# CONVERGENCE OF ADAPTIVE FEM FOR A CLASS OF DEGENERATE CONVEX MINIMIZATION PROBLEMS 

CARSTEN CARSTENSEN*


#### Abstract

A class of degenerate convex minimization problems allows for some adaptive finite element method (AFEM) to compute strongly converging stress approximations. The algorithm AFEM consists of successive loops of the form $$
\text { SOLVE } \rightarrow \text { ESTIMATE } \rightarrow \text { MARK } \rightarrow \text { REFINE }
$$ and employs the bulk criterion. The convergence in $L^{p^{\prime}}\left(\Omega ; \mathbb{R}^{m \times n}\right)$ relies on new sharp strict convexity estimates of degenerate convex minimization problems with $$
\mathcal{J}(v):=\int_{\Omega} W(D v) d x-\int_{\Omega} f v d x \quad \text { for } v \in V:=W_{0}^{1, p}\left(\Omega ; \mathbb{R}^{m}\right)
$$

The class of minimization problems includes strong convex problems and allows applications in an optimal design task, Hencky elastoplasticity, or relaxation of 2 -well problems allowing for microstructures.


## 1. Class of Convex Minimization Problems

This section specifies a class of $C^{1}$ energy densities $W: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ characterized by (H1)-(H2) for some constants $1<p<\infty, 1 \leq r<\infty$, and $0 \leq s<\infty$ with

$$
\max \{(1+s / r) /(1-1 / r), 2 n /(n+2)\} \leq p
$$

through the two-sided growth condition

$$
\begin{equation*}
|F|^{p}-1 \lesssim W(F) \lesssim 1+|F|^{p} \text { for all } F \in \mathbb{R}^{m \times n} \tag{H1}
\end{equation*}
$$

[^0]and the convexity control
\[

$$
\begin{align*}
& \left(1+|A|^{s}+|B|^{s}\right)^{-1}|D W(A)-D W(B)|^{r}  \tag{H2}\\
& \lesssim W(B)-W(A)-D W(A):(B-A) \text { for all } A, B \in \mathbb{R}^{m \times n} .
\end{align*}
$$
\]

Here and throughout "." denotes the scalar product in $\mathbb{R}^{m}$,":" denotes the scalar product in $\mathbb{R}^{m \times n}$, and the expression " $<"$ abbreviates an inequality up to some multiplicative generic constant, i.e., $A \lesssim B$ means $A \leq c B$ with some generic constant $c>0$, which is independent of the arguments $A, B, F$ in (H1)-(H2) (but may depend on $W$ and on the aspect ratio of finite element triangulations).
Finally, $t:=1+s / p$ and the Hölder conjugate $p^{\prime}$ of $p$ satisfy

$$
1<p^{\prime} \leq r / t<\infty, \quad \text { and } \quad 1 / p+1 / p^{\prime}=1
$$

and where $r / t$ and $r /(r-t)$ are conjugate exponents, i.e., $t / r+(r-$ $t) / r=1$.
Section 3 exposes a list of examples with (H1)-(H2). The two-sided growth control (H1) is standard in the form of

$$
|F|^{p} \lesssim W(F)+1 \quad \text { and } \quad W(F) \lesssim 1+|F|^{p} .
$$

By adding a constant to $W(F)$, it could be replaced even by

$$
|F|^{p} \lesssim W(F) \lesssim 1+|F|^{p} .
$$

The convexity control (H2) implies the monotonicity condition

$$
\begin{align*}
& \left(1+|A|^{s}+|B|^{s}\right)^{-1}|D W(A)-D W(B)|  \tag{H3}\\
& \lesssim(D W(A)-D W(B)):(A-B) \quad \text { for all } A, B \in \mathbb{R}^{m \times n}
\end{align*}
$$

from $[10,11]$. Under some conditions, (H2) is in fact equivalent to (H3) [15, 16].
Given such energy density $W: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n}, n=2,3$, and some right-hand side $f \in L^{p^{\prime}}\left(\Omega ; \mathbb{R}^{m}\right)$, define $\mathcal{J}: V \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
\mathcal{J}(v):=\int_{\Omega} W(D v) d x-\int_{\Omega} f \cdot v d x \quad \text { for } v \in V:=W_{0}^{1, p}\left(\Omega ; \mathbb{R}^{m}\right) \tag{1.1}
\end{equation*}
$$

Throughout this paper, $D v(x)$ denotes the $m \times n$ functional matrix of $V$ at $x$ and we adapt standard notation on Lebesgue and Sobolev spaces, e.g., $W_{0}^{1, p}(\Omega)$ denotes the subset of functions in $W^{1, p}(\Omega)$ with trace zero on the boundary $\partial \Omega$ of $\Omega$.
The minimization problem reads: Seek minimizers in $\mathcal{J}$ in $V$, written

$$
\begin{equation*}
u \in \arg \min _{v \in V} \mathcal{J}(v) . \tag{1.2}
\end{equation*}
$$

The existence of minimizers $u$ or $u_{\ell}$ of (1.1) in $V$ or some closed subspace $V_{\ell}$ of $V$ is guaranteed under (H1)-(H2) while, in general, their uniqueness fails. However, the respective exact and discrete stress

$$
\sigma:=D W(D u) \quad \text { and } \quad \sigma_{\ell}=D W\left(D u_{\ell}\right) \in L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)
$$

is unique [11], i.e., $\sigma$ and $\sigma_{\ell}$ do not depend on the choice of $u$ and $u_{\ell}$ amongst the set of exact and discrete minimizers. The smoothness of $\sigma \in W_{l o c}^{1, p}\left(\Omega ; \mathbb{R}^{m \times n}\right)$ has been analysed in $[10,16]$, while the smoothness of $u$ is open (recall that $u$ may be non-unique). Therefore the a priori error estimate (valid for any choice of $u \in \operatorname{argmin} J$ )

$$
\left\|\sigma-\sigma_{\ell}\right\|_{L^{q}\left(\Omega ; \mathbb{R}^{m \times n}\right)} \lesssim \min _{v_{\ell} \in V_{\ell}}\left\|u-v_{\ell}\right\|_{V}
$$

although it may be regarded as quasi-optimal convergent, has its limitations. The a posteriori error estimates for $\left\|\sigma-\sigma_{\ell}\right\|_{L^{q}\left(\Omega ; \mathbb{R}^{m \times n}\right)}$ known from the literature even face some reliability-efficiency gap [9], cf. Section 2 and Remark 2.1 below. Surprisingly, this does not prevent the design of convergent adaptive mesh-refining algorithms.

