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CONVERGENCE OF APPROXIMATING PATHS TO SOLUTIONS OF VARIATIONAL INEQUALITIES INVOLVING NON-LIPSCHITZIAN MAPPINGS

Jong Soo Jung† and Daya Ram Sahu ‡

Abstract. Let X be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of $X, T : C \to X$ a continuous pseudocontractive mapping, and $A : C \to C$ a continuous strongly pseudocontractive mapping. We show the existence of a path $\{x_t\}$ satisfying $x_t = tAx_t + (1-t)Tx_t,\ t \in (0,1)$ and prove that $\{x_t\}$ converges strongly to a fixed point of T, which solves the variational inequality involving the mapping A. As an application, we give strong convergence of the path $\{x_t\}$ defined by $x_t = tAx_t + (1-t)(2I-T)x_t$ to a fixed point of firmly pseudocontractive mapping T.

1. Introduction

Let X be a real Banach space with dual X^* and T be a mapping with domain $D(T)$ and range $R(T)$ in X. Following Morales [12], the mapping T is called *strongly pseudocontractive* if for some constant $k < 1$ and for all $x, y \in D(T)$,

$$(\lambda - k)||x - y|| \leq ||(\lambda I - T)(x) - (\lambda I - T)(y)|| \tag{1}$$

for all $\lambda > k$; while T is called a *pseudocontraction* if (1) holds for $k = 1$. The mapping T is called *Lipschitzian* if there exists $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y|| \text{ for all } x, y \in D(T).$$

Otherwise, the mapping is called non-Lipschitzian. The Lipschitzian mapping T is called nonexpansive if $L = 1$ and is called a contraction if $L < 1$. Every nonexpansive mapping is a pseudocontractive. The converse is not true. The example, $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}, x \in [0,1]$ is a continuous pseudocontraction which is

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not nonexpansive. Indeed,

$$\left|T\left(\frac{1}{4^3}\right) - T\left(\frac{1}{2^3}\right)\right| = \left|\left(\frac{15}{16}\right)^{\frac{3}{2}} - \left(\frac{3}{4}\right)^{\frac{3}{2}}\right| = \frac{|(15)^{\frac{3}{2}} - (12)^{\frac{3}{2}}|}{64}$$
$$> \frac{7}{64} = \left|\frac{1}{4^3} - \frac{1}{2^3}\right|.$$

A mapping T with domain $D(T)$ and range $R(T)$ in X is called *firmly pseudocontractive* if for all $x, y \in D(T)$,

$$\|x - y\| \leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|$$

for all $\lambda > 0$. Following Kato [10], we are able to find an equivalent definition for firmly pseudocontractive operators. An operator $T : D(T) \to R(T)$ is firmly pseudocontractive if and only if for every $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \|x-y\|^2,$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping which is defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2, \|j\| = \|u\|\}$$

(see Browder [2] and Kato [10]). It is an immediate consequence of the Hahn-Banach theorem that $J(u)$ is nonempty for each $u \in X$.

The firmly pseudocontractive mappings are characterized by the fact that a mapping T is firmly pseudocontractive if and only if the mapping $f = T - I$ is accretive (see Lemma 5).

The concept of firmly pseudocontractive mapping was introduced by Sharma and Sahu [20]. The mapping $T : D(T) \to R(T)$ is firmly pseudocontractive if and only $2I - T$ is pseudocontractive (see Lemma 5).

In [15], Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping which is a unique solution of a variational inequality in a Hilbert space. He proved the following theorem:

Theorem M (Theorem 2.1, Moudafi [15]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ a contraction mapping. Let $\{x_n\}$ be the sequence defined by the scheme*

$$x_n = \frac{1}{1+\varepsilon_n} Tx_n + \frac{\varepsilon_n}{1+\varepsilon_n} fx_n,$$

where ε_n is a sequence $(0,1)$ with $\varepsilon_n \to 0$. Then $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (I - f)\tilde{x}, \tilde{x} - x\rangle \leq 0 \text{ for all } x \in F(T). \tag{2}$$

In other word, $\tilde{x}$ is the unique fixed point of $P_{F(T)}f$.

Recently, Xu [22] extended the viscosity approximation method proposed by Moudafi [15] for a nonexpansive mappings in a uniformly smooth Banach space.

