Aplikace matematiky

Jaroslav Haslinger; Ivan Hlaváček

Convergence of a finite element method based on the dual variational formulation

Aplikace matematiky, Vol. 21 (1976), No. 1, 43–65

Persistent URL: http://dml.cz/dmlcz/103621

CONVERGENCE OF A FINITE ELEMENT METHOD BASED ON THE DUAL VARIATIONAL FORMULATION

JAROSLAV HASLINGER and IVAN HLAVÁČEK

(Received February 7, 1975)

INTRODUCTION

In practice, one often meets problems, when the cogradient of the solution is more important than the solution itself (e.g. in elastostatics, the stresses are more interesting than the displacements). There are three possibilities how to calculate the cogradient:

(i) an indirect method, based on the minimum potential energy principle. We have to find the solution and then to evaluate its cogradient (Compatible models);

(ii) to apply the dual variational formulation (principle of minimum complementary energy), which yields the cogradient directly (Equilibrium models);

(iii) to apply a mixed variational principle (of the Hellinger-Reissner type), enabling to get both the solution and its cogradient simultaneously (Hybrid models).

In the present paper, we study a particular procedure of the class (ii), starting from a dual variational formulation. A number of articles has been written on the dual finite element analysis in elastostatics (see e.g. the works [1], [2] by B. Fraeijs de Veubeke or V. B. Watwood, Jr., B. J. Hartz [3]) and in general potential problems (B. Fraeijs de Veubeke, M. Hogge [4]). To the authors' knowledge, however, no theoretical convergence result has been presented for the problems mentioned above. This was done only for a special class of problems by J. P. Aubin, H. G. Burchard [5] and J. Vacek [6], who studied the equations with an "absolute term" (i.e. $A(u) + a_0 u = f$, $a_0 \neq 0$). The dual (also "conjugate") method for equations of this type can be extended to parabolic problems, where the role of the absolute term $a_0 u$ is played by the time-derivative $\partial u/\partial t$ (see I. Hlaváček [7], H. Gajewski [13]).

We concentrated our effort to the convergence analysis of a simplest finite element "equilibrium model", applying the piecewise linear polynomials to the solution of a mixed boundary value problem for one second order elliptic equation without the absolute term.

In Section 1 we employ the method of orthogonal projection to derive the principle of minimum complementary energy in a "week form" and its finite-dimensional analogue. In Section 2 a subspace of triangular finite elements is constructed, which consists of piecewise linear vector-functions such that their divergence vanishes in the whole domain under consideration in the sense of distributions. The subspace is applied to the dual variational method in Section 3 with special regard to the approximation of an inhomogeneous natural boundary condition. As a result, the procedure is shown to be second order correct in h (the maximal side of all triangles), provided the exact solution is sufficiently smooth. The last Section 4 contains some a posteriori extimates, following from the use of both the primal and the dual finite element method (cf. also F. Grenacher [8], W. Prager, J. L. Synge [9]).

The authors will present analogous results for n-dimensional domains elsewhere (see [12]).

1. SETTING OF THE PROBLEM

Let $\Omega \subset R_n$ be a bounded polyhedral domain. Consider a differential operator

$$\mathscr{A}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right),$$

where

$$a_{ij} \in L_\infty(\Omega)^1), \quad a_{ij}(x) = a_{ji}(x), \quad i, j = 1, \ldots, n, \tag{1.1}$$

$$\exists \alpha = \text{const} > 0, \tag{1.2}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geqq \alpha\|\xi\|^2 \quad \forall \xi \in E_n$$

holds almost everywhere in Ω.

Suppose that the boundary Γ of Ω consists of two disjoint sets Γ_u, Γ_g, which are either empty or open in Γ and a set $\mathscr{R}$ of zero $(n-1)$-dimensional measure

$$\Gamma = \Gamma_u \cup \Gamma_g \cup \mathscr{R}, \quad \Gamma_u \cap \Gamma_g = \emptyset, \quad \text{mes}_{n-1}\,\mathscr{R} = 0.$$

Moreover, let Γ_u be *non-empty*.[2])

We shall solve the following problem

$$\begin{aligned} \mathscr{A}u &= f \quad &&\text{in } \Omega \\ u &= \bar{u} \quad &&\text{on } \Gamma_u \\ \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j} n_i &= g \quad &&\text{on } \Gamma_g, \end{aligned} \tag{1.3}$$

[1]) I.e. a_{ij} are bounded measurable functions on Ω.

[2]) If Γ_u is empty, we have to introduce subspaces $V_p \subset V$ of "normalized" test functions to obtain uniqueness of solution and the generalized Friedrichs inequality.

where $f \in L_2(\Omega)$, $\bar{u} \in W^{1,2}(\Omega)$ and $g \in L_2(\Gamma_g)$ are given functions, n_i being the components of the unit outward normal to Γ.

Let us denote

$$V = \{v \mid v \in W^{1,2}(\Omega)\,,\ v = 0 \text{ on } \Gamma_u\}\,.$$

A function $u \in W^{1,2}(\Omega)$ will be called a *weak solution* of the problem (1.3) if

$$u - \bar{u} \in V\,, \tag{1.4}$$

$$a(u, v) = \int_\Omega fv \, \mathrm{d}x + \int_{\Gamma_g} gv \, \mathrm{d}\Gamma \quad \forall v \in V\,,$$

where

$$a(u, v) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x$$

From (1.1), (1.2) and the Lax-Milgram theorem the existence and uniqueness of u follows. Moreover, it holds

$$\|u\|_{W^{1,2}(\Omega)} \leqq C(\|f\|_{L_2(\Omega)} + \|\bar{u}\|_{W^{1,2}(\Omega)} + \|g\|_{L_2(\Gamma_g)})\,. \tag{1.5}$$

Introducing the functional of potential energy

$$\mathscr{L}(v) = \tfrac{1}{2}a(v, v) - \int_\Omega fv \, \mathrm{d}x - \int_{\Gamma_g} gv \, \mathrm{d}\Gamma\,,$$

we may formulate the *minimum potential energy principle*

$$\mathscr{L}(u) = \min_{v \in \bar{u} + V} \mathscr{L}(v)$$

where $\bar{u} + V$ denotes the set of all sums $\bar{u} + v$, $v \in V$.

Let $H = [L_2(\Omega)]^n$ with the norm

$$\|\lambda\|^2 = \sum_{i=1}^n \|\lambda_i\|^2 = \sum_{i=1}^n \int_\Omega \lambda_i^2 \, \mathrm{d}x\,, \quad \lambda = (\lambda_1, \ldots, \lambda_n)$$

Introduce a bilinear form

$$(\lambda, \mu)_H = \sum_{i,j=1}^n \int_\Omega b_{ij}(x)\, \lambda_i \mu_j \, \mathrm{d}x\,, \quad \lambda \in H\,, \quad \mu \in H\,, \tag{1.6}$$

where b_{ij} are the entries of the matrix $[a^{-1}]$ inverse to $[a]$.

Making use of (1.1), (1.2), we can see that (1.6) defines a scalar product on H, and such constants $c_1 > 0$, $c_2 > 0$ exist that

$$c_2\|\lambda\| \leqq \|\lambda\|_H \leqq c_1\|\lambda\| \quad \forall \lambda \in H\,, \tag{1.6'}$$

where

$$\|\lambda\|_H^2 = (\lambda, \lambda)_H$$

Hence the space H with the scalar product (1.6) is a Hilbert space.

Let us define

$$H_1 = \left\{\lambda \mid \lambda \in H,\ \exists v \in V : \lambda_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j}\right\},$$

$$H_2 = \{\lambda \mid \lambda \in H,\ B(\lambda, v) = 0\ \forall v \in V\},$$

$$B(\lambda, v) = \sum_{i=1}^{n} \int_\Omega \lambda_i \frac{\partial v}{\partial x_i}\, dx\,. \tag{1.7}$$

Lemma 1.1.

1° *H_1 and H_2 are (closed) subspaces of H;*
2° *$H_1 \perp H_2$;*
3° *$H = H_1 \oplus H_2$*

Proof

1° Let $\lambda^m \in H_1$, $\lambda^m \to \lambda$ in H. Then we may write

$$\lambda_i^m = \sum_{j=1}^{n} a_{ij} \frac{\partial v_m}{\partial x_j}, \quad v_m \in V,$$

$$\|\lambda^m - \lambda^p\|_H^2 = \int_\Omega \sum_{i,j=1}^{n} b_{ij}(\lambda_i^m - \lambda_i^p)(\lambda_j^m - \lambda_j^p)\, dx = \tag{1.8}$$

$$= \int_\Omega \sum_{i,j=1}^{n} b_{ij} \left(\sum_{l=1}^{n} a_{il} \frac{\partial}{\partial x_l}(v_m - v_p)\right)\left(\sum_{k=1}^{n} a_{jk} \frac{\partial}{\partial x_k}(v_m - v_p)\right) dx \geqq$$

$$\geqq \alpha \int_\Omega |\text{grad}\,(v_m - v_p)|^2\, dx \geqq c\|v_m - v_p\|_{W^{1,2}(\Omega)}^2$$

where the definition of $[b_{ij}]$, (1.2) and the generalized Friedrichs inequality for $v \in V$ (see [10]) have been used. Hence $v_n \to v$ in V,

$$\lambda_1 = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j}, \quad \lambda \in H_1\,.$$

By virtue of (1.6) and the definition (1.7), the set H_2 is closed. The linearity is obvious.

