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Convergence in Hölder norms with applications to Monte Carlo methods in infinite dimensions

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We show that if a sequence of piecewise affine linear processes converges in the strong sense with a positive rate to a stochastic process that is strongly Hölder continuous in time, then this sequence converges in the strong sense even with respect to much stronger Hölder norms and the convergence rate is essentially reduced by the Hölder exponent. Our first application hereof establishes pathwise convergence rates for spectral Galerkin approximations of stochastic partial differential equations. Our second application derives strong convergence rates of multilevel Monte Carlo approximations of expectations of Banach-space-valued stochastic processes.

1. Introduction

In this article we study convergence rates for general stochastic processes in Hölder norms. In particular, in the main results of this work (see Corollaries 2.8 and 2.9) we reveal estimates for uniform Hölder errors of general stochastic processes. In this introductory section we now sketch these results and thereafter outline several applications of the general estimates, which can be found in subsequent sections of this article (see Corollaries 2.11, 4.5 and 5.15). To illustrate the key results of this work, we consider the following framework throughout this section. Let $T \in (0,\infty)$ be a real number, let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, let $\big(E,\|\cdot\|_E\big)$ be an $\mathbb{R}$-Banach space and for every function

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$f\colon [0,T] \to E$ and every natural number $N \in \mathbb{N} = \{1,2,3,\ldots\}$ let $[f]_N\colon [0,T] \to E$ be the function that satisfies for all $n \in \{0,1,\ldots,N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$[f]_N(t) = \big(n+1-\tfrac{tN}{T}\big)\cdot f\big(\tfrac{nT}{N}\big) + \big(\tfrac{tN}{T}-n\big)\cdot f\big(\tfrac{(n+1)T}{N}\big) \tag{1.1}$$

(the piecewise affine linear interpolation of $f|_{\{0,T/N,2T/N,\ldots,(N-1)T/N,T\}}$; cf. (1.19) below).

Theorem 1.1 Assume the above setting. Then for all $p \in (1,\infty)$, $\varepsilon \in (1/p, 1]$, $\alpha \in [0, \varepsilon - 1/p)$ there exists $C \in (0,\infty)$ such that it holds for all $\beta \in [\varepsilon, 1]$, $N \in \mathbb{N}$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths that

$$\begin{aligned}&\Big(\mathbb{E}\Big[\|X - [Y]_N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\\ &\quad\le CN^{\varepsilon}\bigg(\sup_{n\in\{0,1,\ldots,N\}}\big\|X_{\frac{nT}{N}} - Y_{\frac{nT}{N}}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} + N^{-\beta}\|X\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\bigg).\end{aligned} \tag{1.2}$$

The Hölder and $\mathscr{L}^p$-norms in (1.2) are to be understood in the usual sense (see Section 1.1 below for details). Theorem 1.1 is a direct consequence of the more general result in Corollary 2.9, which establishes an estimate similar to (1.2) also for the case of nonequidistant time grids. Moreover, Corollary 2.8 provides an estimate similar to (1.2) but with $\big(\mathbb{E}\big[\|X - Y\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big]\big)^{1/p}$ instead of $\big(\mathbb{E}\big[\|X-[Y]_N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big]\big)^{1/p}$ on the left-hand side and with an appropriate Hölder norm of Y occurring on the right-hand side. Theorem 1.1 has a number of applications in the numerical approximation of stochastic processes, as the next corollary, Corollary 1.2, clarifies. Corollary 1.2 follows immediately from Theorem 1.1.

Corollary 1.2 Assume the above setting, let $\beta \in (0,1]$ and let $X\colon [0,T] \times \Omega \to E$ and $Y^N\colon [0,T] \times \Omega \to E$, $N \in \mathbb{N}$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths that satisfy for all $p \in (1,\infty)$ that $\forall N \in \mathbb{N}\colon Y^N = [Y^N]_N$ and

$$\|X\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}}\bigg[N^\beta \sup_{n\in\{0,1,\ldots,N\}}\big\|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\bigg] < \infty. \tag{1.3}$$

Then it holds for all $p, \varepsilon \in (0,\infty)$ that

$$\sup_{N\in\mathbb{N}}\bigg[N^{\beta-\varepsilon}\bigg(\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_t - Y^N_t\|^p_E\Big]\bigg)^{1/p}\bigg] < \infty. \tag{1.4}$$

It is assumed in (1.3) that a sequence of affine linearly interpolated $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $(Y^N)_{N\in\mathbb{N}}$ converges for every $p \in (1,\infty)$ in $\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)$ to an $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic process X with a positive rate uniformly on all grid points and that this process X admits corresponding temporal Hölder regularity. Corollary 1.2 then shows that these assumptions are sufficient to obtain convergence for every $p \in (1,\infty)$ in the uniform $L^p(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm with essentially the same rate. Corollary 2.11 implies this result as a special case and includes the case of nonequidistant time grids. Moreover, Corollary 2.11 proves an analogous conclusion for convergence in uniform Hölder norms, where the obtained convergence rate is reduced by the considered

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Hölder exponent. Corollary 2.12 demonstrates how this principle can be applied to Euler–Maruyama approximations for stochastic differential equations (SDEs) with globally Lipschitz coefficients. Arguments related to Corollary 2.11 can be found in Bally *et al.* (1995, Lemma A1) and Cox & van Neerven (2013, second display on p. 325).

Corollary 1.2 is particularly useful for the study of stochastic partial differential equations (SPDEs). In general, a solution of an SPDE fails to be a semimartingale. As a consequence, Doob's maximal inequality cannot be applied to obtain estimates with respect to the $L^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. However, convergence rates with respect to the $C([0,T], \|\cdot\|_{L^2(\mathbb{P};\|\cdot\|_E)})$-norm are often feasible and Corollary 1.2 can then be applied to obtain convergence rates with respect to the $L^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. Estimates with respect to the $L^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm are useful for using standard localisation arguments in order to extend results for SPDEs with globally Lipschitz continuous nonlinearities to results for SPDEs with nonlinearities that are only Lipschitz continuous on bounded sets. We demonstrate this in Corollary 4.5 in the case of pathwise convergence rates for Galerkin approximations. To be more specific, Corollary 4.5 proves essentially sharp pathwise convergence rates for spatial Galerkin and noise approximations for a large class of SPDEs with nonglobally Lipschitz continuous nonlinearities. For example, Corollary 4.5 applies to stochastic Burgers, stochastic Ginzburg–Landau, stochastic Kuramoto–Sivashinsky and Cahn–Hilliard–Cook equations.

Another prominent application of Corollary 1.2 is multilevel Monte Carlo methods in Banach spaces. For a random variable $X \in \mathscr{L}^2(\mathbb{P}; \|\cdot\|_E)$ convergence in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_E)$ of Monte Carlo approximations of the expectation $\mathbb{E}[X] \in E$ has only been established if E has so-called (Rademacher) type p for some $p \in (1, 2]$ and in this case the convergence rate is given by $1 - {}^1\!/p$ (see, e.g., Heinrich, 2001 or Corollary 5.12). However, the space $C([0,T], E)$ fails to have type p for any $p \in (1, 2]$. If X has more sample path regularity, this problem can nevertheless be bypassed. More precisely, if it holds for some $\alpha \in (0, 1]$, $p \in ({}^1\!/\alpha, \infty)$ that $X \in \mathscr{L}^2(\mathbb{P}; \|\cdot\|_{\mathcal{W}^{\alpha,p}([0,T],E)})$, then Monte Carlo approximations of $\mathbb{E}[X] \in \mathcal{W}^{\alpha,p}([0,T], E)$ have been shown to converge in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{\mathcal{W}^{\alpha,p}([0,T],E)})$ with rate $1 - {}^1\!/\min\{2,p\}$ and, by the Sobolev embedding theorem, also converge in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$ with the same rate. Here, for any real numbers $\alpha \in (0, 1]$, $p \in ({}^1\!/\alpha, \infty)$ we denote by $\mathcal{W}^{\alpha,p}([0,T], E)$ the Sobolev space with regularity parameter α and integrability parameter p of continuous functions from $[0,T]$ to E. Informally speaking, in order to gain control over the variances appearing in multilevel Monte Carlo approximations it is therefore sufficient for the approximations to converge with respect to the $L^2(\mathbb{P}; \|\cdot\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)})$-norm for some $\alpha \in (0, 1]$. For more details, we refer the reader to Section 5 and, in particular, to Corollary 5.15, which formalises this approach for the case of multilevel Monte Carlo approximations of expectations of Banach-space-valued stochastic processes.

Finally we mention a few results in the literature that employ some findings from this article. In particular, Corollary 2.10 in this article is applied in the proof of Jentzen & Pušnik (2015, Corollary 6.3) to prove uniform convergence in probability for spatial spectral Galerkin approximations of stochastic evolution equations (SEEs) with semiglobally Lipschitz continuous coefficients (see Jentzen & Pušnik, 2015, Proposition 6.4). Moreover, Corollary 4.4 in this article is employed in Cox *et al.* (2013, Sections 5.2 and 5.3) for transferring initial value regularity results for finite-dimensional SDEs to the case of infinite-dimensional SPDEs using the examples of the stochastic Burgers equation and the Cahn–Hilliard–Cook equation. Furthermore, Corollary 2.11 in this article is used in the proof of Hutzenthaler *et al.* (2018a, Corollary 5.2) to establish essentially sharp uniform strong convergence rates for spatial spectral Galerkin approximations of linear stochastic heat equations.

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1.1 *Notation*

In this subsection we introduce some of the notation that we use throughout this article. For two sets A and B we denote by $\mathbb{M}(A,B)$ the set of all mappings from A to B. For two sets A and B and a mapping $f \in \mathbb{M}(A,B)$ we denote by $f(A) \subseteq B$ the image of f. For measurable spaces $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ we denote by $\mathscr{M}(\mathscr{F}_1, \mathscr{F}_2)$ the set of all $\mathscr{F}_1/\mathscr{F}_2$-measurable mappings from Ω_1 to Ω_2. For topological spaces $(E, \mathscr{E})$ and $(F, \mathscr{F})$ we denote by $\mathscr{B}(E)$ the Borel σ-algebra on $(E, \mathscr{E})$ and we denote by $C(E,F)$ the set of all continuous functions from E to F. We denote by $|\cdot| : \mathbb{R} \to [0,\infty)$ the absolute value function on $\mathbb{R}$. We denote by $\Gamma\colon (0,\infty) \to (0,\infty)$ the gamma function, that is, we denote by $\Gamma\colon (0,\infty) \to (0,\infty)$ the function that satisfies for all $x \in (0,\infty)$ that $\Gamma(x) = \int_0^\infty t^{(x-1)}\, e^{-t}\,\mathrm{d}t$. We denote by $\mathscr{E}_r\colon [0,\infty) \to [0,\infty)$, $r \in (0,\infty)$, the mappings that satisfy for all $r \in (0,\infty)$, $x \in [0,\infty)$ that

$$\mathscr{E}_r[x] = \left(\sum_{n=0}^{\infty} \frac{x^{2n}(\Gamma(r))^n}{\Gamma(nr+1)} \right)^{\frac{1}{2}} = \left(1 + \frac{x^2\,\Gamma(r)}{\Gamma(r+1)} + \frac{x^4(\Gamma(r))^2}{\Gamma(2r+1)} + \ \ldots \ \right)^{\frac{1}{2}} \tag{1.5}$$

(cf. Henry, 1981, Chapter 7). As a notational device to condense the statements and proofs of many results in this article in a mathematically rigorous way, we next introduce the notion of an extendedly seminormed vector space, which, roughly speaking, corresponds to a vector space with a seminorm-type function that is allowed to attain infinity. For a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a $\mathbb{K}$-vector space V and a mapping $\|\cdot\| : V \to [0,\infty]$ that satisfies for all $v, w \in \{u \in V\colon \|u\| < \infty\}$, $\lambda \in \mathbb{K}$ that $\|\lambda v\| = \sqrt{[\mathrm{Re}(\lambda)]^2 + [\mathrm{Im}(\lambda)]^2}\,\|v\|$ and $\|v + w\| \leq \|v\| + \|w\|$ we call $\|\cdot\|$ an extended seminorm on V and we call $(V, \|\cdot\|)$ an extendedly seminormed vector space. For a measure space $(\Omega, \mathscr{F}, \mu)$, a measurable space $(S, \mathscr{S})$, a set $R \subseteq S$ and a function $f\colon \Omega \to R$ we denote by $[f]_{\mu,\mathscr{S}}$ the set given by

$$[f]_{\mu,\mathscr{S}} = \big\{ g \in \mathscr{M}(\mathscr{F}, \mathscr{S})\colon (\exists\, A \in \mathscr{F}\colon \mu(A) = 0 \text{ and } \{\omega \in \Omega : f(\omega) \neq g(\omega)\} \subseteq A) \big\}. \tag{1.6}$$

For a measure space $(\Omega, \mathscr{F}, \mu)$, a normed vector space $(V, \|\cdot\|_V)$ and real numbers $p \in [0,\infty)$, $q \in (0,\infty)$ we denote by $\mathscr{L}^0(\mu; \|\cdot\|_V)$ the set given by

$$\mathscr{L}^0(\mu; \|\cdot\|_V) = \big\{ f \in \mathbb{M}(\Omega, V) : f \text{ is } (\mathscr{F}, \|\cdot\|_V)\text{-strongly measurable} \big\}, \tag{1.7}$$

we denote by $\|\cdot\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} : \mathscr{L}^0(\mu; \|\cdot\|_V) \to [0,\infty]$ the mapping that satisfies for all $f \in \mathscr{L}^0(\mu; \|\cdot\|_V)$ that

$$\|f\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} = \left[\int_\Omega \|f(\omega)\|_V^q \, \mu(\mathrm{d}\omega) \right]^{1/q} \in [0,\infty], \tag{1.8}$$

we denote by $\mathscr{L}^q(\mu; \|\cdot\|_V)$ the set given by

$$\mathscr{L}^q(\mu; \|\cdot\|_V) = \left\{ f \in \mathscr{L}^0(\mu; \|\cdot\|_V)\colon\ \|f\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} < \infty \right\}, \tag{1.9}$$

we denote by $L^p(\mu; \|\cdot\|_V)$ the set given by

$$L^p(\mu; \|\cdot\|_V) = \big\{ \{g \in \mathscr{L}^0(\mu; \|\cdot\|_V) : \mu(f \neq g) = 0\} \subseteq \mathscr{L}^0(\mu; \|\cdot\|_V) : f \in \mathscr{L}^p(\mu; \|\cdot\|_V) \big\} \tag{1.10}$$

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and we denote by $\|\cdot\|_{L^q(\mu;\|\cdot\|_V)}\colon L^0(\mu;\|\cdot\|_V) \to [0,\infty]$ the function that satisfies for all $f \in \mathscr{L}^0(\mu;\|\cdot\|_V)$ that

$$\big\|\{g \in \mathscr{L}^0(\mu;\|\cdot\|_V)\colon \mu(f \neq g) = 0\}\big\|_{L^q(\mu;\|\cdot\|_V)} = \|f\|_{\mathscr{L}^q(\mu;\|\cdot\|_V)} \in [0,\infty]. \tag{1.11}$$

Note that for every $p \in [1,\infty)$, every measure space $(\Omega,\mathscr{F},\mu)$ and every normed vector space $(V,\|\cdot\|_V)$ it holds that $\big(\mathscr{L}^0(\mu;\|\cdot\|_V),\|\cdot\|_{\mathscr{L}^p(\mu;\|\cdot\|_V)}\big)$ and $\big(L^0(\mu;\|\cdot\|_V),\|\cdot\|_{L^p(\mu;\|\cdot\|_V)}\big)$ are extendedly seminormed vector spaces. For a real number $T \in [0,\infty)$, a measurable space $(S,\mathscr{S})$, a normed vector space $(V,\|\cdot\|_V)$ and a mapping $X\colon [0,T]\times S \to V$ that satisfies for all $t \in [0,T]$ that $X_t\colon S \to V$ is an $(\mathscr{S},\|\cdot\|_V)$-strongly measurable mapping we call X an $(\mathscr{S},\|\cdot\|_V)$-strongly measurable stochastic process. For a metric space (M,d), an extendedly seminormed vector space $(E,\|\cdot\|)$, a real number $r \in [0,1]$ and a set $A \subseteq (0,\infty)$ we denote by $|\cdot|_{\mathscr{C}^{r,A}(M,\|\cdot\|)}, |\cdot|_{\mathscr{C}^r(M,\|\cdot\|)}, \|\cdot\|_{C(M,\|\cdot\|)}, \|\cdot\|_{\mathscr{C}^r(M,\|\cdot\|)}\colon \mathbb{M}(M,E) \to [0,\infty]$ the mappings that satisfy for all $f \in \mathbb{M}(M,E)$ that

$$|f|_{\mathscr{C}^{r,A}(M,\|\cdot\|)} = \sup\!\Big(\Big\{\tfrac{\|f(e_1)-f(e_2)\|}{|d(e_1,e_2)|^r}\colon e_1,e_2 \in M, d(e_1,e_2) \in A\Big\} \cup \{0\}\Big) \in [0,\infty], \tag{1.12}$$

$$|f|_{\mathscr{C}^r(M,\|\cdot\|)} = |f|_{\mathscr{C}^{r,(0,\infty)}(M,\|\cdot\|)} \in [0,\infty], \tag{1.13}$$

$$\|f\|_{C(M,\|\cdot\|)} = \sup(\{\|f(e)\| : e \in M\} \cup \{0\}) \in [0,\infty], \tag{1.14}$$

$$\|f\|_{\mathscr{C}^r(M,\|\cdot\|)} = \|f\|_{C(M,\|\cdot\|)} + |f|_{\mathscr{C}^r(M,\|\cdot\|)} \in [0,\infty] \tag{1.15}$$

and we denote by $\mathscr{C}^r(M,\|\cdot\|)$ the set given by

$$\mathscr{C}^r(M,\|\cdot\|) = \Big\{f \in C(M,E)\colon \|f\|_{\mathscr{C}^r(M,\|\cdot\|)} < \infty\Big\}. \tag{1.16}$$

For Hilbert spaces $(H_i,\langle\cdot,\cdot\rangle_{H_i},\|\cdot\|_{H_i})$, $i \in \{1,2\}$, we denote by $(\mathrm{HS}(H_1,H_2),\langle\cdot,\cdot\rangle_{\mathrm{HS}(H_1,H_2)},\|\cdot\|_{\mathrm{HS}(H_1,H_2)})$ the Hilbert space of Hilbert–Schmidt operators from H_1 to H_2. For a real number $T \in (0,\infty)$ we denote by $\mathscr{P}_T$ the set given by

$$\mathscr{P}_T = \big\{\theta \subseteq [0,T]\colon \{0,T\} \subseteq \theta \text{ and } \#(\theta) < \infty\big\}. \tag{1.17}$$

We denote by $d_{\max}, d_{\min}\colon \cup_{T\in(0,\infty)}\mathscr{P}_T \to \mathbb{R}$ the functions that satisfy for all $\theta = \{\theta_0,\theta_1,\ldots,\theta_{\#(\theta)-1}\} \in \cup_{T\in(0,\infty)}\mathscr{P}_T$ with $\theta_0 < \theta_1 < \cdots < \theta_{\#(\theta)-1}$ that

$$d_{\max}(\theta) = \max_{j\in\{1,2,\ldots,\#(\theta)-1\}} |\theta_j - \theta_{j-1}| \qquad \text{and} \qquad d_{\min}(\theta) = \min_{j\in\{1,2,\ldots,\#(\theta)-1\}} |\theta_j - \theta_{j-1}|. \tag{1.18}$$

For a normed vector space $(E,\|\cdot\|_E)$, an element $\theta = \{\theta_0,\theta_1,\ldots,\theta_{\#(\theta)-1}\} \in \cup_{T\in(0,\infty)}\mathscr{P}_T$ with $\theta_0 < \theta_1 < \cdots < \theta_{\#(\theta)-1}$ and a function $f\colon [0,\theta_{\#(\theta)-1}] \to E$ we denote by $[f]_\theta\colon [0,\theta_{\#(\theta)-1}] \to E$ the piecewise affine linear interpolation of $f|_{\{\theta_0,\theta_1,\ldots,\theta_{\#(\theta)-1}\}}$, that is, we denote by $[f]_\theta\colon [0,\theta_{\#(\theta)-1}] \to E$ the function that satisfies for all $j \in \{1,2,\ldots,\theta_{\#(\theta)-1}\}$, $s \in [\theta_{j-1},\theta_j]$ that

$$[f]_\theta(s) = \frac{(\theta_j - s)f(\theta_{j-1})}{(\theta_j - \theta_{j-1})} + \frac{(s-\theta_{j-1})f(\theta_j)}{(\theta_j - \theta_{j-1})}. \tag{1.19}$$

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2. Convergence in Hölder norms for Banach-space-valued stochastic processes

2.1 *Error bounds for the Hölder norm*

LEMMA 2.1 (An interpolation-type inequality). Consider the notation in Section 1.1, let $(E, \|\cdot\|_E)$ be a normed vector space, let (M, d) be a metric space, let $f\colon M \to E$ be a function and let $c \in (0, \infty)$, $\alpha, \beta, \gamma \in [0, 1]$ satisfy $\alpha \le \beta \le \gamma$. Then

$$|f|_{\mathscr{C}^\beta(M,\|\cdot\|_E)} \le \max\Big\{c^{\alpha-\beta}\, |f|_{\mathscr{C}^{\alpha,(c,\infty)}(M,\|\cdot\|_E)}, c^{\gamma-\beta}\, |f|_{\mathscr{C}^{\gamma,(0,c]}(M,\|\cdot\|_E)}\Big\} \tag{2.1}$$

and

$$|f|_{\mathscr{C}^\beta(M,\|\cdot\|_E)} \le \max\Big\{c^{\alpha-\beta}\, |f|_{\mathscr{C}^{\alpha,[c,\infty)}(M,\|\cdot\|_E)}, c^{\gamma-\beta}\, |f|_{\mathscr{C}^{\gamma,(0,c)}(M,\|\cdot\|_E)}\Big\}. \tag{2.2}$$

Proof. First of all, note that it holds for all $e_1, e_2 \in M$ with $d(e_1, e_2) \in (c, \infty)$ that

$$\frac{\|f(e_1) - f(e_2)\|_E}{|d(e_1, e_2)|^\beta} \le |d(e_1, e_2)|^{\alpha-\beta}\, |f|_{\mathscr{C}^{\alpha,(c,\infty)}(M,\|\cdot\|_E)} \le c^{\alpha-\beta}\, |f|_{\mathscr{C}^{\alpha,(c,\infty)}(M,\|\cdot\|_E)}. \tag{2.3}$$

In addition, observe that it holds for all $e_1, e_2 \in M$ with $d(e_1, e_2) \in (0, c]$ that

$$\frac{\|f(e_1) - f(e_2)\|_E}{|d(e_1, e_2)|^\beta} \le |d(e_1, e_2)|^{\gamma-\beta}\, |f|_{\mathscr{C}^{\gamma,(0,c]}(M,\|\cdot\|_E)} \le c^{\gamma-\beta}\, |f|_{\mathscr{C}^{\gamma,(0,c]}(M,\|\cdot\|_E)}. \tag{2.4}$$

Combining (2.3) and (2.4) shows (2.1). The proof of (2.2) is analogous. This finishes the proof of Lemma 2.1. □

LEMMA 2.2 (Approximation error for affine linear interpolation). Consider the notation in Section 1.1, let $T \in (0, \infty)$, $\theta \in \mathscr{P}_T$, $\alpha \in [0, 1]$, let $(E, \|\cdot\|_E)$ be a normed vector space and let $f\colon [0, T] \to E$ be a function. Then

$$\big\|f - [f]_\theta\big\|_{C([0,T],\|\cdot\|_E)} \le \left|\tfrac{d_{\max}(\theta)}{2}\right|^\alpha |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}. \tag{2.5}$$

Proof. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \ldots, \theta_N \in [0, T]$ be the real numbers that satisfy $0 = \theta_0 < \theta_1 < \cdots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \ldots, \theta_N\}$, let $s \in [0, T] \setminus \theta$, let $j \in \{1, 2, \ldots, N\}$ be the natural number such that $s \in (\theta_{j-1}, \theta_j)$ and let $g\colon [0, 1] \to \mathbb{R}$ be the function that satisfies for all $u \in [0, 1]$ that

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$g(u) = (1-u)\, u^\alpha + u\,(1-u)^\alpha$. Observe that the concavity of the function $[0,\infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ shows for all $u \in [0,1]$ that

$$\begin{aligned} 2^\alpha g(u) &= (1-u)\,(2u)^\alpha + u\,(2(1-u))^\alpha \le ((1-u)\,2u + u\,2(1-u))^\alpha \\ &= (4u(1-u))^\alpha = (1-(2u-1)^2)^\alpha \le 1. \end{aligned} \tag{2.6}$$

Note that this proves that

$$\begin{aligned} \big\|f(s) - [f]_\theta(s)\big\|_E &\le \tfrac{(\theta_j - s)}{(\theta_j-\theta_{j-1})} \big\|f(s) - f(\theta_{j-1})\big\|_E + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \big\|f(s) - f(\theta_j)\big\|_E \\ &\le \tfrac{(\theta_j - s)}{(\theta_j-\theta_{j-1})} \big(s-\theta_{j-1}\big)^\alpha\, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \big(\theta_j - s\big)^\alpha\, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &= \Big(\tfrac{(\theta_j - s)}{(\theta_j-\theta_{j-1})} \big(\tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})}\big)^\alpha + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \big(\tfrac{(\theta_j - s)}{(\theta_j-\theta_{j-1})}\big)^\alpha\Big) \big(\theta_j - \theta_{j-1}\big)^\alpha\, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &= g\big(\tfrac{s-\theta_{j-1}}{\theta_j-\theta_{j-1}}\big)\, \big(\theta_j - \theta_{j-1}\big)^\alpha\, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \le \big(\tfrac{\theta_j-\theta_{j-1}}{2}\big)^\alpha\, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}. \end{aligned} \tag{2.7}$$

The proof of Lemma 2.2 is thus completed. □

The next result, Corollary 2.3, provides estimates for the Hölder norm differences of two functions by using the difference of the two functions on suitable grid points. Corollary 2.3 is a consequence of Lemmas 2.1 and 2.2.