Theorem X (Theorem 4.1, Xu [22]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, $f \in \Pi_C$ the set of all contractions on C and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then the path $\{x_t\}$ defined by*

$$x_t = tfx_t + (1-t)Tx_t, \quad t \in (0,1)$$

converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) = \lim_{t \to 0^+} x_t, \quad f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality:

$$\langle (I-f)Q(f), J(Q(f)-v)\rangle \leq 0, \quad f \in \Pi_C \text{ and } v \in F(T).$$

It is well known that for certain applications the Lipschitzian assumption of mapping becomes a rather strong condition. In view of this the following natural question arises:

Question. *Is it possible to replace contraction mapping f involving in variational inequality* (2) *by a non-Lipschitzian mapping A?*

Motivated and inspired by the above question, we will consider a more general situation. In this paper our purpose is to prove that in reflexive Banach space X, for pseudocontractive mapping T, the path $\{x_t\}$ defined by

$$x_t = tAx_t + (1-t)Tx_t$$

converges strongly to a fixed point of T, which solves the certain variational inequality involving non-Lipschitzian mapping A. Using our results, we derive strong convergence theorems for firmly pseudocontractive mappings. Our results generalize and improve the results of Jung and Kim [9], Morales [13], Morales and Jung [14], Moudafi [15], O'Hara, Pillay, and Xu [16], Reich [18], Schu [19], Sharma and Sahu [20], and Xu [21, 22].

2. Preliminaries and lemmas

Recall that a Banach space X is said to be *smooth* provided the limit

$$\lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $S = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable.* It is said to be *uniformly Gâteaux differentiable* if for each $y \in$ S, this limit is attained uniformly for $x \in S$. It is well known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm (see e.g., [3]).

When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Suppose that the duality mapping J is single valued. Then J is said to be *weakly sequentially continuous* if, for each $\{x_n\} \in X$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

A Banach space X is said to satisfy *Opial's condition* (see for example [17]) if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$ we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } y \in X.$$

It is well-known that, if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition.

Let X be a Banach space and let T be a mapping with domain $D(T)$ and range $R(T)$ in X. The mapping T is said to be *demiclosed* at a point $p \in D(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $z \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tz = p$. The mapping T is said to be *demicontinuous* if, whenever a sequence $\{x_n\}$ in C converges strongly to $x \in C$, then $\{Tx_n\}$ converges weakly to Tx. The set of fixed point of T will be denoted by $F(T)$.

Let C be a convex subset of X, D a nonempty subset of C, and P a retraction from C onto D, that is, $Px = x$ for each $x \in D$. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If the sunny retraction P is also nonexpansive, then D is said to be a *sunny nonexpansive retract* of C.

Let C be a nonempty closed convex subset of a Banach space X. For $x \in C$, let

$$I_C(x) = \{y \in X : y = x + \lambda(z - x), z \in C \text{ and } \lambda \geq 0\}.$$

$I_C(x)$ is called the *inward set* of $x \in C$ with respect to C (see, for example [5]). $I_C(x)$ is a convex set containing C. A mapping $T : C \to X$ is said to be satisfying the *inward condition* if $Tx \in I_C(x)$ for all $x \in C$, T is also said to be satisfying the *weakly inward condition* if for each $x \in C, Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). It is well-known (Lemma 18.1, Deimling [5]) that $T : C \to X$ is weakly inward if and only if $\lim_{\lambda\to 0^+} \lambda^{-1} d((1-\lambda)x + \lambda Tx, C) = 0$ for all $x \in C$, where d denotes the distance to C.

Recall that a Banach limit LIM is a bounded linear functional on l^∞ such that

$$\|LIM\| = 1, \liminf_{n\to\infty} \ t_n \leq LIM_n t_n \leq \limsup_{n\to\infty} \ t_n,$$

and $LIM_n \ t_n = LIM_n \ t_{n+1}$ for all $t_n \in l^\infty$.

In what follows, we shall make use of the following lemmas.

Lemma 1 (Corollary 5.1, Cioranescu [3]). *If X is a smooth Banach space, then any duality mapping on X is norm to weak* continuous.*

Lemma 2 (Lemma 13.1, Goebel and Reich [6]). *Let C be a convex subset of a smooth Banach space X, D a non-empty subset of C and P a retraction from C onto D. Then the following are equivalent:*

(a) *P is a sunny and nonexpansive;*

(b) $\langle x - Px, J(z - Px)\rangle \leq 0$ *for all* $x \in C, z \in D$;
(c) $\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$ *for all* $x, y \in C$.