## 2. AFEM

This section describes the adaptive mesh-refining strategy, proposed in this paper and states the main result.
2.1. Outline. Given an initial coarse mesh $\mathcal{I}_{0}$, an adaptive finite element method (AFEM) successively generates a sequence of meshes $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots$ and associated discrete subspaces

$$
\begin{equation*}
V_{0} \nsubseteq V_{1} \subsetneq \cdots \nRightarrow V_{\ell} \subsetneq V_{\ell+1} \nsubseteq \cdots \stackrel{\subsetneq}{\nexists} V \tag{2.1}
\end{equation*}
$$

with discrete problems $\left(P_{0}\right),\left(P_{1}\right),\left(P_{2}\right), \ldots$ and discrete solutions $u_{0}$, $u_{1}, u_{2}, \ldots$ and discrete stresses $\sigma_{0}, \sigma_{1}, \sigma_{2}, \ldots$ steered by refinement rules and indicators. A typical loop from $V_{\ell}$ to $V_{\ell+1}$ (at the frozen level $\ell$ ) consists of the steps

$$
\begin{equation*}
\text { SOLVE } \rightarrow \text { ESTIMATE } \rightarrow \text { MARK } \rightarrow \text { REFINE } \tag{2.2}
\end{equation*}
$$

explained in the following Subsections.
2.2. Input. Input a shape-regular triangulation $\mathcal{T}_{0}$ of $\Omega \subset \mathbb{R}^{n}$ into closed triangles (if $n=2$ ) or closed tetrahedra (if $n=3$ ) with associated first-order finite element space $V_{0}$; suppose each element domain in $\mathcal{T}_{0}$ (and furthermore in $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots$ ) has at least one vertex in the interior of $\Omega$, put level $\ell:=0$.
A triangulation $\mathcal{T}_{\ell}$ is regular if two distinct closed-element domains are either disjoint or their intersection is one common vertex, one common
edge (or, if $n=3$ possibly one common face). For simplicity, all triangulations in the paper will be regular. Those common faces are called sides $\mathcal{E}_{\ell}$, if $n=3$. For $n=2, \mathcal{E}_{\ell}$ are the interior edges.
2.3. SOLVE. Given the triangulation $\mathcal{T}_{\ell}$ with set of interior sides $\mathcal{E}_{\ell}$ and interior nodes $\mathcal{K}_{\ell}$, the piecewise affine space $\mathcal{P}_{1}\left(\mathcal{T}_{\ell}\right)$ reads

$$
\begin{aligned}
\mathcal{P}_{1}\left(\mathcal{T}_{\ell} ; \mathbb{R}^{m}\right):= & \left\{v \in L^{\infty}\left(\Omega ; \mathbb{R}^{m}\right): \forall T \in \mathcal{T}_{\ell},\left.v\right|_{T} \in \mathcal{P}_{1}\left(T ; \mathbb{R}^{m}\right)\right\} ; \\
\mathcal{P}_{1}\left(T ; \mathbb{R}^{m}\right):= & \left\{v \in C^{\infty}\left(T ; \mathbb{R}^{m}\right): \exists A \in \mathbb{R}^{m \times n} \exists b \in \mathbb{R}^{m}\right. \\
& \forall x \in T: v(x)=A x+b\} .
\end{aligned}
$$

The discrete space $V_{\ell}:=V \cap \mathcal{P}_{1}\left(\mathcal{T}_{\ell} ; \mathbb{R}^{m}\right)$ is the first-order finite element space and allows for a nodal basis $\left(\varphi_{z}: z \in \mathcal{K}_{\ell}\right)$. Then the step SOLVE reads: Solve the nonlinear discrete problem

$$
\begin{equation*}
u_{\ell} \in \arg \min _{v_{\ell} \in V_{\ell}} \mathcal{J}\left(v_{\ell}\right) \quad \text { and set } \quad \sigma_{\ell}:=D W\left(D u_{\ell}\right) . \tag{2.3}
\end{equation*}
$$

The $\mathbb{R}^{m \times n}$-valued stress $\sigma_{\ell}$ is piecewise constant with respect to $\mathcal{T}_{\ell}$.
2.4. ESTIMATE. Given any interior side $E \in \mathcal{E}_{\ell}$ with measure $|E|$, and normal unit vector $\nu_{E}$, compute the jump

$$
J_{E}:=\left[\sigma_{\ell}\right]_{E} \nu_{E} \in \mathbb{R}^{m}
$$

of the discrete normal stresses $\sigma_{\ell} \nu_{E}$ over $E$, where

$$
\left[\sigma_{\ell}\right]_{E}(x):=\lim _{T_{+} \ni a \rightarrow x} \sigma_{\ell}(a)-\lim _{T_{-} \rightarrow b \rightarrow x} \sigma_{\ell}(b)
$$

for all $x \in E=\partial T_{+} \cap \partial T_{-}$, and by convention, $\nu_{E}$ is exterior to $T_{+}$. Then define

$$
\begin{equation*}
\eta_{\ell}:=\left(\sum_{E \in \mathcal{E}_{\ell}} \eta_{E}^{p^{\prime}}\right)^{1 / p^{\prime}} \quad \text { with } \quad \eta_{E}:=h_{E}^{1 / p^{\prime}}|E|^{1 / p^{\prime}}\left|J_{E}\right| \quad \text { for } E \in \mathcal{E}_{\ell} . \tag{2.4}
\end{equation*}
$$

It is essentially known from $[9,11]$ that $\eta_{\ell}$ is a reliable a posteriori error estimator in the sense that

$$
\begin{equation*}
\left\|\sigma-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{r} \lesssim \eta_{\ell}+\operatorname{osc}_{\ell}, \tag{2.5}
\end{equation*}
$$

cf. Lemma 4.2 below. Here and throughout, osc $C_{\ell}$ denotes data oscillations. Given any connected open nonvoid $\omega \subset \Omega$, let

$$
\begin{equation*}
\operatorname{osc}(f, \omega)^{p^{\prime}}:=\operatorname{diam}(\omega)^{p^{\prime}}\left\|f-f_{\omega}\right\|_{L^{p^{\prime}}(\omega)}^{p^{\prime}} \text { with } f_{\omega}:=|\omega|^{-1} \int_{\omega} f d x \tag{2.6}
\end{equation*}
$$

the integral mean of $f$ over $\omega$. For each node $z$ in the triangulation $\mathcal{T}_{\ell}$ with nodal basis function $\varphi_{z} \in V_{\ell}$, let $\omega_{z}:=\{x \in \Omega: \varphi(x)>0\}$ denote
the patch of $z$. Then, recall $\mathcal{K}_{\ell}$ denotes the set of all interior nodes,

$$
\begin{equation*}
\operatorname{osc}_{\ell}^{p^{\prime}}:=\sum_{z \in \mathcal{K}_{\ell}} \operatorname{osc}\left(f, \omega_{z}\right)^{p^{\prime}} \tag{2.7}
\end{equation*}
$$

Since osc ${ }_{\ell}$ depends on the given data and explicitly on $\mathcal{T}_{\ell}$, it can easily be made arbitrarily small by additional refinement steps. This data oscillation control allows for $\lim _{\ell \rightarrow \infty}$ osc $_{\ell}=0$; cf. [17, 22] for algorithmic details.