Corollary 2.3 Consider the notation in Section 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, $\beta \in [0,1]$, $\alpha \in [0,\beta]$, let $(E, \|\cdot\|_E)$ be a normed vector space and let $f, g\colon [0,T] \to E$ be functions. Then

$$\begin{aligned} &|f-g|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\le \tfrac{2}{|d_{\max}(\theta)|^\alpha} \Big[\sup_{t\in\theta} \|f(t) - g(t)\|_E + \tfrac{|d_{\max}(\theta)|^\beta}{2^\beta} \big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}\big)\Big] \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\|f-g\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\le \Big[\tfrac{2}{|d_{\max}(\theta)|^\alpha} + 1\Big]\Big[\sup_{t\in\theta} \|f(t) - g(t)\|_E + \tfrac{|d_{\max}(\theta)|^\beta}{2^\beta} \big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}\big)\Big]. \end{aligned} \tag{2.9}$$

Proof. Lemma 2.1 and the triangle inequality ensure that

$$\begin{aligned} &|f-g|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\le \max\Big\{|d_{\max}(\theta)|^{-\alpha}\, |f-g|_{\mathscr{C}^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)}, |d_{\max}(\theta)|^{\beta-\alpha}\, |f-g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}\Big\} \\ &\le \max\Big\{2\,|d_{\max}(\theta)|^{-\alpha}\, \|f-g\|_{C([0,T],\|\cdot\|_E)}, |d_{\max}(\theta)|^{\beta-\alpha} \Big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}\Big)\Big\}. \end{aligned} \tag{2.10}$$

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In addition, observe that Lemma 2.2 and the triangle inequality demonstrate that

$$\begin{aligned}\|f-g\|_{C([0,T],\|\cdot\|_E)} &\le \big\|f-[f]_\theta\big\|_{C([0,T],\|\cdot\|_E)} + \big\|[f]_\theta-[g]_\theta\big\|_{C([0,T],\|\cdot\|_E)} + \big\|[g]_\theta-g\big\|_{C([0,T],\|\cdot\|_E)} \\ &\le \sup_{t\in\theta}\|f(t)-g(t)\|_E + \left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta \Big(|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + |g|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}\Big). \end{aligned} \tag{2.11}$$

Inserting (2.11) into (2.10) yields inequality (2.8). Moreover, adding inequalities (2.8) and (2.11) results in inequality (2.9). This finishes the proof of Corollary 2.3. □

LEMMA 2.4 Consider the notation in Section 1.1, let $(E,\|\cdot\|_E)$ be a normed vector space, let $T,c\in(0,\infty)$, $\alpha\in[0,1]$, $\theta\in\mathscr{P}_T$, $N\in\mathbb{N}$, $\theta_0,\dots,\theta_N\in[0,T]$ satisfy $0=\theta_0<\cdots<\theta_N=T$ and $\theta=\{\theta_0,\dots,\theta_N\}$ and let $f\colon[0,T]\to E$ be a function. Then

$$\big|[f]_\theta\big|_{\mathscr{C}^{\alpha,(0,c]}([0,T],\|\cdot\|_E)} \le \frac{c^{1-\alpha}}{d_{\min}(\theta)}\left[\sup_{j\in\{1,2,\dots,N\}}\|f(\theta_j)-f(\theta_{j-1})\|_E\right]. \tag{2.12}$$

Proof. Observe that it holds for all $s,t\in[0,T]$ with $t-s\in(0,c]$ that

$$\begin{aligned}\frac{\|[f]_\theta(t)-[f]_\theta(s)\|_E}{|t-s|^\alpha} &= \frac{\big\|\int_{(s,t)\setminus\theta}([f]_\theta)'(u)\,\mathrm{d}u\big\|_E}{|t-s|^\alpha} \\ &\le \frac{|t-s|\left[\sup_{u\in(s,t)\setminus\theta}\|([f]_\theta)'(u)\|_E\right]}{|t-s|^\alpha} \\ &\le |t-s|^{1-\alpha}\left[\sup_{j\in\{1,2,\dots,N\}}\frac{\|f(\theta_j)-f(\theta_{j-1})\|_E}{|\theta_j-\theta_{j-1}|}\right] \\ &\le \frac{c^{1-\alpha}}{d_{\min}(\theta)}\left[\sup_{j\in\{1,2,\dots,N\}}\|f(\theta_j)-f(\theta_{j-1})\|_E\right].\end{aligned} \tag{2.13}$$

This completes the proof of Lemma 2.4. □

LEMMA 2.5 Consider the notation in Section 1.1, let $(E,\|\cdot\|_E)$ be a normed vector space, let $T\in(0,\infty)$, $\alpha\in[0,1]$, $\theta\in\mathscr{P}_T$ and let $f\colon[0,T]\to E$ be a function. Then $\big|[f]_\theta\big|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \le |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}$.

Proof. Throughout this proof let $N\in\mathbb{N}$, $\theta_0,\theta_1,\dots,\theta_N\in[0,T]$ be the real numbers that satisfy $0=\theta_0<\theta_1<\cdots<\theta_N=T$ and $\theta=\{\theta_0,\theta_1,\dots,\theta_N\}$ and let $n\colon[0,T]\to\mathbb{N}$ and $\rho\colon[0,T]\to[0,1]$ be the functions that satisfy for all $t\in[0,T]$ that

$$n(t)=\min\big\{k\in\{1,2,\dots,N\}\colon t\in[\theta_{k-1},\theta_k]\big\} \qquad\text{and}\qquad \rho(t)=\frac{t-\theta_{n(t)-1}}{\theta_{n(t)}-\theta_{n(t)-1}}. \tag{2.14}$$

Note that it holds for all $t\in[0,T]$ that

$$[f]_\theta(t)=(1-\rho(t))\cdot f(\theta_{n(t)-1})+\rho(t)\cdot f(\theta_{n(t)}) = f(\theta_{n(t)-1})+\rho(t)\cdot\big(f(\theta_{n(t)})-f(\theta_{n(t)-1})\big). \tag{2.15}$$

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Hence, we obtain for all $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $n(t_1) = n(t_2)$ that

$$
\begin{aligned}
\big\|[f]_\theta(t_1) - [f]_\theta(t_2)\big\|_E &= \big\| \big[\big(1-\rho(t_1)\big)\cdot f(\theta_{n(t_1)-1}) + \rho(t_1)\cdot f(\theta_{n(t_1)})\big] \\
&\quad - \big[\big(1-\rho(t_2)\big)\cdot f(\theta_{n(t_1)-1}) + \rho(t_2)\cdot f(\theta_{n(t_1)})\big] \big\|_E \\
&= \big\| \big(\rho(t_2)-\rho(t_1)\big)\cdot f(\theta_{n(t_1)-1}) + \big(\rho(t_1)-\rho(t_2)\big)\cdot f(\theta_{n(t_1)}) \big\|_E \\
&= \big|\rho(t_1)-\rho(t_2)\big| \cdot \big\| f(\theta_{n(t_1)-1}) - f(\theta_{n(t_1)}) \big\|_E \\
&\le \big|\rho(t_1)-\rho(t_2)\big| \, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|\theta_{n(t_1)-1} - \theta_{n(t_1)}\big|^\alpha \qquad (2.16)\\
&= \big|\rho(t_1)-\rho(t_2)\big|^{1-\alpha} \, |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|\big(\rho(t_1)-\rho(t_2)\big)\cdot\big(\theta_{n(t_1)} - \theta_{n(t_1)-1}\big)\big|^\alpha \\
&\le |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|\big(\rho(t_1)-\rho(t_2)\big)\cdot\big(\theta_{n(t_1)} - \theta_{n(t_1)-1}\big)\big|^\alpha \\
&= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|t_1 - \theta_{n(t_1)-1} - \big(t_2 - \theta_{n(t_1)-1}\big)\big|^\alpha \\
&= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|t_1 - t_2\big|^\alpha.
\end{aligned}
$$

Moreover, (2.15) ensures for all $t_1, t_2 \in [0, T]$ with $n(t_1) < n(t_2)$ that

$$
\begin{aligned}
&\big\|[f]_\theta(t_1) - [f]_\theta(t_2)\big\|_E = \big\| \big[\big(1-\rho(t_1)\big)\cdot f(\theta_{n(t_1)-1}) + \rho(t_1)\cdot f(\theta_{n(t_1)})\big] \\
&\quad - \big[\big(1-\rho(t_2)\big)\cdot f(\theta_{n(t_2)-1}) + \rho(t_2)\cdot f(\theta_{n(t_2)})\big] \big\|_E \\
&\le \big(1-\rho(t_1)\big)\big(1-\rho(t_2)\big) \big\| f(\theta_{n(t_1)-1}) - f(\theta_{n(t_2)-1}) \big\|_E + \rho(t_1)\,\rho(t_2) \big\| f(\theta_{n(t_1)}) - f(\theta_{n(t_2)}) \big\|_E \\
&\quad + \big(1-\rho(t_1)\big)\rho(t_2) \big\| f(\theta_{n(t_1)-1}) - f(\theta_{n(t_2)}) \big\|_E + \rho(t_1)\big(1-\rho(t_2)\big) \big\| f(\theta_{n(t_1)}) - f(\theta_{n(t_2)-1}) \big\|_E \\
&\le |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big\{ \big(1-\rho(t_1)\big)\big(1-\rho(t_2)\big) |\theta_{n(t_1)-1} - \theta_{n(t_2)-1}|^\alpha + \rho(t_1)\,\rho(t_2)\, |\theta_{n(t_1)} - \theta_{n(t_2)}|^\alpha \\
&\quad + \big(1-\rho(t_1)\big)\rho(t_2)\, |\theta_{n(t_1)-1} - \theta_{n(t_2)}|^\alpha + \rho(t_1)\big(1-\rho(t_2)\big) |\theta_{n(t_1)} - \theta_{n(t_2)-1}|^\alpha \Big\}. \qquad (2.17)
\end{aligned}
$$

The concavity of the function $(-\infty, 0] \ni x \mapsto |x|^\alpha \in \mathbb{R}$ hence proves for all $t_1, t_2 \in [0, T]$ with $n(t_1) < n(t_2)$ that

$$
\begin{aligned}
&\big\|[f]_\theta(t_1) - [f]_\theta(t_2)\big\|_E \\
&\le |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big| \big(1-\rho(t_1)\big)\big(1-\rho(t_2)\big)\big(\theta_{n(t_1)-1} - \theta_{n(t_2)-1}\big) + \rho(t_1)\,\rho(t_2)\big(\theta_{n(t_1)} - \theta_{n(t_2)}\big) \\
&\quad + \big(1-\rho(t_1)\big)\rho(t_2)\big(\theta_{n(t_1)-1} - \theta_{n(t_2)}\big) + \rho(t_1)\big(1-\rho(t_2)\big)\big(\theta_{n(t_1)} - \theta_{n(t_2)-1}\big) \Big|^\alpha \\
&= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big| \big(1-\rho(t_1)\big)\theta_{n(t_1)-1} + \rho(t_1)\,\theta_{n(t_1)} - \big(1-\rho(t_2)\big)\theta_{n(t_2)-1} - \rho(t_2)\,\theta_{n(t_2)} \Big|^\alpha \\
&= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \qquad (2.18)\\
&\quad \cdot \Big| \big\{\theta_{n(t_1)-1} + \rho(t_1)\big[\theta_{n(t_1)} - \theta_{n(t_1)-1}\big]\big\} - \big\{\theta_{n(t_2)-1} + \rho(t_2)\big[\theta_{n(t_2)} - \theta_{n(t_2)-1}\big]\big\} \Big|^\alpha \\
&= |f|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \, \big|t_1 - t_2\big|^\alpha.
\end{aligned}
$$

Combining this and (2.16) completes the proof of Lemma 2.5. □

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LEMMA 2.6 (Approximations by piecewise affine linear functions). Consider the notation in Section 1.1, let $(E, \|\cdot\|_E)$ be a normed vector space, let $T \in (0,\infty)$, $\alpha \in [0,1]$, $\beta \in [\alpha,1]$, $\theta \in \mathscr{P}_T$ and let $f, g\colon [0,T] \to E$ be functions. Then

$$\left|f - [g]_\theta\right|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq \tfrac{2\,|d_{\max}(\theta)|^{1-\alpha}}{d_{\min}(\theta)} \sup_{t\in\theta} \|f(t) - g(t)\|_E + 2\left|d_{\max}(\theta)\right|^{\beta-\alpha} |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \tag{2.19}$$

and

$$\begin{aligned} &\left\|f - [g]_\theta\right\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \\ &\leq \left(\tfrac{2\,|d_{\max}(\theta)|^{1-\alpha}}{d_{\min}(\theta)} + 1\right) \sup_{t\in\theta} \|f(t) - g(t)\|_E + \left(\tfrac{2}{|d_{\max}(\theta)|^\alpha} + \tfrac{1}{2^\beta}\right) \left|d_{\max}(\theta)\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)}. \end{aligned} \tag{2.20}$$

Proof. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \ldots, \theta_N \in [0,T]$ be the real numbers that satisfy $0 = \theta_0 < \theta_1 < \cdots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \ldots, \theta_N\}$. Note that Lemma 2.1 implies that

$$\begin{aligned} \left|f - [g]_\theta\right|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq \max\Big\{ &\left|d_{\max}(\theta)\right|^{-\alpha} \left|f - [g]_\theta\right|_{\mathscr{C}^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)}, \\ &\left|d_{\max}(\theta)\right|^{\beta-\alpha} \left|f - [g]_\theta\right|_{\mathscr{C}^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \Big\}. \end{aligned} \tag{2.21}$$

Next note that Lemma 2.2 ensures that

$$\begin{aligned} \left|f - [g]_\theta\right|_{\mathscr{C}^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)} &\leq 2\left\|f - [g]_\theta\right\|_{C([0,T],\|\cdot\|_E)} \\ &\leq 2\left\|f - [f]_\theta\right\|_{C([0,T],\|\cdot\|_E)} + 2\left\|[f]_\theta - [g]_\theta\right\|_{C([0,T],\|\cdot\|_E)} \\ &\leq 2\left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + 2\cdot\sup_{t\in\theta}\|f(t) - g(t)\|_E \\ &\leq 2\left|d_{\max}(\theta)\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + 2\cdot\frac{d_{\max}(\theta)}{d_{\min}(\theta)}\cdot\sup_{t\in\theta}\|f(t) - g(t)\|_E. \end{aligned} \tag{2.22}$$

Moreover, observe that Lemmas 2.4 and 2.5 imply that

$$\begin{aligned} &\left|f - [g]_\theta\right|_{\mathscr{C}^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \leq \left|f - [f]_\theta\right|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + \left|[f - g]_\theta\right|_{\mathscr{C}^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \\ &\leq |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + \left|[f]_\theta\right|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} \\ &\quad + \tfrac{|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} \left[\sup_{j\in\{1,2,\ldots,N\}} \left\|\left[f(\theta_j) - g(\theta_j)\right] - \left[f(\theta_{j-1}) - g(\theta_{j-1})\right]\right\|_E\right] \\ &\leq 2\,|f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + \tfrac{2}{|d_{\max}(\theta)|^\beta}\cdot\tfrac{d_{\max}(\theta)}{d_{\min}(\theta)}\cdot\sup_{t\in\theta}\|f(t) - g(t)\|_E. \end{aligned} \tag{2.23}$$

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Substituting (2.23) and (2.22) into (2.21) proves (2.19). It thus remains to prove estimate (2.20). For this, note that Lemma 2.2 yields that

$$\begin{aligned}\left\|f - [g]_\theta\right\|_{C([0,T],\|\cdot\|_E)} &\le \left\|f - [f]_\theta\right\|_{C([0,T],\|\cdot\|_E)} + \left\|[f]_\theta - [g]_\theta\right\|_{C([0,T],\|\cdot\|_E)} \\ &\le \left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta |f|_{\mathscr{C}^\beta([0,T],\|\cdot\|_E)} + \sup_{t\in\theta} \|f(t) - g(t)\|_E .\end{aligned} \tag{2.24}$$

Combining (2.19) and (2.24) shows (2.20). The proof of Lemma 2.6 is thus completed. □

2.2 *Upper error bounds for stochastic processes with Hölder continuous sample paths*

We now turn to the result announced in the introduction that provides convergence of stochastic processes in Hölder norms given convergence on the grid points. For this, we first recall the Kolmogorov–Chentsov continuity theorem; cf., e.g., Revuz & Yor (1999, Theorem I.2.1 and its proof).

THEOREM 2.7 (Kolmogorov–Chentsov continuity theorem). Consider the notation in Section 1.1. There exists a function $\varXi = (\varXi_{T,p,\alpha,\beta})_{T,p,\alpha,\beta\in\mathbb{R}} \colon \mathbb{R}^4 \to \mathbb{R}$ such that for every $T \in [0,\infty)$, $p \in (1,\infty)$, $\beta \in (1/p, 1]$, every Banach space $(E, \|\cdot\|_E)$, every probability space $(\varOmega, \mathscr{F}, \mathbb{P})$ and every $X \in \mathscr{C}^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})$ there exists an $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic process $Y \colon [0,T] \times \varOmega \to E$ with continuous sample paths such that it holds for every $\alpha \in [0, \beta - 1/p)$ that

$$\begin{aligned}&\left(\mathbb{E}\Big[\|Y\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\right)^{1/p} \le \varXi_{T,p,\alpha,\beta}\, \|X\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty \quad \text{and} \\ &\forall\, t \in [0,T] \colon \mathbb{P}(X_t = Y_t) = 1.\end{aligned} \tag{2.25}$$

The next result, Corollary 2.8, follows directly from Corollary 2.3 (with $T = T$, $\theta = \theta$, $\beta = \gamma$, $\alpha = \beta$, $E = L^p(\mathbb{P}; \|\cdot\|_E)$, $f = ([0,T] \ni t \mapsto \{Z \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E) \colon \mathbb{P}(Z \neq X_t - Y_t) = 0\} \in L^p(\mathbb{P}; \|\cdot\|_E))$, $g = 0$ for $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta, 1]$ and $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \varOmega \to E$ with $\forall\, t \in [0,T] \colon \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} < \infty$ in the notation of Corollary 2.3) and the Kolmogorov–Chentsov continuity theorem (see Theorem 2.7 above).

COROLLARY 2.8 (Grid point approximations). Consider the notation in Section 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, let $(\varOmega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\big(E, \|\cdot\|_E\big)$ be a Banach space. Then

(i) it holds for all $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta, 1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y \colon [0,T] \times \varOmega \to E$ that

$$\begin{aligned}\|X - Y\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} &\le \left(2\left|d_{\max}(\theta)\right|^{-\beta} + 1\right)\Bigg[\sup_{t\in\theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + \left|d_{\max}(\theta)\right|^\gamma \Big(|X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\Big)\Bigg]\end{aligned} \tag{2.26}$$

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(ii) and it holds for all $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $\alpha \in [0, \beta - 1/p)$, $\gamma \in [\beta, 1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths that

$$
\begin{aligned}
\left(\mathbb{E}\left[\|X - Y\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p} &\le \Xi_{T,p,\alpha,\beta}\, \|X - Y\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\
&\le \Xi_{T,p,\alpha,\beta}\left(2\left|d_{\max}(\theta)\right|^{-\beta} + 1\right)\left[\sup_{t\in\theta}\|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right. \\
&\quad \left. + \left|d_{\max}(\theta)\right|^{\gamma}\left(|X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\right)\right].
\end{aligned}
\tag{2.27}
$$

The next result, Corollary 2.9, follows directly from Lemma 2.6 (with $E = L^p(\mathbb{P}; \|\cdot\|_E)$, $T = T$, $\alpha = \beta$, $\beta = \gamma$, $\theta = \theta$, $f = ([0,T] \ni t \mapsto \{Z \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E)\colon \mathbb{P}(Z \neq X_t - X_0) = 0\} \in L^p(\mathbb{P}; \|\cdot\|_E))$, $g = ([0,T] \ni t \mapsto \{Z \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E)\colon \mathbb{P}(Z \neq [Y]_\theta(t) - X_0) = 0\} \in L^p(\mathbb{P}; \|\cdot\|_E))$ for $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta, 1]$ and $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with $\sup_{t\in\theta}\|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} + |X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ in the notation of Lemma 2.6) and the Kolmogorov–Chentsov continuity theorem (see Theorem 2.7 above).

Corollary 2.9 (Piecewise affine linear stochastic processes). Consider the notation in Section 1.1, let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $(E, \|\cdot\|_E)$ be a Banach space. Then

(i) it holds for all $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta,1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ that

$$
\begin{aligned}
&\left\|X - [Y]_\theta\right\|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \le \left[\frac{2\,|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} + 1\right]\sup_{t\in\theta}\|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\
&\quad + \left[2\,|d_{\max}(\theta)|^{-\beta} + 2^{-\gamma}\right]\left|d_{\max}(\theta)\right|^{\gamma}\, |X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}
\end{aligned}
\tag{2.28}
$$

(ii) and it holds for all $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $\alpha \in [0, \beta - 1/p)$, $\gamma \in [\beta, 1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths that

$$
\begin{aligned}
&\left(\mathbb{E}\left[\|X - [Y]_\theta\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p} \le \Xi_{T,p,\alpha,\beta}\left(\left[\frac{2\,|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} + 1\right]\sup_{t\in\theta}\|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right. \\
&\quad \left. + \left[2\,|d_{\max}(\theta)|^{-\beta} + 2^{-\gamma}\right]\left|d_{\max}(\theta)\right|^{\gamma}\, |X|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\right).
\end{aligned}
\tag{2.29}
$$

In (2.29) in Corollary 2.9 we assume besides other assumptions that α is strictly smaller than γ. In general, this assumption cannot be omitted. To give an example, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $W\colon [0,1] \times \Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion with continuous sample paths. Then it clearly holds for all $p \in [1,\infty)$ that $\|W\|_{\mathscr{C}^{1/2}([0,1],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} < \infty$. However, the fact that the sample paths of the Brownian motion are $\mathbb{P}$-a.s. not $1/2$-Hölder continuous (cf., e.g., Revuz & Yor, 1999, Theorem I.2.7 and Arcones, 1995, Corollary 3.1) ensures that it holds for all $\theta \in \mathscr{P}_1$, $p \in (0,\infty)$ that $\mathbb{E}\left[\|W - [W]_\theta\|^p_{\mathscr{C}^{1/2}([0,1],|\cdot|)}\right] = \infty$. The following corollary is related to Bally *et al.* (1995, Lemma A1).

