\eta(dz) &= \beta \int_Z \left[p_a \lambda(z) f'(k) + 1 - \delta\right] \eta(dz) \\ &= \beta \left[p_a f'(k) + 1 - \delta\right] \geq \beta \left[p_a f'(k_a^w) + 1 - \delta\right] = 1.\end{aligned}$$

Claim 4: If $k \geq k_m(z^{**})$, then $1 \geq \beta \int_Z y'(k,z)\eta(dz)$.

Proof: For all $z \in Z$, we have $k \geq k_m(z^{**}) \geq k_m(z)$ and $k \geq k_m(z^{**}) \geq k_m^w$. Hence,

$$\begin{aligned}\beta \int_Z y'(k,z)\eta(dz) &= \beta \int_Z \left[p_m \theta(z) g'(k) + 1 - \delta\right] \eta(dz) \\ &= \beta \left[p_m g'(k) + 1 - \delta\right] \leq \beta \left[p_m g'(k_m^w) + 1 - \delta\right] = 1.\end{aligned}$$

Claim 2 of Proposition 2 and Claim 4 in this appendix establish that $k_{\underline{z}}^{\max} < k_m(z^{**})$, while Claim 1 of Proposition 2 and Claim 3 in this appendix prove that $k_{\overline{z}}^{\min} > k_a(z^*)$.

B.6 Proof of Proposition 3

We will prove that, for all $k \in [k_a^w, k_m^w]$, the following must be true: $\beta \int_Z y'(k,z)\eta(dz) = 1$. Once that is established, the results of the proposition follow from Claims 1 and 2 in the proof of Proposition 2.

Fix any $k \in [k_a^w, k_m^w]$. We then have:

$$\begin{aligned}
\beta \int_Z y'(k,z)\eta(dz) &= \beta \int_Z \left[p_a \lambda(z) f'(k_a^w) + 1 - \delta\right] \eta(dz) \\
&= \beta \left[p_a f'(k_a^w) + 1 - \delta\right] = 1.
\end{aligned}$$

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