Remark 2.1. The upper bound in (2.5) is not sharp, the estimator $\eta_{\ell}$ is not efficient, because of $r>1$. This is called reliability-efficiency gap [9].
2.5. MARK. Select a subset $\mathcal{M}_{\ell}$ of $\mathcal{E}_{\ell}$ in the current triangulation $\mathcal{T}_{\ell}$ with

$$
\begin{equation*}
\eta_{\ell}^{p^{\prime}} \lesssim \sum_{E \in \mathcal{M}_{\ell}} \eta_{E}^{p^{\prime}} \tag{2.8}
\end{equation*}
$$

Given a parameter $0<\Theta<1$ the selection condition (2.8) results from choosing sufficiently many sides $E$ with bigger $\eta_{E}$ in $\mathcal{M}_{\ell}$ such that the bulk criterion [13, 17, 18, 22] holds:

$$
\Theta \eta_{\ell}^{p^{\prime}} \leq \sum_{E \in \mathcal{M}_{\ell}} \eta_{E}^{p^{\prime}}
$$

This is easily arranged with some greedy algorithm.
2.6. REFINE. Refine the triangulation $\mathcal{T}_{\ell}$ and design a refined shaperegular triangulation $\mathcal{T}_{\ell+1}$ such that each interior side $E=\partial T_{+} \cap \partial T_{-} \in$ $\mathcal{M}_{\ell}$ is refined in $\mathcal{T}_{\ell+1}$, for $T_{+}, T_{-} \in \mathcal{T}_{\ell}$ and $T_{+} \cup T_{-}$includes at least one new node on $E$ and at least one new node in the interior of either $T_{+}$

red

blue (left)


3 bisections

blue (right)


5 bisections

Figure 2.1. Possible refinements of a triangle in REFINE of AFEM. The 5 bisections allow for an interior node property.
or $T_{-}$. For $n=2$ the inner node property is easily depicted with 5 bisections as in Figure 2.1. More details on the shape-regular refinement strategies can be found in [6].
2.7. Output. The AFEM computes a sequence of discrete stresses $\sigma_{0}, \sigma_{1}, \sigma_{2}, \ldots$ in $L^{p^{\prime}}\left(\Omega ; \mathbb{R}^{m \times n}\right)$ as approximations to $\sigma:=D W(D u)$. The main result of this paper is the strong convergence of the stresses.

Theorem 2.1 (Convergence Theorem). Suppose (H1)-(H2) and

$$
\lim _{\ell \rightarrow \infty} \operatorname{osc}_{\ell}=0
$$

Then the sequence of stress fields $\sigma_{0}, \sigma_{1}, \sigma_{2}, \ldots$ converges strongly towards the exact stress field $\sigma$ in $L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)$.

The technical proof is postponed to Section 4, after the motivating list of examples in Section 3.

## 3. Examples and Applications

This section briefly summarizes a few applications with explicit proofs of (H1)-(H2) and hence with a convergent AFEM.
3.1. Uniformly Convex Minimization. Uniformly convex $\mathcal{C}^{1}$ function $W: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ with globally Lipschitz continuous derivative $D W$, i.e., for all $A, B \in \mathbb{R}^{m \times n}$ there holds

$$
\begin{aligned}
|A-B|^{2} & \lesssim D W(A):(A-B)-W(A)+W(B) \\
|D W(A)-D W(B)| & \lesssim|A-B| .
\end{aligned}
$$

This implies (H1)-(H2) with $p=2=r$ and $s=0$ and, thus, the class (i) is included in class (ii). Simple examples are $W(F)=\varphi(|\operatorname{sym} F|)|F|^{2}$ for proper $\mathcal{C}^{2}$ functions $\varphi$ (cf., e.g., [23, Sections 62.3, 62.8-9] and [15, Exercise 1.7 on page 21]).
3.2. Nonlinear Laplacian. The $p$-Laplacian satisfies (H1)-(H2) for any $2 \leq p<\infty$ and $r=2, s=p-2$.

Lemma 3.1. Given $1 \leq p<\infty$ define the function $W: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ by $W(A):=|A|^{p} / p$. Then there exist a constant $c_{1}=c(p)$ such that for all $A, B \in \mathbb{R}^{m \times n}$ there holds

$$
\begin{aligned}
&|D W(A)-D W(B)|^{2} \leq c_{1}\left(|A|^{p-2}+|B|^{p-2}\right) \\
& \times(W(B)-W(A)-D W(A ; B-A)) .
\end{aligned}
$$

Proof. Given $A, B \in \mathbb{R}^{m \times n}$ with $A \neq B$ set $a:=|A|$ and $b:=|B|$. A quick check verifies that the assertion holds for either $a=0$ or $b=0$ with $c_{1}=\max \{p, q\}$. It is therefore assumed that $a b>0$ in the sequel and $c:=A: B /(a b)$. Then $0<t:=b / a<\infty$. The left- and right-hand side of the assertion vanish for $a=b$ and $c=+1$. This situation is therefore excluded in the sequel. Then,

$$
\begin{aligned}
W(B)-W(A)-D W(A ; B-A) & =b^{p} / p-a^{p} / p-a^{p-1}(c b-a) \\
& =b^{p} / p+a^{p} / q-a^{p-1} b c
\end{aligned}
$$

is strictly positive (non-negativity immediately follows from Young's inequality and $-1 \leq c \leq 1)$. Since

$$
|D W(A)-D W(B)|^{2}=a^{2(p-1)}+b^{2(p-1)}-2 c a^{p-1} b^{p-1}
$$

The quotient of the left- and the right-hand side of the assertion reads

$$
\begin{aligned}
\frac{a^{2(p-1)}+b^{2(p-1)}-2 c a^{p-1} b^{p-1}}{\left(a^{p-2}+b^{p-2}\right)\left(b^{p} / p+a^{p} / q-a^{p-1} b c\right)} & =\frac{1+t^{2(p-1)}-2 c t^{p-1}}{\left(1+t^{p-2}\right)\left(t^{p} / p+1 / q-c t\right)} \\
& =: f(t, c) .
\end{aligned}
$$

A direct calculation verifies that $\partial f / \partial c$ as a function of $c$ has one sign (which depends on $t$ and $p$ ) and hence is monotone increasing or decreasing. Therefore

$$
\max _{-1 \leq c \leq 1} f(t, c)=\max \{f(t, 1), f(t,-1)\}
$$

and the assertion reads $f(t, 1) \leq c_{1}$ and $f(t,-1) \leq c_{1}$ for all $0<t<\infty$. The case $c=+1$ is the crucial one because $t^{p} / p+1 / q-t$ vanishes for $t=1$. Hospital's rule yields $f(1,1)=0$. Since $f(0,1)=q$ and $\lim _{t \rightarrow \infty} f(t, 1)=p$, one deduces from continuity of $f(t, 1)$ in $t$ that

$$
\sup _{0<t<\infty} f(t, 1)=: c_{1}<\infty .
$$

The analysis for $c=-1$ is simpler and hence omitted.
3.3. Optimal Design Problem. Let $0<t_{1}<t_{2}$ and $0<\mu_{2}<\mu_{1}$ be positive real numbers with $t_{1} \mu_{1}=t_{2} \mu_{2}$ and consider a convex $C^{1}$ function $\psi:[0, \infty) \rightarrow \mathbb{R}$ with $\psi(0)=0$ and

$$
\psi^{\prime}(t):= \begin{cases}\mu_{1} t & \text { for } 0 \leq t \leq t_{1} \\ t_{1} \mu_{1}=t_{2} \mu_{2} & \text { for } t_{1} \leq t \leq t_{2}, \\ \mu_{2} t & \text { for } t_{2} \leq t .\end{cases}
$$

The energy density $W(A):=\psi(|A|), A \in \mathbb{R}^{n}$, results from a relaxation process [14]. It satisfies (H1)-(H2) with $p=r=2$ and $s=0$. Details can be found in [2].
3.4. Scalar 2-Well Problem. The scalar convexified 2-well energy density $W$ results from a relaxation in nonconvex minimization problems allowing for microstructures [11]. It satisfies (H1)-(H2) with $p=4$ and $r=2=s$.