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COROLLARY 2.10 ($\mathscr{L}^p$-convergence in Hölder norms for a fixed $p \in [1,\infty)$). Consider the notation in Section 1.1, let $T \in (0,\infty)$, $p \in [1,\infty)$, $\beta \in [0,1]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\big(E, \|\cdot\|_E\big)$ be a Banach space and let $Y^N \colon [0,T] \times \Omega \to E$, $N \in \mathbb{N}_0$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths that satisfy $\limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ and $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$. Then

(i) it holds that $|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \le \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$,

(ii) it holds for all $\alpha \in [0,1] \cap (-\infty,\beta)$ that $\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0$

(iii) and it holds for all $\alpha \in [0,1] \cap (-\infty, \beta - 1/p)$ that

$$\limsup_{N\to\infty} \mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] = 0. \tag{2.30}$$

Proof. Throughout this proof let $\theta^n \in \mathscr{P}_T$, $n \in \mathbb{N}$, be the sequence that satisfies for all $n \in \mathbb{N}$ that $\theta^n = \{0, \frac{T}{n}, \frac{2T}{n}, \ldots, \frac{(n-1)T}{n}, T\} \in \mathscr{P}_T$. Observe that the assumption that $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$ and the assumption that $\limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ ensure that

$$\begin{aligned}
|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} &= \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \left[\frac{\|Y_t^0 - Y_s^0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \\
&= \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \left[\frac{\limsup_{N\to\infty} \|(Y_t^N - Y_s^N) + (Y_t^0 - Y_t^N) + (Y_s^N - Y_s^0)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \\
&\le \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \limsup_{N\to\infty} \left[\frac{\|Y_t^N - Y_s^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \le \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty.
\end{aligned} \tag{2.31}$$

This establishes item (i). In the next step we prove item (ii). We apply Corollary 2.8 (i) to obtain for all $\alpha \in [0,\beta]$, $n, N \in \mathbb{N}$ that

$$\begin{aligned}
\|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} &\le \big(2\,\big|d_{\max}(\theta^n)\big|^{-\alpha} + 1\big)\Bigg[\sup_{t\in\theta^n} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\
&\quad + \big|d_{\max}(\theta^n)\big|^{\beta}\big(|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\big)\Bigg] \\
&\le \left(\tfrac{2\,T^{-\alpha}}{n^{-\alpha}} + 1\right) \sup_{t\in\theta^n} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\
&\quad + \left(\tfrac{2\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{T^\beta}{n^\beta}\right)\big(|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\big).
\end{aligned} \tag{2.32}$$

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Item (i) and the assumption that $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$ hence imply for all $\alpha \in [0,\beta]$, $n \in \mathbb{N}$ that

$$\begin{aligned}\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} &\le \left[\tfrac{2\,T^{-\alpha}}{n^{-\alpha}} + 1\right]\left[\limsup_{N\to\infty}\sup_{t\in\theta^n} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right] \\ &\quad + \left[\tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\ &= \left[\tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty.\end{aligned} \tag{2.33}$$

Hence, we obtain for all $\alpha \in [0,1] \cap (-\infty,\beta)$ that

$$\begin{aligned}\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} &= \limsup_{n\to\infty}\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\ &\le \left[\limsup_{n\to\infty} \tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \limsup_{n\to\infty} \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0.\end{aligned} \tag{2.34}$$

This shows item (ii). It thus remains to establish item (iii) to complete the proof of Corollary 2.10. For this we apply the first inequality in Corollary 2.8 (ii) to obtain for all $r \in (1/p,\infty) \cap (-\infty,\beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\left(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\right)^{1/p} \le \varXi_{T,p,\alpha,r}\, \|Y^0 - Y^N\|_{\mathscr{C}^r([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}. \tag{2.35}$$

This and item (ii) imply for all $r \in (1/p,\infty) \cap (-\infty,\beta)$, $\alpha \in [0, r - 1/p)$ that

$$\limsup_{N\to\infty} \mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] \le (\varXi_{T,p,\alpha,r})^p \limsup_{N\to\infty} \|Y^0 - Y^N\|^p_{\mathscr{C}^r([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0. \tag{2.36}$$

This establishes item (iii). The proof of Corollary 2.10 is thus completed. □

The next result, Corollary 2.11, is a consequence of Corollaries 2.8 and 2.9.

COROLLARY 2.11 (Convergence rates with respect to Hölder norms). Consider the notation in Section 1.1, let $T \in (0,\infty)$, $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $(\theta^N)_{N\in\mathbb{N}} \subseteq \mathscr{P}_T$ satisfy $\limsup_{N\to\infty} d_{\max}(\theta^N) = 0$, let $(\varOmega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\big(E, \|\cdot\|_E\big)$ be a Banach space, let $Y^N\colon [0,T] \times \varOmega \to E$, $N \in \mathbb{N}_0$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths that satisfy $Y_0^0 \in \mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)$ and

$$|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \left[\big|d_{\max}(\theta^N)\big|^{-\beta} \sup_{t\in\theta^N} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right] < \infty \tag{2.37}$$

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and assume $\big(\big[\sup_{N\in\mathbb{N}} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty\big]$ or $\big[\sup_{N\in\mathbb{N}} {}^{d_{\max}(\theta^N)}/_{d_{\min}(\theta^N)} < \infty$ and $\forall N \in \mathbb{N}\colon Y^N = [Y^N]_{\theta^N}\big]\big)$. Then it holds for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0,\infty)$ that

$$\sup_{N\in\mathbb{N}} \left[\mathbb{E}\Big[\|Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] + \big|d_{\max}(\theta^N)\big|^{-(\beta-\alpha-1/p-\varepsilon)} \Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \Big)^{1/p} \right] < \infty. \tag{2.38}$$

Proof. Throughout this proof let $c_0 \in [0,\infty)$, $c_1, c_2 \in [0,\infty]$ be the extended real numbers given by

$$\begin{aligned} c_0 &= |Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \left[|d_{\max}(\theta^N)|^{-\beta} \sup_{t\in\theta^N} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \right], \\ c_1 &= \sup_{N\in\mathbb{N}} \left[\frac{d_{\max}(\theta^N)}{d_{\min}(\theta^N)} \right] \qquad \text{and} \qquad c_2 = \sup_{N\in\mathbb{N}} |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}. \end{aligned} \tag{2.39}$$

Next we observe that Corollary 2.8 (ii) ensures for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned} &\Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \Big)^{1/p} \leq \Xi_{T,p,\alpha,r} \Big(2|d_{\max}(\theta^N)|^{-r} + 1 \Big) \Big[\sup_{t\in\theta^N} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\ &\quad + |d_{\max}(\theta^N)|^\beta \big(|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \big) \Big] \\ &\leq \Xi_{T,p,\alpha,r} \Big(2|d_{\max}(\theta^N)|^{(\beta-r)} + |d_{\max}(\theta^N)|^\beta \Big) \Big[c_0 + |Y^N|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \Big] \\ &\leq \Xi_{T,p,\alpha,r} \Big(2|d_{\max}(\theta^N)|^{(\beta-r)} + |d_{\max}(\theta^N)|^\beta \Big) \big[c_0 + c_2 \big] \\ &= \Xi_{T,p,\alpha,r} \Big(2 + |d_{\max}(\theta^N)|^r \Big) |d_{\max}(\theta^N)|^{(\beta-r)} \big[c_0 + c_2 \big]. \end{aligned} \tag{2.40}$$

This implies for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$ that

$$\sup_{N\in\mathbb{N}} \left[\big|d_{\max}(\theta^N)\big|^{-(\beta-r)} \Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \Big)^{1/p} \right] \leq \Xi_{T,p,\alpha,r} \big(2 + T^r\big) \big[c_0 + c_2\big]. \tag{2.41}$$

Hence, we obtain for all $\alpha \in [0, \beta - 1/p)$, $r \in (\alpha + 1/p, \beta]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left[\big|d_{\max}(\theta^N)\big|^{-(\beta-\alpha-1/p-[r-\alpha-1/p])} \Big(\mathbb{E}\Big[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big] \Big)^{1/p} \right] \\ &\quad \leq \Xi_{T,p,\alpha,\alpha+1/p+[r-\alpha-1/p]} \,(3+T)\, \big(c_0 + c_2\big). \end{aligned} \tag{2.42}$$

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This shows for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \beta - \alpha - 1/p]$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-\varepsilon)}\left(\mathbb{E}\left[\|Y^0 - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p}\right]\\&\quad\le \varXi_{T,p,\alpha,\alpha+1/p+\varepsilon}\,(3+T)\left(c_0+c_2\right).\end{aligned} \tag{2.43}$$

In the next step we note that Corollary 2.9 (ii) proves for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned}&\left(\mathbb{E}\left[\left\|Y^0 - [Y^N]_{\theta^N}\right\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p} \le \varXi_{T,p,\alpha,r}\left(\left[\frac{2\,|d_{\max}(\theta^N)|^{1-r}}{d_{\min}(\theta^N)}+1\right]\sup_{t\in\theta^N}\|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right.\\&\left.\quad+\left[2\,|d_{\max}(\theta^N)|^{-r}+2^{-\beta}\right]|d_{\max}(\theta^N)|^{\beta}|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\right).\end{aligned} \tag{2.44}$$

This implies for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned}&\left(\mathbb{E}\left[\left\|Y^0 - [Y^N]_{\theta^N}\right\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p}\\&\quad\le c_0|d_{\max}(\theta^N)|^{\beta}\,\varXi_{T,p,\alpha,r}\left(\frac{2\,|d_{\max}(\theta^N)|^{1-r}}{d_{\min}(\theta^N)}+1+2\,|d_{\max}(\theta^N)|^{-r}+2^{-\beta}\right)\\&\quad\le 2\,c_0|d_{\max}(\theta^N)|^{\beta}\,\varXi_{T,p,\alpha,r}\left(\left[c_1+1\right]|d_{\max}(\theta^N)|^{-r}+1\right).\end{aligned} \tag{2.45}$$

Hence, we obtain for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-r)}\left(\mathbb{E}\left[\left\|Y^0 - [Y^N]_{\theta^N}\right\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p}\right]\\&\quad\le 2\,c_0\,\varXi_{T,p,\alpha,r}\left(c_1+1+T^r\right)\le 2\,c_0\,\varXi_{T,p,\alpha,r}\left(2+T+c_1\right).\end{aligned} \tag{2.46}$$

This shows for all $\alpha \in [0, \beta - 1/p)$, $r \in (\alpha + 1/p, \beta]$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-[r-\alpha-1/p])}\left(\mathbb{E}\left[\left\|Y^0 - [Y^N]_{\theta^N}\right\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p}\right]\\&\quad\le 2\,c_0\,\varXi_{T,p,\alpha,\alpha+1/p+[r-\alpha-1/p]}\left(2+T+c_1\right).\end{aligned} \tag{2.47}$$

This establishes for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \beta - \alpha - 1/p]$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-(\beta-\alpha-1/p-\varepsilon)}\left(\mathbb{E}\left[\left\|Y^0 - [Y^N]_{\theta^N}\right\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right]\right)^{1/p}\right]\\&\quad\le 2\,c_0\,\varXi_{T,p,\alpha,\alpha+1/p+\varepsilon}\left(2+T+c_1\right).\end{aligned} \tag{2.48}$$

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Combining (2.43) and (2.48) proves for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \infty)$ that

$$\sup_{N\in\mathbb{N}} \left[\left| d_{\max}(\theta^N) \right|^{-(\beta-\alpha-1/p-\varepsilon)} \left(\mathbb{E}\left[\| Y^0 - Y^N \|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \right] \right)^{1/p} \right] < \infty. \tag{2.49}$$

In addition, note that the assumption that $Y_0^0 \in \mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)$, the assumption that $|Y^0|_{\mathscr{C}^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$, the assumption that Y^0 has continuous sample paths and Theorem 2.7 ensure for all $\alpha \in [0, \beta - 1/p)$ that $\mathbb{E}\big[\|Y^0\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big] < \infty$. This and (2.49) complete the proof of Corollary 2.11. □

The next result, Corollary 2.12, illustrates Corollary 2.11 through a simple example. For this, note that standard results for the Euler–Maruyama method show under suitable hypotheses for every $p \in [2, \infty)$, $\beta \in [0, 1/2]$ that condition (2.37) in Corollary 2.11 with uniform time steps is satisfied (cf., e.g., Kloeden & Platen, 1992, Section 10.6). The convergence rate established in Corollary 2.12 (see (2.52) below) is essentially sharp; see Proposition 2.14. Corollary 2.12 is related to Cox & van Neerven (2010, Theorem 1.2) and Cox & van Neerven (2013, Theorem 1.1).

Corollary 2.12 (Euler–Maruyama method). Consider the notation in Section 1.1, let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be an m-dimensional standard $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be globally Lipschitz continuous functions, let $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(\mathbb{R}^d)$-adapted stochastic process with continuous sample paths that satisfies $\forall p \in [1, \infty)\colon \mathbb{E}\big[\|X_0\|^p_{\mathbb{R}^d}\big] < \infty$ and that satisfies for all $t \in [0, T]$ that

$$[X_t]_{\mathbb{P},\mathscr{B}(\mathbb{R}^d)} = \left[X_0 + \int_0^t \mu(X_s)\,\mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(\mathbb{R}^d)} + \int_0^t \sigma(X_s)\,\mathrm{d}W_s \tag{2.50}$$

and let $Y^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings that satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$ that $Y_0^N = X_0$ and

$$Y_t^N = Y^N_{\frac{nT}{N}} + \big(t - \tfrac{nT}{N}\big) \cdot \mu\big(Y^N_{\frac{nT}{N}}\big) + \big(\tfrac{tN}{T} - n\big) \cdot \sigma\big(Y^N_{\frac{nT}{N}}\big)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big). \tag{2.51}$$

Then it holds for all $\alpha \in [0, 1/2)$, $\varepsilon \in (0, \infty)$, $p \in [1, \infty)$ that

$$\sup_{N\in\mathbb{N}} \left[N^{\frac{1}{2}-\alpha-\varepsilon} \left(\mathbb{E}\left[\|X - Y^N\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathbb{R}^d})} \right] \right)^{1/p} \right] < \infty. \tag{2.52}$$

2.3 *Lower error bounds for stochastic processes with Hölder continuous sample paths*

In this subsection we comment on the optimality of the convergence rate provided by Corollaries 2.11 and 2.12. In particular, in the setting of Corollary 2.12, Müller-Gronbach (2002, Theorem 3) shows in the case $\alpha = 0$ that there exists a class of SDEs for which the factors $N^{1/2-\varepsilon}$, $N \in \mathbb{N}$, on the left-hand side of estimate (2.52) can at best—up to a constant—be replaced by the factors $\frac{N^{1/2}}{\log(N)}$, $N \in \mathbb{N}$. In Proposition 2.14 we show for every $\alpha \in [0, 1/2)$ in the simple case of $\mu = 0$ and $\sigma = (\mathbb{R} \ni x \mapsto 1 \in \mathbb{R})$ in Corollary 2.12 that the factors $N^{1/2-\alpha-\varepsilon}$, $N \in \mathbb{N}$, on the left-hand side of estimate (2.52) can at

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best—up to a constant—be replaced by the factors $N^{1/2-\alpha}$, $N \in \mathbb{N}$. Our proof of Proposition 2.14 uses the following elementary lemma.

LEMMA 2.13 Consider the notation in Section 1.1, let $T \in (0,\infty)$, $p \in [1,\infty)$, $\alpha \in [0,1]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a normed vector space and let $X\colon [0,T] \times \Omega \to E$ be an $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic process with continuous sample paths. Then

$$\max\Big\{|X|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}, 2^{(1/p-1)}\,\|X\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\Big\} \leq \Big(\mathbb{E}\Big[\|X\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}. \tag{2.53}$$

The proof of Lemma 2.13 is clear. Instead we now present the promised proposition on the optimality of the convergence rate estimate in Corollary 2.12.

PROPOSITION 2.14 Consider the notation in Section 1.1, let $T \in (0,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion with continuous sample paths and let $W^N\colon [0,T]\times\Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be mappings that satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$W_t^N = \big(n+1-\tfrac{tN}{T}\big)\cdot W_{\frac{nT}{N}} + \big(\tfrac{tN}{T}-n\big)\cdot W_{\frac{(n+1)T}{N}}. \tag{2.54}$$

Then it holds for all $\alpha \in [0, 1/2]$, $p \in [1,\infty)$, $N \in \{2,3,4,\ldots\}$ that

$$\|W-W^N\|_{C([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} = \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{2\sqrt{N}}, \tag{2.55}$$

$$\frac{|W-W^N|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}}{N^{(\alpha-\frac{1}{2})}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} = \frac{\big(\frac{1}{2}-\alpha\big)^{(\frac{1}{2}-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \in \Big[\frac{1}{\sqrt{2}}, 1\Big], \tag{2.56}$$

$$\frac{\|W-W^N\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}}{N^{(\alpha-\frac{1}{2})}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} = \frac{T^\alpha}{2N^\alpha} + \frac{\big(\frac{1}{2}-\alpha\big)^{(\frac{1}{2}-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \in \Big[\frac{1}{\sqrt{2}}, \frac{2+T^\alpha}{2}\Big], \tag{2.57}$$

$$\frac{\Big(\mathbb{E}\Big[\|W-W^N\|^p_{\mathscr{C}^\alpha([0,T],|\cdot|)}\Big]\Big)^{1/p}}{N^{(\alpha-\frac{1}{2})}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} \geq \frac{\big(\frac{1}{2}-\alpha\big)^{(\frac{1}{2}-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \geq \frac{1}{\sqrt{2}}. \tag{2.58}$$

Proof. Throughout this proof let $f\colon [0,1/2] \to (0,\infty)$ be the function that satisfies for all $x \in [0,1/2]$ that $f(x) = \frac{(1/2-x)^{(1/2-x)}}{2^x\,(1-x)^{(1-x)}}$ and let $g_\alpha\colon (0,1]^2 \to \mathbb{R}$, $\alpha \in [0,1/2]$, be the functions that satisfy for all $x, y \in (0,1]$, $\alpha \in [0,1/2]$ that

$$g_\alpha(x,y) = \frac{x\,(1-x) + y\,(1-y)}{(x+y)^{2\alpha}}. \tag{2.59}$$

We first prove (2.55). For this, observe that it holds for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$\begin{aligned} W_t - W_t^N &= W_t - \big(n+1-\tfrac{tN}{T}\big)\cdot W_{\frac{nT}{N}} - \big(\tfrac{tN}{T}-n\big)\cdot W_{\frac{(n+1)T}{N}} \\ &= \big(n-\tfrac{tN}{T}\big)\cdot\Big(W_{\frac{(n+1)T}{N}} - W_t\Big) + \big(n+1-\tfrac{tN}{T}\big)\cdot\Big(W_t - W_{\frac{nT}{N}}\Big). \end{aligned} \tag{2.60}$$

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This and the fact that

$$\forall N \in \mathbb{N} \colon \forall t \in \left(0, \tfrac{T}{N}\right) \colon \forall p \in [1, \infty) \colon \quad \left\| \frac{W_t - W_t^N}{\|W_t - W_t^N\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}} \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} = \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \tag{2.61}$$

imply that it holds for all $N \in \mathbb{N}$, $p \in [1, \infty)$ that

$$\begin{aligned}
\left\| W - W^N \right\|_{C([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} &= \sup_{t \in [0,T]} \left\| W_t - W_t^N \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&= \sup_{t \in [0,\frac{T}{N}]} \left\| W_t - W_t^N \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} = \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\sup_{t \in [0,\frac{T}{N}]} \left\| W_t - W_t^N \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)} \right] \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\sup_{t \in [0,\frac{T}{N}]} \left\| \tfrac{tN}{T} \cdot \left(W_t - W_{\frac{T}{N}} \right) + \left(1 - \tfrac{tN}{T}\right) \cdot W_t \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)} \right] \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\sup_{t \in [0,\frac{T}{N}]} \left[\left(\tfrac{tN}{T}\right)^2 \cdot \left(\tfrac{T}{N} - t\right) + \left(1 - \tfrac{tN}{T}\right)^2 \cdot t \right]^{\frac{1}{2}} \right] \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}} \left[\sup_{t \in [0,1]} \sqrt{\left(t^2 \cdot (1-t) + (1-t)^2 \cdot t\right)} \right] \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}} \left[\sup_{t \in [0,1]} \sqrt{t \cdot (1-t)} \right] = \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{2\sqrt{N}}.
\end{aligned} \tag{2.62}$$

This establishes (2.55). In the next step we prove (2.56). For this, observe that (2.60) shows for all $N \in \{2, 3, 4, \ldots\}$, $n \in \{1, 2, \ldots, N-1\}$, $t_1 \in [0, \frac{T}{N}]$, $t_2 \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $p \in [1, \infty)$ that

$$\begin{aligned}
&\left\| \left(W_{t_2} - W_{t_2}^N \right) - \left(W_{t_1} - W_{t_1}^N \right) \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad = \Big\| \left(n - \tfrac{t_2 N}{T} \right) \cdot \left(W_{\frac{(n+1)T}{N}} - W_{t_2} \right) + \left(n + 1 - \tfrac{t_2 N}{T} \right) \cdot \left(W_{t_2} - W_{\frac{nT}{N}} \right) \\
&\qquad + \tfrac{t_1 N}{T} \cdot \left(W_{\frac{T}{N}} - W_{t_1} \right) + \left(\tfrac{t_1 N}{T} - 1 \right) \cdot W_{t_1} \Big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad = \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \Big[\left(n - \tfrac{t_2 N}{T} \right)^2 \left(\tfrac{(n+1)T}{N} - t_2 \right) \\
&\qquad + \left(n + 1 - \tfrac{t_2 N}{T} \right)^2 \left(t_2 - \tfrac{nT}{N} \right) + \tfrac{(t_1)^2 N^2}{T^2} \left(\tfrac{T}{N} - t_1 \right) + \left(\tfrac{t_1 N}{T} - 1 \right)^2 t_1 \Big]^{\frac{1}{2}} \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}} \Big[\left(\tfrac{t_2 N}{T} - n \right) \left(n + 1 - \tfrac{t_2 N}{T} \right) + \tfrac{t_1 N}{T} \left(1 - \tfrac{t_1 N}{T} \right) \Big]^{\frac{1}{2}}.
\end{aligned} \tag{2.63}$$

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Moreover, (2.54) ensures for all $N \in \mathbb{N}$, $t_1, t_2 \in [0, \frac{T}{N}]$, $p \in [1, \infty)$ with $t_1 < t_2$ that

$$
\begin{aligned}
&\left\| \left(W_{t_2} - W^N_{t_2}\right) - \left(W_{t_1} - W^N_{t_1}\right) \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&= \left\| \left(W_{t_2} - \tfrac{t_2 N}{T} \cdot W_{\frac{T}{N}}\right) - \left(W_{t_1} - \tfrac{t_1 N}{T} \cdot W_{\frac{T}{N}}\right) \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left\| W_{t_2} - W_{t_1} + \tfrac{(t_1-t_2)N}{T} \cdot W_{\frac{T}{N}} \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \\
&\quad \cdot \left\| \left(1 + \tfrac{(t_1-t_2)N}{T}\right) \cdot \left(W_{t_2} - W_{t_1}\right) + \tfrac{(t_1-t_2)N}{T} \cdot \left(W_{\frac{T}{N}} - W_{t_2}\right) + \tfrac{(t_1-t_2)N}{T} \cdot W_{t_1} \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left[\left| 1 + \tfrac{(t_1-t_2)N}{T} \right|^2 \cdot \left(t_2 - t_1\right) + \tfrac{|t_1-t_2|^2 N^2}{T^2} \cdot \left(\tfrac{T}{N} - t_2\right) + \tfrac{|t_1-t_2|^2 N^2}{T^2} \cdot t_1 \right]^{\frac{1}{2}} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \cdot \left[1 + \tfrac{2(t_1-t_2)N}{T} + \tfrac{(t_1-t_2)^2 N^2}{T^2} + \tfrac{|t_1-t_2| N^2}{T^2} \cdot \left(\tfrac{T}{N} + t_1 - t_2\right) \right]^{\frac{1}{2}} \cdot \left(t_2 - t_1\right)^{\frac{1}{2}} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \cdot \left(1 + \tfrac{(t_1-t_2)N}{T}\right)^{\frac{1}{2}} \cdot \left(t_2 - t_1\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.64}
$$

Combining (2.63) and (2.64) proves for all $N \in \{2, 3, 4, \ldots\}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$
\begin{aligned}
&\left| W - W^N \right|_{\mathscr{C}^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} = \sup_{t_1, t_2 \in [0,T],\, t_1 < t_2} \frac{\left\| \left(W_{t_2} - W^N_{t_2}\right) - \left(W_{t_1} - W^N_{t_1}\right) \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{|t_1 - t_2|^\alpha} \\
&= \sup_{t_1 \in [0, \frac{T}{N}],\, t_2 \in [0,T],\, t_1 < t_2} \frac{\left\| \left(W_{t_2} - W^N_{t_2}\right) - \left(W_{t_1} - W^N_{t_1}\right) \right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{|t_1 - t_2|^\alpha} \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left| \max \left\{ \sup_{\substack{t_1, t_2 \in [0, \frac{T}{N}], \\ t_1 < t_2}} \frac{\left(1 + \frac{(t_1-t_2)N}{T}\right)}{(t_2-t_1)^{(2\alpha-1)}}, \sup_{\substack{t_1 \in [0, \frac{T}{N}], \\ t_2 \in (\frac{T}{N}, \frac{2T}{N}]}} \frac{T\left[\left(\frac{t_2 N}{T} - 1\right)\left(2 - \frac{t_2 N}{T}\right) + \frac{t_1 N}{T}\left(1 - \frac{t_1 N}{T}\right)\right]}{N\,(t_2-t_1)^{2\alpha}} \right\} \right|^{\frac{1}{2}}.
\end{aligned} \tag{2.65}
$$

This implies for all $N \in \{2, 3, 4, \ldots\}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$
\begin{aligned}
&\left| W - W^N \right|_{\mathscr{C}^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left| \left|\tfrac{T}{N}\right|^{(1-2\alpha)} \max \left\{ \sup_{x \in (0,1]} \frac{(1-x)}{x^{(2\alpha-1)}}, \sup_{\substack{x \in [0,1], \\ y \in (1,2]}} \frac{(y-1)\,(2-y) + x\,(1-x)}{(y-x)^{2\alpha}} \right\} \right|^{\frac{1}{2}} \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\tfrac{T}{N}\right|^{\left(\frac{1}{2}-\alpha\right)} \left[\max \left\{ \sup_{x \in (0,1]} \frac{x\,(1-x)}{x^{2\alpha}}, \sup_{\substack{x \in [0,1], \\ y \in (0,1]}} \frac{y\,(2-(y+1)) + x\,(1-x)}{([y+1]-[1-x])^{2\alpha}} \right\} \right]^{\frac{1}{2}}.
\end{aligned} \tag{2.66}
$$