2° Let $\lambda' \in H_1$, $\lambda'' \in H_2$. Then we have

$$(\lambda', \lambda'')_H = \int_\Omega \sum_{i,j=1}^{n} b_{ij} \left(\sum_{k=1}^{n} a_{ik} \frac{\partial v}{\partial x_k}\right) \lambda_j''\, dx = B(\lambda'', v) = 0\,,$$

using the definition of $[b_{ij}]$ and H_2.

3° Let $H_1^\perp$ denote the orthogonal complement of H_1 in H. From the point 2° it follows that $H_2 \subset H_1^\perp$. Conversely, let $z \in H_1^\perp$. Then

$$\int_\Omega \sum_{i,j=1}^n b_{ij} \left(\sum_{k=1}^n a_{ik} \frac{\partial v}{\partial x_k} \right) z_j \, dx = 0 \quad \forall v \in V .$$

The properties of $[b_{ij}]$ yield that

$$\int_\Omega \sum_{j=1}^n \frac{\partial v}{\partial x_j} z_j \, dx = 0 \quad \forall v \in V ,$$

Hence $z \in H_2$. Q.E.D.

Let us introduce

$$(1.9) \qquad \lambda_i(v) = \sum_{j=1}^n a_{ij} \frac{\partial v}{\partial x_j}, \quad \lambda(v) = (\lambda_1(v), \ldots, \lambda_n(v)), \quad v \in W^{1,2}(\Omega)$$

$$\Lambda_{f,g} = \left\{ \lambda \mid \lambda \in H, B(\lambda, v) = \int_\Omega fv \, dx + \int_{\Gamma_g} gv \, d\Gamma, \ \forall v \in V \right\}.$$

The dual variational formulation is the following

Theorem 1.1. (*Principle of minimum complementary energy*).

Let $B(\lambda, v)$ and $\lambda(v)$ be defined through (1.7) and (1.9), respectively. Let $u \in W^{1,2}(\Omega)$ be the weak solution of the problem (1.3). Then the functional

$$\mathcal{S}(\lambda) = \tfrac{1}{2} \int_\Omega \sum_{i,j=1}^n b_{ij} \lambda_i \lambda_j \, dx - B(\lambda, \bar{u})$$

attains its minimum on the set $\Lambda_{f,g}$, if and only if $\lambda = \lambda(u)$.[1])

Proof. (cf. [11]). Let us set $u = \bar{u} + w$, $w \in V$. Then $\lambda(w) \in H_1$ and $\lambda - \lambda(u) \in H_2$ for all $\lambda \in \Lambda_{f,g}$.

The functional

$$\mathcal{J}(\lambda) = \|\lambda - \lambda(\bar{u})\|_H^2 = \|(\lambda - \lambda(u)) + (\lambda(u) - \lambda(\bar{u}))\|_H^2 = \\ = \|\lambda - \lambda(u)\|_H^2 + \|\lambda(w)\|_H^2$$

attains its minimum on $\Lambda_{f,g}$, iff $\lambda = \lambda(u)$.

Rearranging leads to the formula

$$\mathcal{J}(\lambda) - \|\lambda(\bar{u})\|_H^2 = \|\lambda\|_H^2 - 2(\lambda, \lambda(\bar{u}))_H = \\ = \int_\Omega \sum_{i,j=1}^n b_{ij} \lambda_i \lambda_j \, dx - 2B(\lambda, \bar{u}) = 2\mathcal{S}(\lambda),$$

which yields the assertion of the theorem. Q.E.D.

[1]) It holds that $-\mathcal{S}(\lambda(u)) = \mathcal{L}(u) + \int_\Omega f\bar{u} \, dx + \int_{\Gamma g} g\bar{u} \, d\Gamma.$

We obtained a variational problem of a minimum of a quadratic functional on a closed *convex set* $\Lambda_{f,g} \subset H$. As usual, the problem can be transformed into a similar one, but on a *linear space* $\Lambda_{0,0} = H_2$. Let $\bar{\lambda} \in \Lambda_{f,g}$ be a fixed vector-function. Then

$$\Lambda_{f,g} = \bar{\lambda} + \Lambda_{0,0} = \bar{\lambda} + H_2 ,$$

i.e., every $\lambda \in \Lambda_{f,g}$ can be written in the form $\lambda = \bar{\lambda} + \chi$, $\chi \in H_2$. Then

$$\mathscr{S}(\lambda) = \mathscr{S}(\bar{\lambda} + \chi) = \tfrac{1}{2}\int_\Omega \sum_{i,j=1}^{n} b_{ij}\chi_i\chi_j \, dx + \int_\Omega \sum_{i,j=1}^{n} b_{ij}\bar{\lambda}_i\chi_j \, dx - B(\chi, \bar{u}) + N(\bar{\lambda}, \bar{u}) ,$$

where $N(\bar{\lambda}, \bar{u})$ depends only on $\bar{\lambda}$, $\bar{u}$. Let us set

$$\Phi(\chi) = \tfrac{1}{2}(\chi, \chi)_H - F(\chi) , \tag{1.10}$$

where

$$F(\chi) = -\int_\Omega \sum_{i,j=1}^{n} b_{ij}\bar{\lambda}_i\chi_j \, dx + B(\chi, \bar{u}) .$$

Then we may replace the minimum problem of Theorem 1.1 for $\mathscr{S}(\lambda)$ by an equivalent problem:
to find $\chi^\circ \in H_2$ such that

$$\Phi(\chi^0) = \min_{\chi \in H_2} \Phi(\chi) . \tag{1.11}$$

Let $h \in (0, 1\rangle$ and let $\{V_h\}$ be a system of finite-dimensional subspaces of H_2. We define the following procedure:
to find $\chi_h^0 \in V_h$ such that

$$\Phi(\chi_h^0) = \min_{\chi \in V_h} \Phi(\chi) . \tag{1.12}$$

Theorem 1.2. *To every* $h \in (0, 1\rangle$ *there exists precisely one* $\chi_h^0 \in V_h$ *satisfying* (1.12). *It holds*

$$\|\chi^0 - \chi_h^0\|_H \leqq \inf_{\chi \in V_h} \|\chi^0 - \chi\|_H \leqq c_1 \inf_{\chi \in V_h} \|\chi^0 - \chi\| \tag{1.13}$$

Proof. The element χ_h^0, satisfying (1.12), is characterized through the relation

$$D\Phi(\chi_h^0, \chi) = 0 \quad \forall \chi \in V_h ,$$

where $D\Phi(\chi_h^0, \chi)$ denotes the value of Gâteaux differential of Φ at the "point" χ_h^0 and in the "direction" χ. We have

$$D\Phi(\chi_h^0, \chi) = (\chi_h^0, \chi)_H - F(\chi) ,$$

consequently

$$(1.14) \qquad (\chi_h^0, \chi)_H = F(\chi) \quad \forall \chi \in V_h .$$

It is readily seen that (1.14) is equivalent with a system of linear equations with a positive definite matrix. Therefore precisely one $\chi_h^0 \in V_h$ exists, satisfying (1.14).

Similarly, by virtue of (1.11), it holds

$$(1.15) \qquad (\chi^0, \chi)_H = F(\chi) \quad \forall \chi \in H_2 .$$

Substracting (1.14) from (1.15) yields that

$$(\chi^0 - \chi_h^0, \chi)_H = 0 \quad \forall \chi \in V_h.$$

Next let $\chi \in V_h$ be an arbitrary element. Then we may write

$$\|\chi^0 - \chi_h^0\|_H^2 = (\chi^0 - \chi_h^0, \chi^0 - \chi)_H + (\chi^0 - \chi_h^0, \chi - \chi_h^0)_H =$$
$$= (\chi^0 - \chi_h^0, \chi^0 - \chi)_H \leqq \|\chi^0 - \chi_h^0\|_H \, \|\chi^0 - \chi\|_H$$

Dividing by $\|\chi^0 - \chi_h^0\|_H$ and taking the infimum of the right-hand side leads to (1.13), if we use also (1.6'). Q.E.D.

2. THE SUBSPACES $\mathcal{N}_h$

We shall restrict ourselves to the plane case ($n = 2$). Let K be a triangle with vertices a_1, a_2, a_3 and set $a_4 = a_1$. We introduce the following notation:

$$\mathbf{W} = [W^{1,2}(K)]^2 = \{\mathbf{u} = (u_1, u_2),\ u_i \in W^{1,2}(K),\ i = 1,2\} ,$$

$\mathbf{C} = [C(K)]^2$ with the norm

$$\|\mathbf{u}\|_{\mathbf{C}} = \max_{\substack{i=1,2\\ x\in K}} |u_i(x)|$$

$\mathbf{C}^2(K) = [C^2(K)]^2$ with the seminorm

$$\|\mathbf{u}\|_{\mathbf{C}^2(K)} = \max_{\substack{i,j,k=1,2\\ x\in K}} \left|\frac{\partial^2 u_i(x)}{\partial x_j\, \partial x_k}\right| .$$

$\mathbf{n}$ is the outward unit normal to ∂K; thus $\mathbf{n} = \mathbf{n}(x, y) = \mathbf{n}^{(i)} \in R_2$ is constant on the side $a_i a_{i+1}$, ($i = 1, 2, 3$) of the triangle K.