Proposition 3.2. Given distinct $F_{1}$ and $F_{2}$ in $\mathbb{R}^{n}$ set $A:=\left(F_{2}-\right.$ $\left.F_{1}\right) / 2 \neq 0$ and $B:=\left(F_{1}+F_{2}\right) / 2$ where $(\cdot)_{+}:=\max \{0, \cdot\}$ and $(\cdot)_{+}^{2}:=$ $\max \{0, \cdot\}^{2}$. For any $F \in \mathbb{R}^{n}$ let

$$
W(F):=\left(|F-B|^{2}-|A|^{2}\right)_{+}^{2}+4\left(|A|^{2}|F-B|^{2}-(A \cdot(F-B))^{2}\right) .
$$

Then for any $F, G \in \mathbb{R}^{n}$ with $\xi:=\left(|F-B|^{2}-|A|^{2}\right)_{+}$and $\eta:=(\mid G-$ $\left.\left.B\right|^{2}-|A|^{2}\right)_{+}$there holds

$$
\begin{aligned}
\mid D W(G) & -\left.D W(F)\right|^{2} \\
& \leq 32\left(|A|^{2}+\xi+\eta\right)(W(G)-W(F)-D W(F) \cdot(G-F))
\end{aligned}
$$

The proof of Proposition 3.2 is based on two lemmas.
Lemma 3.3. Given $A, B \in \mathbb{R}^{n}$ let $W(F):=\left(|F-B|^{2}-|A|^{2}\right)^{2}$. For any $F$ and $G$ in $\mathbb{R}^{n}$ let

$$
\xi:=\left(|F-B|^{2}-|A|^{2}\right)_{+} \quad \text { and } \quad \eta:=\left(|G-B|^{2}-|A|^{2}\right)_{+} .
$$

Then there holds

$$
\begin{aligned}
\mid D W(F) & -\left.D W(G)\right|^{2} \\
& \leq 32\left(|A|^{2}+\xi+\eta\right)(W(G)-W(F)-D W(F) \cdot(G-F))
\end{aligned}
$$

Proof. Let $U:=F-B, V:=G-B, a:=|A|$ and notice that $D W(F)=$ $4 \xi U$ and $D W(G)=4 \eta V$. In the first case suppose that both, $\xi=$ $|U|^{2}-a^{2}$ and $\eta=|V|^{2}-a^{2}$, are positive. Utilizing

$$
D W(F)-D W(G)=4(\xi U-\eta V)=4 \xi(U-V)+4(\xi-\eta) V
$$

one obtains

$$
1 / 32|D W(F)-D W(G)|^{2} \leq \xi^{2}|U-V|^{2}+(\xi-\eta)^{2}|V|^{2} .
$$

Since $|V|^{2}=\eta+a^{2}$ this proves

$$
\text { (3.1) } 1 / 32|D W(F)-D W(G)|^{2} \leq\left(a^{2}+\xi+\eta\right)\left(\xi|U-V|^{2}+(\xi-\eta)^{2}\right)
$$

On the other hand, the preceeding situation allows the direct calculation of

$$
\begin{aligned}
& W(G)-W(F)-D W(F) \cdot(F-G) \\
& =\eta^{2}-\xi^{2}+4 \xi U \cdot(U-V) \\
& =\eta^{2}-\xi^{2}+2 \xi\left(|U|^{2}-|V|^{2}\right)+2 \xi|U-V|^{2} \\
& =2 \xi|U-V|^{2}+(\xi-\eta)^{2} .
\end{aligned}
$$

The combination with (3.1) shows the assertion in the present first case of positive $\xi$ and $\eta$. For $\xi=0<\eta=|V|^{2}-a^{2}$ the assertion reads

$$
16 \eta^{2}|V|^{2} \leq 32\left(a^{2}+\eta\right) \eta^{2}
$$

which follows immediately from $|V|^{2} \leq\left(a^{2}+\eta\right)$. In the remaining case $\eta=a<\xi=|U|^{2}-a^{2}$, whence $|V| \leq a<|U|$, the assertion reads

$$
16 \xi^{2}|U|^{2} \leq 32\left(a^{2}+\xi\right)\left(4 \xi U \cdot(U-V)-\xi^{2}\right)
$$

This is equivalent to

$$
\xi^{2}|U|^{2} \leq 2\left(a^{2}+\xi\right)\left(\xi^{2}+2 \xi\left(a^{2}-|V|^{2}\right)+2 \xi|U-V|^{2}\right)
$$

and hence follows from $|U|^{2}=a^{2}+\xi$ and $0 \leq a^{2}-|V|^{2}$.
Lemma 3.4. Let $S$ be a symmetric and positive semidefinite real $n \times n$ matrix with spectral radius $\varrho(S)$ and pseudo inverse $S^{+}$and induced seminorm $|\cdot|_{S^{+}}$, i.e.,

$$
|F|_{S^{+}}:=\left(F \cdot S^{+} F\right)^{1 / 2} \quad \text { for all } F \in \mathbb{R}^{n} .
$$

Then the function $W: \mathbb{R}^{n} \rightarrow \mathbb{R}$ defined by

$$
W(F):=1 / 2 F \cdot S F \quad \text { for } F \in \mathbb{R}^{n}
$$

satisfies

$$
\begin{aligned}
\varrho(S)^{-1}|D W(F)-D W(G)|^{2} & \leq|D W(F)-D W(G)|_{S^{+}}^{2} \\
& =(F-G) \cdot S(F-G) \\
& =2(W(G)-W(F)-(S F) \cdot(G-F)) .
\end{aligned}
$$

Proof. Since $S$ is symmetric, $S=S S^{+} S$, and so $D W(F)=S F$ satisfies

$$
|S(F-G)|^{2} \leq \varrho(S)\left|S^{1 / 2}(F-G)\right|^{2}=\varrho(S)|S(F-G)|_{S^{+}}^{2}
$$

The remaining identity results from

$$
1 / 2(F-G) \cdot S(F-G)=W(G)-W(F)+F \cdot S(F-G)
$$

Proof of Proposition 3.2. Notice that $W(F)$ is the sum of the two energy densities of the aforegoing lemmas. Indeed, let $A^{0}:=A /|A|$ and define the symmetric and positive semidefinite matrix $S:=1-A^{0} \otimes A^{0}$. Then

$$
4\left(|A|^{2}|F-B|^{2}-(A \cdot(F-B))^{2}\right)=4|A|^{2}|F-B|_{S}^{2} .
$$

Observe the upper bound of $S$

$$
|D W(G)-D W(F)|^{2} \leq 32|\xi U-\eta V|^{2}+32|A|^{4}|U-V|_{S}^{2}
$$

is estimated in Lemma 3.3 and Lemma 3.4, respectively. This concludes the proof.
3.5. Vectorial 2-Well Problem. Given two distinct wells $E_{1}$ and $E_{2}$ in $\mathbb{R}_{\mathrm{sym}}^{n \times n}$ with minimal energies $W_{1}^{0}$ and $W_{2}^{0}$ in $\mathbb{R}$, we consider the quadratic elastic energies

$$
W_{j}(E):=1 / 2\left(E-E_{j}\right): \mathbb{C}\left(E-E_{j}\right)+W_{j}^{0} \quad \text { for all } E \in \mathbb{R}_{\mathrm{sym}}^{n \times n} .
$$