Lemma 3 (Lemma 1, Ha and Jung [8]). *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $\{x_n\}$ a bounded sequence in X. Let LIM be a Banach limit and $y \in C$. Then*

$$LIM_n\|x_n - y\|^2 = \min_{z\in C} LIM_n\|x_n - z\|^2$$

if and only if

$$LIM_n\langle x - y, J(x_n - y)\rangle \leq 0 \ for\ all\ x \in C.$$

Lemma 4 (Theorem 10.3, Goebel and Kirk [7]). *Let X be a reflexive Banach space which satisfies Opial condition, C a nonempty closed convex subset of X and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping.*

Lemma 5 (Lemma 2.2, Sharma and Sahu [20]). *Let X be a Banach space and T a mapping with domain and range in X. Then following are equivalent:*

(a) *T is firmly pseudocontractive;*
(b) *$2I - T$ is pseudocontractive;*
(c) *$T - I$ is accretive.*

Lemma 6 (Corollary 1, Deimling [4]). *Let C be a nonempty closed subset of a Banach space X and $T : C \to X$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$ satisfying*

$$\lim_{\lambda\to 0^+} \lambda^{-1} d((1 - \lambda)x + \lambda Tx, C) = 0 \text{ for all } x \in C,$$

where d denotes the distance to C (equivalently, the weakly inward condition under additional assumption that C is convex). Then T has a unique fixed point.

Lemma 7. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let $A : C \to C$ be a continuous strongly pseudocontractive with constant $k \in [0, 1)$. Then variational inequality problem $VIP(I - A, C)$:*

$$to\ find\ u \in C\ such\ that\ \langle (I - A)u, J(u - x)\rangle \leq 0\ for\ all\ x \in C$$

has at most one solution.

Proof. Let x^* and y^* be two distinct solutions of $VIP(I - A, C)$. Then

$$\langle x^* - Ax^*, J(x^* - y^*)\rangle \leq 0 \quad \text{and} \quad \langle y^* - Ay^*, J(y^* - x^*)\rangle \leq 0.$$

Adding these inequalities, we get

$$\langle x^* - y^* - (Ax^* - Ay^*), J(x^* - y^*)\rangle \leq 0,$$

which implies that

$$\|x^* - y^*\|^2 \leq \langle Ax^* - Ay^*, J(x^* - y^*)\rangle \leq k\|x^* - y^*\|^2,$$

a contradiction. Therefore, $x^* = y^*$. □

3. Main results

Before proving main results we need the following propositions:

Proposition 1. *Let C be a nonempty closed convex subset of a normed space X. Let $A : C \to C$ be a mapping and $T : C \to X$ another mapping satisfying the weakly inward condition. Then for each $\lambda \in (0,1)$, the mapping $T_\lambda^A : C \to X$ defined by*

$$T_\lambda^A x = (1-\lambda)Ax + \lambda Tx, \quad x \in C$$

satisfies the weakly inward condition.

Proof. Let $x \in C$ and $\varepsilon > 0$. Since T is weakly inward, there exists $y \in I_C(x)$ such that $\|y - Tx\| \leq \varepsilon$, and since C is convex, there exists t_0 such that $z_t := (1-t)x + ty \in C$ for $0 < t \leq t_0$. For these t we have

$$d((1-t)x + tTx, C) \leq \|(1-t)x + tTx - z_t\| \leq t\varepsilon.$$

Moreover, since C is convex,

$$w_t = \frac{(1-t+\lambda t)x + (1-\lambda)tAx + \lambda z_t}{1+\lambda} \in C$$

for all $\lambda \in (0,1)$ whenever $t \in (0,1)$. Set $\alpha := \frac{t}{1+\lambda}$ and let $t \in (0,1)$. Then we have

$$\begin{aligned}
&d((1-\alpha)x + \alpha T_\lambda^A x, C)\\
&\leq \|(1-\alpha)x + \alpha T_\lambda^A x - w_t\|\\
&= \|(1+\lambda-t)x + tT_\lambda^A x - (1+\lambda)w_t\|/(1+\lambda)\\
&= \|(1+\lambda-t)x + t[(1-\lambda)Ax + \lambda Tx] - (1+\lambda)w_t\|/(1+\lambda)\\
&= \frac{\lambda}{1+\lambda}\|(1-t)x + tTx - z_t\| \leq \frac{t}{1+\lambda}\lambda\varepsilon,
\end{aligned}$$

and hence $\lim_{\alpha \to 0^+} \alpha^{-1} d((1-\alpha)x + \alpha T_\lambda^A x, C) = 0$. By (Lemma 18.1, Deimling [5]), T_λ^A satisfies the weakly inward condition. □