For $\mathbf{u} \in \mathbf{W}$ we define the outward flux

$$(2.1) \qquad T_i \mathbf{u} = \mathbf{u}\big|_{a_i a_{i+1}} \cdot \mathbf{n}^{(i)} = \bar{u}_1 n_1^{(i)} + \bar{u}_2 n_2^{(i)},\ \mathbf{n}^{(i)} \approx (n_1^{(i)}, n_2^{(i)}) ,$$

where $\bar{u}_j$ are the traces of u_j on $a_i a_{i+1}$.

By $P_k(M)$, with k a non-negative integer, we denote the set of polynomials of the order at most k, defined on the set $M \subset R_2$.

Definition 2.1. *We say that* $\lambda_1^{(i)}, \lambda_2^{(i)}$, $(i = 1, 2, 3)$, *are basic linear functions of the side* $a_i a_{i+1}$, *if*

$$\lambda_k^{(i)} \in P_1(a_i a_{i+1}), \; (k = 1, 2)$$

$$\lambda_1^{(i)}(a_i) = 1\,, \qquad \lambda_1^{(i)}(a_{i+1}) = 0\,,$$

$$\lambda_2^{(i)}(a_i) = 0\,, \qquad \lambda_2^{(i)}(a_{i+1}) = 1\,.$$

From the definition it follows immediately that

$$\lambda_1^{(i)} + \lambda_2^{(i)} = 1 \quad \text{on} \quad a_i a_{i+1}\,, \tag{2.2}$$

$$\int_{a_i a_{i+1}} \lambda_k^{(i)} \, \mathrm{d}s = \tfrac{1}{2} l_i\,, \quad k = 1, 2, \tag{2.3}$$

where l_i denotes the length of $a_i a_{i+1}$.

Henceforth we shall use the notation

$$\int_{a_i a_{i+1}} u \,.\, v \, \mathrm{d}\Gamma = [u, v]_i\,, \quad u, v \in L_2(a_i a_{i+1})$$

Lemma 2.1. *Let* $\gamma_i, \delta_i \in R_1$, $(i = 1, 2, 3)$ *be given. Then precisely one* $\mathbf{u} \in [P_1(K)]^2$ *exists such that*

$$T_i \, \mathbf{u}(a_i) = \gamma_i\,, \quad T_i \, \mathbf{u}(a_{i+1}) = \delta_i\,, \quad (i = 1, 2, 3)\,. \tag{2.4}$$

Proof. Denote $\mathbf{u} = (u_1, u_2)$, $u_i \in P_1(K)$. It is sufficient to prove that the six values $\{u_j(a_i)\}_{i=1}^3$, $j = 1, 2$ are determined uniquely by (2.4). Let e.g. $i = 1$. Then

$$T_1 \, \mathbf{u}(a_1) = \mathbf{u}(a_1) \,.\, \mathbf{n}^{(1)} = \gamma_1$$

$$T_3 \, \mathbf{u}(a_1) = \mathbf{u}(a_1) \,.\, \mathbf{n}^{(3)} = \delta_3\,.$$

As $\mathbf{n}^{(1)}, \mathbf{n}^{(3)}$ are linearly independent vectors, $\mathbf{u}(a_1)$ is uniquely determined. The same holds at the remaining vertices. Q.E.D.

Theorem 2.1. *Let* $\mathbf{u} \in \mathbf{W}$. *Then the equations*

$$[T_i \, \mathbf{u}, \lambda_k^{(i)}]_i = \alpha_i [\lambda_1^{(i)}, \lambda_k^{(i)}]_i + \beta_i [\lambda_2^{(i)}, \lambda_k^{(i)}]_i\,, \quad k = 1, 2 \tag{j}$$

$$\Pi \, \mathbf{u}(a_i) \,.\, \mathbf{n}^{(i)} = \alpha_i \tag{jj}$$

$$\Pi \, \mathbf{u}(a_{i+1}) \,.\, \mathbf{n}^{(i)} = \beta_i\,,$$

with $i = 1, 2, 3$, *define an operator* $\Pi \in \mathscr{L}(\mathbf{W}; [P_1(K)]^2 \cap \mathscr{L}(\mathbf{C}; [P_1(K)]^2)$.[1])

[1]) $\mathscr{L}(X, Y)$ denotes the space of linear bounded mappings of X into Y.

Proof. The numbers α_i, β_i are determined uniquely by the system (j). Indeed, the (Gramm's) matrix $A^{(i)} = ([\lambda_j^{(i)}, \lambda_k^{(i)}])$, $j, k = 1, 2$ is regular. The lemma 2.1 yields the existence and uniqueness of $\mathbf{w} \in [P_1(K)]^2$ such that

$$T_i \mathbf{w}(a_i) = \alpha_i \,, \quad T_i \mathbf{w}(a_{i+1}) = \beta_i \,.$$

Let us set $\Pi \mathbf{u} = \mathbf{w}$. The mapping Π is obviously linear. Let $\mathbf{u}_n \to \mathbf{u}$ in $\mathbf{W}$ and denote by α_i^n, β_i^n and α_i, β_i the solutions of (j) with the left-hand sides $[T_i \mathbf{u}_n, \lambda_k^{(i)}]_i$ and $[T_i \mathbf{u}, \lambda_k^{(i)}]_i$, respectively. Using the trace theorem and the Cramer's rule, we obtain that $\alpha_i^n \to \alpha_i$, $\beta_i^n \to \beta_i$. The rest of the proof is obvious as well as the fact that $\Pi \in \mathscr{L}(\mathbf{C}; [P_1(K)]^2)$. Q.E.D.

By virtue of Theorem 2.1 there exists a $c > 0$ such that

$$\|\Pi \mathbf{u}\|_{\mathbf{C}} \leqq c \|\mathbf{u}\|_{\mathbf{C}} \,.$$

In the following we present an estimate for c.

Theorem 2.2. *Let Π be defined through* (j), (jj). *Then*

$$\|\Pi \mathbf{u}\|_{\mathbf{C}} \leqq \frac{6\sqrt{2}}{\sin \alpha} \|\mathbf{u}\|_{\mathbf{C}} \,, \tag{2.5}$$

holds for all $\mathbf{u} \in \mathbf{C}$, where α is the minimal interior angle of K.

Proof. A direct calculation yields that

$$\det A^{(i)} = \tfrac{1}{12} l_i^2 \,, \quad |\alpha_i| \leqq 3\sqrt{(2)}\, \|\mathbf{u}\|_{\mathbf{C}} \,, \quad |\beta_i| \leqq 3\sqrt{(2)}\, \|\mathbf{u}\|_{\mathbf{C}} \,.$$

Next from (jj) it follows e.g. for $\Pi\, \mathbf{u}(a_2) = (w_1(a_2), w_2(a_2))$

$$|w_1(a_2)| = \left| \begin{vmatrix} \beta_1 & n_2^{(1)} \\ \alpha_2 & n_2^{(2)} \end{vmatrix} \cdot \begin{vmatrix} n_1^{(1)} & n_2^{(1)} \\ n_1^{(2)} & n_2^{(2)} \end{vmatrix}^{-1} \right| \leqq \frac{6\sqrt{2}}{\sin \alpha} \|\mathbf{u}\|_{\mathbf{C}} \,,$$

because the det $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)})$ is equal to the sinus of the angle between $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$. Similar estimates hold for the remaining values of $\mathbf{w}$ at the vertices. Q.E.D.

Let us define

$$\mathscr{M}(K) = \{v = (v_1, v_2),\ v_j \in P_1(K),\ j = 1, 2, \operatorname{div} \mathbf{v} = 0\} \,.$$

It is easy to see that $\dim \mathscr{M}(K) = 5$,

$$\mathscr{M}(K) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} x \\ -y \end{pmatrix} \right\},$$

$$\mathbf{v} \in \mathscr{M}(K) \Leftrightarrow \mathbf{v} = (\gamma_1 + \gamma_2 x + \gamma_3 y, \delta_1 + \delta_2 x - \gamma_2 y) \,. \tag{2.6}$$

Lemma 2.2.

$$\mathbf{v} \in \mathscr{M}(K) \Leftrightarrow \mathbf{v} \in [P_1(K)]^2 \;\&\; \int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}\Gamma = 0 \,.$$

Proof. Let $\mathbf{v} \in \mathscr{M}(K)$. Then

$$\operatorname{div} \mathbf{v} = 0 \quad \text{on} \quad K \Rightarrow \int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}\Gamma = 0$$

by virtue of the Green's theorem.

Let $\mathbf{v} = (\gamma_1 + \gamma_2 x + \gamma_3 y, \delta_1 + \delta_2 x + \delta_3 y)$ and

$$\int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}\Gamma = \int_K \operatorname{div} \mathbf{v} \,\mathrm{d}x \,\mathrm{d}y = 0 \,. \tag{2.7}$$

Then $\operatorname{div} \mathbf{v} = \gamma_2 + \delta_3$ and (2.7) yields

$$(\gamma_2 + \delta_3) \operatorname{mes} K = 0 \Rightarrow \gamma_2 = -\delta_3 \,.$$

Using also (2.6) we obtain that $\mathbf{v} \in \mathscr{M}(K)$. Q.E.D.

Lemma 2.3.

$$\mathbf{v} \in \mathscr{M}(K) \Leftrightarrow \mathbf{v} \in [P_1(K)]^2 \;\&\; \sum_{i=1}^{3} (\alpha_i + \beta_1)\, l_i = 0 \,,$$

where $\alpha_i = T_i \mathbf{v}(a_i)$, $\beta_i = T_i \mathbf{v}(a_{i+1})$.