Energy minimization leads to an optimal choice of the configuration of the two phases, and so the strain energy density $\tilde{W}$ is modelled by the minimum

$$
\tilde{W}(E)=\min \left\{W_{1}(E), W_{2}(E)\right\} \quad \text { for all } E \in \mathbb{R}_{\mathrm{sym}}^{n \times n} .
$$

The two wells (transformation strains) are said to be compatible if

$$
\begin{equation*}
E_{1}=E_{2}+1 / 2(a \otimes b+b \otimes a) \quad \text { for some } a, b \in \mathbb{R}^{n} \tag{3.2}
\end{equation*}
$$

Then the constant $\gamma=1 / 2\left|E_{2}-E_{1}\right|_{\mathbb{C}}^{2}$ and the quasiconvexification $W$ of $\tilde{W}=\left\{W_{1}, W_{2}\right\}[14]$ is given by

$$
W(E)=\left\{\begin{array}{lc}
W_{2}(E) & \text { if } W_{2}(E)+\gamma \leq W_{1}(E) \\
\frac{1}{2}\left(W_{2}(E)+W_{1}(E)\right)-\frac{1}{4 \gamma}\left(W_{2}(E)-W_{1}(E)\right)^{2}-\frac{\gamma}{4} \\
& \text { if }\left|W_{2}(E)-W_{1}(E)\right| \leq \gamma, \\
W_{1}(E) & \text { if } W_{1}(E)+\gamma \leq W_{2}(E)
\end{array}\right.
$$

The convex $W$ satisfies (H1)-(H2) with $p=2=r$ and $s=0$.
Proposition 3.5. In the compatible case (3.2) there holds, for all $A, B \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$,

$$
1 / 2|D W(A)-D W(B)|_{\mathbb{C}^{-1}}^{2} \leq W(B)-W(A)-D W(A):(B-A)
$$

Proof. A translation of the argument in $W$ allows us to assume, without loss of generality, that $E_{1}+E_{2}=0$. For $E \in \mathbb{R}_{\text {sym }}^{n \times n}$, let

$$
\begin{aligned}
\varphi(E) & :=\gamma^{-1}\left(W_{2}(E)-W_{1}(E)\right) \\
\psi(E) & :=\max \{-1, \min \{1, \varphi(E)\}\} .
\end{aligned}
$$

As in [12] one deduces, for $E \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$ and $\gamma \varphi(E)=2\left(\mathbb{C} E_{1}\right): E+W_{2}^{0}-$ $W_{1}^{0}$,

$$
D W(E)=\mathbb{C} E-\psi(E) \mathbb{C} E_{1}
$$

and observes that $\psi(E)=\varphi(E)$ for $E \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$ with $-1 \leq \varphi(E) \leq 1$. The proof of the proposition starts with the discussion of

$$
\begin{equation*}
\gamma / 2(\psi(B)-\psi(A))(\psi(A)-\varphi(A)) \geq 0 \tag{3.3}
\end{equation*}
$$

In fact, $\psi(A) \neq \varphi(A)$ implies either $\psi(A)=1<\varphi(A)$ [notice $\psi(B)-$ $1 \leq 0]$ or $\psi(A)=-1>\varphi(A)$ [notice $\psi(B)+1 \geq 0]$ and in each case (3.3) follows. Algebraic manipulations will show in the sequel that (3.3) is equivalent to the assertion. Abbreviate $\sigma:=D W(A)$ and $\tau:=D W(B)$ to compute the left-hand side of the assertion, namely

$$
1 / 2|\sigma-\tau|_{\mathbb{C}^{-1}}^{2}=1 / 2(\tau-\sigma): \mathbb{C}^{-1}(\tau+\sigma)+(\sigma-\tau): \mathbb{C}^{-1} \sigma
$$

With $\mathbb{C}^{-1}(\sigma-\tau)=A-B-\psi(A) E_{1}+\psi(B) E_{1}$, this reads

$$
\begin{aligned}
\sigma:(A-B)-1 / 2 \mid \sigma & -\left.\tau\right|_{\mathbb{C}^{-1}} ^{2} \\
& =(\psi(A)-\psi(B)) E_{1}: \sigma-1 / 2|\tau|_{C^{-1}}^{2}+1 / 2|\sigma|_{\mathbb{C}^{-1}}^{2} .
\end{aligned}
$$

The definition of $\sigma$ and $\tau$ and $\gamma / 2=\left|E_{1}\right|_{\mathbb{C}}^{2}$ show

$$
\begin{aligned}
& 1 / 2|\sigma|_{\mathbb{C}^{-1}}^{2}-1 / 2|\tau|_{\mathbb{C}^{-1}}^{2}=1 / 2|A|_{\mathbb{C}}^{2}-1 / 2|B|_{\mathbb{C}}^{2}+\gamma / 4\left(\psi(A)^{2}-\psi(B)^{2}\right) \\
&-\psi(A) A: \mathbb{C} E_{1}+\psi(B) B: \mathbb{C} E_{1}
\end{aligned}
$$

It is a lengthy but direct verification that $W(E), E \in \mathbb{R}_{\text {sym }}^{n \times n}$, can be written as

$$
W(E)=1 / 2 E: \mathbb{C} E+1 / 2\left(W_{1}^{0}+W_{2}^{0}\right)+\gamma / 4 \psi(E)(\psi(E)-2 \varphi(E))
$$

The combination of the preceeding three identities [the last applied to $E=A$ and $E=B]$ shows

$$
\begin{aligned}
& W(B)-W(A)+\sigma:(A-B)-1 / 2|\sigma-\tau|_{\mathbb{C}^{-1}}^{2} \\
= & (\psi(A)-\psi(B))\left(E_{1}: \mathbb{C} A-\psi(A) \gamma / 2\right) \\
& -\psi(A) A: \mathbb{C} E_{1}+\psi(B) B: \mathbb{C} E_{1} \\
& +\gamma / 2 \varphi(A) \psi(A)-\gamma / 2 \varphi(B) \psi(B) \\
= & -\gamma / 2 \psi(A)^{2}+\gamma / 2 \psi(A) \psi(B)-\psi(B) E_{1}: \mathbb{C}(A-B) \\
& +\gamma / 2 \varphi(A) \psi(A)-\gamma / 2 \varphi(B) \psi(B) .
\end{aligned}
$$

Since $E_{1}: \mathbb{C}(A-B)=\gamma / 2(\varphi(A)-\varphi(B))$ shows that the preceeding expression equals the left-hand side of (3.3).