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Hence, we obtain for all $N \in \{2, 3, 4, \ldots\}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$\begin{aligned}
&\left| W - W^N \right|_{\mathscr{C}^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})} \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\frac{T}{N}\right|^{\left(\frac{1}{2}-\alpha\right)} \left[\max\left\{ \sup_{y\in(0,1]} \frac{y\,(1-y)}{y^{2\alpha}}, \sup_{x\in[0,1]} \sup_{y\in(0,1]} \frac{x\,(1-x)+y\,(1-y)}{(x+y)^{2\alpha}} \right\} \right]^{\frac{1}{2}} \\
&= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\frac{T}{N}\right|^{\left(\frac{1}{2}-\alpha\right)} \left[\sup_{x\in[0,1]} \sup_{y\in(0,1]} \frac{x\,(1-x)+y\,(1-y)}{(x+y)^{2\alpha}} \right]^{\frac{1}{2}} \qquad (2.67) \\
&= N^{(\alpha-\frac{1}{2})}\, T^{-\alpha}\, \|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \left[\sup_{x,y\in(0,1]} g_\alpha(x,y) \right]^{\frac{1}{2}}.
\end{aligned}$$

To complete the proof of (2.56), we study a few properties of the functions g_α, $\alpha \in [0, 1/2]$. Note that it holds for all $x, y \in [0, 1]$ that

$$x(1-x) + y(1-y) = (x+y)\Big(1 - \frac{x+y}{2}\Big) - \frac{(x-y)^2}{2} \le 2\Big(\frac{x+y}{2}\Big)\Big(1 - \frac{x+y}{2}\Big). \qquad (2.68)$$

In addition, observe that it holds for all $\alpha \in [0, 1/2]$, $z \in (0, 1)$ that

$$\tfrac{\partial}{\partial z}\big(z^{(1-2\alpha)}\,(1-z)\big) = (1-2\alpha)\,z^{-2\alpha}\,(1-z) - z^{(1-2\alpha)} = -2\,(1-\alpha)\,\big[z - \tfrac{\frac{1}{2}-\alpha}{1-\alpha}\big] z^{-2\alpha}. \qquad (2.69)$$

Combining this with (2.68) ensures for all $\alpha \in [0, 1/2]$, $x, y \in (0, 1]$ that

$$\begin{aligned}
g_\alpha(x,y) &= \frac{x(1-x)+y(1-y)}{(x+y)^{2\alpha}} \le 2^{(1-2\alpha)} \Big(\frac{x+y}{2}\Big)^{1-2\alpha} \Big(1 - \frac{x+y}{2}\Big) \\
&\le 2^{(1-2\alpha)} \sup_{z\in(0,1]} \big[z^{(1-2\alpha)}(1-z)\big] = 2^{(1-2\alpha)} \big[\tfrac{\frac{1}{2}-\alpha}{1-\alpha}\big]^{(1-2\alpha)} \big[1 - \tfrac{\frac{1}{2}-\alpha}{1-\alpha}\big] \qquad (2.70) \\
&= 2^{-2\alpha} \big[\tfrac{\frac{1}{2}-\alpha}{1-\alpha}\big]^{(1-2\alpha)} \big[\tfrac{1}{1-\alpha}\big] = \Big[\tfrac{(\frac{1}{2}-\alpha)^{(\frac{1}{2}-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}}\Big]^2 = [f(\alpha)]^2.
\end{aligned}$$

This proves for all $\alpha \in [0, 1/2]$ that

$$[f(\alpha)]^2 = \sup_{z\in(0,1]} \big[(2\,z)^{(1-2\alpha)}(1-z)\big] = \sup_{x\in(0,1]} g_\alpha(x,x) \le \sup_{x,y\in(0,1]} g_\alpha(x,y) \le [f(\alpha)]^2. \qquad (2.71)$$

This shows for all $\alpha \in [0, 1/2]$ that

$$\sup_{x,y\in(0,1]} g_\alpha(x,y) = \sup_{x\in(0,1]} g_\alpha(x,x) = [f(\alpha)]^2. \qquad (2.72)$$

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Next note that it holds for all $\alpha \in (0, 1/2)$ that

$$f(\alpha) = \exp\Big(\Big(\tfrac{1}{2} - \alpha\Big) \cdot \ln\Big(\tfrac{1}{2} - \alpha\Big) + (\alpha - 1) \cdot \ln(1 - \alpha) - \alpha \cdot \ln(2)\Big). \tag{2.73}$$

Moreover, observe that it holds for all $\alpha \in (0, 1/2)$ that

$$\begin{aligned} &\tfrac{\partial}{\partial \alpha} \Big(\Big(\tfrac{1}{2} - \alpha\Big) \cdot \ln\Big(\tfrac{1}{2} - \alpha\Big) + (\alpha - 1) \cdot \ln(1 - \alpha) - \alpha \cdot \ln(2)\Big) \\ &= -\ln\Big(\tfrac{1}{2} - \alpha\Big) - 1 + \ln(1 - \alpha) + 1 - \ln(2) = \ln(1 - \alpha) - \ln\Big(\tfrac{1}{2} - \alpha\Big) - \ln(2) \\ &= \ln\Big(\tfrac{1-\alpha}{1-2\alpha}\Big) > 0. \end{aligned} \tag{2.74}$$

This and (2.73) ensure that f is strictly increasing. Equation (2.72) hence proves for all $\alpha \in [0, 1/2]$ that

$$\sup_{x,y \in (0,1]} g_\alpha(x, y) = \sup_{x \in (0,1]} g_\alpha(x, x) = [f(\alpha)]^2 \in \Big[|f(0)|^2, \big|f(\tfrac{1}{2})\big|^2\Big] = \Big[\tfrac{1}{2}, 1\Big]. \tag{2.75}$$

Putting this into (2.67) establishes (2.56). Combining (2.55) with (2.56) proves (2.57). Moreover, (2.56) and Lemma 2.13 imply (2.58). The proof of Proposition 2.14 is thus completed. □

3. Basic results for mild solutions of SEEs

In this section we collect a number of elementary results for mild solution processes of SEEs, most of which are well known. Note that throughout this article we denote for every $\mathbb{R}$-Hilbert space $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ by $L(H)$ the set of all bounded linear operators from H to H.

3.1 *Temporal regularity of solutions of SEEs*

Proposition 3.1 Consider the notation in Section 1.1; let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H; let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$; let $A\colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} \big|\lambda_h \langle h, v\rangle_H\big|^2 < \infty\big\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$; let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., Sell & You, 2002, Section 3.7); let $T \in (0,\infty)$, $p \in [2,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0, 1)$, $\beta \in [\gamma - \eta/2, \gamma]$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_\beta))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U,H_\beta)})} < \infty$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process and let $X\colon [0, T] \times \Omega \to H_\gamma$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic process that satisfies for all $t \in [0, T]$ that $\sup_{s\in[0,T]} \|X_s\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})} < \infty$ and

$$\begin{aligned} [X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} = &\bigg[e^{tA} X_0 + \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-u)A} F(X_u)\|_{H_\gamma} \mathrm{d}u < \infty\}}\, e^{(t-s)A} F(X_s)\, \mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ &+ \int_0^t e^{(t-s)A} B(X_s)\, \mathrm{d}W_s. \end{aligned} \tag{3.1}$$

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Then it holds for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that $\inf_{s\in(0,T]} \mathbb{P}(X_s \in H_r) = 1$ and

$$\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\|(X_{t_1} - e^{t_1 A}X_0)\,\mathbb{1}_{\{X_{t_1}\in H_r\}} - (X_{t_2} - e^{t_2 A}X_0)\,\mathbb{1}_{\{X_{t_2}\in H_r\}}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1 - t_2|^{\varepsilon}} \right) \\
&\leq \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1+\gamma-\eta-r-\varepsilon)}}{(1+\gamma-\eta-r-\varepsilon)} \\
&\quad + \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right] \frac{\sqrt{p\,(p-1)}\;T^{(\frac{1}{2}+\beta-r-\varepsilon)}}{(1+2\beta-2r-2\varepsilon)^{\frac{1}{2}}} < \infty.
\end{aligned} \tag{3.2}$$

Proof. Note that the fact that it holds for all $u \in [0,1]$ that

$$\sup_{t\in(0,T]} t^u \|(-A)^u e^{tA}\|_{L(H)} \leq 1 \quad \text{and} \quad \sup_{t\in(0,T]} t^{-u} \|(-A)^{-u}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \leq 1 \tag{3.3}$$

(cf., e.g., Renardy & Rogers, 2004, Lemma 12.36) implies that it holds for all $r \in [\gamma, 1+\gamma-\eta)$, $\varepsilon \in [0, 1+\gamma-\eta-r)$, $t_1 \in [0,T)$, $t_2 \in (t_1, T]$ that

$$\begin{aligned}
&\Bigg\| \int_0^{t_1} \mathbb{1}_{\{\int_0^{t_1} \|e^{(t_1-u)A}F(X_u)\|_{H_r}\mathrm{d}u<\infty\}}\, e^{(t_1-s)A}F(X_s)\,\mathrm{d}s \\
&\qquad - \int_0^{t_2} \mathbb{1}_{\{\int_0^{t_2} \|e^{(t_2-u)A}F(X_u)\|_{H_r}\mathrm{d}u<\infty\}}\, e^{(t_2-s)A}F(X_s)\,\mathrm{d}s \Bigg\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})} \\
&\leq \int_{t_1}^{t_2} \left\| e^{(t_2-s)A}F(X_s) \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})} \mathrm{d}s + \int_0^{t_1} \left\| e^{(t_1-s)A}\big(\mathrm{Id}_{H_{\gamma-\eta}} - e^{(t_2-t_1)A}\big)F(X_s) \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})} \mathrm{d}s \\
&\leq \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \left[\int_{t_1}^{t_2} (t_2-s)^{\gamma-\eta-r}\mathrm{d}s + \int_0^{t_1} (t_1-s)^{\gamma-\eta-r-\varepsilon}(t_2-t_1)^{\varepsilon}\mathrm{d}s \right] \\
&= \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \left[\frac{(t_2-t_1)^{(1+\gamma-\eta-r)}}{(1+\gamma-\eta-r)} + \frac{(t_2-t_1)^{\varepsilon}(t_1)^{(1+\gamma-\eta-r-\varepsilon)}}{(1+\gamma-\eta-r-\varepsilon)} \right] \qquad (3.4) \\
&\leq \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \left[\frac{2\,T^{(1+\gamma-\eta-r-\varepsilon)}(t_2-t_1)^{\varepsilon}}{(1+\gamma-\eta-r-\varepsilon)} \right].
\end{aligned}$$

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Moreover, (3.3) ensures for all $r \in [\gamma, 1/2+\beta)$, $\varepsilon \in [0, 1/2+\beta-r)$, $t_1 \in [0,T)$, $t_2 \in (t_1, T]$ that

$$\begin{aligned}
&\left\| \int_0^{t_1} e^{(t_1-s)A} B(X_s)\,\mathrm{d}W_s - \int_0^{t_2} e^{(t_2-s)A} B(X_s)\,\mathrm{d}W_s \right\|^2_{L^p(\mathbb{P};\|\cdot\|_{H_r})} \\
&\leq \tfrac{p\,(p-1)}{2} \int_{t_1}^{t_2} \left\| e^{(t_2-s)A} B(X_s) \right\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_r)})} \mathrm{d}s \\
&\quad + \tfrac{p\,(p-1)}{2} \int_0^{t_1} \left\| e^{(t_1-s)A} \big(\mathrm{Id}_{H_\beta} - e^{(t_2-t_1)A}\big) B(X_s) \right\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_r)})} \mathrm{d}s \\
&\leq \tfrac{p\,(p-1)}{2} \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right]^2 \\
&\quad \cdot \left[\int_{t_1}^{t_2} (t_2-s)^{(2\beta-2r)} \mathrm{d}s + \int_0^{t_1} (t_1-s)^{(2\beta-2r-2\varepsilon)} (t_2-t_1)^{2\varepsilon} \mathrm{d}s \right] \\
&\leq \tfrac{p\,(p-1)}{2} \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right]^2 \left[\frac{2\,T^{(1+2\beta-2r-2\varepsilon)} (t_2-t_1)^{2\varepsilon}}{(1+2\beta-2r-2\varepsilon)} \right].
\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) completes the proof of Proposition 3.1. □

COROLLARY 3.2 Consider the notation in Section 1.1; let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H; let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$; let $A\colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$; let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$; let $T \in (0,\infty)$, $p \in [2,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $\beta \in [\gamma - \eta/2, \gamma]$, $\delta \in [\gamma,\infty)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_\beta))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} < \infty$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process and let $X\colon [0,T]\times\Omega \to H_\gamma$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic process that satisfies for all $t \in [0,T]$ that $\sup_{s\in[0,T]} \|X_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty$, $X_0(\Omega) \subseteq H_\delta$, $\mathbb{E}[\|X_0\|^p_{H_\delta}] < \infty$ and

$$\begin{aligned}
[X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} &= \left[e^{tA} X_0 + \int_0^t \mathbb{1}_{\{\int_0^t \|e^{(t-u)A} F(X_u)\|_{H_\gamma} \mathrm{d}u < \infty\}}\, e^{(t-s)A} F(X_s)\,\mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&\quad + \int_0^t e^{(t-s)A} B(X_s)\,\mathrm{d}W_s.
\end{aligned} \tag{3.6}$$

Then it holds for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that $\inf_{s\in(0,T]} \mathbb{P}(X_s \in H_r) = 1$ and

$$\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{|\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}} \|\mathbb{1}_{\{X_{t_1}\in H_r\}} X_{t_1} - \mathbb{1}_{\{X_{t_2}\in H_r\}} X_{t_2}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^{\varepsilon}} \right) \\
&\leq \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} + \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1+\gamma-\eta-\min\{\delta,r+\varepsilon\})}}{(1+\gamma-\eta-r-\varepsilon)} \\
&\quad + \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right] \frac{\sqrt{p\,(p-1)}\, T^{(\frac{1}{2}+\beta-\min\{\delta,r+\varepsilon\})}}{(1+2\beta-2r-2\varepsilon)^{\frac{1}{2}}} < \infty.
\end{aligned} \tag{3.7}$$

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Proof. The fact that $\forall\, u \in [0,1]\colon \big(\sup_{t\in(0,T]} t^u \|(-A)^u e^{tA}\|_{L(H)} \le 1$ and $\sup_{t\in(0,T]} t^{-u}\|(-A)^{-u}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \le 1\big)$ ensures for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that

$$
\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\| |\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}} (e^{t_1A}X_0 - e^{t_2A}X_0) \|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^{\varepsilon}} \right) \\
&\le \sup_{\substack{t_1,t_2\in[0,T],\\ t_1< t_2}} \left(\frac{\| |t_1|^{\max\{r+\varepsilon-\delta,0\}} (-A)^{r-\min\{\delta,r+\varepsilon\}} (e^{t_1A} - e^{t_2A}) \|_{L(H)} \, \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})}}{|t_1-t_2|^{\varepsilon}} \right) \\
&\le \sup_{\substack{t_1,t_2\in(0,T],\\ t_1< t_2}} \left(|t_1|^{\max\{r+\varepsilon-\delta,0\}} \|(-A)^{r+\varepsilon-\min\{\delta,r+\varepsilon\}}\, e^{t_1A}\|_{L(H)} \, \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} \right) \qquad (3.8) \\
&= \sup_{t_1\in(0,T]} \left(|t_1|^{\max\{r+\varepsilon-\delta,0\}} \|(-A)^{\max\{r+\varepsilon-\delta,0\}}\, e^{t_1A}\|_{L(H)} \, \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} \right) \\
&\le \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})}.
\end{aligned}
$$

Combining this with the triangle inequality and Proposition 3.1 proves for all $r \in [\gamma, \min\{1+\gamma-\eta, 1/2+\beta\})$, $\varepsilon \in [0, \min\{1+\gamma-\eta-r, 1/2+\beta-r\})$ that $\inf_{s\in(0,T]} \mathbb{P}(X_s \in H_r) = 1$ and

$$
\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\big|\min\{t_1,t_2\}\big|^{\max\{r+\varepsilon-\delta,0\}} \|\mathbb{1}_{\{X_{t_1}\in H_r\}} X_{t_1} - \mathbb{1}_{\{X_{t_2}\in H_r\}} X_{t_2}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^{\varepsilon}} \right) \\
&\le \sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\| |\min\{t_1,t_2\}|^{\max\{r+\varepsilon-\delta,0\}} (e^{t_1A}X_0 - e^{t_2A}X_0) \|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^{\varepsilon}} \right) + T^{\max\{r+\varepsilon-\delta,0\}} \\
&\quad \cdot \sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\|(X_{t_1} - e^{t_1A}X_0)\, \mathbb{1}_{\{X_{t_1}\in H_r\}} - (X_{t_2} - e^{t_2A}X_0)\, \mathbb{1}_{\{X_{t_2}\in H_r\}}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_r})}}{|t_1-t_2|^{\varepsilon}} \right) \qquad (3.9) \\
&\le \|X_0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\delta,r+\varepsilon\}}})} + \left[\sup_{s\in[0,T]} \|F(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1+\gamma-\eta-\min\{\delta,r+\varepsilon\})}}{(1+\gamma-\eta-r-\varepsilon)} \\
&\quad + \left[\sup_{s\in[0,T]} \|B(X_s)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\beta)})} \right] \frac{\sqrt{p\,(p-1)}\, T^{(\frac{1}{2}+\beta-\min\{\delta,r+\varepsilon\})}}{(1+2\beta-2r-2\varepsilon)^{\frac{1}{2}}} < \infty.
\end{aligned}
$$

The proof of Corollary 3.2 is thus completed. □

3.2 *A priori bounds for solutions of SEEs*

LEMMA 3.3 Consider the notation in Section 1.1 and let $\mathbb{B}\colon (0,\infty)^2 \to (0,\infty)$ and $\mathrm{E}_\eta\colon [0,\infty) \to [0,\infty)$, $\eta \in (-\infty,1)$, be the functions that satisfy for all $\eta \in (-\infty,1)$, $x,y \in (0,\infty)$, $z \in [0,\infty)$ that

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$\mathbb{B}(x,y)=\int_0^1 t^{(x-1)}(1-t)^{(y-1)}\,dt$ and $\mathrm{E}_\eta(z)=1+\sum_{n=1}^\infty z^n\prod_{k=0}^{n-1}\mathbb{B}(1-\eta,k(1-\eta)+1)$. Then it holds for all $\eta\in(-\infty,1)$, $x\in[0,\infty)$ that $\sqrt{\mathrm{E}_\eta(x^2)}=\mathscr{E}_{(1-\eta)}(x)$.

Proof. Note that the fact that $\forall\, x,y\in(0,\infty)\colon \mathbb{B}(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ implies that it holds for all $\eta\in(-\infty,1)$, $x\in[0,\infty)$ that

$$\begin{aligned}\mathrm{E}_\eta(x^2)&=1+\sum_{n=1}^{\infty}(x^2)^n\prod_{k=0}^{n-1}\mathbb{B}(1-\eta,k(1-\eta)+1)\\&=1+\sum_{n=1}^{\infty}x^{2n}\prod_{k=0}^{n-1}\frac{\Gamma(1-\eta)\,\Gamma(k(1-\eta)+1)}{\Gamma((k+1)(1-\eta)+1)}=1+\sum_{n=1}^{\infty}\frac{x^{2n}[\Gamma(1-\eta)]^n}{\Gamma(n(1-\eta)+1)}\\&=\sum_{n=0}^{\infty}\frac{x^{2n}[\Gamma(1-\eta)]^n}{\Gamma(n(1-\eta)+1)}=[\mathscr{E}_{(1-\eta)}(x)]^2.\end{aligned}\tag{3.10}$$

The proof of Lemma 3.3 is thus completed. □

PROPOSITION 3.4 (*A priori* bounds). Consider the notation in Section 1.1; let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$ and $(U,\langle\cdot,\cdot\rangle_U,\|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H}\subseteq H$ be a nonempty orthonormal basis of H; let $\lambda\colon\mathbb{H}\to\mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}}\lambda_h<0$; let $A\colon D(A)\subseteq H\to H$ be the linear operator that satisfies $D(A)=\left\{v\in H\colon\sum_{h\in\mathbb{H}}\left|\lambda_h\langle h,v\rangle_H\right|^2<\infty\right\}$ and $\forall\, v\in D(A):Av=\sum_{h\in\mathbb{H}}\lambda_h\langle h,v\rangle_H h$; let $(H_r,\langle\cdot,\cdot\rangle_{H_r},\|\cdot\|_{H_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $-A$; let $T\in(0,\infty)$, $p\in[2,\infty)$, $\gamma\in\mathbb{R}$, $\eta\in[0,1)$, $F\in C(H_\gamma,H_{\gamma-\eta})$, $B\in C(H_\gamma,\mathrm{HS}(U,H_{\gamma-\eta/2}))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}+|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}<\infty$; let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_t)_{t\in[0,T]})$-Wiener process and let $X\colon[0,T]\times\Omega\to H_\gamma$ be an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic process that satisfies for all $t\in[0,T]$ that $\sup_{s\in[0,T]}\|X_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}<\infty$ and

$$\begin{aligned}[X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)}=&\left[e^{tA}X_0+\int_0^t\mathbb{1}_{\{\int_0^t\|e^{(t-u)A}F(X_u)\|_{H_\gamma}\,du<\infty\}}\,e^{(t-s)A}F(X_s)\,ds\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\\&+\int_0^t e^{(t-s)A}B(X_s)\,dW_s.\end{aligned}\tag{3.11}$$

Then

$$\begin{aligned}&\sup_{t\in[0,T]}\left\|\max\left\{1,\|X_t\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\le\sqrt{2}\left\|\max\left\{1,\|X_0\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\&\cdot\mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\left(\sup_{v\in H_\gamma}\frac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right)+\sqrt{T^{1-\eta}p(p-1)}\left(\sup_{v\in H_\gamma}\frac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\right]<\infty.\end{aligned}\tag{3.12}$$

Proof. The Burkholder–Davis–Gundy-type inequality in Da Prato & Zabczyk (1992, Lemma 7.7), the fact that $\forall\, u\in[0,1]:\sup_{t\in(0,T]}t^u\|(-A)^u e^{tA}\|_{L(H)}\le 1$ and Hölder's inequality imply that it holds for

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all $t \in [0,T]$ that

$$
\begin{aligned}
&\big\| \max\{1, \|X_t\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad\le \big\| \max\{1, \|X_0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} + \int_0^t \big\| e^{(t-s)A} F(X_s)\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \,\mathrm{d}s \\
&\qquad + \sqrt{\frac{p\,(p-1)}{2}} \bigg[\int_0^t \big\| e^{(t-s)A} B(X_s)\big\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\gamma)})} \,\mathrm{d}s \bigg]^{1/2} \\
&\quad\le \big\| \max\{1, \|X_0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} + \bigg[\frac{t^{(1-\eta)}}{(1-\eta)} \int_0^t (t-s)^{-\eta} \, \|F(X_s)\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \,\mathrm{d}s \bigg]^{1/2} \\
&\qquad + \sqrt{\frac{p\,(p-1)}{2}} \bigg[\int_0^t (t-s)^{-\eta} \, \|B(X_s)\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} \,\mathrm{d}s \bigg]^{1/2} \\
&\quad\le \big\| \max\{1, \|X_0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} + \bigg[\int_0^t (t-s)^{-\eta} \, \big\| \max\{1, \|X_s\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \,\mathrm{d}s \bigg]^{1/2} \\
&\qquad \cdot \bigg[\sqrt{\tfrac{T^{(1-\eta)}}{(1-\eta)}} \bigg(\sup_{v\in H_\gamma} \tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}} \bigg) + \sqrt{\tfrac{p\,(p-1)}{2}} \bigg(\sup_{v\in H_\gamma} \tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}} \bigg) \bigg].
\end{aligned}
\tag{3.13}
$$

This and the fact that $\forall\, a,b \in \mathbb{R}\colon (a+b)^2 \le 2\,(a^2+b^2)$ prove for all $t \in [0,T]$ that

$$
\begin{aligned}
&\big\| \max\{1, \|X_t\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \le 2\, \big\| \max\{1, \|X_0\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad + \int_0^t (t-s)^{-\eta} \, \big\| \max\{1, \|X_s\|_{H_\gamma}\}\big\|^2_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \,\mathrm{d}s \\
&\quad \cdot \bigg[\sqrt{\tfrac{2\,T^{(1-\eta)}}{(1-\eta)}} \bigg(\sup_{v\in H_\gamma} \tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}} \bigg) + \sqrt{p\,(p-1)} \bigg(\sup_{v\in H_\gamma} \tfrac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}} \bigg) \bigg]^2.
\end{aligned}
\tag{3.14}
$$

For example, Andersson *et al.* (2015, Lemma 2.6) and Lemma 3.3 hence complete the proof of Proposition 3.4. □

3.3 *A strong perturbation estimate for SEEs*

PROPOSITION 3.5 (Perturbation estimate). Consider the notation in Section 1.1; let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H; let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$; let $A\colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} \big|\lambda_h \langle h, v\rangle_H\big|^2 < \infty\big\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$; let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$; let $T \in [0,\infty)$, $p \in [2,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process

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and let $X^1, X^2\colon [0,T]\times\Omega\to H_\gamma$ be $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-predictable stochastic processes that satisfy $\max_{k\in\{1,2\}}\sup_{s\in[0,T]}\|X^k_s\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}<\infty$. Then

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|X^1_t-X^2_t\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}}+\sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\right]\\
&\quad\cdot\sqrt{2}\sup_{t\in[0,T]}\left\|\left[X^1_t-\int_0^t \mathbb{1}_{\{\int_0^t\|e^{(t-r)A}F(X^1_r)\|_{H_\gamma}\,\mathrm{d}r<\infty\}}e^{(t-s)A}F(X^1_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.\\
&\quad-\int_0^t e^{(t-s)A}B(X^1_s)\,\mathrm{d}W_s-\left\{\left[X^2_t-\int_0^t \mathbb{1}_{\{\int_0^t\|e^{(t-r)A}F(X^2_r)\|_{H_\gamma}\,\mathrm{d}r<\infty\}}e^{(t-s)A}F(X^2_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.\\
&\quad\left.\left.-\int_0^t e^{(t-s)A}B(X^2_s)\,\mathrm{d}W_s\right\}\right\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}<\infty.
\end{aligned}\tag{3.15}$$

Proof. Throughout this proof we assume w.l.o.g. that $T\neq 0$ and throughout this proof let $\mathscr{A}\colon H_{\gamma+1}\subseteq H_\gamma\to H_\gamma$ be the linear operator that satisfies for all $v\in H_{\gamma+1}$ that $\mathscr{A}v=\sum_{h\in\mathbb{H}}\lambda_h\langle(-\lambda_h)^{-\gamma}h,v\rangle_{H_\gamma}(-\lambda_h)^{-\gamma}h$. Observe that $(H_{r+\gamma},\langle\cdot,\cdot\rangle_{H_{r+\gamma}},\|\cdot\|_{H_{r+\gamma}})$, $r\in\mathbb{R}$, is a family of interpolation spaces associated to $-\mathscr{A}$. This, Lemma 3.3 and Andersson *et al.* (2015, Proposition 2.7) show for all $\varepsilon\in(0,\infty)$ that

$$\begin{aligned}
&\sup_{t\in(0,T]}\left\|X^1_t-X^2_t\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}}\sup_{t\in(0,T]}t^\eta\|(-\mathscr{A})^\eta e^{t\mathscr{A}}\|_{L(H_\gamma)}\right.\\
&\quad\left.+\sqrt{T^{1-\eta}p(p-1)}\left(|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}+\varepsilon\right)\sup_{t\in(0,T]}t^{\eta/2}\|(-\mathscr{A})^{\eta/2}e^{t\mathscr{A}}\|_{L(H_\gamma)}\right]\\
&\quad\cdot\sqrt{2}\sup_{t\in(0,T]}\left\|\left[X^1_t-\int_0^t \mathbb{1}_{\{\int_0^t\|e^{(t-r)\mathscr{A}}F(X^1_r)\|_{H_\gamma}\,\mathrm{d}r<\infty\}}e^{(t-s)\mathscr{A}}F(X^1_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.\\
&\quad-\int_0^t e^{(t-s)\mathscr{A}}B(X^1_s)\,\mathrm{d}W_s-\left\{\left[X^2_t-\int_0^t \mathbb{1}_{\{\int_0^t\|e^{(t-r)\mathscr{A}}F(X^2_r)\|_{H_\gamma}\,\mathrm{d}r<\infty\}}e^{(t-s)\mathscr{A}}F(X^2_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.\\
&\quad\left.\left.-\int_0^t e^{(t-s)\mathscr{A}}B(X^2_s)\,\mathrm{d}W_s\right\}\right\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}.
\end{aligned}\tag{3.16}$$

The fact that $\forall\,u\in[0,1]\colon \sup_{t\in(0,T]}t^u\|(-\mathscr{A})^u e^{t\mathscr{A}}\|_{L(H_\gamma)}\leq 1$ hence proves (3.15). The proof of Proposition 3.5 is thus completed. $\square$

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3.4 *Existence of continuous solutions of SEEs*

The next result, Proposition 3.6, proves the existence of continuous solution processes of SPDEs (see, e.g., van Neerven *et al.*, 2008, Theorem 7.1 for a similar result in a more general framework).