Proposition 2. *Let C be a nonempty closed convex subset of a Banach space X. Let $A : C \to C$ be a continuous strongly pseudocontractive with constant $k \in [0,1)$ and $T : C \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. Then*

(a) *for each $t \in (0,1)$, there exists unique solution $x_t \in C$ of equation*

$$x = tAx + (1-t)Tx, \tag{3}$$

(b) *Moreover, if v is a fixed point of T, then for each $t \in (0,1)$, there exists $j(x_t - v) \in J(x_t - v)$ such that*

$$\langle x_t - Ax_t, j(x_t - v)\rangle \leq 0,$$

(c) *$\{x_t\}$ is bounded.*

Proof. (a) For each $t \in (0,1)$, the mapping $T_t^A : C \to X$ defined by

$$T_t^A x = tAx + (1-t)Tx, \;\; x \in C$$

is continuous strongly pseudocontractive with constant $1 - t(1-k) \in (0,1)$. Indeed, for $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned}\langle T_t^A x - T_t^A y, j(x-y)\rangle &= t\langle Ax - Ay, j(x-y)\rangle \\ &\quad + (1-t)\langle Tx - Ty, j(x-y)\rangle \\ &\leq tk\|x-y\|^2 + (1-t)\|Tx - Ty\|\|x-y\| \\ &\leq (1 - t(1-k))\|x-y\|^2.\end{aligned}$$

From Proposition 1, T_t^A satisfies the weakly inward condition. Thus, by Lemma 6, there exists a unique fixed point $x_t \in C$ of T_t^A such that

$$x_t = tAx_t + (1-t)Tx_t. \tag{4}$$

(b) Suppose that v is a fixed point of T. Since T is pseudocontractive, for $j(x_t - v) \in J(x_t - v)$, we have

$$\begin{aligned}\langle x_t - Tx_t, j(x_t - v)\rangle &= \langle x_t - v + Tv - Tx_t, j(x_t - v)\rangle \\ &= \|x_t - v\|^2 - \langle Tx_t - Tv, j(x_t - v)\rangle \geq 0.\end{aligned}$$

Hence from (4) we have

$$\begin{aligned}\langle x_t - Ax_t, j(x_t - v)\rangle &= (1-t)\langle Tx_t - Ax_t, j(x_t - v)\rangle \\ &\leq (1-t)\langle Tx_t - x_t + x_t - Ax_t, j(x_t - v)\rangle,\end{aligned}$$

which implies that

$$\langle x_t - Ax_t, j(x_t - v)\rangle \leq 0.$$

(c) By strong pseudocontractivity of A, there exists $j(x_t - v) \in J(x_t - v)$ such that

$$\langle Ax_t - Av, j(x_t - v)\rangle \leq k\|x_t - v\|^2.$$

Using Proposition 2(b), we obtain

$$\begin{aligned}\|x_t - v\|^2 &= \langle x_t - v, j(x_t - v)\rangle \\ &= \langle x_t - Ax_t, j(x_t - v)\rangle + \langle Ax_t - Av, j(x_t - v)\rangle \\ &\quad + \langle Av - v, j(x_t - v)\rangle \\ &\leq k\|x_t - v\|^2 + \langle Av - v, j(x_t - v)\rangle.\end{aligned}$$

Thus,

$$\|x_t - v\|^2 \leq \frac{1}{1-k}\langle Av - v, j(x_t - v)\rangle, \tag{5}$$

which yields

$$\|x_t - v\| \leq \frac{1}{1-k}\|Av - v\|.$$

Therefore, $\{x_t\}$ is bounded. □

Theorem 1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$ and $T : C \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Suppose also that the set*

$$E = \{x \in C : Tx = \lambda x + (1 - \lambda)Ax \text{ for some } \lambda > 1\}$$

is bounded. For $t \in (0, 1)$, let $\{x_t\}$ be the path defined by (4). Then we have the following:

(a) $\lim_{t\to 0^+} x_t = \tilde{x}$ *exists,*

(b) $\tilde{x}$ *is a fixed point of T and it is the unique solution of the variational inequality:*

$$\langle (I - A)\tilde{x}, J(\tilde{x} - v)\rangle \leq 0 \text{ for all } v \in F(T).$$

Proof. (a) It follows from Theorem 6 of [11] that the mapping $2I - T$ has a nonexpansive inverse, denoted by g, which maps C into itself with $F(T) = F(g)$. By Proposition 2(c), $\{x_t\}$ is bounded and hence, the sets $\{Tx_t : t \in (0, 1)\}$ and $\{Ax_t : t \in (0, 1)\}$ are also bounded. By (4), we have

$$\|x_t - Tx_t\| = t\|Ax_t - Tx_t\| \to 0 \text{ as } t \to 0^+,$$

which implies that

$$x_t - gx_t \to 0 \text{ as } t \to 0^+. \tag{6}$$

Since X is reflexive, there exists a weakly convergent subsequence $\{x_{t_n}\} \subseteq \{x_t\}$ such that $x_{t_n} \rightharpoonup z$, where $\{t_n\}$ is a sequence in $(0, 1)$ such that $t_n \to 0$ as $n \to \infty$.

Now define the function $\varphi : C \to \mathbb{R}$ by

$$\varphi(x) := LIM_n\|x_n - x\|^2, \quad x \in C.$$

Since X is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and φ is continuous convex function, by Theorem 1.2 of [1, p. 79] we have that the set

$$M := \{y \in C : \varphi(y) = \inf_{x\in C} \varphi(x)\} \tag{7}$$

is nonempty. M is also closed convex and bounded. Moreover, M is invariant under g. In fact, we have for each $y \in M$,

$$\begin{aligned}\varphi(gy) &= LIM_n\|x_n - gy\|^2\\ &= LIM_n\|gx_n - gy\|^2\\ &\leq LIM_n\|x_n - y\|^2 = \varphi(y).\end{aligned}$$

So, by the hypothesis, there exists a fixed point u of g in M. By Lemma 3, we have

$$LIM_n\langle z, J(x_n - u)\rangle \leq 0 \text{ for all } z \in C.$$

In particular,

$$LIM_n\langle Au - u, J(x_n - u)\rangle \leq 0. \tag{8}$$

Observe that

$$\begin{aligned}\|x_n - u\|^2 =&\langle x_n - Ax_n, J(x_n - u)\rangle + \langle Ax_n - Au, J(x_n - u)\rangle \\ &+ \langle Au - u, J(x_n - u)\rangle.\end{aligned}$$

By pseudocontractivity of T,

$$(1-k)\|x_n - u\|^2 \leq \langle x_n - Ax_n, J(x_n - u)\rangle + \langle Au - u, J(x_n - u)\rangle.$$

From (8) and Proposition 2(b), we obtain

$$LIM_n\|x_n - u\|^2 \leq 0.$$

Therefore, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to u$. Assume that there is another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \tilde{u}$. Since $x_n - gx_n \to 0$, it follows that $\tilde{u} \in F(g)$. Using Proposition 2(b), we have that

$$\langle x_t - Ax_t, J(x_t - v)\rangle \leq 0 \text{ for all } v \in F(T). \tag{9}$$

By norm to weak* uniform continuity of J, we obtain

$$\langle u - Au, J(u - \tilde{u})\rangle \leq 0 \quad \text{and} \quad \langle \tilde{u} - A\tilde{u}, J(\tilde{u} - u)\rangle \leq 0.$$

Adding these two inequalities yields that

$$\langle u - \tilde{u} + A\tilde{u} - Au, J(u - \tilde{u})\rangle \leq 0.$$

This implies that

$$\|u - \tilde{u}\|^2 \leq k\|u - \tilde{u}\|^2.$$

Since $k \in [0, 1)$, it follows that $u = \tilde{u}$. Thus, $\{x_n\}$ converges strongly to u.