Remark 3.1. The immediate corollary (H3) of Proposition 3.5 is known from $[10,12]$ and fundamental for error analysis and regularity.
3.6. Hencky elastoplasticity with hardening. One time step within an elastoplastic evolution problem leads to Hencky's model. For various hardening laws and von-Mises yield conditions, an elimination of internal variables [1] leads to the energy function

$$
\begin{equation*}
W(E):=\frac{1}{2} E: \mathbb{C} E-\frac{1}{4 \mu} \max \left\{0,|\operatorname{dev} \mathbb{C} E|-\sigma_{y}\right\}^{2} /(1+\eta) \tag{3.4}
\end{equation*}
$$

for $E \in \mathbb{R}_{\text {sym }}^{n \times n}$. Here we adopt notation of the previous section and $\mathbb{C}$ is the fourth-order elasticity tensor, $\sigma_{y}>0$ is the yield stress, and $\eta>0$ is the modulus of hardening. The model of perfect plasticity corresponds to $\eta=0$ [21]. For $\eta>0$ there holds (H1)-(H2) for $p=2=r$ and $s=0$.

Proposition 3.6. For all $A, B \in \mathbb{R}_{\text {sym }}^{n \times n}$ there holds

$$
{ }^{1} / 2|D W(A)-D W(B)|_{\mathbb{C}^{-1}}^{2} \leq W(B)-W(A)-D W(A):(B-A)
$$

Proof. Set $\psi(x):=1-\max \left\{0,1-\sigma_{y} /(2 \mu x)\right\} /(1+\eta)$ to define the continuous and monotone decreasing function $\psi:[0, \infty) \rightarrow(\eta /(1+$ $\eta), 1]$ which satisfies
$D W(E)=(\lambda+2 \mu / n) \operatorname{tr}(E) \mathbf{1}+2 \mu \psi(|\operatorname{dev} E|) \operatorname{dev} E \quad$ for all $E \in \mathbb{R}_{\text {sym }}^{n \times n}$.
Given $A, B \in \mathbb{R}_{\text {sym }}^{n \times n}$, the following abbreviations will be used throughout the remaining part of the proof:

$$
\begin{array}{rlrl}
\sigma:=D W(A), & & a:=|\operatorname{dev} A|, & \\
\tau:=\psi(a), \\
\tau W(B), & & b:=|\operatorname{dev} B|, & \beta:=\psi(b) .
\end{array}
$$

Then the assertion reads

$$
\delta:=W(B)-W(A)+\sigma:(A-B)-1 / 2|\sigma-\tau|_{\mathbb{C}^{-1}}^{2} \geq 0
$$

In the first three steps one computes $\delta$. The aforementioned formulae for $D W(A)$ and $D W(B)$ and elementary calculations with the third formula of Binomi yield in step one that

$$
\begin{aligned}
& \sigma: \mathbb{C}^{-1}(\sigma-\tau)-1 / 2|\sigma-\tau|_{\mathbb{C}^{-1}}^{2} \\
& =1 / 2|\sigma|_{\mathbb{C}^{-1}}^{2}-1 / 2|\tau|_{\mathbb{C}^{-1}}^{2} \\
& =(\lambda / 2+\mu / n)\left(\operatorname{tr}(A)^{2}-\operatorname{tr}(B)^{2}\right)+\mu\left(\alpha^{2} a^{2}-\beta^{2} b^{2}\right)
\end{aligned}
$$

Step two employs the definition of $\psi$ to rewrite the energy as

$$
W(E)=1 / 2|E|_{\mathbb{C}}^{2}-(1+\eta) \mu(1-\psi(|\operatorname{dev} E|))^{2}|\operatorname{dev} E|^{2}
$$

for all $E \in \mathbb{R}_{\text {sym }}^{n \times n}$. Step three employs the above formulae for $\sigma$ and $\tau$ to estimate
$\sigma:(A-B)-\sigma: \mathbb{C}^{-1}(\sigma-\tau)=2 \mu \alpha \operatorname{dev} A:((1-\alpha) \operatorname{dev} A-(1-\beta) \operatorname{dev} B)$.
The Cauchy inequality, leads to

$$
\sigma:(A-B)-\sigma: \mathbb{C}^{-1}(\sigma-\tau) \geq 2 \mu \alpha(1-\alpha) a^{2}-2 \mu \alpha(1-\beta) a b
$$

The left-hand sides considered in the first three steps add up to $\delta$ and so lead to a lower bound of $\delta$. Elementary manipulations with this
lower bound in step four of the proof yield the estimate

$$
\begin{aligned}
\delta / \mu \geq & \alpha^{2} a^{2}-\beta^{2} b^{2}+b^{2}-a^{2}+(1+\eta)(1-\alpha)^{2} a^{2}-(1+\eta)(1-\beta)^{2} b^{2} \\
& +2 \alpha(1-\alpha) a^{2}-2 \alpha(1-\beta) a b \\
= & \eta(1-\alpha)^{2} a^{2}-\eta(1-\beta)^{2} b^{2}+2(1-\beta) b(\beta b-\alpha a) \\
= & \eta((1-\alpha) a-(1-\beta) b)^{2} \\
& +2(1-\beta) b((1+\eta)(\beta b-\alpha a)-\eta(b-a)) .
\end{aligned}
$$

Step five concerns the function $g(x):=x \psi(x)$ which satisfies $g^{\prime}(x)=1$ and $g^{\prime}(x)=\eta /(1+\eta)$ for $2 \mu x<\sigma_{y}$ and $\sigma_{y}<2 \mu x$, respectively. For $a \leq b$, this and the fundamental theorem of calculus show

$$
\begin{equation*}
\eta(b-a) \leq(1+\eta) \int_{a}^{b} g^{\prime}(x) d x=(1+\eta)(\beta b-\alpha a) \tag{3.5}
\end{equation*}
$$

This concludes the proof of $\delta \geq 0$ in this case. In the case $b<a$, the above lower bound of $\delta$ shows $\delta \geq 0$ if $\beta=1$. Hence it remains to consider $b<a$ and $\beta<1$ which implies $\sigma_{y}<2 \mu b$ and so $g^{\prime}(x)=$ $\eta /(1+\eta)$ for all $b<x<a$. This yields equality in (3.5) and so proves $\delta \geq 0$.

Remark 3.2. Although (H2) holds for $\eta=0$ as well, the linear growth condition yields a different functional analytical setting in $B D(\Omega)$ [21].

## 4. Proof of Convergence

This section provides a proof of Theorem 2.1 on the convergence of the stress fields in $L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)$. Throughout this section, the focus is on the energy difference

$$
\delta_{\ell}:=\mathcal{J}\left(u_{\ell}\right)-\mathcal{J}(u) \geq 0 .
$$

Due to (2.1), the sequence $\left(\delta_{\ell}\right)_{\ell}$ is monotone decreasing, and hence convergent to some limit $\delta \geq 0$. It is essential to prove $\delta=0$, which is not known in the beginning of the proof.