PROPOSITION 3.6 Consider the notation in Section 1.1; let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H; let $T \in (0,\infty)$, $p \in [2,\infty)$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process; let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$; let $A \colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \big\{v \in H \colon \sum_{h\in\mathbb{H}} \big|\lambda_h \langle h, v\rangle_H\big|^2 < \infty\big\}$ and $\forall\, v \in D(A) \colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$; let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ and let $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$, $\xi \in \mathscr{M}(\mathscr{F}_0, \mathscr{B}(H_\gamma))$ satisfy $|F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\eta}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$. Then there exists an $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic process $X \colon [0,T]\times\Omega \to H_\gamma$ with continuous sample paths that satisfies for all $t \in [0,T]$ that $[X_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \big[e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\,\mathrm{d}s\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} B(X_s)\,\mathrm{d}W_s$ and

$$\begin{aligned} &\sup_{t\in[0,T]} \Big\|\max\big\{1, \|X_t\|_{H_\gamma}\big\}\Big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \le \sqrt{2}\,\Big\|\max\big\{1, \|\xi\|_{H_\gamma}\big\}\Big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &\quad\cdot \mathscr{E}_{(1-\eta)}\Bigg[\frac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\Bigg(\sup_{v\in H_\gamma} \frac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\Bigg) + \sqrt{T^{1-\eta}p(p-1)}\Bigg(\sup_{v\in H_\gamma} \frac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\Bigg)\Bigg]. \end{aligned} \tag{3.17}$$

Proof. Throughout this proof let $\Omega_n \in \mathscr{F}_0$, $n \in \mathbb{N}_0$, be the sets that satisfy for all $n \in \mathbb{N}_0$ that $\Omega_n = \{\|\xi\|_{H_\gamma} < n\}$ and let $\xi_n \colon \Omega \to H_\gamma$, $n \in \mathbb{N}$, be the mappings that satisfy for all $n \in \mathbb{N}$ that $\xi_n = \xi\,\mathbb{1}_{\Omega_n}$. Note that it holds for all $q \in [0,\infty)$, $n \in \mathbb{N}$ that $\mathbb{E}\big[\|\xi_n\|_{H_\gamma}^q\big] \le n^q < \infty$. For example, Jentzen & Kloeden (2012, Theorem 5.1), Proposition 3.1 and the Kolmogorov–Chentsov continuity theorem (see Theorem 2.7) hence ensure that there exist $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes with continuous sample paths $X^n \colon [0,T]\times\Omega \to H_\gamma$, $n \in \mathbb{N}$, that satisfy for all $n \in \mathbb{N}$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s^n\|_{H_\gamma}^p\big] < \infty$ and

$$[X_t^n]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \bigg[e^{tA}\xi_n + \int_0^t e^{(t-s)A} F(X_s^n)\,\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} B(X_s^n)\,\mathrm{d}W_s. \tag{3.18}$$

Observe that it holds for all $k \in \mathbb{N}$, $n, m \in \{k, k+1, \ldots\}$, $t \in [0,T]$ that

$$\begin{aligned} [(X_t^n - X_t^m)\,\mathbb{1}_{\Omega_k}]_{\mathbb{P},\mathscr{B}(H_\gamma)} &= \bigg[\int_0^t e^{(t-s)A}\big[F(\mathbb{1}_{\Omega_k} X_s^n) - F(\mathbb{1}_{\Omega_k} X_s^m)\big]\mathbb{1}_{\Omega_k}\mathrm{d}s\bigg]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ &\quad + \int_0^t e^{(t-s)A}\big[B(\mathbb{1}_{\Omega_k} X_s^n) - B(\mathbb{1}_{\Omega_k} X_s^m)\big]\mathbb{1}_{\Omega_k}\mathrm{d}W_s. \end{aligned} \tag{3.19}$$

Jentzen & Kurniawan (2015, Proposition 2.1) hence shows for all $k \in \mathbb{N}$, $n, m \in \{k, k+1, \ldots\}$ that

$$\sup_{t\in[0,T]} \big\|(X_t^n - X_t^m)\,\mathbb{1}_{\Omega_k}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} = 0. \tag{3.20}$$

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This implies that

$$\mathbb{P}\Bigg(\forall\, k \in \mathbb{N}\colon \forall\, n, m \in \{k, k+1, \ldots\}\colon \mathbb{1}_{\Omega_k}\Bigg[\sup_{t\in[0,T]} \big\|X_t^n - X_t^m\big\|_{H_\gamma}\Bigg] = 0\Bigg) = 1. \tag{3.21}$$

Next let $Y\colon [0,T]\times\Omega \to H_\gamma$ be the mapping that satisfies for all $(t,\omega) \in [0,T]\times\Omega$ that

$$Y_t(\omega) = \sum_{n=1}^{\infty} X_t^n(\omega)\cdot \mathbb{1}_{\Omega_n\setminus\Omega_{n-1}}(\omega). \tag{3.22}$$

Note that it holds for all $n\in\mathbb{N}$ that

$$\begin{aligned}
&\mathbb{1}_{\Omega_n} \sup_{t\in[0,T]} \|Y_t - X_t^n\|_{H_\gamma} = \sup_{t\in[0,T]} \|Y_t\, \mathbb{1}_{\Omega_n} - X_t^n\, \mathbb{1}_{\Omega_n}\|_{H_\gamma} \\
&= \sup_{t\in[0,T]} \left\|\left[\sum_{k=1}^{n} X_t^k\, \mathbb{1}_{\Omega_k\setminus\Omega_{k-1}}\right] - X_t^n\, \mathbb{1}_{\Omega_n}\right\|_{H_\gamma} = \sup_{t\in[0,T]} \left\|\sum_{k=1}^{n} (X_t^k - X_t^n)\, \mathbb{1}_{\Omega_k\setminus\Omega_{k-1}}\right\|_{H_\gamma} \\
&= \sum_{k=1}^{n} \left[\mathbb{1}_{\Omega_k} \sup_{t\in[0,T]} \|X_t^k - X_t^n\|_{H_\gamma}\right] \mathbb{1}_{\Omega_k\setminus\Omega_{k-1}}.
\end{aligned} \tag{3.23}$$

This and (3.21) show that

$$\mathbb{P}\Bigg(\forall\, n\in\mathbb{N}\colon \mathbb{1}_{\Omega_n} \sup_{t\in[0,T]} \|Y_t - X_t^n\|_{H_\gamma} = 0\Bigg) = 1. \tag{3.24}$$

Hence, we obtain for all $n\in\mathbb{N}$, $t\in[0,T]$ that

$$\begin{aligned}
&[Y_t\, \mathbb{1}_{\Omega_n}]_{\mathbb{P},\mathscr{B}(H_\gamma)} = [X_t^n\, \mathbb{1}_{\Omega_n}]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&= \left(\left[e^{tA}\xi_n + \int_0^t e^{(t-s)A} F(X_s^n)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} B(X_s^n)\,\mathrm{d}W_s\right) \mathbb{1}_{\Omega_n} \\
&= \left(\left[e^{tA}\xi + \int_0^t e^{(t-s)A}\, \mathbb{1}_{\Omega_n} F(X_s^n)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}\, \mathbb{1}_{\Omega_n} B(X_s^n)\,\mathrm{d}W_s\right) \mathbb{1}_{\Omega_n} \\
&= \left(\left[e^{tA}\xi + \int_0^t e^{(t-s)A} F(Y_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} B(Y_s)\,\mathrm{d}W_s\right) \mathbb{1}_{\Omega_n}.
\end{aligned} \tag{3.25}$$

This implies for all $t\in[0,T]$ that

$$[Y_t]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \left[e^{tA}\xi + \int_0^t e^{(t-s)A} F(Y_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} B(Y_s)\,\mathrm{d}W_s. \tag{3.26}$$

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Next note that (3.24) and Proposition 3.4 ensure for all $n \in \mathbb{N}$ that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| \max\left\{1, \|Y_t \mathbb{1}_{\Omega_n}\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} = \sup_{t\in[0,T]} \left\| \max\left\{1, \|X_t^n \mathbb{1}_{\Omega_n}\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\leq \sup_{t\in[0,T]} \left\| \max\left\{1, \|X_t^n\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \leq \sqrt{2} \left\| \max\left\{1, \|\xi_n\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad \cdot \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\left(\sup_{v\in H_\gamma} \frac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right) + \sqrt{T^{1-\eta}p(p-1)}\left(\sup_{v\in H_\gamma} \frac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\right].
\end{aligned} \tag{3.27}$$

This and Fatou's lemma imply for all $t \in [0, T]$ that

$$\begin{aligned}
&\left\| \max\left\{1, \|Y_t\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} = \left\| \liminf_{n\to\infty} \max\left\{1, \|Y_t \mathbb{1}_{\Omega_n}\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\leq \liminf_{n\to\infty} \left\| \max\left\{1, \|Y_t \mathbb{1}_{\Omega_n}\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \leq \sqrt{2} \left\| \max\left\{1, \|\xi\|_{H_\gamma}\right\}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\
&\quad \cdot \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\left(\sup_{v\in H_\gamma} \frac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right) + \sqrt{T^{1-\eta}p(p-1)}\left(\sup_{v\in H_\gamma} \frac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\right].
\end{aligned} \tag{3.28}$$

The proof of Proposition 3.6 is thus completed. □

3.5 *Uniqueness of left-continuous solutions of SEEs with semiglobally Lipschitz continuous coefficients*

The proof of the next result, Proposition 3.7, is similar to the proof of Da Prato & Zabczyk (1992, Theorem 7.4) (also see, e.g., van Neerven *et al.*, 2008, Lemma 8.2 for an analogous result in a more general framework).

PROPOSITION 3.7 (Local solutions). Consider the notation in Section 1.1; let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H; let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$; let $A\colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \left\{v \in H\colon \sum_{h\in\mathbb{H}} \left|\lambda_h \langle h, v\rangle_H\right|^2 < \infty\right\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$; let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$; let $T \in (0,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0, 1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$ satisfy for all bounded sets $E \subseteq H_\gamma$ that $|F|_E|_{\mathscr{C}^1(E,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_E|_{\mathscr{C}^1(E,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process; let $\tau_k\colon \Omega \to [0, T]$, $k \in \{1, 2\}$, be $(\mathscr{F}_t)_{t\in[0,T]}$-stopping times and let $X^k\colon [0, T] \times \Omega \to H_\gamma$, $k \in \{1, 2\}$, be $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes with left-continuous and bounded sample paths that satisfy for all $k \in \{1, 2\}$, $t \in [0, T]$ that

$$\begin{aligned}
\left[X_t^k \mathbb{1}_{\{t\leq\tau_k\}}\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \Bigg(&\left[e^{tA}X_0^k + \int_0^t \mathbb{1}_{\{s<\tau_k\}}\, e^{(t-s)A} F(X_s^k)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&+ \int_0^t \mathbb{1}_{\{s<\tau_k\}}\, e^{(t-s)A} B(X_s^k)\,\mathrm{d}W_s\Bigg)\mathbb{1}_{\{t\leq\tau_k\}}.
\end{aligned} \tag{3.29}$$

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Then $\mathbb{P}\big(\forall\, t \in [0,T]\colon \mathbb{1}_{\{X_0^1=X_0^2\}}\, X^1_{\min\{t,\tau_1,\tau_2\}} = \mathbb{1}_{\{X_0^1=X_0^2\}}\, X^2_{\min\{t,\tau_1,\tau_2\}}\big) = 1$.

Corollary 3.8 is an immediate consequence of Proposition 3.7.

COROLLARY 3.8 (Continuous solutions). Consider the notation in Section 1.1; let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces; let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H; let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$; let $A\colon D(A) \subseteq H \to H$ be the linear operator that satisfies $D(A) = \big\{v \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\big\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$; let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$; let $T \in (0,\infty)$, $\gamma \in \mathbb{R}$, $\eta \in [0,1)$, $F \in C(H_\gamma, H_{\gamma-\eta})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\eta/2}))$ satisfy for all bounded sets $E \subseteq H_\gamma$ that $|F|_E|_{\mathscr{C}^1(E,\|\cdot\|_{H_{\gamma-\eta}})} + |B|_E|_{\mathscr{C}^1(E,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})} < \infty$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$; let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process and let $X^k\colon [0,T] \times \Omega \to H_\gamma$, $k \in \{1,2\}$, be $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes with continuous sample paths that satisfy for all $k \in \{1,2\}$, $t \in [0,T]$ that

$$[X_t^k]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \left[e^{tA}X_0^1 + \int_0^t e^{(t-s)A}F(X_s^k)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}B(X_s^k)\,\mathrm{d}W_s. \tag{3.30}$$

Then $\mathbb{P}\big(\forall\, t \in [0,T]\colon X_t^1 = X_t^2\big) = 1$.

4. Convergence in Hölder norms for Galerkin approximations

4.1 *Setting*

Consider the notation in Section 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a nonempty orthonormal basis of H, let $T, \iota \in (0,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$-Wiener process, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator that satisfies

$$D(A) = \left\{v \in H\colon\ \textstyle\sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty\right\} \tag{4.1}$$

and that satisfies for all $v \in D(A)$ that

$$Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h, \tag{4.2}$$

let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $\gamma \in \mathbb{R}$, $\alpha \in [0,1)$, $\beta \in [0, 1/2)$, $\chi \in [\beta, 1/2)$, $F \in C(H_\gamma, H_{\gamma-\alpha})$, $B \in C(H_\gamma, \mathrm{HS}(U, H_{\gamma-\beta}))$ satisfy for all bounded sets $E \subseteq H_\gamma$ that

$$|F|_E|_{\mathscr{C}^1(E,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_E|_{\mathscr{C}^1(E,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} < \infty, \tag{4.3}$$

let $\mathbb{H}_N \subseteq \mathbb{H}$, $N \in \mathbb{N}_0$, be sets that satisfy $\mathbb{H}_0 = \mathbb{H}$ and $\sup_{N\in\mathbb{N}} N^\iota \sup\big(\{1/|\lambda_h|\colon h \in \mathbb{H}\backslash\mathbb{H}_N\} \cup \{0\}\big) < \infty$, let $P_N \in L(H_{\min\{0,\gamma-1\}})$, $N \in \mathbb{N}_0$, and $\mathscr{P}_N \in L(U)$, $N \in \mathbb{N}_0$, be linear operators that satisfy for

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all $N \in \mathbb{N}_0$, $v \in H$ that

$$P_N(v) = \sum_{h \in \mathbb{H}_N} \langle h, v \rangle_H \, h \tag{4.4}$$

and let $X^N \colon [0, T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}_0$, be $(\mathscr{F}_t)_{t \in [0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes with continuous sample paths that satisfy for all $N \in \mathbb{N}_0$, $t \in [0, T]$ that

$$\left[X_t^N\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \left[e^{tA} P_N X_0^0 + \int_0^t e^{(t-s)A} P_N F(X_s^N)\, \mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B(X_s^N) \mathscr{P}_N \, \mathrm{d}W_s. \tag{4.5}$$

4.2 *Strong convergence in Hölder norms for Galerkin approximations of SEEs with globally Lipschitz continuous nonlinearities*

The next lemma, Lemma 4.1, follows directly from, e.g., Proposition 3.6 and, e.g., Corollary 3.8.

LEMMA 4.1 Assume the setting in Section 4.1, let $p \in [2, \infty)$, $\eta \in [\max\{\alpha, 2\beta\}, 1)$, $N \in \mathbb{N}_0$ and assume that

$$\mathbb{E}\big[\|X_0^0\|_{H_\gamma}^p\big] + |F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} < \infty. \tag{4.6}$$

Then

$$\begin{aligned} \sup_{t \in [0,T]} \left\| \max\{1, \|X_t^N\|_{H_\gamma}\} \right\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} &\leq \sqrt{2} \left\| \max\{1, \|X_0^0\|_{H_\gamma}\} \right\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \\ &\cdot \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}} \left(\sup_{v \in H_\gamma} \tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1, \|v\|_{H_\gamma}\}} \right) + \sqrt{T^{1-\eta} p(p-1)} \left(\sup_{v \in H_\gamma} \tfrac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})}}{\max\{1, \|v\|_{H_\gamma}\}} \right) \right]. \end{aligned} \tag{4.7}$$

LEMMA 4.2 Assume the setting in Section 4.1, let $p \in [2, \infty)$, $\eta \in [\max\{\alpha, 2\beta\}, 1)$, $N \in \mathbb{N}_0$ and assume that

$$\mathbb{E}\big[\|X_0^0\|_{H_\gamma}^p\big] + |F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} < \infty. \tag{4.8}$$

Then

$$\begin{aligned} \sup_{t \in [0,T]} \left\| X_t^0 - X_t^N \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})} &\leq \Bigg[\sqrt{2} \sup_{t \in [0,T]} \left\| (P_0 - P_N) X_t^0 \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})} \\ &+ \tfrac{T^{\frac{1}{2}-\chi}\sqrt{p(p-1)}}{\sqrt{1-2\chi}} \left(1 + \sup_{t \in [0,T]} \|X_t^N\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})} \right) \left(\sup_{v \in H_\gamma} \frac{\|B(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U, H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}} \right) \Bigg] \\ &\cdot \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\, |F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta} p(p-1)}\, |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\eta/2})})} \|\mathscr{P}_0\|_{L(U)} \right] < \infty. \end{aligned} \tag{4.9}$$

Proof. First of all, observe that Lemma 4.1 ensures that

$$\sup_{t \in [0,T]} \max\left\{ \|X_t^0\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})}, \|X_t^N\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{H_\gamma})} \right\} < \infty. \tag{4.10}$$

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We can hence apply Proposition 3.5 to obtain that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| X_t^0 - X_t^N \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|P_N F(\cdot)|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,\left|P_N B(\cdot)\mathscr{P}_0\right|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\right] \\
&\quad \cdot \sqrt{2} \sup_{t\in[0,T]} \left\| \left[X_t^0 - \int_0^t e^{(t-s)A} P_N F(X_s^0)\,\mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(H_\gamma)} - \int_0^t e^{(t-s)A} P_N B(X_s^0)\mathscr{P}_0\,\mathrm{d}W_s \right. \\
&\quad \left. + \left[\int_0^t e^{(t-s)A} P_N F(X_s^N)\,\mathrm{d}s - X_t^N \right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B(X_s^N)\mathscr{P}_0\,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\|\mathscr{P}_0\|_{L(U)}\right] \\
&\quad \cdot \sqrt{2} \sup_{t\in[0,T]} \left\| [(P_0 - P_N)X_t^0]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B(X_s^N)(\mathscr{P}_0 - \mathscr{P}_N)\,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}.
\end{aligned}
\tag{4.11}
$$

The Burkholder–Davis–Gundy-type inequality in Da Prato & Zabczyk (1992, Lemma 7.7) and the fact that $\forall\, u \in [0,1]\colon\ \sup_{t\in(0,T]} t^u \|(-A)^u e^{tA}\|_{L(H)} \leq 1$ hence imply that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| X_t^0 - X_t^N \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\|\mathscr{P}_0\|_{L(U)}\right] \\
&\quad \cdot \sqrt{2} \left[\sup_{t\in[0,T]} \left\| (P_0 - P_N)X_t^0 \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right. \\
&\quad \left. + \sqrt{\tfrac{p\,(p-1)}{2}} \left[\sup_{t\in[0,T]} \int_0^t \left\| e^{(t-s)A} B(X_s^N)(\mathscr{P}_0 - \mathscr{P}_N) \right\|^2_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_\gamma)})} \mathrm{d}s \right]^{\frac{1}{2}} \right] \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\eta/2})})}\|\mathscr{P}_0\|_{L(U)}\right] \\
&\quad \cdot \sqrt{2} \left[\sup_{t\in[0,T]} \left\| (P_0 - P_N)X_t^0 \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right. \\
&\quad \left. + \sqrt{\tfrac{p\,(p-1)}{2} \left[\sup_{t\in[0,T]} \int_0^t (t-s)^{-2\chi}\,\mathrm{d}s \right]} \sup_{s\in[0,T]} \left\| B(X_s^N)[\mathscr{P}_0 - \mathscr{P}_N] \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\chi})})} \right].
\end{aligned}
\tag{4.12}
$$

This and (4.10) complete the proof of Lemma 4.2. □

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COROLLARY 4.3 Assume the setting in Section 4.1, let $\vartheta \in [0, \min\{1-\alpha, 1/2-\beta\})$, $p \in [2,\infty)$ and assume that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ and

$$\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] + |F|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})} < \infty, \tag{4.13}$$

$$\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{N^{\iota\vartheta}\, \|B(v)(\mathscr{P}_0 - \mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right] < \infty. \tag{4.14}$$

Then

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(\|F(X_t^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\alpha}})} + \|B(X_t^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\chi})})}\big) < \infty \tag{4.15}$$

and

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(N^{\iota\vartheta}\, \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big) < \infty. \tag{4.16}$$

Proof. Combining the assumptions that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ and $\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$ with, e.g., Proposition 3.6 and, e.g., Corollary 3.8 ensures that $\forall\, t \in [0,T]\colon \mathbb{P}\big(X_t^0 \in H_{\gamma+\vartheta}\big) = 1$ and $\sup_{t\in[0,T]} \mathbb{E}\big[\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$. This and the assumption that $\sup_{N\in\mathbb{N}} N^\iota \sup\big(\{1/|\lambda_h|\colon h \in \mathbb{H}\backslash\mathbb{H}_N\} \cup \{0\}\big) < \infty$ imply that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left[N^{\iota\vartheta} \left\|(P_0 - P_N)X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right] \\
&\leq \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left[N^{\iota\vartheta} \left\|(-A)^{-\vartheta}(P_0|_{H_\gamma} - P_N|_{H_\gamma})\right\|_{L(H_\gamma)} \left\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}\right] \\
&\leq \left[\sup_{N\in\mathbb{N}} N^{\iota\vartheta} \left\|(-A)^{-1}(\mathrm{Id}_{H_\gamma} - P_N|_{H_\gamma})\right\|_{L(H_\gamma)}^{\vartheta}\right] \left[\sup_{t\in[0,T]} \left\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}\right] \\
&= \left[\sup_{N\in\mathbb{N}} N^{\iota\vartheta} \left[\sup\big(\{1/|\lambda_h|\colon h \in \mathbb{H}\backslash\mathbb{H}_N\} \cup \{0\}\big)\right]^{\vartheta}\right] \left[\sup_{t\in[0,T]} \left\|\mathbb{1}_{\{X_t^0\in H_{\gamma+\vartheta}\}} X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}\right] \\
&< \infty.
\end{aligned} \tag{4.17}$$

In addition, observe that the triangle inequality, (4.13) and (4.14) ensure that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}_0} \sup_{v\in H_\gamma} \left(\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{\max\{1,\|v\|_{H_\gamma}\}}\right) \leq 2\left(\sup_{N\in\mathbb{N}_0} \sup_{v\in H_\gamma} \frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right) \\
&\leq 2\left(\left[\sup_{v\in H_\gamma} \frac{\|B(v)\mathscr{P}_0\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right] + \left[\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \frac{\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right]\right) \\
&\leq 2\left(\|\mathscr{P}_0\|_{L(U)}\left[\sup_{v\in H_\gamma} \frac{\|B(v)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right] + \left[\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \frac{N^{\iota\vartheta}\,\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right]\right) < \infty.
\end{aligned} \tag{4.18}$$

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Again, (4.13) and Lemma 4.1 hence establish that

$$\sup_{N\in\mathbb{N}_0}\sup_{t\in[0,T]}\|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}<\infty. \tag{4.19}$$

This and (4.18) prove that

$$\begin{aligned}&\sup_{N\in\mathbb{N}_0}\sup_{t\in[0,T]}\|B(X_t^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\chi})})}\\&\leq\left(1+\sup_{N\in\mathbb{N}_0}\sup_{t\in[0,T]}\|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right)\left(\sup_{N\in\mathbb{N}_0}\sup_{v\in H_\gamma}\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right)<\infty.\end{aligned} \tag{4.20}$$

In the next step we combine (4.19), (4.17) and (4.14) with Lemma 4.2 to obtain that

$$\sup_{N\in\mathbb{N}_0}\sup_{t\in[0,T]}\left(N^{\iota\vartheta}\,\|X_t^0-X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right)<\infty. \tag{4.21}$$

Furthermore, observe that (4.19) ensures that $\sup_{N\in\mathbb{N}_0}\sup_{t\in[0,T]}\|F(X_t^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\alpha}})}<\infty$. This, (4.20) and (4.21) complete the proof of Corollary 4.3. □

The next result, Corollary 4.4, proves strong convergence rates in Hölder norms for spatial spectral Galerkin approximations of SEEs with globally Lipschitz continuous nonlinearities. Note in the setting of Corollary 4.4 that, e.g., Becker *et al.* (2018, Theorem 1.1 and Lemma 2.6) show in the case $\iota=2$, $\delta=0$ that the convergence rate established in (4.23) is essentially sharp (cf., e.g., Conus *et al.*, 2019, Lemma 7.2).