Proof. We may write

$$T_i\mathbf{v} = T_i \mathbf{v}(a_i)\, \lambda_1^{(i)} + T_i \mathbf{v}(a_{i+1})\, \lambda_2^{(i)} = \alpha_i \lambda_1^{(i)} + \beta_i \lambda_2^{(i)} \,,$$

$$\int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}\Gamma = \sum_{i=1}^{3} \int_{a_i a_{i+1}} T_i \mathbf{v} \,\mathrm{d}\Gamma = \sum_{i=1}^{3} (\alpha_i + \beta_i) \frac{l_i}{2}$$

Consequently,

$$\int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}\Gamma = 0 \Leftrightarrow \sum_{i=1}^{3} (\alpha_i + \beta_i)\, l_i = 0$$

and from Lemma 2.2 the assertion follows. Q.E.D.

Let us set

$$\mathbf{U} = \{\mathbf{v} \in \mathbf{W},\ \operatorname{div} \mathbf{v} = 0\} \tag{2.8}$$

Theorem 2.3. *Let the mapping Π be defined through the relations* (j), (jj), *as in Theorem* 2.1. *Then $\Pi \in \mathscr{L}(\mathbf{U}; \mathscr{M}(K))$ and*

$$\Pi \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in [P_1(K)]^2 \,. \tag{2.9}$$

Proof. Adding the equations (j) for $k = 1, 2$ yields, by virtue of (2.2):

$$\int_{a_i a_{i+1}} T_i \boldsymbol{u} \, \mathrm{d}s = [T_i \boldsymbol{u}, \lambda_1^{(i)} + \lambda_2^{(i)}]_i = \alpha_i [\lambda_1^{(i)}. 1]_i + \beta_i [\lambda_2^{(i)}, 1]_i = \tfrac{1}{2} l_i (\alpha_i + \beta_i),$$

$$\boldsymbol{u} \in \boldsymbol{U} \Rightarrow 0 = \int_{\partial K} \boldsymbol{u} . \boldsymbol{n} \, \mathrm{d}\Gamma = \sum_{i=1}^{3} \int_{a_i a_{i+1}} T_i \boldsymbol{u} \, \mathrm{d}\Gamma = \tfrac{1}{2} \sum_{i=1}^{3} (\alpha_i + \beta_1) \, l_i .$$

Hence $\boldsymbol{u} \in \mathscr{M}(K)$ follows from Lemma 2.3. The mapping Π is linear and continuous, being the restriction of $\boldsymbol{W}$ onto $\boldsymbol{U}$.

Let $\boldsymbol{v} \in [P_i(K)]^2$. A direct calculation from (j) implies $\alpha_i = T_i \, \boldsymbol{v}(a_i)$, $\beta_i = T_i \, \boldsymbol{v}(a_{i+1})$. Then using (jj) and Lemma 2.1 we obtain $\Pi \boldsymbol{v} = \boldsymbol{v}$. Q.E.D.

Theorem 2.4. Let $\boldsymbol{u} \in \boldsymbol{C}^2(K)$. Then

$$\|\boldsymbol{u} - \Pi \boldsymbol{u}\|_{\boldsymbol{C}} \leqq 4\left(1 + \frac{6\sqrt{2}}{\sin \alpha}\right) h^2 \|\boldsymbol{u}\|_{\boldsymbol{C}^2(K)} , \tag{2.10}$$

where $h = \operatorname{diam} K$ and α is the minimal interior angle of K.

Proof. Let $\boldsymbol{x}_0 \in K$ be an arbitrary point. Taylor's theorem implies for $\boldsymbol{x} \in K$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}_0) + D \, \boldsymbol{u}(\boldsymbol{x}_0) (\boldsymbol{x} - \boldsymbol{x}_0) + D^2 \, \boldsymbol{u}(\Theta) (\boldsymbol{x} - \boldsymbol{x}_0)^2 , \tag{2.11}$$

where $\Theta \in \boldsymbol{x}_0 \boldsymbol{x}$.

Applying Π to (2.11), using its linearity and (2.9), we obtain

$$\Pi \, \boldsymbol{u}(\boldsymbol{x}) = \Pi \, \boldsymbol{u}(\boldsymbol{x}_0) + \Pi(D \, \boldsymbol{u}(\boldsymbol{x}_0) (\boldsymbol{x} - \boldsymbol{x}_0)) + \Pi(D^2 \, \boldsymbol{u}(\Theta) (\boldsymbol{x} - \boldsymbol{x}_0)^2) =$$

$$= \boldsymbol{u}(\boldsymbol{x}_0) + D \, \boldsymbol{u}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) + \Pi(D^2 \, \boldsymbol{u}(\Theta) (\boldsymbol{x} - \boldsymbol{x}_0)^2) .$$

Consequently, we have by virtue of (2.5)

$$\|\boldsymbol{u} - \Pi \boldsymbol{u}\|_{\boldsymbol{C}} \leqq \|D^2 \, \boldsymbol{u}(\Theta) (\boldsymbol{x} - \boldsymbol{x}_0)^2\|_{\boldsymbol{C}} +$$

$$+ \|\Pi D^2 \, \boldsymbol{u}(\Theta) (\boldsymbol{x} - \boldsymbol{x}_0)^2\|_{\boldsymbol{C}} \leqq 4\left(1 + \frac{6\sqrt{2}}{\sin \alpha}\right) h^2 \|\boldsymbol{u}\|_{\boldsymbol{C}^2(K)} . \text{ Q.E.D.}$$

Let $\Omega \in R_2$ be a bounded polygonal domain, $h \in (0, 1 \rangle$, $\mathscr{T}_h$ a triangulation of $\bar{\Omega}$, satisfying the usual requirements concerning the mutual position of two triangles. Suppose that

$$h = \max \operatorname{diam} K \quad \forall K \in \mathscr{T}_h .$$

Denote by Π_K the mapping, defined on $K \in \mathscr{T}_h$ by the conditions (j), (jj) of Theorem 2.1. Let $K, K' \in \mathscr{T}_h$ be two adjacent triangles, with a common side $a_i a_{i+1}$ and $\boldsymbol{v} \in [W^{1,2}(K \cup K')]^2$. The function $T_i \boldsymbol{v}$, defined by (2.1) with respect to the triangle K

(or K') will be denoted by $T_{i,K}\mathbf{v}$ (or $T_{i,K'}\mathbf{v}$, respectively). We say that *the condition* (R) *is satisfied on the common side* a_ia_{i+1}, if

$$T_{i,K}\mathbf{v} + T_{i,K'}\mathbf{v} = 0 \quad \text{on} \quad a_ia_{i+1} . \tag{2.12}$$

Let us define

$$\mathbf{U}(\Omega) = \{\mathbf{v} \in [W^{1,2}(\Omega)]^2 , \quad \operatorname{div} \mathbf{v} = 0\} ,$$

$\mathcal{N}_h(\Omega) = \{\mathbf{v},\ \mathbf{v}|_K \in \mathcal{M}(K)\ \forall K \in \mathcal{T}_h, \text{(R) is satisfied on each common side of any pair } K, K' \text{ of adjacent triangles of } \mathcal{T}_h\}$.

For $\mathbf{v} \in \mathbf{U}(\Omega)$ we define a mapping r_h by the relations

$$r_h\mathbf{v}|_K = \Pi_K\mathbf{v} \quad \forall K \in \mathcal{T}_h . \tag{2.13}$$

We say that a *family* $\{\mathcal{T}_h\}$, $h \in (0, 1\rangle$ *of triangulations of* Ω *is regular*, if there exists a constant $\alpha_0 > 0$ independent of h and such that all interior angles of the triangles of $\mathcal{T}_h \in \{\mathcal{T}_h\}$ are not less than α_0.

Theorem 2.5. *Let* $\{\mathcal{T}_h\}$, $h \in (0, 1\rangle$ *be a regular family of triangulations of* Ω. *Then*

$$r_h \in \mathcal{L}(\mathbf{U}(\Omega) ; \ \mathcal{N}_h(\Omega)) , \tag{2.14}$$

$$\|\mathbf{u} - r_h\mathbf{u}\|_{[L_2(\Omega)]^2} \leqq ch^2\|\mathbf{u}\|_{[C^2(\bar{\Omega})]^2} \quad \forall \mathbf{u} \in [C^2(\bar{\Omega})]^2 , \tag{2.15}$$

where c *does not depend on* h, $\mathbf{u}$.