Lemma 4.1. There holds

$$
\left\|\sigma_{\ell+1}-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{r} \lesssim \delta_{\ell}-\delta_{\ell+1} .
$$

Proof. The two-sided growth conditions in (H1) lead in [11] to the boundedness of discrete minimizers in $W^{1, p}$ and show

$$
\begin{equation*}
\int_{\Omega}\left(1+\left|D u_{\ell}\right|^{s}+\left|D u_{\ell+1}\right|^{s}\right)^{p / s} d x \lesssim 1 \tag{4.1}
\end{equation*}
$$

Since $\sigma_{\ell+1}$ satisfies the discrete Euler-Lagrange equations, there holds

$$
\int_{\Omega} \sigma_{\ell+1}: D\left(u_{\ell}-u_{\ell+1}\right) d x=\int_{\Omega} f \cdot\left(u_{\ell}-u_{\ell+1}\right) d x
$$

Therefore,

$$
\begin{aligned}
\delta_{\ell}-\delta_{\ell+1} & =\int_{\Omega}\left(W\left(D u_{\ell}\right)-W\left(D u_{\ell+1}\right)-f \cdot\left(u_{\ell}-u_{\ell+1}\right)\right) d x \\
& =\int_{\Omega}\left(W\left(D u_{\ell}\right)-W\left(D u_{\ell+1}\right)-\sigma_{\ell+1}: D\left(u_{\ell}-u_{\ell+1}\right)\right) d x
\end{aligned}
$$

An application of (H2) with $A=D u_{\ell+1}(x)$ and $B=D u_{\ell}(x)$ leads to an estimate for all $x$ in $\Omega$. The integral of those inequalities reads

$$
\begin{align*}
& \int_{\Omega}\left(1+\left|D u_{\ell}\right|^{s}+\left|D u_{\ell+1}\right|^{s}\right)^{-1}\left|\sigma_{\ell}-\sigma_{\ell+1}\right|^{r} d x \\
& \quad \lesssim \int_{\Omega}\left(W\left(D u_{\ell}\right)-W\left(D u_{\ell+1}\right)-\sigma_{\ell+1}: D\left(u_{\ell}-u_{\ell+1}\right)\right) d x  \tag{4.2}\\
& \quad=\delta_{\ell}-\delta_{\ell+1}
\end{align*}
$$

The Hölder inequality with $t$ and $t^{\prime}=1+p / s, 1 / t+1 / t^{\prime}=1$, plus (4.1) with $t^{\prime} / t=p / s$ lead to

$$
\begin{aligned}
&\left\|\sigma_{\ell+1}-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{r / t}=\int_{\Omega}\left(1+\left|D u_{\ell}\right|^{s}+\left|D u_{\ell+1}\right|^{s}\right)^{-1 / t}\left|\sigma_{\ell}-\sigma_{\ell+1}\right|^{r / t} \\
& \times\left(1+\left|D u_{\ell}\right|^{s}+\left|D u_{\ell+1}\right|^{s}\right)^{1 / t} d x \\
& \lesssim\left(\int_{\Omega}\left(1+\left|D u_{\ell}\right|^{s}+\left|D u_{\ell+1}\right|^{s}\right)^{-1}\left|\sigma_{\ell}-\sigma_{\ell+1}\right|^{r} d x\right)^{1 / t}
\end{aligned}
$$

The combination of this estimate with (4.2) proves the lemma.

Lemma 4.2. There holds (2.5), namely

$$
\left\|\sigma-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{r} \lesssim \eta_{\ell}+\operatorname{osc}_{\ell} .
$$

Proof. In slightly different notation, it is proven in [11] that

$$
\begin{equation*}
\left\|\sigma-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{r} \lesssim \eta_{\ell}+\left\|h_{\mathcal{T}_{\ell}} f\right\|_{L^{p^{\prime}}(\Omega)} . \tag{4.3}
\end{equation*}
$$

It is known since $[19,20]$ that the volume contribution $\left\|h_{\mathcal{T}_{\ell}} f\right\|_{L^{p^{\prime}}(\Omega)}$ can be controlled by $\eta_{\ell}+$ osc $_{\ell}$ and so (4.3) leads to the assertion; cf. [9] for one particular case. The main arguments are recalled here for convenient reading. A triangle inequality yields, for each free node $z$, that

$$
\begin{equation*}
\|f\|_{L^{p^{\prime}}\left(\omega_{z}\right)} \leq\left\|f-f_{\omega_{z}}\right\|_{L^{p^{\prime}}\left(\omega_{z}\right)}+\left|f_{\omega_{z}}\right|\left|\omega_{z}\right|^{1 / p^{\prime}} \tag{4.4}
\end{equation*}
$$

The integral mean equals

$$
\begin{equation*}
f_{\omega_{z}}\left|\omega_{z}\right| \approx \int_{\Omega} \varphi_{z} f_{\omega_{z}} d x=\int_{\Omega} \varphi_{z}\left(f-f_{\omega_{z}}\right) d x+\int_{\Omega} \varphi_{z} f d x \tag{4.5}
\end{equation*}
$$

The combination of (4.4)-(4.5) plus a Hölder inequality shows

$$
\begin{equation*}
\|f\|_{L^{p^{\prime}\left(\omega_{z}\right)}} \lesssim\left\|f-f_{\omega_{z}}\right\|_{L^{p^{\prime}\left(\omega_{z}\right)}}+\left|\omega_{z}\right|^{-1 / p}\left|\int_{\Omega} \varphi_{z} f d x\right| \tag{4.6}
\end{equation*}
$$

On the other hand, the discrete Euler-Lagrange equations show for the $j$-th component $f_{j}$ of $f$ and the components $\sigma_{\ell, j}:=\left(\sigma_{\ell, j_{1}}, \ldots, \sigma_{\ell, j_{n}}\right)$ of $\sigma_{\ell}$, that

$$
\begin{equation*}
\int_{\Omega} \varphi_{z} f_{j} d x=\int_{\Omega} \sigma_{\ell, j} \cdot \nabla \varphi_{z} d x=\sum_{E \in \mathcal{E}} \int_{E}\left(\left[\sigma_{\ell, j}\right] \cdot \nu_{E}\right) \varphi_{z} d s \tag{4.7}
\end{equation*}
$$

with an elementwise integration by parts. Let $\mathcal{E}(z):=\{E \in \mathcal{E}: z \in$ $E\}$ denote the set of sides which contribute in (4.7). Then for all $j=1,2, \ldots, m$ components in (4.7) it follows that

$$
\begin{equation*}
\left|\int_{\Omega_{z}} f \varphi_{z} d x\right| \leq\left(\sum_{E \in \mathcal{E}(z)} \eta_{E}^{p^{\prime}}\right)^{1 / p^{\prime}}\left(\sum_{E \in \mathcal{E}(z)} h_{E}^{-p / p^{\prime}}\left\|\varphi_{z}\right\|_{L^{p}(E)}^{p}\right)^{1 / p} \tag{4.8}
\end{equation*}
$$

Since the last factor in (4.8) is proportional to $h_{z}^{n / p-1}$ for $h_{z}=\operatorname{diam}\left(\omega_{z}\right)$, (4.7)-(4.8) yield

$$
\begin{equation*}
\left|\omega_{z}\right|^{-p^{\prime} / p}\left|\int_{\Omega} f \varphi_{z} d x\right|^{p^{\prime}} \lesssim h_{z}^{-p^{\prime}} \sum_{E \in \mathcal{E}(z)} \eta_{E}^{p^{\prime}} \tag{4.9}
\end{equation*}
$$