COROLLARY 4.4 Assume the setting in Section 4.1, let $\vartheta\in(0,\min\{1-\alpha,1/2-\beta\})$, $p\in(1/\vartheta,\infty)$ and assume that $X_0^0(\Omega)\subseteq H_{\gamma+\vartheta}$, $\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big]<\infty$, $|F|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})}<\infty$, $|B|_{\mathscr{C}^1(H_\gamma,\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})}<\infty$ and

$$\sup_{N\in\mathbb{N}}\sup_{v\in H_\gamma}\left[\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})}+N^{\iota\vartheta}\,\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right]<\infty. \tag{4.22}$$

Then it holds for all $\delta\in[0,\vartheta-1/p)$, $\varepsilon\in(0,\infty)$ that

$$\sup_{N\in\mathbb{N}}\left[\mathbb{E}\Big[\|X^N\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}^p\Big]+N^{\iota(\vartheta-\delta-1/p-\varepsilon)}\Big(\mathbb{E}\Big[\|X^0-X^N\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}^p\Big]\Big)^{1/p}\right]<\infty. \tag{4.23}$$

Proof. Throughout this proof let $\eta\in\mathbb{R}$ be the real number given by $\eta=\max\{\alpha,2\beta\}$ and let $\theta^N\in\mathscr{P}_T$, $N\in\mathbb{N}$, be a sequence of sets such that

$$\sup_{N\in\mathbb{N}}\left[\frac{d_{\max}(\theta^N)}{N^{-\iota}}+\frac{N^{-\iota}}{d_{\min}(\theta^N)}\right]<\infty. \tag{4.24}$$

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In particular, this ensures that $\limsup_{N\to\infty} d_{\max}(\theta^N) = 0$. In addition, Corollary 4.3 proves that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}}\left[\left|d_{\max}(\theta^N)\right|^{-\vartheta}\sup_{t\in\theta^N}\|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right]\\
&\leq \left[\sup_{N\in\mathbb{N}}\frac{|d_{\max}(\theta^N)|^{-\vartheta}}{N^{\iota\vartheta}}\right]\left(\sup_{N\in\mathbb{N}}\sup_{t\in\theta^N}N^{\iota\vartheta}\,\|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right)\\
&\leq \left[\sup_{N\in\mathbb{N}}\frac{N^{-\iota}}{d_{\min}(\theta^N)}\right]^{\vartheta}\left(\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}N^{\iota\vartheta}\,\|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right) < \infty.
\end{aligned}\tag{4.25}$$

Next note that, e.g., Corollary 3.2 shows for all $N\in\mathbb{N}_0$, $\varepsilon\in(0,\min\{1-\eta, 1/2-\beta\})$ that

$$\begin{aligned}
&\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}}\left(\frac{\left|\min\{t_1,t_2\}\right|^{\max\{\gamma+\varepsilon-(\gamma+\vartheta),0\}}\|X_{t_1}^N - X_{t_2}^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}}{\left|t_1-t_2\right|^{\varepsilon}}\right)\\
&\leq \|X_0^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\gamma+\vartheta,\gamma+\varepsilon\}}})} + \left[\sup_{s\in[0,T]}\left\|P_N F(X_s^N)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})}\right]\frac{2\,T^{(1+\gamma-\eta-\min\{\gamma+\vartheta,\gamma+\varepsilon\})}}{(1-\eta-\varepsilon)}\\
&\quad + \left[\sup_{s\in[0,T]}\left\|P_N B(X_s^N)\mathscr{P}_N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})}\right]\frac{\sqrt{p\,(p-1)}\,T^{(\frac{1}{2}+\gamma-\beta-\min\{\gamma+\vartheta,\gamma+\varepsilon\})}}{(1-2\beta-2\varepsilon)^{\frac{1}{2}}} < \infty.
\end{aligned}\tag{4.26}$$

This and the fact that $\min\{1-\eta, 1/2-\beta\} = \min\{1-\max\{\alpha,2\beta\}, 1/2-\beta\} = \min\{1-\alpha, 1/2-\beta\} > \vartheta > 0$ imply that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}_0}\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}}\left(\frac{\|X_{t_1}^N - X_{t_2}^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}}{\left|t_1-t_2\right|^{\vartheta}}\right)\\
&\leq \sup_{N\in\mathbb{N}_0}\|X_0^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} + \left[\sup_{N\in\mathbb{N}_0}\sup_{s\in[0,T]}\|F(X_s^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})}\right]\frac{2\,T^{(1-\eta-\vartheta)}}{(1-\eta-\vartheta)}\\
&\quad + \left[\sup_{N\in\mathbb{N}_0}\sup_{s\in[0,T]}\|B(X_s^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})}\right]\frac{\sqrt{p\,(p-1)}\,T^{(\frac{1}{2}-\beta-\vartheta)}}{(1-2\beta-2\vartheta)^{\frac{1}{2}}}.
\end{aligned}\tag{4.27}$$

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Corollary 4.3 and estimate (4.22) hence prove that

$$\begin{aligned}&\sup_{N\in\mathbb{N}_0}|X^N|_{\mathscr{C}^{\vartheta}([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})})}\\&\leq \|X_0^0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})}+\left[\sup_{N\in\mathbb{N}_0}\sup_{s\in[0,T]}\|F(X_s^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})}\right]\frac{2\,T^{(1-\eta-\vartheta)}}{(1-\eta-\vartheta)}\\&\quad+\left[\sup_{N\in\mathbb{N}_0}\sup_{s\in[0,T]}\|B(X_s^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{\mathrm{HS}(U,H_{\gamma-\beta})})}\right]\frac{\sqrt{p\,(p-1)}\,T^{(\frac{1}{2}-\beta-\vartheta)}}{(1-2\beta-2\vartheta)^{\frac{1}{2}}}<\infty.\end{aligned} \tag{4.28}$$

This, (4.25) and the fact that $\vartheta\in(1/p,1]$ allow us to apply Corollary 2.11 to obtain for all $\delta\in[0,\vartheta-1/p)$, $\varepsilon\in(0,\infty)$ that

$$\begin{aligned}\sup_{N\in\mathbb{N}}\Bigg[&\mathbb{E}\Big[\|X^N\|^p_{\mathscr{C}^{\delta}([0,T],\|\cdot\|_{H_\gamma})}\Big]\\&+\left|d_{\max}(\theta^N)\right|^{-(\vartheta-\delta-1/p-\varepsilon)}\Big(\mathbb{E}\Big[\|X^0-X^N\|^p_{\mathscr{C}^{\delta}([0,T],\|\cdot\|_{H_\gamma})}\Big]\Big)^{1/p}\Bigg]<\infty.\end{aligned} \tag{4.29}$$

Combining this with the fact that $\sup_{N\in\mathbb{N}}\Big[\frac{d_{\max}(\theta^N)}{N^{-\iota}}\Big]<\infty$ completes the proof of Corollary 4.4. □

4.3 *Almost-sure convergence in Hölder norms for Galerkin approximations of SEEs with semiglobally Lipschitz continuous nonlinearities*

The proof of the following corollary employs a standard localisation argument; see, e.g., Gyöngy (1998) and Printems (2001).

COROLLARY 4.5 Assume the setting in Section 4.1, let $\vartheta\in(0,\min\{1-\alpha,1/2-\beta\})$, assume that $\mathbb{P}\big(X_0^0\in H_{\gamma+\vartheta}\big)=1$ and assume for all nonempty bounded sets $E\subseteq H_\gamma$ that

$$\sup_{N\in\mathbb{N}}\sup_{v\in E}\left[\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})}+N^{\iota\vartheta}\,\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right]<\infty. \tag{4.30}$$

Then it holds for all $\delta\in[0,\vartheta)$, $\varepsilon\in(0,\infty)$ that

$$\mathbb{P}\Bigg(\sup_{N\in\mathbb{N}}\big[N^{\iota(\vartheta-\delta-\varepsilon)}\,\|X^0-X^N\|_{\mathscr{C}^{\delta}([0,T],\|\cdot\|_{H_\gamma})}\big]<\infty\Bigg)=1. \tag{4.31}$$

Proof. Throughout this proof we assume w.l.o.g. that $X_0^0(\Omega)\subseteq H_{\gamma+\vartheta}$, let $\delta\in[0,\vartheta)$, let $\phi_{r,M}\colon H_r\to H_r$, $r\in\mathbb{R}$, $M\in(0,\infty)$, be the mappings that satisfy for all $r\in\mathbb{R}$, $M\in(0,\infty)$, $v\in H_r$ that

$$\phi_{r,M}(v)=v\cdot\min\left\{1,\frac{M+1}{1+\|v\|_{H_r}}\right\}, \tag{4.32}$$

let $\xi_M\colon\Omega\to H_\gamma$, $M\in\mathbb{N}$, be the mappings that satisfy for all $M\in\mathbb{N}$ that $\xi_M=\phi_{\gamma+\vartheta,M}(X_0^0)$, let $F_M\colon H_\gamma\to H_{\gamma-\alpha}$, $M\in\mathbb{N}$, and $B_M\colon H_\gamma\to\mathrm{HS}(U,H_{\gamma-\beta})$, $M\in\mathbb{N}$, be the mappings that satisfy for all

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$M \in \mathbb{N}$ that $F_M = F \circ \phi_{\gamma,M}$ and $B_M = B \circ \phi_{\gamma,M}$ and let $S_M \subseteq H_\gamma$, $M \in \mathbb{N}$, be the sets that satisfy for all $M \in \mathbb{N}$ that $S_M = \{v \in H_\gamma : \|v\|_{H_\gamma} \leq M + 1\}$. Observe that it holds for all $v, w \in H_\gamma$, $M \in \mathbb{N}$ that

$$
\begin{aligned}
&\left\|\phi_{\gamma,M}(v) - \phi_{\gamma,M}(w)\right\|_{H_\gamma} \\
&= \left\| \frac{v\,(1 + \|w\|_{H_\gamma}) \min\{1 + \|v\|_{H_\gamma}, M+1\} - w\,(1 + \|v\|_{H_\gamma}) \min\{1 + \|w\|_{H_\gamma}, M+1\}}{(1 + \|v\|_{H_\gamma})\,(1 + \|w\|_{H_\gamma})} \right\|_{H_\gamma} \\
&\leq \|v - w\|_{H_\gamma} \\
&\quad + \left\| \frac{w\left[(1 + \|w\|_{H_\gamma}) \min\{1 + \|v\|_{H_\gamma}, M+1\} - (1 + \|v\|_{H_\gamma}) \min\{1 + \|w\|_{H_\gamma}, M+1\}\right]}{(1 + \|v\|_{H_\gamma})\,(1 + \|w\|_{H_\gamma})} \right\|_{H_\gamma} \\
&\leq \|v - w\|_{H_\gamma} \\
&\quad + \frac{\left|(1 + \|w\|_{H_\gamma}) \min\{1 + \|v\|_{H_\gamma}, M+1\} - (1 + \|v\|_{H_\gamma}) \min\{1 + \|w\|_{H_\gamma}, M+1\}\right|}{(1 + \|v\|_{H_\gamma})}.
\end{aligned}
\tag{4.33}
$$

This ensures for all $v, w \in H_\gamma$, $M \in \mathbb{N}$ that

$$
\begin{aligned}
&\left\|\phi_{\gamma,M}(v) - \phi_{\gamma,M}(w)\right\|_{H_\gamma} \\
&\leq \|v - w\|_{H_\gamma} + \frac{\left|\|w\|_{H_\gamma} - \|v\|_{H_\gamma}\right| \min\{1 + \|v\|_{H_\gamma}, M+1\}}{(1 + \|v\|_{H_\gamma})} \\
&\quad + \frac{(1 + \|v\|_{H_\gamma}) \left|\min\{1 + \|v\|_{H_\gamma}, M+1\} - \min\{1 + \|w\|_{H_\gamma}, M+1\}\right|}{(1 + \|v\|_{H_\gamma})} \\
&\leq \|v - w\|_{H_\gamma} + \left|\|w\|_{H_\gamma} - \|v\|_{H_\gamma}\right| \\
&\quad + \left|\min\{1 + \|v\|_{H_\gamma}, M+1\} - \min\{1 + \|w\|_{H_\gamma}, M+1\}\right| \\
&\leq 3\,\|v - w\|_{H_\gamma}\,.
\end{aligned}
\tag{4.34}
$$

Hence, we obtain for all $M \in \mathbb{N}$ that $|\phi_{\gamma,M}|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_\gamma})} \leq 3$. This, the fact that $\forall\, M \in \mathbb{N}\colon |F|_{S_M}|_{\mathscr{C}^1(S_M, \|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{S_M}|_{\mathscr{C}^1(S_M, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} + |\phi_{\gamma,M}|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_\gamma})} < \infty$ and the fact that $\forall\, M \in \mathbb{N}\colon \phi_{\gamma,M}(H_\gamma) \subseteq S_M$ ensure that it holds for all $M \in \mathbb{N}$, $p \in [1, \infty)$ that

$$
|F_M|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{H_{\gamma-\alpha}})} + |B_M|_{\mathscr{C}^1(H_\gamma, \|\cdot\|_{\mathrm{HS}(U, H_{\gamma-\beta})})} + \mathbb{E}\big[\|\xi_M\|^p_{H_{\gamma+\vartheta}}\big] < \infty. \tag{4.35}
$$

For example, Proposition 3.6 hence proves that there exist $(\mathscr{F}_t)_{t\in[0,T]}/\mathscr{B}(H_\gamma)$-adapted stochastic processes $\mathscr{X}^{N,M}\colon [0, T] \times \Omega \to H_\gamma$, $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, with continuous sample paths such that it

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holds for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0,T]$ that

$$\begin{aligned} \big[\mathscr{X}_t^{N,M}\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} = &\left[e^{tA}P_N\xi_M + \int_0^t e^{(t-s)A}P_N F_M(\mathscr{X}_s^{N,M})\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ &+ \int_0^t e^{(t-s)A}P_N B_M(\mathscr{X}_s^{N,M})\mathscr{P}_N\,\mathrm{d}W_s \end{aligned} \tag{4.36}$$

(cf., e.g., van Neerven *et al.*, 2008, Theorem 7.1). We now introduce a bit more notation. Let $\tau_{N,M}\colon \Omega \to [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$, be the mappings that satisfy for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\tau_{N,M} = \min\Big\{T\,\mathbb{1}_{\{\|X_0^0\|_{H_{\gamma+\vartheta}} \leq M\}}, \inf\Big(\Big\{t \in [0,T]\colon \|\mathscr{X}_t^{N,M}\|_{H_\gamma} \geq M\Big\} \cup \{T\}\Big)\Big\}, \tag{4.37}$$

let $\Upsilon \in \mathscr{F}$ be the set given by

$$\begin{aligned} &\Upsilon = \\ &\Big[\cap_{N\in\mathbb{N}_0} \cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\ldots\}} \{\tau_{N,m} = T\}\Big] \cap \Big[\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big\{\|\mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty\Big\}\Big] \\ &\cap \Big[\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big(\Big\{\|X_0^0\|_{H_{\gamma+\vartheta}} > M\Big\} \cup \Big\{\forall\, t \in [0,T]\colon \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = X^N_{\min\{t,\tau_{N,M}\}}\Big\}\Big)\Big] \\ &\cap \Big[\cap_{M,n\in\mathbb{N}} \Big\{\sup_{N\in\mathbb{N}} \big(N^{\iota(\vartheta-\delta-1/n)}\, \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\big) < \infty\Big\}\Big], \end{aligned} \tag{4.38}$$

let $\mathscr{M}\colon \Upsilon \to \mathbb{N}$ be the mapping that satisfies for all $\omega \in \Upsilon$ that

$$\mathscr{M}(\omega) = \min\big\{M \in \mathbb{N} \cap \big(\|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}, \infty\big)\colon \forall\, m \in \{M, M+1, \ldots\}\colon \tau_{0,m}(\omega) = T\big\} \tag{4.39}$$

and let $\mathscr{N}\colon \Upsilon \to \mathbb{N}$ be the mapping that satisfies for all

$$\omega \in \Upsilon \subseteq \Big\{\mathfrak{w} \in \Omega\colon \Big[\forall\, M \in \mathbb{N}\colon \limsup_{N\to\infty} \|\mathscr{X}^{0,M}(\mathfrak{w}) - \mathscr{X}^{N,M}(\mathfrak{w})\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} = 0\Big]\Big\}$$

that

$$\mathscr{N}(\omega) = \min\Big\{N \in \mathbb{N}\colon \sup_{n\in\{N,N+1,\ldots\}} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{n,2\mathscr{M}(\omega)}(\omega)\|_{C([0,T],\|\cdot\|_{H_\gamma})} < 1\Big\}. \tag{4.40}$$

Observe that (4.38) ensures for all $\omega \in \Upsilon$, $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0, \tau_{N,M}(\omega)]$ with $M \geq \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}$ that

$$\mathscr{X}_t^{N,M}(\omega) = X_t^N(\omega). \tag{4.41}$$

This, the fact that $\forall\, \omega \in \Upsilon, N \in \mathbb{N}_0\colon \exists\, M \in \mathbb{N}\colon \forall\, m \in \{M, M+1, \ldots\}\colon \tau_{N,m}(\omega) = T$ and the fact that $\forall\, \omega \in \Upsilon, N \in \mathbb{N}_0, m \in \mathbb{N}\colon \|\mathscr{X}^{N,m}(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty$ prove that it holds for all $\omega \in \Upsilon$, $N \in \mathbb{N}_0$ that

$$\|X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty. \tag{4.42}$$

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Next note that (4.39) ensures for all $\omega \in \Upsilon$, $M \in \{\mathscr{M}(\omega), \mathscr{M}(\omega)+1, \ldots\}$ that

$$\tau_{0,M}(\omega) = T \qquad \text{and} \qquad M \geq \mathscr{M}(\omega) > \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}. \tag{4.43}$$

This and (4.41) show for all $\omega \in \Upsilon$, $M \in \{\mathscr{M}(\omega), \mathscr{M}(\omega)+1, \ldots\}$, $t \in [0,T]$ that

$$\mathscr{X}_t^{0,M}(\omega) = X_t^0(\omega) = \mathscr{X}_t^{0,\mathscr{M}(\omega)}(\omega). \tag{4.44}$$

This, (4.43) and (4.37) prove for all $\omega \in \Upsilon$ that

$$\sup_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} = \sup_{t\in[0,T]} \|\mathscr{X}_t^{0,\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \leq \mathscr{M}(\omega). \tag{4.45}$$

The triangle inequality and (4.40) hence ensure for all $\omega \in \Upsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \ldots\}$ that

$$\begin{aligned} &\sup_{t\in[0,T]} \|\mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \\ &\leq \sup_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} + \sup_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \\ &< \sup_{t\in[0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} + 1 \leq \mathscr{M}(\omega) + 1 \leq 2\mathscr{M}(\omega). \end{aligned} \tag{4.46}$$

This and the fact that $\forall\, \omega \in \Upsilon\colon \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}} < \mathscr{M}(\omega) \leq 2\mathscr{M}(\omega)$ prove for all $\omega \in \Upsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \ldots\}$ that $\tau_{N,2\mathscr{M}(\omega)}(\omega) = T$. Again the fact that $\forall\, \omega \in \Upsilon\colon \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}} < \mathscr{M}(\omega) \leq 2\mathscr{M}(\omega)$ and (4.41) hence show for all $\omega \in \Upsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \ldots\}$, $t \in [0,T]$ that $\mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega) = X_t^N(\omega)$. This and (4.44) prove for all $\omega \in \Upsilon$, $\varepsilon \in (0,\infty)$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\leq \sup_{N\in\{1,2,\ldots,\mathscr{N}(\omega)\}} N^{\iota\vartheta} \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\quad + \sup_{N\in\{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\ldots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\leq [\mathscr{N}(\omega)]^{\iota\vartheta} \left[\|X^0(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} + \sup_{N\in\{1,2,\ldots,\mathscr{N}(\omega)\}} \|X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \right] \\ &\quad + \sup_{N\in\{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\ldots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{N,2\mathscr{M}(\omega)}(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}. \end{aligned} \tag{4.47}$$

Combining this with (4.42) and (4.38) ensures for all $\omega \in \Upsilon$, $\varepsilon \in (0,\infty)$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0(\omega) - X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\leq [\mathscr{N}(\omega)]^{\iota\vartheta} \textstyle\sum_{N=0}^{\mathscr{N}(\omega)} \|X^N(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\quad + \sup_{N\in\{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\ldots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{N,2\mathscr{M}(\omega)}(\omega)\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty. \end{aligned} \tag{4.48}$$

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It thus remains to prove that $\mathbb{P}(\Upsilon) = 1$ to complete the proof of Corollary 4.5. For this, observe that assumption (4.30) shows for all $M \in \mathbb{N}$ that

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\sup_{v\in H_\gamma}\left[\frac{\|B_M(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})} + N^{\iota\vartheta}\,\|B_M(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right]\\&\le \sup_{N\in\mathbb{N}}\sup_{v\in S_M}\left[\frac{\|B(v)\mathscr{P}_N\|_{\mathrm{HS}(U,H_{\gamma-\beta})} + N^{\iota\vartheta}\,\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{\mathrm{HS}(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right] < \infty.\end{aligned} \tag{4.49}$$

Corollary 4.4 hence proves for all $p \in (1/\vartheta, \infty)$, $r \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$, $M \in \mathbb{N}$ that

$$\sup_{N\in\mathbb{N}}\left[\mathbb{E}\Big[\|\mathscr{X}^{N,M}\|^p_{\mathscr{C}^r([0,T],\|\cdot\|_{H_\gamma})}\Big] + N^{\iota(\vartheta-r-\varepsilon)}\Big(\mathbb{E}\Big[\|\mathscr{X}^{0,M}-\mathscr{X}^{N,M}\|^p_{\mathscr{C}^r([0,T],\|\cdot\|_{H_\gamma})}\Big]\Big)^{1/p}\right] < \infty. \tag{4.50}$$