Proof. With regard to Theorems 2.1, 2.3, (2.14) will follow, if the condition (R) (2.12) is verified. Let $K, K' \in \mathcal{T}_h$, $K \cap K' = a_ia_{i+1}$. As $T_{i,K}(\Pi_K\mathbf{u})$, $T_{i,K'}(\Pi_{K'}\mathbf{u}) \in P_1(a_ia_{i+1})$, it suffices to show that

$$T_{i,K}(\Pi_K\mathbf{u})(a_j) + T_{i,K'}(\Pi_{K'}\mathbf{u})(a_j) = 0 \quad \text{for} \quad j = i , \quad i + 1 .$$

The latter equation, however, is an immediate consequence of (j), (jj), because $\mathbf{n}_K^{(i)} = -\mathbf{n}_{K'}^{(i)}$, $T_{i,K}\mathbf{u} = -T_{i,K'}\mathbf{u}$ and

$$\Pi_K\, \mathbf{u}(a_i) \,.\, \mathbf{n}_K^{(i)} = \alpha_i , \quad \Pi_K\, \mathbf{u}(a_i) \,.\, \mathbf{n}_{K'}^{(i)} = -\alpha_i$$

$$\Pi_K\, \mathbf{u}(a_{i+1}) \,.\, \mathbf{n}_K^{(i)} = \beta_i , \quad \Pi_{K'}\, \mathbf{u}(a_{i+1}) \,.\, \mathbf{n}_{K'}^{(i)} = -\beta_i .$$

As the estimate (2.15) is concerned, we obtain, making use of Theorem 2.4

$$\|\mathbf{u} - r_h\mathbf{u}\|^2_{[L_2(\Omega)]^2} = \sum_{K\in\mathcal{T}_h} \|\mathbf{u} - r_h\mathbf{u}\|^2_{[L_2(K)]^2} = \sum_{K\in\mathcal{T}_h} \|\mathbf{u} - \Pi_K\mathbf{u}\|^2_{[L_2(K)]^2} \leqq$$

$$\leqq \sum_{K\in\mathcal{T}_h} \operatorname{mes} K \,.\, C^2h^4 \,.\, \|\mathbf{u}\|^2_{C^2(K)} \leqq C^2h^4 \operatorname{mes} \Omega \,.\, \|\mathbf{u}\|^2_{[C_2(\bar{\Omega})]^2} ,$$

where $C = 4\left(1 + \dfrac{6\sqrt{2}}{\sin \alpha_0}\right)$. Q.E.D.

Remark 2.1 Any function $\mathbf{v} \in \mathcal{N}_h(\Omega)$ satisfies the equation $\operatorname{div} \mathbf{v} = 0$ on Ω in the sense of distributions.

Indeed, let us take $\varphi \in \mathcal{D}(\Omega)$ (i.e., infinitely differentiable function with compact support in Ω). Then

$$\langle \operatorname{div} \mathbf{v}, \varphi \rangle = - \int_\Omega \mathbf{v} \cdot \operatorname{grad} \varphi \, \mathrm{d}x \, \mathrm{d}y = - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{v} \cdot \operatorname{grad} \varphi \, \mathrm{d}x \, \mathrm{d}y = 0 \,,$$

if we use integration by parts and the definition of $\mathcal{N}_h(\Omega)$ (condition (R)).

3. APPLICATION OF SUBSPACES $\mathcal{N}_h(\Omega)$ TO THE DUAL VARIATIONAL FORMULATION

Consider a polygonal domain $\Omega \in R_2$. Let $\{\mathcal{T}_h\}$, $h \in (0, 1\rangle$, be a regular family of triangulations of Ω, satisfying moreover the following requirement: if a part of Γ_g belongs to a side of $K \in \mathcal{T}_h$, then $\bar{\Gamma}_g$ covers the whole side.

Define

$$V_h = \mathcal{N}_h(\Omega) \cap H_2 = \{\mathbf{v} \in \mathcal{N}_h(\Omega) \,, \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_g\} \,.$$

Let $\chi^0 \in H_2$ be such that

$$\Phi(\chi^0) = \min_{\chi \in H_2} \Phi(\chi) \tag{3.1}$$

and $\chi_h^0 \in V_h$ such that

$$\Phi(\chi_h^0) = \min_{\chi \in V_h} \Phi(\chi) \,.$$

Theorem 3.1.[1]) *Let* $\chi^0 \in [C^2(\bar{\Omega})]^2$. *Then for any regular family of triangulations* $\{\mathcal{T}_h\}$ *it holds*

$$\|\chi^0 - \chi_h^0\|_H \leqq c h^2 \|\chi^0\|_{[C^2(\bar{\Omega})]^2} \,,$$

where c is independent of h, χ^0.

Proof. We shall verify that $r_h \chi^0 \in V_h$.

$$\chi^0 \in [C^2(\bar{\Omega})]^2 \cap H_2 \Leftrightarrow \int_\Omega \sum_{i=1}^2 \chi_i^0 \frac{\partial v}{\partial x_i} \, \mathrm{d}x = 0 \quad \forall v \in V \Leftrightarrow$$

$$\Leftrightarrow \operatorname{div} \chi^0 = 0 \quad \text{in} \quad \Omega \,, \quad \chi^0 \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_g \,.$$

Hence $\chi^0 \in \mathbf{U}(\Omega)$ and Theorem 2.5 yields that $r_h \chi^0 \in \mathcal{N}_h(\Omega)$. Consequently, it suffices to prove $(r_h \chi^0) \cdot \mathbf{n} = 0$ on Γ_g. Let $a_i a_{i+1} \subset \bar{\Gamma}_g$ be a side of a boundary triangle

[1]) Using the Bramble-Hilbert Lemma, one can prove that $\|\chi^0 - \chi_h^0\|_H \leqq c h^2 |\chi^0|_{2,\Omega}$ if $\chi^0 \in [W^{2,2}(\Omega)]^2$, where $|\,.\,|_{2,\Omega}$ denotes the seminorm of second derivatives.

$K \in \mathscr{T}_h$. As $\chi^0 . \mathbf{n}^{(i)} = 0$ on $a_i a_{i+1}$, from (j) $\alpha_i = \beta_i = 0$ follows and (jj) results in

$$\Pi_K \chi^0(a_i) . \mathbf{n}^{(i)} = \Pi_K \chi^0(a_{i+1}) . \mathbf{n}^{(i)} = 0 .$$

Consequently, $(\Pi_K \chi^0) . \mathbf{n}^{(i)} = 0$ on $a_i a_{i+1}$ and $r_h \chi^\circ \in V_h$. Using Theorem 1.2 and 2.5, we obtain

$$\|\chi^0 - \chi_h^0\|_H \leqq c_1 \|\chi^0 - r_h \chi^0\| \leqq ch^2 \|\chi^0\|_{[C^2(\bar{\Omega})]^2} . \quad \text{Q.E.D.}$$

Corollary. *Let the suppositions of Theorem* 3.1 *hold. Then for* $\lambda^0 = \bar{\lambda} + \chi^0$, $\lambda_h^0 = \bar{\lambda} + \chi_h^0$ *we have*

$$\|\lambda^0 - \lambda_h^0\|_H = O(h^2) .$$

Proof. Obviously, we have

$$\|\lambda^0 - \lambda_h^0\|_H = \|\chi^0 - \chi_h^0\|_H$$

and we make use of Theorem 3.1. Q.E.D.

Let us recall once more the transformation of the problem $\mathscr{S}(\lambda) = \min.$ on the set $\Lambda_{f,g}$ into the problem (3.1). We supposed that an element $\bar{\lambda} \in \Lambda_{f,g}$ is available. In praxis, however, difficulties may occur in finding this function. Therefore, let us suppose that we are able to construct a $\tilde{\lambda} \in \Lambda_{f,\gamma}$, where γ is close to g in some sense. Let us solve the problem

$$\Phi_\gamma(\chi_\gamma^0) = \min_{\chi \in H_2} \Phi_\gamma(\chi) \tag{3.2}$$

where

$$\Phi_\gamma(\chi) = \tfrac{1}{2}\|\chi\|_H^2 + \int_\Omega \sum_{i,j=1}^n b_{ij} \tilde{\lambda}_i \chi_j \, dx - B(\chi, \bar{u}) .$$

Let us set

$$\lambda^0 = \bar{\lambda} + \chi^0 \tag{3.3}$$

$$\lambda_\gamma^0 = \tilde{\lambda} + \chi_\gamma^0 \tag{3.4}$$

and seek an estimate for $\|\lambda^0 - \lambda_\gamma^0\|_H$.

Theorem 3.2. *Let* $g, \gamma \in L_2(\Gamma_g)$. *Then*

$$\|\chi^0 - \chi_\gamma^0\|_H \leqq c\|g - \gamma\|_{L_2(\Gamma_g)} .$$

Proof. The principle of minimum complementary energy (Theorem 1.1) implies that $\lambda^0 = \lambda(u)$, where $\lambda(u)$ is defined by (1.8) and u is the weak solution of the primal problem (1.3). Similarly, $\lambda^0 = \lambda(u_\gamma)$, where u_γ is the weak solution of the modified problem (1.3) with g replaced by γ on Γ_g. From (1.4) and (1.5) it follows

$$\|u - u_\gamma\|_{W^{1,2}(\Omega)} \leqq c\|g - \gamma\|_{L_2(\Gamma_g)} .$$

Finally, we have

$$\|\lambda^0 - \lambda^0_\gamma\|_H = \|\lambda(u - u_\gamma)\|_H \leqq c\|u - u_\gamma\|_{W^{1,2}(\Omega)} \leqq c'\|g - \gamma\|_{L_2(\Gamma_g)} . \qquad \text{Q.E.D.}$$

Let $\Gamma_g = \bigcup_{j=1}^{m} a_j a_{j+1}$. Suppose that we have found $\lambda^{(1)} \in H$ such that $\operatorname{div} \lambda^{(1)} = -f$, (note that this problem is easy to solve by integration) and set

$$\tilde{g} = g - \lambda^{(1)} . \boldsymbol{n} \quad \text{on} \quad \Gamma_g .$$

Let $\tilde{g}$ be such that the values $\tilde{g}(a_j)$ are well defined for $j = 1, 2, \ldots, m + 1$ and find a function $\lambda^{(2)} \in \mathcal{N}_h(\Omega)$ such that at all vertices $a_j \in \bar{\Gamma}_g$

$$\varphi(a_j) \equiv (\lambda^{(2)} . \boldsymbol{n})(a_j) = \tilde{g}(a_j) , \quad j = 1, 2, 3, \ldots, m + 1 .$$

In other words, the function $\varphi(x) = (\lambda^{(2)} . \boldsymbol{n})(x)$ represents the Lagrange linear interpolate of the function $\tilde{g}$ on every $a_j a_{j+1} \in \Gamma_g$.