Since $\mathcal{E}(z)$, for free nodes $z \in \mathcal{K}$, have a finite overlap, the combination of (4.6) and (4.9) shows

$$
\left\|h_{\mathcal{T}_{\ell}} f\right\|_{L^{p^{\prime}(\Omega)}}^{p^{\prime}} \approx \sum_{z \in \mathcal{K}} h_{z}^{p^{\prime}}\|f\|_{L^{p^{\prime}}\left(\omega_{z}\right)}^{p^{\prime}} \lesssim \operatorname{osc}_{\ell}(f)^{p^{\prime}}+\eta_{\ell} .
$$

This and (4.3) proof the assertion.
Remark 4.1. The condition that each element has at least one vertex, which is a free node, leads to $\Omega=\bigcup_{z \in \mathcal{K}} \omega_{z}$ in the proof of Lemma 4.2. This can be generalised by enlarging $\omega_{z}$ to $\Omega_{z}$ by some elements near the boundary. We refer to [5, 4, 7, 8] for details.

Lemma 4.3. For any $E \in \mathcal{M}_{\ell}$ with $E=\partial T_{+} \cup \partial T_{-}$for $T_{+}, T_{-} \in \mathcal{T}_{\ell}$ and $\omega_{E}=\operatorname{int}\left(T_{+} \cup T_{-}\right)$there holds

$$
\eta_{E} \lesssim\left\|\sigma_{\ell+1}-\sigma_{\ell}\right\|_{L^{p^{\prime}}\left(\omega_{E} ; \mathbb{R}^{m \times n}\right)}+\left\|f-f_{\omega_{E}}\right\|_{L^{p^{\prime}}\left(\omega_{E} ; \mathbb{R}^{m}\right)}
$$

Proof. REFINE allows for nodal basis functions $\varphi_{E}$ of a new node $\operatorname{mid}(E)$ in $E$ and $\psi_{E}$ of a new node $\operatorname{mid}\left(\omega_{E}\right)$ in either $T_{+}$or $T_{-}$, with respect to the finer triangulation $\mathcal{T}_{\ell+1}$ and $E, T_{+}, T_{-}$from $\mathcal{T}_{\ell}$. Then, there exists some linear combination

$$
V_{E}:=\alpha \varphi_{E}+\beta \psi_{E} \in V_{\ell+1} \cap W_{0}^{1, p}\left(\omega_{E} ; \mathbb{R}^{m}\right)
$$

with the following conditions

$$
\int_{E} v_{E} d s=|E|, \int_{\omega_{E}} v_{E} d x=0,\left\|v_{E}\right\|_{V} \approx h_{E}^{-1}\left|\omega_{E}\right|^{1 / p}
$$

The construction of such $V_{E}$ is the same as in linear problems [3, 13, $17,18,22$ ] and hence the remaining details are neglected and the subsequent outline is kept brief. Since $J_{E}$ is constant along $E$

$$
|E| J_{E}=\int_{E}\left(\left[\sigma_{\ell}\right] \nu_{E}\right) \cdot v_{E} d s=\int_{\omega_{E}} \sigma_{\ell}: D v_{E} d x
$$

Since $v_{E} \in V_{\ell+1}$ and $\sigma_{\ell+1}$ satisfy the discrete Euler-Lagrange equations,

$$
\int_{\omega_{E}} \sigma_{\ell}: D v_{E} d x=\int_{\omega_{E}}\left(\sigma_{\ell}-\sigma_{\ell+1}\right): D v_{E} d x+\int_{\omega_{E}}\left(f-f_{\omega_{E}}\right) \cdot v_{E} d x
$$

with the constant integral mean $f_{\omega_{E}}$ of $f$ over $\omega_{E}$. The combination of the above identity with Friedrichs inequality $\left\|v_{E}\right\|_{L^{p}\left(\omega_{E} ; \mathbb{R}^{m}\right)} \lesssim h_{E}\left\|v_{E}\right\|_{V}$ proves

$$
\begin{aligned}
\eta_{E}=h_{E}^{1 / p^{\prime}}|E|^{1 / p^{\prime}}\left|J_{E}\right| \lesssim h_{E}^{1 / p^{\prime}}|E|^{1 / p} & \left(\left\|\sigma_{\ell}-\sigma_{\ell+1}\right\|_{L^{p^{\prime}}\left(\omega_{E} ; \mathbb{R}^{m \times n}\right)}\right. \\
& \left.+h_{\omega_{E}}\left\|f-f_{\omega_{E}}\right\|_{L^{p^{\prime}}\left(\omega_{E} ; \mathbb{R}^{m}\right)}\right)\left\|v_{E}\right\|_{V}
\end{aligned}
$$

Proof of Theorem 2.1. Notice that the patches have a finite overlap and

$$
\sum_{E \in \mathcal{E}_{\ell}} h_{E}^{p^{\prime}}\left\|f-f_{\omega_{E}}\right\|_{L^{p^{\prime}}\left(\omega_{E} ; \mathbb{R}^{m}\right)} \lesssim \operatorname{osc}_{\ell}^{p^{p^{\prime}}}
$$

Hence Lemma 4.3 leads to

$$
\sum_{E \in \mathcal{M}} \eta_{E}^{p^{\prime}} \lesssim\left\|\sigma_{\ell+1}-\sigma_{\ell}\right\|_{L^{p^{\prime}}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{p^{\prime}}+\operatorname{osc}_{\ell}^{p^{\prime}} .
$$

This, (2.8) in MARK and Lemma 4.2 show

$$
\begin{align*}
\left\|\sigma-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{p^{\prime}} & \lesssim \eta_{\ell}^{p^{\prime}}+\operatorname{osc}_{\ell}^{p^{\prime}} \\
& \lesssim \sum_{E \in \mathcal{M}_{\ell}} \eta_{E}^{p^{\prime}}+\operatorname{osc}_{\ell}^{p^{\prime}}  \tag{4.10}\\
& \lesssim\left\|\sigma_{\ell+1}-\sigma_{\ell}\right\|_{L^{p^{\prime}}\left(\Omega ; \mathbb{R}^{m \times n}\right)}^{p^{\prime}}+\operatorname{osc}_{\ell}^{p^{\prime}}
\end{align*}
$$

Since $\left(\delta_{\ell}\right) \rightarrow \delta$, the right-hand side in Lemma 4.1 converges to zero, i.e.,

$$
\lim _{\ell \rightarrow \infty}\left\|\sigma_{\ell+1}-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}=0 .
$$

Since $p^{\prime} \leq r / t$ and $|\Omega| \lesssim 1$, the right-hand side in (4.10) tends to zero as $\ell \rightarrow \infty$. This proves the claimed strong convergence

$$
\lim _{\ell \rightarrow \infty}\left\|\sigma-\sigma_{\ell}\right\|_{L^{r / t}\left(\Omega ; \mathbb{R}^{m \times n}\right)}=0
$$

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Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

E-mail address: cc@math.hu-berlin.de


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