A standard Borel–Cantelli-type argument (see, e.g., Kloeden & Neuenkirch, 2007, Lemma 2.1) hence ensures for all $\varepsilon \in (0,\infty)$, $M \in \mathbb{N}$ that

$$\mathbb{P}\Big(\sup\nolimits_{N\in\mathbb{N}}\big(N^{\iota(\vartheta-\delta-\varepsilon)}\|\mathscr{X}^{0,M}-\mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\big) < \infty\Big) = 1. \tag{4.51}$$

Hence, we obtain that

$$\mathbb{P}\Big(\forall M,n\in\mathbb{N}\colon\ \sup\nolimits_{N\in\mathbb{N}}\big[N^{\iota(\vartheta-\delta-1/n)}\,\|\mathscr{X}^{0,M}-\mathscr{X}^{N,M}\|_{\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})}\big] < \infty\Big) = 1. \tag{4.52}$$

In addition, (4.50) proves for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ that $\mathbb{P}\big(\mathscr{X}^{N,M} \in \mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})\big) = 1$. This, in turn, ensures that

$$\mathbb{P}\Big(\forall M\in\mathbb{N}, N\in\mathbb{N}_0\colon\ \mathscr{X}^{N,M}\in\mathscr{C}^\delta([0,T],\|\cdot\|_{H_\gamma})\Big) = 1. \tag{4.53}$$

Next observe that it holds for all $t \in [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\begin{aligned}&\big[\mathscr{X}^{N,M}_t - e^{tA}P_N\mathscr{X}^{0,M}_0\big]_{\mathbb{P},\mathscr{B}(H_\gamma)}\mathbb{1}_{\{t\le\tau_{N,M}\}}\\&=\left(\left[\int_0^t e^{(t-s)A}P_NF_M(\mathscr{X}^{N,M}_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}P_NB_M(\mathscr{X}^{N,M}_s)\mathscr{P}_N\,\mathrm{d}W_s\right)\mathbb{1}_{\{t\le\tau_{N,M}\}}\\&=\left(\left[\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}}\,e^{(t-s)A}P_NF_M(\mathscr{X}^{N,M}_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.\\&\quad\left.+\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}}e^{(t-s)A}P_NB_M(\mathscr{X}^{N,M}_s)\mathscr{P}_N\,\mathrm{d}W_s\right)\mathbb{1}_{\{t\le\tau_{N,M}\}}\\&=\left(\left[\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}}\,e^{(t-s)A}P_NF(\mathscr{X}^{N,M}_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)}\right.\\&\quad\left.+\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}}\,e^{(t-s)A}P_NB(\mathscr{X}^{N,M}_s)\mathscr{P}_N\mathrm{d}W_s\right)\mathbb{1}_{\{t\le\tau_{N,M}\}}.\end{aligned} \tag{4.54}$$

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For example, Proposition 3.7 hence shows for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ that

$$\mathbb{P}\Big(\forall\, t \in [0,T]\colon \mathbb{1}_{\{\mathscr{X}_0^{N,M}=X_0^N\}} \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = \mathbb{1}_{\{\mathscr{X}_0^{N,M}=X_0^N\}} X^N_{\min\{t,\tau_{N,M}\}}\Big) = 1 \tag{4.55}$$

(cf., e.g., van Neerven *et al.*, 2008, Lemma 8.2). This implies for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ that

$$\mathbb{P}\big(\{\|X_0^0\|_{H_{\gamma+\vartheta}} > M\} \cup \big\{\forall\, t \in [0,T]\colon \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = X^N_{\min\{t,\tau_{N,M}\}}\big\}\big) = 1. \tag{4.56}$$

Hence, we obtain that

$$\mathbb{P}\Big(\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big[\{\|X_0^0\|_{H_{\gamma+\vartheta}} > M\} \cup \big\{\forall\, t \in [0,T]\colon \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = X^N_{\min\{t,\tau_{N,M}\}}\big\}\Big]\Big) = 1. \tag{4.57}$$

In the next step we combine this with (4.37) to obtain for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\mathbb{P}\Big(\tau_{N,M} = \min\Big\{T\,\mathbb{1}_{\{\|X_0^0\|_{H_{\gamma+\vartheta}} \leq M\}}, \inf\Big(\big\{t \in [0,T]\colon \|X_t^N\|_{H_\gamma} \geq M\big\} \cup \{T\}\Big)\Big\}\Big) = 1. \tag{4.58}$$

This shows for all $N \in \mathbb{N}_0$, $M_1, M_2 \in \mathbb{N}$ with $M_1 \leq M_2$ that $\mathbb{P}\big(\tau_{N,M_1} \leq \tau_{N,M_2}\big) = 1$. This, (4.58) and the fact that $\forall\, \omega \in \Omega, N \in \mathbb{N}_0\colon \sup_{t\in[0,T]} \|X_t^N(\omega)\|_{H_\gamma} < \infty$ imply that it holds for all $N \in \mathbb{N}_0$ that $\mathbb{P}\big(\cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\ldots\}} \{\tau_{N,m} = T\}\big) = 1$. This, in turn, proves that

$$\mathbb{P}\big(\cap_{N\in\mathbb{N}_0} \cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\ldots\}} \{\tau_{N,m} = T\}\big) = 1. \tag{4.59}$$

Combining (4.59), (4.53), (4.57) and (4.52) proves that $\mathbb{P}(\Upsilon) = 1$. The proof of Corollary 4.5 is thus completed. □

5. Cubature methods in Banach spaces

We first discuss in Section 5.1 a number of preliminary definitions related to the Monte Carlo method in Banach spaces. In Section 5.2 we present an elementary error estimate for the Monte Carlo method in Corollary 5.12. In Section 5.3 we then illustrate how expectations of Banach-space-valued functions of stochastic processes can be approximated.

5.1 *Preliminaries*

As mentioned in the introduction, the rate of convergence of Monte Carlo approximations in a Banach space depends on the so-called *type* of the Banach space; cf., e.g., Ledoux & Talagrand (1991, Section 9.2). In order to define the type of a Banach space, we first reconsider a few concepts from the literature.

Definition 5.1 Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let J be a set and let $r_j\colon \Omega \to \{-1,1\}$, $j \in J$, be a family of independent random variables with $\forall\, j \in J\colon \mathbb{P}\big(r_j = 1\big) = \mathbb{P}\big(r_j = -1\big)$. Then we say that $(r_j)_{j\in J}$ is a $\mathbb{P}$-Rademacher family.

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DEFINITION 5.2 Let $p \in (0,\infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space. Then we denote by $\mathscr{T}_p(E) \in [0,\infty]$ the extended real number given by

$$\mathscr{T}_p(E) = \sup\left(\left\{ c \in [0,\infty) \colon \begin{array}{c} \exists \text{ probability space } (\Omega, \mathscr{F}, \mathbb{P}) \colon \\ \exists\ \mathbb{P}\text{-Rademacher family } (r_j)_{j\in\mathbb{N}} \colon \\ \exists\, k \in \mathbb{N} \colon \exists\, x_1, x_2, \dots, x_k \in E\backslash\{0\} \colon \\ c = \frac{\left(\mathbb{E}\left[\|\sum_{j=1}^k r_j x_j\|_E^p\right]\right)^{1/p}}{\left(\sum_{j=1}^k \|x_j\|_E^p\right)^{1/p}} \end{array} \right\} \cup \{0\}\right) \tag{5.1}$$

and we call $\mathscr{T}_p(E)$ the type-p constant of E.

DEFINITION 5.3 Let $p \in (0,\infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space that satisfies $\mathscr{T}_p(E) < \infty$. Then we say that $(E, \|\cdot\|_E)$ has type p (we say that E has type p).

Note that it holds for all $p \in (0,\infty)$, all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with type p, all probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$, all $\mathbb{P}$-Rademacher families $(r_j)_{j\in\mathbb{N}}$ and all $k \in \mathbb{N}$, $x_1, x_2, \dots, x_k \in E$ that

$$\left\| \sum_{j=1}^k r_j x_j \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \le \mathscr{T}_p(E) \left(\sum_{j=1}^k \|x_j\|_E^p \right)^{1/p}. \tag{5.2}$$

In addition, observe that it holds for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$, all probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$, all $\mathbb{P}$-Rademacher families $(r_j)_{j\in\mathbb{N}}$ and all $p \in [2,\infty)$, $k \in \mathbb{N}$, $x \in E\backslash\{0\}$ that

$$\mathscr{T}_p(E) \ge \frac{\left\| \sum_{j=1}^k r_j x \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{\left[\sum_{j=1}^k \|x\|_E^p\right]^{1/p}} \ge \frac{\|x\|_E \left\| \sum_{j=1}^k r_j \right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}}{k^{1/p}\, \|x\|_E} = \frac{k^{1/2}\, \|x\|_E}{k^{1/p}\, \|x\|_E} = k^{(1/2-1/p)}. \tag{5.3}$$

In particular, it holds for all $p \in (2,\infty)$ and all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with $E \neq \{0\}$ that $\mathscr{T}_p(E) = \infty$. Furthermore, observe that Jensen's inequality together with the fact that it holds for all normed $\mathbb{R}$-vector spaces $(E, \|\cdot\|_E)$ and all $p \in (0,\infty)$, $q \in [p,\infty)$, $k \in \mathbb{N}$, $x_1, \dots, x_k \in E$ that

$$\left(\sum_{j=1}^k \|x_j\|_E^q\right)^{1/q} \le \left(\sum_{j=1}^k \|x_j\|_E^p\right)^{1/p} \tag{5.4}$$

ensures that it holds for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ and all $p, q \in (0,\infty)$ with $p \le q$ that $\mathscr{T}_p(E) \le \mathscr{T}_q(E)$. Hence, it holds for every $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ that the function $(0,\infty) \ni p \mapsto \mathscr{T}_p(E) \in [0,\infty]$ is nondecreasing. This and the triangle inequality ensure for all $p \in (0,1]$ and all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with $E \neq \{0\}$ that $\mathscr{T}_p(E) = 1$. In particular, note that it holds for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ that $\sup_{p\in(0,1]} \mathscr{T}_p(E) \le 1 < \infty$. Additionally, observe that it holds for all $p \in (0,2]$ and all $\mathbb{R}$-Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ with $H \neq \{0\}$ that $\mathscr{T}_p(H) = 1$. Furthermore, we note that it holds for every probability space $(\Omega, \mathscr{F}, \mathbb{P})$, every $p, q \in [1,\infty)$ and every $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ with type q that $L^p(\mathbb{P}; \|\cdot\|_E)$ has type $\min\{p,q\}$; cf., e.g., Hytönen *et al.* (2017, Proposition 7.1.4), Ledoux & Talagrand (1991, Section 9.2) or Albiac & Kalton (2006, Theorem 6.2.14). In particular, it holds for every $p \in [1,\infty)$ and every probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that $L^p(\mathbb{P}; |\cdot|)$ has type $\min\{p, 2\}$.

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DEFINITION 5.4 Let $p, q \in (0, \infty)$. Then we denote by $\mathscr{K}_{p,q} \in [0, \infty]$ the extended real number given by

$$\mathscr{K}_{p,q} = \sup \left\{ c \in [0,\infty) \colon \begin{array}{c} \exists\ \mathbb{R}\text{-Banach space } (E, \|\cdot\|_E) \colon \\ \exists \text{ probability space } (\Omega, \mathscr{F}, \mathbb{P}) \colon \\ \exists\ \mathbb{P}\text{-Rademacher family } (r_j)_{j\in\mathbb{N}} \colon \exists\, k \in \mathbb{N} \colon \\ \exists\, x_1, x_2, \ldots, x_k \in E\backslash\{0\} \colon c = \frac{\left(\mathbb{E}\left[\| \sum_{j=1}^k r_j x_j \|_E^p\right]\right)^{1/p}}{\left(\mathbb{E}\left[\| \sum_{j=1}^k r_j x_j \|_E^q\right]\right)^{1/q}} \end{array} \right\} \tag{5.5}$$

and we call $\mathscr{K}_{p,q}$ the (p, q)–Kahane–Khintchine constant.

The celebrated *Kahane–Khintchine inequality* asserts that it holds for all $p, q \in (0, \infty)$ that $\mathscr{K}_{p,q} < \infty$; see, e.g., Albiac & Kalton (2006, Theorem 6.2.5). Observe that Jensen's inequality ensures for all $p, q \in (0, \infty)$ with $p \le q$ that $\mathscr{K}_{p,q} = 1$. The nontrivial assertion of the Kahane–Khintchine inequality is the fact that it holds for all $p, q \in (0, \infty)$ with $p > q$ that $\mathscr{K}_{p,q} < \infty$. In our analysis below we also use the following two abbreviations.

DEFINITION 5.5 Let $p, q \in (0, \infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space. Then we denote by $\Theta_{p,q}(E) \in [0, \infty]$ the extended real number given by $\Theta_{p,q}(E) = 2\mathscr{T}_q(E)\mathscr{K}_{p,q}$.

DEFINITION 5.6 Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $p \in (0, \infty)$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space and let $X \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$. Then we denote by $\sigma_{p,E}(X) \in [0, \infty]$ the extended real number given by $\sigma_{p,E}(X) = \left(\mathbb{E}\left[\|X - \mathbb{E}[X]\|_E^p\right]\right)^{1/p}$.

5.2 *Monte Carlo methods in Banach spaces*

In this subsection we collect a few elementary results on sums of random variables with values in Banach spaces. Note that for every probability space $(\Omega, \mathscr{F}, \mathbb{P})$, every normed $\mathbb{R}$-vector space $(E, \|\cdot\|_E)$ and every $\mathscr{F}/\mathscr{B}(E)$-measurable mapping $\xi \colon \Omega \to E$ we denote by $\xi(\mathbb{P})$ the pushforward measure on $\mathscr{B}(E)$ of $\mathbb{P}$ under ξ. The next result, Lemma 5.7, can be found, e.g., in Ledoux & Talagrand (1991, Section 2.2).

LEMMA 5.7 (Symmetrisation lemma). Consider the notation in Section 1.1, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\xi, \tilde{\xi} \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E)$ be independent mappings that satisfy $\mathbb{E}\big[\|\tilde{\xi}\|_E\big] < \infty$ and $\mathbb{E}[\tilde{\xi}] = 0$ and let $\varphi \colon [0, \infty) \to [0, \infty)$ be a convex and nondecreasing function. Then

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big]. \tag{5.6}$$

Proof. Jensen's inequality ensures that

$$\begin{aligned} \mathbb{E}\big[\varphi(\|\xi\|_E)\big] = \mathbb{E}\big[\varphi(\|\xi - \mathbb{E}[\tilde{\xi}]\|_E)\big] &= \int_\Omega \varphi\left(\left\| \int_\Omega \xi(\omega) - \tilde{\xi}(\tilde{\omega})\, \mathbb{P}(\mathrm{d}\tilde{\omega}) \right\|_E\right) \mathbb{P}(\mathrm{d}\omega) \\ &\le \int_\Omega \varphi\left(\int_\Omega \|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E\, \mathbb{P}(\mathrm{d}\tilde{\omega}) \right) \mathbb{P}(\mathrm{d}\omega) \\ &\le \int_\Omega \int_\Omega \varphi(\|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E)\, \mathbb{P}(\mathrm{d}\tilde{\omega})\, \mathbb{P}(\mathrm{d}\omega) \end{aligned} \tag{5.7}$$

$$= \int_\Omega \int_\Omega \mathbb{1}_{\overline{\xi(\Omega)}^E \times \overline{\tilde{\xi}(\Omega)}^E}\big(\xi(\omega), \tilde{\xi}(\tilde{\omega})\big)\, \varphi(\|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E)\, \mathbb{P}(\mathrm{d}\tilde{\omega})\, \mathbb{P}(\mathrm{d}\omega)$$
$$= \int_E \int_E \mathbb{1}_{\overline{\xi(\Omega)}^E \times \overline{\tilde{\xi}(\Omega)}^E}(x, y)\, \varphi(\|x - y\|_E)\, \big(\tilde{\xi}(\mathbb{P})\big)(\mathrm{d}y)\big(\xi(\mathbb{P})\big)(\mathrm{d}x)$$
$$= \int_{E\times E} \mathbb{1}_{\overline{\xi(\Omega)}^E \times \overline{\tilde{\xi}(\Omega)}^E}(x, y)\, \varphi(\|x - y\|_E)\, \big((\xi, \tilde{\xi})(\mathbb{P})\big)(\mathrm{d}x, \mathrm{d}y) = \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big].$$

This completes the proof of Lemma 5.7. □

COROLLARY 5.8 (Symmetrisation corollary). Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\xi, \tilde{\xi} \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$ be independent and identically distributed mappings that satisfy $\mathbb{E}[\xi] = 0$ and let $\varphi\colon [0, \infty) \to [0, \infty)$ be a convex and nondecreasing function. Then

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big] \le \mathbb{E}\big[\varphi(2\, \|\xi\|_E)\big]. \tag{5.8}$$

Proof. Lemma 5.7 shows that

$$\begin{aligned}
\mathbb{E}\big[\varphi(\|\xi\|_E)\big] &\le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi\|_E + \|\tilde{\xi}\|_E)\big] = \mathbb{E}\big[\varphi(\tfrac{1}{2}\, 2\, \|\xi\|_E + \tfrac{1}{2}\, 2\, \|\tilde{\xi}\|_E)\big] \\
&\le \mathbb{E}\big[\tfrac{1}{2}\, \varphi(2\, \|\xi\|_E) + \tfrac{1}{2}\, \varphi(2\, \|\tilde{\xi}\|_E)\big] = \tfrac{1}{2}\, \mathbb{E}\big[\varphi(2\, \|\xi\|_E)\big] + \tfrac{1}{2}\, \mathbb{E}\big[\varphi(2\, \|\tilde{\xi}\|_E)\big] \\
&= \tfrac{1}{2}\, \mathbb{E}\big[\varphi(2\, \|\xi\|_E)\big] + \tfrac{1}{2}\, \mathbb{E}\big[\varphi(2\, \|\xi\|_E)\big] = \mathbb{E}\big[\varphi(2\, \|\xi\|_E)\big].
\end{aligned} \tag{5.9}$$

The proof of Corollary 5.8 is thus completed. □

As a straightforward application we obtain the following randomisation result; cf., e.g., Ledoux & Talagrand (1991, Lemma 6.3).

LEMMA 5.9 (Randomisation). Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $k \in \mathbb{N}$, let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, k\}$, satisfy for all $j \in \{1, \ldots, k\}$ that $\mathbb{E}[\xi_j] = 0$ and let $r_j\colon \Omega \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be a $\mathbb{P}$-Rademacher family such that $\xi_1, \xi_2, \ldots, \xi_k, r_1, r_2, \ldots, r_k$ are independent. Then it holds for all $p \in [1, \infty)$ that

$$\Big\| \sum_{j=1}^{k} \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \le 2 \Big\| \sum_{j=1}^{k} r_j \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)}. \tag{5.10}$$

Proof. Throughout this proof let $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P}) = (\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F}, \mathbb{P} \otimes \mathbb{P})$, let $\mathbf{r}_j\colon \boldsymbol{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be the mappings that satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \boldsymbol{\Omega}$, $j \in \{1, \ldots, k\}$ that $\mathbf{r}_j(\boldsymbol{\omega}) = r_j(\omega)$ and let $\boldsymbol{\xi}_j\colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\tilde{\boldsymbol{\xi}}_j\colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, be the mappings that satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \boldsymbol{\Omega}$, $j \in \{1, \ldots, k\}$ that $\boldsymbol{\xi}_j(\boldsymbol{\omega}) = \xi_j(\omega)$ and $\tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega}) = \xi_j(\tilde{\omega})$. The fact that

$$\{0, 1\} \times \{1, \ldots, k\} \ni (i, j) \mapsto \begin{cases} \boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j & : i = 0, \\ \mathbf{r}_j & : i = 1 \end{cases} \tag{5.11}$$

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is a family of independent mappings and the fact that $\forall j \in \{1,\dots,k\}\colon (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)(\mathbf{P}) = (\tilde{\boldsymbol{\xi}}_j - \boldsymbol{\xi}_j)(\mathbf{P})$ prove for all $p \in [1,\infty)$ that

$$
\begin{aligned}
&\int_{\boldsymbol{\Omega}} \left\| \sum_{j=1}^{k} \mathbf{r}_j(\boldsymbol{\omega})\big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big] \right\|_E^p \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) \\
&= \int_{\boldsymbol{\Omega}} \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^{1}[(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}}\big(\mathbf{r}_j(\boldsymbol{\omega})\big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big]\big)\, \mathbf{r}_j(\boldsymbol{\omega})\big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big] \right\|_E^p \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) \\
&= \int_{(\{-1,1\}\times E)^k} \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^{1}[(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}}(z_j x_j)\, z_j x_j \right\|_E^p \\
&\quad \big((\mathbf{r}_1, \boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1, \cdots, \mathbf{r}_k, \boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}z_1, \mathrm{d}x_1, \cdots, \mathrm{d}z_k, \mathrm{d}x_k) \\
&= \int_{\{-1,1\}} \int_E \cdots \int_{\{-1,1\}} \int_E \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^{1}[(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}}(z_j x_j)\, z_j x_j \right\|_E^p \\
&\quad \big((\boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}x_k)\big((\mathbf{r}_k)(\mathbf{P})\big)(\mathrm{d}z_k) \cdots \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1)(\mathbf{P})\big)(\mathrm{d}x_1)\big((\mathbf{r}_1)(\mathbf{P})\big)(\mathrm{d}z_1) \\
&= \int_E \cdots \int_E \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^{1}[(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}}(x_j)\, x_j \right\|_E^p \\
&\quad \big((\boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}x_k) \cdots \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1)(\mathbf{P})\big)(\mathrm{d}x_1) \\
&= \int_{E^k} \left\| \sum_{j=1}^{k} \mathbb{1}^E_{\overline{\cup_{i=0}^{1}[(-1)^i(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j)](\boldsymbol{\Omega})}}(x_j)\, x_j \right\|_E^p \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1, \dots, \boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P})\big)(\mathrm{d}x_1, \dots, \mathrm{d}x_k) \\
&= \int_{\boldsymbol{\Omega}} \left\| \sum_{j=1}^{k} \big[\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\big] \right\|_E^p \mathbf{P}(\mathrm{d}\boldsymbol{\omega}).
\end{aligned}
\tag{5.12}
$$

Furthermore, the fact that $\sum_{j=1}^k \boldsymbol{\xi}_j\colon \boldsymbol{\Omega} \to E$ and $\sum_{j=1}^k \tilde{\boldsymbol{\xi}}_j\colon \boldsymbol{\Omega} \to E$ are independent, the facts that $\int_{\boldsymbol{\Omega}} \|\sum_{j=1}^k \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\|_E\, \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) < \infty$ and $\int_{\boldsymbol{\Omega}} \sum_{j=1}^k \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega})\, \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) = 0$, Lemma 5.7 and (5.12) imply that it holds for all $p \in [1,\infty)$ that

$$
\begin{aligned}
&\left\| \sum_{j=1}^{k} \xi_j \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = \left\| \sum_{j=1}^{k} \boldsymbol{\xi}_j \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \le \left\| \sum_{j=1}^{k} (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j) \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \\
&= \left\| \sum_{j=1}^{k} \mathbf{r}_j(\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j) \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \le \left\| \sum_{j=1}^{k} \mathbf{r}_j \boldsymbol{\xi}_j \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} + \left\| \sum_{j=1}^{k} \mathbf{r}_j \tilde{\boldsymbol{\xi}}_j \right\|_{\mathscr{L}^p(\mathbf{P};\|\cdot\|_E)} \\
&= 2 \left\| \sum_{j=1}^{k} r_j \xi_j \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}.
\end{aligned}
\tag{5.13}
$$

The proof of Lemma 5.9 is thus completed. □

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The next result, Proposition 5.10, is the key to estimating the statistical error term in the Banach-space-valued Monte Carlo method in the next subsection. Proposition 5.10 is similar to, e.g., Ledoux & Talagrand (1991, Proposition 9.11).