If we set $\tilde{\lambda} = \lambda^{(1)} + \lambda^{(2)}$, then

$$\tilde{\lambda} \in \Lambda_{f,\gamma} , \quad \gamma = \varphi + \lambda^{(1)} . \boldsymbol{n} ,$$

because

$$B(\lambda^{(1)} + \lambda^{(2)}, v) = \int_\Omega fv \, dx + \int_{\Gamma_g} \gamma v \, d\Gamma \ \forall v \in V$$

follows from the definition of $\lambda^{(1)}$, $\lambda^{(2)}$, integrating by parts.

Remark 3.1. The problem to find $\lambda^{(2)}$, satisfying the conditions mentioned above, is a problem of linear algebra. If α_j, β_j have the sense of Lemma 2.3, i.e. $\alpha_j = T_j\lambda^{(2)}(a_j)$, $\beta_j = T_j\lambda^{(2)}(a_{j+1})$ on the side $a_j a_{j+1}$ of a triangle $K \in \mathcal{T}_h$, then we have to solve the system:

$$\sum_{j=1}^{3} (\alpha_j + \beta_j) \, l_j = 0 \ \forall K \in \mathcal{T}_h .$$

Moreover, to satisfy also the condition (R), we choose the values α_j, β_j on the common side of any two adjacent triangles such that (2.12) holds, i.e., the corresponding parameters differ by the sign only. Finally, at the vertices on $\bar{\Gamma}_g$, the values α_j, β_j must equal to the values of $\tilde{g}$.

Let us derive an estimate for $\|\gamma - g\|_{L_2(\Gamma_g)}$. We have

$$\|\gamma - g\|^2_{L_2(\Gamma_g)} = \sum_{j=1}^{m} \|\varphi + \lambda^{(1)} . \boldsymbol{n} - g\|^2_{L_2(a_j a_{j+1})} =$$

$$= \sum_{j=1}^{m} \|\hat{\Pi} g - \hat{\Pi}(\lambda^{(1)} . \boldsymbol{n}) + \lambda^{(1)} . \boldsymbol{n} - g\|^2_{L_2(a_j a_{j+1})} \leqq$$

$$\leqq 2 \sum_{j=1}^{m} \left(\|\hat{\Pi} g - g\|^2_{L_2(a_j a_{j+1})} + \|\lambda^{(1)} . \boldsymbol{n} - \hat{\Pi}(\lambda^{(1)} . \boldsymbol{n})\|^2_{L_2(a_j a_{j+1})}\right) ,$$

where $\hat{\Pi} v$ denotes the linear Lagrange interpolate of v on the segment $a_j a_{j+1}$.

If e.g. g, $\lambda^{(1)} \,.\, \mathbf{n} \in W^{2,2}(\Gamma_m)$, where Γ_m is any side of the polygon Γ_g, then Theorem 3.2 yields

$$(3.5) \qquad \|\lambda^0 - \lambda_\gamma^0\|_H \leqq c\|g - \gamma\|_{L_2(\Gamma_g)} \leqq ch^2(\sum_m \|g\|_{W^{2,2}(\Gamma_m)} + \sum_m \|\lambda^{(1)} \,.\, \mathbf{n}\|_{W^{2,2}(\Gamma_m)}) \,.$$

Let $\chi_{\gamma,h}^0 \in V_h$ be such that

$$(3.6) \qquad \Phi_\gamma(\chi_{\gamma,h}^0) = \min_{\chi \in V_h} \Phi_\gamma(\chi)$$

and let us set

$$\lambda_{\gamma,h}^0 = \bar{\lambda} + \chi_{\gamma,h}^0 \,.$$

Then we have the following error estimate

Theorem 3.3. *If g, $\lambda^{(1)} \,.\, \mathbf{n} \in W^{2,2}(\Gamma_m)$ for any side Γ_m of Γ_g and $\chi_\gamma^0 \in [C^2(\bar{\Omega})]^2$,*[1]) *then for any regular family of triangulations*

$$\|\lambda^0 - \lambda_{\gamma,h}^0\|_H \leqq Ch^2 \,,$$

where the constant C is independent of h.

Proof. We may write

$$\|\lambda^0 - \lambda_{\gamma,h}^0\|_H \leqq \|\lambda^0 - \lambda_\gamma^0\|_H + \|\lambda_\gamma^0 - \lambda_{\gamma,h}^0\|_H \,,$$

where λ_γ^0 was introduced by (3.4). Using (3.5) and Corollary to Theorem 3.1 (applied to the modified problems (3.2), (3.6)), we obtain the bounds for both terms of the right-hand side. Q.E.D.

4. A POSTERIORI ERROR ESTIMATES, THE HYPERCIRCLE METHOD

Suppose that, besides $\lambda_h^0 = \bar{\lambda} + \chi_h^0$, we have also found a Ritz-Galerkin approximate solution u_{h*}^0 of the primal problem (1.3), i.e. $u_{h*}^0 \in \mathscr{V}_{h*}$, such that

$$(4.1) \qquad \mathscr{L}(u_{h*}^0) = \min_{u \in \bar{u} + \mathscr{V}_{h*}} \mathscr{L}(u)$$

and $\mathscr{V}_{h*} \subset V$ is a finite-dimensional subspace of finite elements.

Then we obtain the following *a posteriori estimates* of errors.

Lemma 4.1. *Let u and u_{h*}^0 be the solutions of the problem (1.3) and (4.1), respectively. Let $\lambda_h^0 = \bar{\lambda} + \chi_h^0$, where $\bar{\lambda} \in \Lambda_{f,g}$ and χ_h^0 is the solution of the problem (1.12). Define $\lambda(v)$ for any $v \in W^{1,2}(\Omega)$ by the relation (1.9). Then it holds*

$$(4.2) \qquad \|u_{h*}^0 - u\|_{W^{1,2}(\Omega)} \leqq C\|\lambda(u_{h*}^0) - \lambda_h^0\| \,,$$

$$(4.3) \qquad \|\lambda_h^0 - \lambda(u)\| \leqq C\|\lambda(u_{h*}^0) - \lambda_h^0\| \,.$$

[1]) It is also sufficient if $\chi_\gamma^0 \in [W^{2,2}(\Omega)]^2$ only.

Proof. As $\lambda(u^0_{h*} - u) = \lambda(u^0_{h*}) - \lambda(u) \in H_1$ and $\lambda(u) - \lambda^0_h \in H_2$, by virtue of Lemma 1.1 we may write

$$\begin{aligned}(4.4)\qquad \|\lambda(u^0_{h*}) - \lambda^0_h\|^2_H &= \|\lambda(u^0_h - u) + \lambda(u) - \lambda^0_h\|^2_H = \\ &= \|\lambda(u^0_{h*} - u)\|^2_H + \|\lambda(u) - \lambda^0_h\|^2_H .\end{aligned}$$

Hence (4.3) follows from (4.4) and (1.6'). An analogue of (1.8) yields that

$$\|\lambda(u^0_{h*} - u)\|^2_H \geqq c\|u^0_{h*} - u\|^2_{W^{1,2}(\Omega)}$$

and (4.2) follows, making use of (4.4) and (1.6'). Q.E.D.

Lemma 4.2. (*Hypercircle*). *Let the assumptions of Lemma* 4.1 *hold. Then*

$$(4.5)\qquad \|\tfrac{1}{2}(\lambda(u^0_{h*}) + \lambda^0_h) - \lambda(u)\|_H = \tfrac{1}{2}\|\lambda(u^0_{h*}) - \lambda^0_h\|_H$$

Proof. We may write, using Lemma 1.1 and (4.4), that

$$\begin{aligned}\|\lambda(u^0_{h*}) + \lambda^0_h - 2\lambda(u)\|^2_H &= \|\lambda(u^0_{h*} - u) + \lambda^0_h - \lambda(u)\|^2_H = \\ = \|\lambda(u^0_{h*}) - \lambda(u)\|^2_H + \|\lambda(u) - \lambda^0_h\|^2_H &= \|\lambda(u^0_{h*}) - \lambda^0_h\|^2_H . \text{ Q.E.D.}\end{aligned}$$

Remark 4.1. The equation (4.5) has the following geometrical sense: the solution $\lambda(u) \in H$ lies on a hypercircle, the center of which is at the point $\frac{1}{2}(\lambda(u^0_h) + \lambda^0_h)$ and the diameter equals the distance $\|\lambda(u^0_{h*}) - \lambda^0_h)\|_H$.

Note that if the center of the hypercircle is taken for an approximation of $\lambda(u)$, then the error measured in H is equal to the radius of the hypercircle, whereas the errors of both λ^0_h and $\lambda(u^0_{h*})$ can only be estimated from above by the diameter, as an immediate consequence of (4.4).

APPENDIX

Let us consider the following mixed boundary value problem, including the Newton condition:

$$\begin{aligned}(\text{A1})\qquad \mathscr{A}u &= f \quad \text{in} \quad \Omega \\ u &= \bar{u} \quad \text{on} \quad \Gamma_u \\ \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j} n_i &= g \quad \text{on} \quad \Gamma_g , \\ \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j} n_i + \alpha u &= \gamma \quad \text{on} \quad \Gamma_N ,\end{aligned}$$

where the boundary Γ consists of at most four disjoint parts

$$\Gamma = \Gamma_u \cup \Gamma_g \cup \Gamma_N \cup \mathscr{R},$$

$\Gamma_N \neq \emptyset$ open in Γ, $\text{mes}_{n-1} \mathscr{R} = 0$, Γ_u and Γ_g are either empty or open in Γ,

$$f \in L_2(\Omega), \quad g \in L_2(\Gamma_g), \quad \gamma \in L_2(\Gamma_N), \quad \bar{u} \in W^{1,2}(\Omega).$$

a_{ij} satisfy (1.1), (1.2) and

$$\alpha \in L_\infty(\Gamma_N), \quad \alpha(x) \geqq \alpha_0 > 0$$

almost everywhere on Γ_N.
Define again

$$V = \{v | v \in W^{1,2}(\Omega), \; v = 0 \text{ on } \Gamma_u\}$$

$$a(u, v) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Gamma_N} \alpha uv \, d\Gamma .$$

The weak solution of the problem (A 1) is a function $u \in W^{1,2}(\Omega)$, satisfying

$$u - \bar{u} \in V,$$

$$a(u, v) = \int_\Omega fv \, dx + \int_{\Gamma_g} gv \, d\Gamma + \int_{\Gamma_N} \gamma v \, d\Gamma \quad \forall v \in V.$$

It is well known, that a unique weak solution of the problem (A.1) exists and that it minimizes the functional

$$\mathscr{L}(v) = \tfrac{1}{2} a(v, v) - \int_\Omega fv \, dx - \int_{\Gamma_g} gv \, d\Gamma - \int_{\Gamma_N} \gamma v \, d\Gamma$$

on the set $\bar{u} \oplus V$.

Definition A1. *Let $v \in W^{1/2,2}(\Gamma)$, $v = O$ on Γ_u. Denote $Zv = w \in V$ an arbitrary extension of v into Ω (see e.g. [10], p. 103 for the extension). Let $M_{f,g} \subset [L_2(\Omega)]^n$ be the set of vector-functions λ, such that the operator $\mathscr{G}_{f,g}(\lambda)$, defined through the relation*

$$\langle \mathscr{G}_{f,g}(\lambda), v \rangle = \int_\Omega (\lambda \operatorname{grad} w - fw) \, dx - \int_{\Gamma_g} gv \, d\Gamma,$$

maps $M_{f,g}$ into $W^{-1/2,2}(\Gamma_N)$ [1]).

Remark 1. From Definition A1 it follows that

$$|\langle \mathscr{G}_{f,g}(\lambda), v \rangle| \leqq c(\lambda) \, \|v\|_{W^{1/2,2}(\Gamma_N)}$$

and the values of $\mathscr{G}_{f,g}(\lambda)$ do not depend on the extension Z.

---

[1]) $W^{-1/2,2}(\Gamma_N)$ denotes the space of linear continuous functionals on $W^{1/2,2}(\Gamma_N)$

Remark 2. Consider $\varphi \in W^{1/2,2}(\Gamma)$ such that $\varphi = 0$ on $\Gamma_u \cup \Gamma_N$, and denote $Z\varphi = \psi$. According to Definition A1 we have

$$\text{(A2)} \qquad \langle \mathscr{G}_{f,g}(\lambda), \varphi \rangle = \int_\Omega (\lambda \,.\, \operatorname{grad} \psi - f\psi) \, dx - \int_{\Gamma g} g\varphi \, d\Gamma = 0$$

Consequently, $\lambda \in M_{f,g}$ satisfies the following conditions

$$\operatorname{div} \lambda + f = 0 \quad \text{in} \quad \Omega$$

$$\sum_{j=1}^{n} \lambda_j n_j = g \quad \text{on} \quad \Gamma_g ,$$

in a weak sense. In fact, integrating by parts in (A2) formally, we obtain

$$\langle \mathscr{G}_{f,g}(\lambda), \varphi \rangle = -\int_\Omega (\operatorname{div} \lambda + f) \psi \, dx + \int_{\Gamma_g} (\sum_{j=1}^{n} \lambda_j n_j - g) \varphi \, d\Gamma = 0 .$$

Definition A2. *Let $\Lambda_{f,g} \subset M_{f,g}$ be the set of all $\lambda \in M_{f,g}$ such that*

$$\mathscr{G}_{f,g}(\lambda) \in L_2(\Gamma_N) .$$

$\Lambda_{f,g}$ will be called the set of admissible functions.

Next let f and g vary all over the space $L_2(\Omega)$ and $L_2(\Gamma_g)$, respectively. Denote

$$\text{(A3)} \qquad M = \bigcup_{\substack{f \in L_2(\Omega) \\ g \in L_2(\Gamma_g)}} \Lambda_{f,g} .$$

Then to every $\lambda \in M$ there exists a pair $\{f(\lambda), g(\lambda)\}$ ($f = -\operatorname{div} \lambda$, $g = \lambda_j n_j$ on Γ_g in a weak sense) such that

$$G(\lambda) \equiv \mathscr{G}_{f(\lambda),g(\lambda)}(\lambda) \in L_2(\Gamma_N) .$$

It is readily seen that M is a linear manifold and G is linear on M.

Let us define a bilinear form

$$\text{(A4)} \qquad (\lambda, \mu)_H = \int_\Omega \sum_{i,j=1}^{n} b_{ij} \lambda_i \mu_j \, dx + \int_{\Gamma_N} \alpha^{-1} G(\lambda) G(\mu) \, d\Gamma$$

on $M \times M$, where b_{ij} are the entries of the matrix $[a^{-1}]$ inverse to $[a]$. Introducing the norm of graph

$$\|\lambda\|_M^2 = \sum_{i=1}^{n} \int_\Omega \lambda_i^2 \, dx + \int_{\Gamma_N} [G(\lambda)]^2 \, d\Gamma ,$$

and using the properties of the coefficients a_{ij}, α, we are led to the inequalities

(A5) $$C_1\|\lambda\|_M^2 \leqq (\lambda, \lambda)_H \leqq c_2\|\lambda\|_M^2 .$$

The bilinear form is symmetric, continuous and positive definite, hence it represents a scalar product. The manifold M with the scalar product (A4) will be denoted by H.

Let us define

$$H_1 = \{\lambda \mid \lambda \in M , \exists v \in V : \lambda_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} \ (i = 1, 2, \ldots, n) ,$$
$$\alpha v + G(\lambda) = 0 \text{ on } \Gamma_N\} ,$$

$H_2 = \Lambda_{0,0}$.

Then we prove the following

Lemma A1.

α) *H_1 and H_2 are closed subspaces of M.*

β) *$H_1 \perp H_2$.*

Proof. Let $\lambda^m \in H_1$, $\lambda^m \to \lambda$ in H. Then we may write (cf. the proof of Lemma 1.1)

$$\|\lambda^m - \lambda^p\|_H^2 \geqq \alpha \int_\Omega |\text{grad}\,(v_m - v_p)|^2 \,\text{d}x + \alpha_0 \int_{\Gamma_N} (v_m - v_p)^2 \,\text{d}\Gamma \geqq c\|v_m - v_p\|_{W^{1,2}(\Omega)}^2$$

using the generalized Friedrichs inequality (see [10]). Hence $v_n \to v$ in V, and from (A5) it follows that

$$\lambda_i = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} , \quad G(\lambda) = -\alpha v , \quad \lambda \in H_1 .$$

Definitions A1 and A2 yield

(A6) $$\langle \mathscr{G}_{0,0}(\lambda^n), v \rangle \equiv \int_{\Gamma_N} G(\lambda^n)\, v \,\text{d}\Gamma = \int_\Omega \lambda^n \,.\, \text{grad}\, w \,\text{d}x$$

for all $v \in W^{1/2,2}(\Gamma_N)$, $w = Zv \in V$, $\lambda^n \in \Lambda_{0,0}$.

If $\lambda^n \to \lambda$ in H, by virtue of (A5) $\lambda^n \to \lambda$ in $[L_2(\Omega)]^n$, $G(\lambda^n) \to \lambda$ in $L_2(\Gamma_N)$. Therefore (A6) holds even for λ, i.e., $\lambda \in \Lambda_{0,0} = H_2$.

Let $\lambda \in H_1$, $\mu \in H_2$. Then we obtain

$$(\lambda, \mu)_H = \int_\Omega \mu \,.\, \text{grad}\, v \,\text{d}x - \int_{\Gamma_N} v\, G(\mu) \,\text{d}\Gamma = 0 ,$$

as $G(\mu) = \mathscr{G}_{0,0}(\mu)$.

Theorem A1. (*Principle of minimum complementary energy*). *Let u be the weak solution of* (A1). *Suppose that $\lambda(\bar{u}) \in M$, and $G(\lambda(\bar{u})) = \gamma - \alpha\bar{u}$.*

Then the functional

$$\mathscr{S}(\lambda) = \tfrac{1}{2}(\lambda, \lambda)_H - (\lambda, \lambda(\bar{u}))_H$$

attains its minimum on the set $\Lambda_{f,g}$ *of admissible functions, if and only if* $\lambda = \lambda(u)$.