PROPOSITION 5.10 (Sums of independent, centred, Banach-space-valued random variables). Let $k \in \mathbb{N}$, $q \in [1,2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, k\}$, be independent mappings that satisfy for all $j \in \{1, \ldots, k\}$ that $\mathbb{E}[\xi_j] = 0$. Then it holds for all $p \in [q, \infty)$ that

$$\Big\| \sum_{j=1}^{k} \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \le \Theta_{p,q}(E) \bigg(\sum_{j=1}^{k} \|\xi_j\|^q_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \bigg)^{1/q}. \tag{5.14}$$

Proof. Throughout this proof let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ be a probability space, let $r_j \colon \tilde{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be a $\tilde{\mathbb{P}}$-Rademacher family and let $\boldsymbol{\xi}_j \colon \Omega \times \tilde{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\mathbf{r}_j \colon \Omega \times \tilde{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be the mappings that satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$, $j \in \{1, \ldots, k\}$ that $\boldsymbol{\xi}_j(\boldsymbol{\omega}) = \xi_j(\omega)$ and $\mathbf{r}_j(\boldsymbol{\omega}) = r_j(\tilde{\omega})$. Lemma 5.9 and the triangle inequality show for all $p \in [q, \infty)$ that

$$\begin{aligned}
\Big\| \sum_{j=1}^{k} \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} &= \Big\| \sum_{j=1}^{k} \boldsymbol{\xi}_j \Big\|_{\mathscr{L}^p(\mathbb{P}\otimes\tilde{\mathbb{P}}; \|\cdot\|_E)} \le 2 \Big\| \sum_{j=1}^{k} \mathbf{r}_j \boldsymbol{\xi}_j \Big\|_{\mathscr{L}^p(\mathbb{P}\otimes\tilde{\mathbb{P}}; \|\cdot\|_E)} \\
&= 2 \bigg(\int_{\Omega} \Big\| \sum_{j=1}^{k} r_j(\cdot)\xi_j(\omega) \Big\|^p_{\mathscr{L}^p(\tilde{\mathbb{P}}; \|\cdot\|_E)} \mathbb{P}(\mathrm{d}\omega) \bigg)^{1/p} \\
&\le 2\mathscr{K}_{p,q} \bigg(\int_{\Omega} \Big\| \sum_{j=1}^{k} r_j(\cdot)\xi_j(\omega) \Big\|^p_{\mathscr{L}^q(\tilde{\mathbb{P}}; \|\cdot\|_E)} \mathbb{P}(\mathrm{d}\omega) \bigg)^{1/p} \\
&\le 2\mathscr{K}_{p,q}\mathscr{T}_q(E) \bigg\| \bigg(\sum_{j=1}^{k} \|\xi_j\|^q_E \bigg)^{1/q} \bigg\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} = 2\mathscr{K}_{p,q}\mathscr{T}_q(E) \Big\| \sum_{j=1}^{k} \|\xi_j\|^q_E \Big\|^{1/q}_{\mathscr{L}^{p/q}(\mathbb{P}; |\cdot|)} \\
&\le 2\mathscr{K}_{p,q}\mathscr{T}_q(E) \bigg(\sum_{j=1}^{k} \|\xi_j\|^q_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \bigg)^{1/q}.
\end{aligned} \tag{5.15}$$

This finishes the proof of Proposition 5.10. □

The result in Corollary 5.11 below is a direct consequence of Proposition 5.10.

COROLLARY 5.11 (Sums of independent Banach-space-valued random variables). Let $M \in \mathbb{N}$, $q \in [1,2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, M\}$, be independent. Then it holds for all $p \in [q, \infty)$ that

$$\sigma_{p,E}\bigg(\sum_{j=1}^{M} \xi_j \bigg) \le \Theta_{p,q}(E) \bigg(\sum_{j=1}^{M} |\sigma_{p,E}(\xi_j)|^q \bigg)^{1/q}. \tag{5.16}$$

COROLLARY 5.12 (Monte Carlo methods in Banach spaces). Let $M \in \mathbb{N}$, $q \in [1,2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, M\}$,

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be independent and identically distributed. Then it holds for all $p \in [q, \infty)$ that

$$\left\| \mathbb{E}[\xi_1] - \frac{1}{M} \sum_{j=1}^{M} \xi_j \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = \frac{\sigma_{p,E}\left(\sum_{j=1}^{M} \xi_j\right)}{M} \leq \frac{\Theta_{p,q}(E)\, \sigma_{p,E}(\xi_1)}{M^{1-1/q}}. \tag{5.17}$$

Results on lower and upper error bounds related to Corollary 5.12 can be found, e.g., in Daun & Heinrich (2013, Theorem 1) and in Heinrich & Hinrichs (2014, Corollary 2). Note that Corollary 5.12 does not imply convergence if the underlying Banach space $(E, \|\cdot\|_E)$ has only type 1, in the sense that it holds for all $q \in (1, \infty)$ that $\mathscr{T}_q(E) = \infty$.

5.3 *Multilevel Monte Carlo methods in Banach spaces*

In many situations the work required to obtain a certain accuracy of an approximation using the Monte Carlo method can be reduced by using a multilevel Monte Carlo method. Heinrich (1998, 2001) was first to observe this and established multilevel Monte Carlo methods concerning convergence in a Banach (function) space. However, these methods do not apply to SDEs. Then Giles (2008) derived the complexity reduction of multilevel Monte Carlo methods for SDEs. The minor contribution of Proposition 5.13 to the literature on multilevel Monte Carlo methods is to combine the approaches of Heinrich (1998) and of Giles (2008) into a single result on multilevel Monte Carlo methods in Banach spaces. The useful observation of Proposition 5.13 generalises the discussion in Heinrich (2001, Section 4).

PROPOSITION 5.13 (Abstract multilevel Monte Carlo methods in Banach spaces). Let $q \in [1, 2]$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space; let $(V_1, \|\cdot\|_{V_1})$ be an $\mathbb{R}$-Banach space with type q; let $(V_2, \|\cdot\|_{V_2})$ be an $\mathbb{R}$-Banach space with $V_1 \subseteq V_2$ continuously; let $v \in V_2$, $L \in \mathbb{N}$, $M_1, \ldots, M_L \in \mathbb{N}$ and for every $\ell \in \{1, \ldots, L\}$ let $D_{\ell,k} \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_{V_1})$, $k \in \{1, \ldots, M_\ell\}$, be independent and identically distributed. Then it holds for all $p \in [q, \infty)$ that

$$\begin{aligned} &\left\| v - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\ &\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}(D_{\ell,1})}{(M_\ell)^{1-1/q}}. \end{aligned} \tag{5.18}$$

Proof. The triangle inequality and Corollary 5.12 imply for all $p \in [q, \infty)$ that

$$\begin{aligned} &\left\| v - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\ &\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] \right\|_{V_2} + \left\| \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \end{aligned} \tag{5.19}$$

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$$\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)} \sum_{\ell=1}^{L} \left\| \mathbb{E}\big[D_{\ell,1}\big] - \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}$$

$$\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}(D_{\ell,1})}{(M_\ell)^{1-1/q}}.$$

This completes the proof of Proposition 5.13. ☐

COROLLARY 5.14 (Multilevel Monte Carlo methods in Banach spaces). Consider the notation in Section 1.1; let $q \in [1,2]$, $L \in \mathbb{N}_0$, $M_0, M_1, \ldots, M_{L+1}, N_0, N_1, \ldots, N_L \in \mathbb{N}$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space; let $(V_i, \|\cdot\|_{V_i})$, $i \in \{1,2\}$, be separable $\mathbb{R}$-Banach spaces such that $(V_1, \|\cdot\|_{V_1})$ has type q and such that $V_1 \subseteq V_2$ continuously; let $(V_3, \|\cdot\|_{V_3})$ be an $\mathbb{R}$-Banach space; let $f \in \mathscr{M}(\mathscr{B}(V_3), \mathscr{B}(V_2))$, $g \in \mathscr{M}(\mathscr{B}(V_3), \mathscr{B}(V_1))$, $X \in \mathscr{M}(\mathscr{F}, \mathscr{B}(V_3))$ satisfy $\mathbb{E}\big[\|f(X)\|_{V_2}\big] < \infty$; for every $n \in \mathbb{N}$ let $Y^{n,l,k} \in \mathscr{M}(\mathscr{F}, \mathscr{B}(V_3))$, $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, satisfy $\mathbb{E}\big[\|g(Y^{n,0,1})\|_{V_1}\big] < \infty$; assume that $Y^{N_0,0,k}$, $k \in \mathbb{N}$, are independent and identically distributed and assume for every $\ell \in \mathbb{N} \cap [0,L]$ that $(Y^{N_{(\ell-1)},\mathfrak{l},k}, Y^{N_\ell,\mathfrak{l},k})$, $k \in \mathbb{N}$, $\mathfrak{l} \in \mathbb{N}_0$, are independent and identically distributed. Then it holds for all $p \in [q, \infty)$ that

$$\begin{aligned}
&\left\| \mathbb{E}[f(X)] - \tfrac{1}{M_0} \sum_{k=1}^{M_0} g(Y^{N_0,0,k}) - \sum_{\ell=1}^{L} \tfrac{1}{M_\ell} \sum_{k=1}^{M_\ell} \Big[g(Y^{N_\ell,\ell,k}) - g(Y^{N_{(\ell-1)},\ell,k}) \Big] \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_2})} \\
&\leq \Big\| \mathbb{E}[f(X)] - \mathbb{E}\Big[g(Y^{N_L,0,1})\Big] \Big\|_{V_2} \\
&\quad + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \left(\frac{\sigma_{p,V_1}\big(g(Y^{N_0,0,1})\big)}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}\Big(g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\Big)}{(M_\ell)^{1-1/q}} \right) \\
&\leq \Big\| \mathbb{E}[f(X)] - \mathbb{E}\Big[g(Y^{N_L,0,1})\Big] \Big\|_{V_2} \\
&\quad + \|\mathrm{Id}_{V_1}\|_{L(V_1,V_2)}\, \Theta_{p,q}(V_1) \left(\frac{2\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=0}^{L} \frac{4\,\|g(Y^{N_\ell,0,1}) - g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(\min\{M_\ell, M_{\ell+1}\})^{1-1/q}} \right).
\end{aligned} \tag{5.20}$$

Proof. Observe that the assumption that for every $\ell \in \mathbb{N} \cap [0,L]$ it holds that $(Y^{N_{(\ell-1)},\mathfrak{l},k}, Y^{N_\ell,\mathfrak{l},k})$, $k \in \mathbb{N}$, $\mathfrak{l} \in \mathbb{N}_0$, are independent ensures that for all $\ell \in \mathbb{N} \cap [0,L]$, $\mathfrak{N} \in \mathbb{N}$, $\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_{\mathfrak{N}} \in \mathbb{N}_0$, $k_1, k_2, \ldots, k_{\mathfrak{N}} \in \mathbb{N}$, $A_1, A_2, \ldots, A_{\mathfrak{N}} \in \mathscr{B}(V_3) \otimes \mathscr{B}(V_3)$ it holds that

$$\begin{aligned}
&\mathbb{P}\Big((Y^{N_{(\ell-1)},\mathfrak{l}_1,k_1}, Y^{N_\ell,\mathfrak{l}_1,k_1}) \in A_1, \ldots, (Y^{N_{(\ell-1)},\mathfrak{l}_{\mathfrak{N}},k_{\mathfrak{N}}}, Y^{N_\ell,\mathfrak{l}_{\mathfrak{N}},k_{\mathfrak{N}}}) \in A_{\mathfrak{N}}\Big) \\
&= \prod_{i=1}^{\mathfrak{N}} \mathbb{P}\Big((Y^{N_{(\ell-1)},\mathfrak{l}_i,k_i}, Y^{N_\ell,\mathfrak{l}_i,k_i}) \in A_i\Big).
\end{aligned} \tag{5.21}$$

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Furthermore, Proposition 5.13 and the identity

$$\mathbb{E}\Big[g(Y^{N_L,0,1})\Big] = \mathbb{E}\Big[g(Y^{N_0,0,1})\Big] + \sum_{\ell=1}^{L} \mathbb{E}\Big[g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\Big] \tag{5.22}$$

imply the first inequality in (5.20). Next note that the triangle inequality demonstrates for all $\xi \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_{V_1})$, $p \in [q, \infty)$ that $\sigma_{p,V_1}(\xi) \leq 2\,\|\xi\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}$. This and again the triangle inequality show for all $p \in [q, \infty)$ that

$$\begin{aligned}
&\frac{\sigma_{p,V_1}\big(g(Y^{N_0,0,1})\big)}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}\Big(g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\Big)}{(M_\ell)^{1-1/q}} \\
&\leq \frac{2\big\|g(Y^{N_0,0,1})\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \frac{2\Big\|g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\Big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_\ell)^{1-1/q}} \\
&\leq \frac{2\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})} + 2\,\big\|g(Y^{N_0,0,1}) - g(X)\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} \\
&\quad + \sum_{\ell=1}^{L} \frac{2\big\|g(Y^{N_\ell,0,1}) - g(X)\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})} + 2\Big\|g(Y^{N_{(\ell-1)},0,1}) - g(X)\Big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_\ell)^{1-1/q}} \\
&\leq \frac{2\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=0}^{L} \frac{4\big\|g(Y^{N_\ell,0,1}) - g(X)\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(\min\{M_\ell, M_{\ell+1}\})^{1-1/q}}.
\end{aligned} \tag{5.23}$$

This implies the second inequality in (5.20). The proof of Corollary 5.14 is thus completed. □

Corollary 5.15 (Convergence of multilevel Monte Carlo approximations). Consider the notation in Section 1.1; let $T \in (0, \infty)$, $\beta \in (0, 1]$, $\alpha \in (0, \beta)$, $c, r \in [0, \infty)$; let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space; let $(E, \|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space with type 2; let $X\colon [0, T] \times \Omega \to E$ be a stochastic process with continuous sample paths that satisfies for all $p \in [1, \infty)$, $\gamma \in [0, \beta)$ that $X \in \mathscr{C}^\gamma\big([0, T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\big)$; for every $N \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ let $Y^{N,\ell,k}\colon [0, T] \times \Omega \to E$ be a stochastic process that satisfies for all $n \in \{0, 1, \ldots, N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $p \in [1, \infty)$, $\rho \in [0, \beta)$ that

$$Y_t^{N,\ell,k} = \big(n + 1 - \tfrac{tN}{T}\big) \cdot Y^{N,\ell,k}_{\frac{nT}{N}} + \big(\tfrac{tN}{T} - n\big) \cdot Y^{N,\ell,k}_{\frac{(n+1)T}{N}}, \tag{5.24}$$

$$\sup_{M \in \mathbb{N}} \sup_{m \in \{0,1,\ldots,M\}} \left(M^\rho \left\| X_{\frac{mT}{M}} - Y^{M,0,1}_{\frac{mT}{M}} \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right) < \infty; \tag{5.25}$$

assume for every $N_1, N_2 \in \mathbb{N}$ that $(Y^{N_1,\ell,k}, Y^{N_2,\ell,k})$, $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, are independent and identically distributed and let $f\colon C([0, T], E) \to C([0, T], E)$ be a $\mathscr{B}\big(C([0, T], E)\big)/\mathscr{B}\big(C([0, T], E)\big)$-measurable function that satisfies for all $v, w \in \mathscr{C}^\alpha([0, T], \|\cdot\|_E)$ that $f(v), f(w) \in \mathscr{C}^\alpha([0, T], \|\cdot\|_E)$ and

$$\|f(v) - f(w)\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \leq c\left(1 + \|v\|^r_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} + \|w\|^r_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\right)\|v - w\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}. \tag{5.26}$$

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Then it holds that

$$\mathbb{E}\Big[\|f(X)\|_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\Big] < \infty; \tag{5.27}$$

it holds for all $p \in [1,\infty)$, $\rho \in [0, \beta-\alpha)$ that

$$\sup_{N\in\mathbb{N}} \left[N^\rho \Big(\mathbb{E}\Big[\|f(X) - f(Y^{N,0,1})\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)} \Big]\Big)^{1/p} \right] < \infty \tag{5.28}$$

and it holds for all $p \in [1,\infty)$, $\gamma \in [0,\alpha)$, $\rho \in [0,\beta-\alpha)$ that

$$\sup_{L\in\mathbb{N}} \left[2^{L\cdot\min\{\rho,1/2\}} L^{-\mathbb{1}_{\{1/2\}}(\rho)} \left| \mathbb{E}\left[\left\| \mathbb{E}[f(X)] - \sum_{k=1}^{2^L} \frac{f(Y^{1,0,k})}{2^L} - \sum_{\ell=1}^{L} \sum_{k=1}^{2^{L-\ell}} \frac{f(Y^{2^\ell,\ell,k}) - f(Y^{2^{(\ell-1)},\ell,k})}{2^{L-\ell}} \right\|^p_{\mathscr{C}^\gamma([0,T],\|\cdot\|_E)} \right] \right|^{1/p} \right] < \infty. \tag{5.29}$$

Proof. Throughout this proof let $\gamma \in [0,\alpha)$, $\delta \in (\gamma, \frac{3\gamma+\alpha}{4})$; let $C^1([0,T],E)$ be the $\mathbb{R}$-vector space of continuously Fréchet differentiable functions from $[0,T]$ to E; let $\|\cdot\|_{C^1([0,T],E)} \colon C^1([0,T],E) \to [0,\infty)$ be the function that satisfies for all $v \in C^1([0,T],E)$ that $\|v\|_{C^1([0,T],E)} = \|v\|_{C([0,T],\|\cdot\|_E)} + \|v'\|_{C([0,T],\|\cdot\|_E)}$; let $\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)$ be the Sobolev space with regularity parameter $(\alpha+\gamma)/2 \in (0,1)$ and integrability parameter $4/(\alpha-\gamma) \in (4,\infty)$ of continuous functions from $[0,T]$ to E; let

$$\|\cdot\|_{\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)} \colon \mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E) \to [0,\infty)$$

be the function that satisfies for all $v \in \mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)$ that

$$\|v\|_{\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)} = \left[\int_0^T \|v(t)\|_E^{\frac{4}{\alpha-\gamma}}\,\mathrm{d}t + \int_0^T \int_0^T \frac{\|v(t)-v(s)\|_E^{\frac{4}{\alpha-\gamma}}}{|t-s|^{\frac{3\alpha+\gamma}{\alpha-\gamma}}}\,\mathrm{d}t\,\mathrm{d}s \right]^{\frac{\alpha-\gamma}{4}}; \tag{5.30}$$

let $V_1, V_2 \subseteq \mathscr{C}^\gamma([0,T],\|\cdot\|_E)$ be the sets given by $V_1 = \mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)$ and

$$V_2 = \left\{ v \in \mathscr{C}^\gamma([0,T],\|\cdot\|_E) \colon \limsup_{n\to\infty} \sup_{s,t\in[0,T],\, 0<|s-t|<1/n} \frac{\|v(s)-v(t)\|_E}{|s-t|^\gamma} = 0 \right\} \tag{5.31}$$

(cf., e.g., Lunardi, 1995, Section 0.2); let $\|\cdot\|_{V_1} \colon V_1 \to [0,\infty)$ be the function given by $\|\cdot\|_{V_1} = \|\cdot\|_{\mathcal{W}^{(\alpha+\gamma)/2,4/(\alpha-\gamma)}([0,T],E)}$; let $\|\cdot\|_{V_2} \colon V_2 \to [0,\infty)$ be the function that satisfies for all $v \in V_2$ that $\|v\|_{V_2} = \|v\|_{\mathscr{C}^\gamma([0,T],\|\cdot\|_E)}$; let $(V_3, \|\cdot\|_{V_3})$ be the $\mathbb{R}$-Banach space given by

$$(V_3, \|\cdot\|_{V_3}) = \Big(C([0,T],E), \|\cdot\|_{C([0,T],\|\cdot\|_E)}\big|_{C([0,T],E)} \Big) \tag{5.32}$$

and let $\mathfrak{f} \colon V_3 \to V_2$ and $g \colon V_3 \to V_1$ be the functions that satisfy for all $v \in V_3$ that $\mathfrak{f}(v) = g(v) = \mathbb{1}_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}(v) f(v)$. Observe that the Kolmogorov–Chentsov continuity theorem (see Theorem 2.7) together with the assumptions that $X \in \cap_{p\in[1,\infty)} \cap_{\eta\in[0,\beta)} \mathscr{C}^\eta\big([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\big)$ and that X has continuous sample paths implies for all $p \in [1,\infty)$ that $\mathbb{E}\big[\|X\|^p_{\mathscr{C}^\alpha([0,T],\|\cdot\|_E)}\big] < \infty$. This, assumption

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(5.26), Hölder's inequality and Corollary 2.11 show for all $p \in [1,\infty)$, $\rho \in [0,\beta-\alpha)$ that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \Big(N^{\rho}\,\mathbb{E}\Big[\|\mathfrak{f}(X)-g(Y^{N,0,1})\|_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big) \\
&\leq \sup_{N\in\mathbb{N}} \Big[N^{\rho}\Big(\mathbb{E}\Big[\|f(X)-f(Y^{N,0,1})\|^{p}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\Big] \\
&\leq \sup_{N\in\mathbb{N}} \Big[N^{\rho}\Big(\mathbb{E}\Big[\Big(c\,\big(1+\|X\|^{r}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}+\|Y^{N,0,1}\|^{r}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\big) \\
&\qquad \cdot\|X-Y^{N,0,1}\|_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big)^{p}\Big]\Big)^{1/p}\Big] \\
&\leq c\Big[1+\Big(\mathbb{E}\Big[\|X\|^{2pr}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/(2p)}+\sup_{N\in\mathbb{N}}\Big(\mathbb{E}\Big[\|Y^{N,0,1}\|^{2pr}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/(2p)}\Big] \\
&\qquad \cdot\sup_{N\in\mathbb{N}}\Big[N^{\rho}\Big(\mathbb{E}\Big[\|X-Y^{N,0,1}\|^{2p}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/(2p)}\Big]<\infty.
\end{aligned} \tag{5.33}$$

Assumption (5.26) also ensures for all $p \in [1,\infty)$ that

$$\begin{aligned}
&\mathbb{E}\Big[\|f(X)\|_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big] \leq \Big(\mathbb{E}\Big[\|f(X)\|^{p}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p} \\
&\leq \|f(0)\|_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}+c\Big[\Big(\mathbb{E}\Big[\|X\|^{p}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}+\Big(\mathbb{E}\Big[\|X\|^{(r+1)p}_{\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E)}\Big]\Big)^{1/p}\Big]<\infty.
\end{aligned} \tag{5.34}$$

Next note that $(V_1,\|\cdot\|_{V_1})$ is a separable $\mathbb{R}$-Banach space with type 2. In addition, the fact that $(C^1([0,T],E),\|\cdot\|_{C^1([0,T],E)})$ is a separable $\mathbb{R}$-Banach space, the fact that $C^1([0,T],E) \subseteq \mathscr{C}^{\gamma}([0,T],\|\cdot\|_E)$ continuously and the fact that

$$\overline{C^1([0,T],E)}^{\mathscr{C}^{\gamma}([0,T],\|\cdot\|_E)} = V_2 \tag{5.35}$$

(cf., e.g., Lunardi, 1995, Proposition 0.2.1) prove that $(V_2,\|\cdot\|_{V_2})$ is a separable $\mathbb{R}$-Banach space. Moreover, the Sobolev embedding theorem proves that $V_1 \subseteq \mathscr{C}^{\delta}([0,T],\|\cdot\|_E)$ continuously. This and the fact that $\mathscr{C}^{\delta}([0,T],\|\cdot\|_E) \subseteq V_2$ continuously establish that $V_1 \subseteq V_2$ continuously. Combining (5.34) with (5.33) and the fact that $\mathscr{C}^{\alpha}([0,T],\|\cdot\|_E) \subseteq V_1$ continuously hence implies for all $p \in [1,\infty)$, $\rho \in [0,\beta-\alpha)$ that $\mathbb{E}\big[\|\mathfrak{f}(X)\|_{V_2}\big]+\sup_{N\in\mathbb{N}}\mathbb{E}\big[\|g(Y^{N,0,1})\|_{V_1}\big]<\infty$, $\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}<\infty$ and

$$\sup_{N\in\mathbb{N}}\Big(N^{\rho}\,\mathbb{E}\Big[\|\mathfrak{f}(X)-g(Y^{N,0,1})\|_{V_2}\Big]\Big)+\sup_{N\in\mathbb{N}}\Big(N^{\rho}\,\|g(X)-g(Y^{N,0,1})\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}\Big)<\infty. \tag{5.36}$$

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Furthermore, observe that it holds for all $L \in \mathbb{N}$, $\rho \in [0, \beta - \alpha) \setminus \left\{\frac{1}{2}\right\}$ that

$$\sum_{\ell=1}^{L} (2^{\ell})^{-\rho}\, 2^{-\frac{1}{2}(L-\ell)} = 2^{-\frac{L}{2}} \sum_{\ell=1}^{L} 2^{(\frac{1}{2}-\rho)\ell} = 2^{-\frac{L}{2}} \frac{1-2^{(\frac{1}{2}-\rho)L}}{2^{\rho-\frac{1}{2}}-1} = 2^{-L\cdot\min\{\rho,\frac{1}{2}\}} \frac{1-2^{-|\frac{1}{2}-\rho|L}}{|1-2^{\rho-\frac{1}{2}}|} \leq \frac{2^{-L\cdot\min\{\rho,\frac{1}{2}\}}}{|1-2^{\rho-\frac{1}{2}}|} \tag{5.37}$$

and

$$\sum_{\ell=1}^{L} (2^{\ell})^{-\frac{1}{2}}\, 2^{-\frac{1}{2}(L-\ell)} = 2^{-\frac{L}{2}} L. \tag{5.38}$$

Combining Corollary 5.14 with (5.36), (5.37) and (5.38) implies (5.29). This finishes the proof of Corollary 5.15. □

Corollary 5.15 can be applied to many SDEs. Under general conditions on the coefficient functions of the SDEs (see, e.g., Hutzenthaler & Jentzen, 2014, Theorem 1.3 and Section 3.1), suitable stopped-tamed Euler approximations (cf. Hutzenthaler *et al.*, 2018b, (6) or Hutzenthaler *et al.*, 2012, (10)) converge in the strong sense with convergence rate $1/2$. We note that the classical Euler–Maruyama approximations do not satisfy condition (5.25) for most SDEs with superlinearly growing coefficients; see Hutzenthaler *et al.* (2011, Theorem 2.1) and Hutzenthaler *et al.* (2013, Theorem 2.1). Moreover, under general conditions on the coefficients it holds that the solution process is strongly $1/2$-Hölder continuous in time. In conclusion, provided that a suitable numerical scheme is employed, Corollary 5.15 can be applied to many SDEs with $\beta = 1/2$.

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