Moreover it holds

$$\text{(A7)} \qquad -\mathscr{S}(\lambda(u)) = \mathscr{L}(u) + \int_\Omega f\bar{u}\,\mathrm{d}x + \int_{\Gamma_g} g\bar{u}\,\mathrm{d}\Gamma + \tfrac{1}{2}\int_{\Gamma_N} \alpha^{-1}\gamma^2\,\mathrm{d}\Gamma\,.$$

Proof. First we show that $\lambda(u) \in \Lambda_{f,g}$. In fact, from the definition of a weak solution it follows

$$\text{(A8)} \qquad \int_\Omega \lambda(u)\,\mathrm{grad}\,v\,\mathrm{d}x + \int_{\Gamma_N} \alpha u v\,\mathrm{d}\Gamma = \int_\Omega fv\,\mathrm{d}x + \int_{\Gamma_g} gv\,\mathrm{d}\Gamma + \int_{\Gamma_N} \gamma v\,\mathrm{d}\Gamma$$

$\forall v \in V$. Consequently, we have

$$\int_\Omega (\lambda(u)\,\mathrm{grad}\,v - fv)\,\mathrm{d}x - \int_{\Gamma_g} gv\,\mathrm{d}\Gamma = \int_{\Gamma_N} (\gamma - \alpha u)\,v\,\mathrm{d}\Gamma =$$

$$= \langle \mathscr{G}_{f,g}(\lambda(u)), v\rangle,\ \mathscr{G}_{f,g}(\lambda(u)) \equiv G(\lambda(u)) = \gamma - \alpha u \in L_2(\Gamma_N)\,,$$

hence $\lambda(u) \in \Lambda_{f,g}$.

Denote $u = \bar{u} + \omega$, $\omega \in V$. Then

$$\text{(A9)} \qquad \lambda(\omega) \in H_1\,.$$

Indeed, insert $\lambda(u) = \lambda(\bar{u}) + \lambda(\omega)$ into (A8), and denote by $f(\bar{u}) = f(\lambda(\bar{u}))$, $g(\bar{u}) = g(\lambda(\bar{u}))$ the functions, corresponding with $\lambda(\bar{u}) \in M$ (see Definition A2). Thus we have

$$\text{(A10)} \qquad \int_\Omega (\lambda(\omega)\,\mathrm{grad}\,v + \lambda(\bar{u})\,\mathrm{grad}\,v)\,\mathrm{d}x + \int_{\Gamma_N} \alpha v(\bar{u} + \omega)\,\mathrm{d}\Gamma =$$

$$= \int_\Omega fv\,\mathrm{d}x + \int_{\Gamma_g} gv\,\mathrm{d}\Gamma + \int_{\Gamma_N} \gamma v\,\mathrm{d}\Gamma\,.$$

and Definition of $G(\lambda(\bar{u}))$ yields that

$$\int_\Omega \lambda(\bar{u})\,\mathrm{grad}\,v\,\mathrm{d}x = \langle G(\lambda(\bar{u})), v\rangle + \int_\Omega f(\bar{u})\,v\,\mathrm{d}x + \int_{\Gamma_g} v\,g(\bar{u})\,\mathrm{d}\Gamma\,.$$

Substituting into (A10), we obtain

$$\int_\Omega [\lambda(\omega)\,\mathrm{grad}\,v - (f - f(\bar{u})\,v]\,\mathrm{d}x - \int_{\Gamma_g} [g - g(\bar{u})]\,v\,\mathrm{d}\Gamma +$$

$$+ \langle G(\lambda(\bar{u})), v\rangle = \int_{\Gamma_N} (\gamma - \alpha\bar{u} - \alpha\omega)\,v\,\mathrm{d}\Gamma,\quad \forall v \in V\,.$$

As $G(\lambda(\bar{u})) = \gamma - \alpha\bar{u} \in L_2(\Gamma_N)$ we deduce

$$G(\lambda(\omega)) = \mathscr{G}_{f_1,g_1}(\lambda(\omega)) = -\alpha\omega\,,$$

where

$$f_1 = f - f(\bar{u})\,, \quad g_1 = g - g(\bar{u})\,.$$

Consequently, $\lambda(\omega)$ belongs to H_1.

Next let $\lambda \in \Lambda_{f,g}$. Then

$$\text{(A11)} \qquad \lambda - \lambda(u) \in H_2\,.$$

In fact, subtracting the relations of Definition A1 for λ and $\lambda(u)$, we come to the following equation

$$\int_\Omega (\lambda - \lambda(u)) \operatorname{grad} w \,\mathrm{d}x = \langle \mathscr{G}_{f,g}(\lambda) - \mathscr{G}_{f,g}(\lambda(u)), v \rangle = \langle \mathscr{G}_{0,0}(\lambda - \lambda(u)), v \rangle\,.$$

Consequently, $\mathscr{G}_{0,0}(\lambda - \lambda(u)) \in L_2(\Gamma_N)$,, $\lambda - \lambda(u) \in \Lambda_{0,0}$.

The rest of the proof is the same as that of Theorem 1.1. being based on (A9), (A11) and the orthogonality (β) of Lemma A1. Q.E.D.

Following the line of thought of Section 1, we can introduce an equivalent variational problem (1.11) and establish an analogue of Theorem 1.2. The only change is that the norms $\|\lambda\|$ should be replaced by $\|\lambda\|_M$. Using the subspaces $\mathscr{N}_h(\Omega)$, we set again $V_h = \{\chi \mid \chi \in \mathscr{N}_h(\Omega), \chi \,.\, \boldsymbol{n} = 0 \text{ on } \Gamma_g\}$, $h \in (0, 1\rangle$.

Then $V_h \subset H_2$ and an exact analogue of Theorem 3.1 holds, as

$$\chi^0 \in [C^2(\bar{\Omega})]^2 \cap H_2 \Rightarrow G\chi^0 \equiv \chi^0 \,.\, \boldsymbol{n} \in C^2(a_j a_{j+1})\,, \quad \forall a_j a_{j+1} \in \Gamma_N\,,$$
$$\|\chi^0 \,.\, \boldsymbol{n} - (r_h\chi^0) \,.\, \boldsymbol{n}\|_{L_2(\Gamma_N)} \leqq ch^2 \|\chi^0\|_{[C^2(\bar{\Omega})]^2}$$

can be proved easily.

Acknowledgments. The authors are indebted to Doc. J. Nečas, DrSc. for some valuable advices concerning the approach used in this Appendix.

References

[1] *B. Fraeijs de Veubeke:* Displacement and equilibrium models in the finite element method. Stress Analysis, ed. by O.C. Zienkiewicz and G. Holister, J. Wiley, 1965, 145—197.

[2] *B. Fraeijs de Veubeke, O. C. Zienkiewicz:* Strain energy bounds in finite-element analysis by slab analogies. J. Strain Analysis 2, (1967) 265—271.

[3] *V. B. Jr. Watwood, B. J. Hartz:* An equilibrium stress field model for finite element solution of two-dimensional elastostatic problems. Int. J. Solids Structures 4, (1968), 857—873.

[4] *B. Fraeijs de Veubeke, M. Hogge:* Dual analysis for heat conduction problems by finite elements. Int. J. Numer. Meth. Eng. 5, (1972), 65—82.

[5] *J. P. Aubin, H. G. Burchard:* Some aspects of the method of the hypercircle applied to elliptic variational problems. Numer. sol. Part. Dif. Eqs. II, SYNSPADE (1970), 1—67.
[6] *J. Vacek:* Dual variational principles for an elliptic partial differential equation. Apl. mat. 21 (1976), 5—27.
[7] *I. Hlaváček:* On a conjugate semi-variational method for parabolic equations. Apl. mat. 18 (1973), 434—444.
[8] *F. Grenacher:* A posteriori error estimates for elliptic partial differential equations. Inst. Fluid Dynamics and Appl. Math., Univ. Maryland, TN-BN-T 43, July 1972.
[9] *W. Prager, J. L. Synge:* Approximations in elasticity based on the concept of function space. Quart. Appl. Math. 5 (1947), 241—269.
[10] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.
[11] *I. Hlaváček:* Variational principles in the linear theory of elasticity for general boundary conditions. Apl. mat. 12 (1967), 425—448.
[12] *J. Haslinger, I. Hlaváček:* Convergence of a dual finite element method in R_n. CMUC 16 (1975), 469—485.
[13] *H. Gajewski:* On conjugate evolution equations and a posteriori error estimates. Proceedings of Internat. Summer School on Nonlinear Operators held in Berlin, 1975.

Souhrn

KONVERGENCE METODY KONEČNYCH PRVKŮ ZALOŽENÉ NA DUÁLNÍ VARIAČNÍ FORMULACI

Jaroslav Haslinger, Ivan Hlaváček

Je studován „rovnovážný" model metody konečných prvků při aplikaci po částech (trojúhelnících) lineárních polynomů na řešení kombinované okrajové úlohy v rovině pro eliptickou diferenciální rovnici druhého řádu. Dokazuje se, že je-li řešení dost hladké, přibližná řešení konvergují s rychlostí h^2.

Author's addresses: Dr. *Jaroslav Haslinger*, Matematicko-fysikální fakulta KU, Sokolovská 83, 186 00 Praha 8; Ing. *Ivan Hlaváček*, CSc., Matematický ústav ČSAV, Žitná 25, 11 567 Praha 1.