We finally prove that the entire net $\{x_t\}$ converges strongly. To this end, we assume that $\{t_{n'}\}$ is another subsequence in $(0, 1)$ such that $x_{t_{n'}} \to u'$ as $t_{n'} \to 0$. By (6), we obtain $u' \in F(T)$. From (9), we have that

$$\langle u - Au, J(u - u')\rangle \leq 0 \text{ and } \langle u' - Au', J(u' - u)\rangle \leq 0.$$

We must have $u = u'$. Therefore, $\{x_t\}$ converges strongly to $u \in F(T)$.

(b) Since $x_t \to u \in F(T)$, it follows from Proposition 2(b) and Lemma 7 that u is a unique point satisfying

$$\langle u - Au, J(u - v)\rangle \leq 0 \text{ for all } v \in F(T).$$

□

Corollary 1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$ and $T : C \to C$ a continuous pseudocontractive mapping. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Suppose also that the set*

$$E = \{x \in C : Tx = \lambda x + (1-\lambda)Ax \text{ for some } \lambda > 1\}$$

is bounded. For $t \in (0,1)$, let $\{x_t\}$ be the path defined by (4). Then we have the following:

(a) $\lim_{t\to 0^+} x_t = \tilde{x}$ *exists,*

(b) $\tilde{x}$ *is a fixed point of T and it is the unique solution of the variational inequality:*

$$\langle (I-A)\tilde{x}, J(\tilde{x}-v)\rangle \leq 0 \text{ for all } v \in F(T).$$

Corollary 2 (Theorem 1, Morales and Jung [14]). *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C nonempty closed convex subset of X and $T : C \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. Suppose every closed convex bounded subset of C has fixed point property for nonexpansive self mappings. If there exists $u_0 \in C$ such that the set*

$$E = \{x \in C : Tx = \lambda x + (1-\lambda)u_0 \text{ for some } \lambda > 1\}$$

is bounded, then the path $\{x_t : t \in (0,1)\}$ defined by

$$x_t = tu_0 + (1-t)Tx_t$$

converges strongly to a fixed point of T.

Proof. In this case the mapping $A : C \to C$ defined by $Ax = u_0$ for all $x \in C$ is continuous strongly pseudocontractive with constant 0. The proof follows from Theorem 1. □

Remark 1. (1) Theorem 1 is also an extension of Theorem 5 of Morales [13] in terms of the space itself and the viscosity type method.

(2) Corollary 1 generalizes the corresponding results in Ha and Jung [8], Moudafi [15], Reich [18], and Xu [22] to ones for pseudocontractive mappings.

(3) Corollary 2 improves Theorem 1 of Xu [21], which is done for nonexpansive mapping and the inwardness condition, as well as Theorem 1 of Jung and Kim [9] for nonexpansive mappings under the additional assumption that C is a sunny nonexpansive retract of X.

(4) In Theorem 1 and Corollary 1, boundedness of the set E can be replaced by the assumption that $F(T) \neq \emptyset$.

We now replace the fixed point property assumption, mentioned in Theorem 1 by imposing certain conditions on the space X or on the mapping T.

Theorem 2. *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0,1)$ and $T : C \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. If T has a fixed point in C, then the path $\{x_t\}$ defined by (4) converges strongly to a fixed point of T, which is a unique solution of variational inequality:*

$$\langle (I-A)\tilde{x}, J(\tilde{x}-v)\rangle \leq 0 \text{ for all } v \in F(T).$$

Proof. To be able to use the argument of the proof of Theorem 1, we just need to show that the set M defined by (7) has a fixed point of g. Since $F(T) = F(g) \neq \emptyset$, let $v \in F(g)$. Then the set M_0 defined by

$$M_0 = \{u \in M : \|u - v\| = \inf_{x \in M} \|x - v\|\}$$

is singleton since X is strictly convex. Let $M_0 = \{u_0\}$ for some $u_0 \in M$. Observe that

$$\|gu_0 - v\| = \|gu_0 - gv\| \leq \|u_0 - v\| = \inf_{x \in M} \|x - v\|.$$

Therefore $gu_0 = u_0$. We now follow the proof of Theorem 1. □

Next we obtain a convergence of path described by (4) in which continuity assumption of operator T is weaken and convexity of C is dispensed.

Theorem 3. *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$ and $T : C \to X$ a demicontinuous pseudocontractive mapping such that the equation*

$$x = tAx + (1 - t)Tx$$

has a solution x_t in C for each $t \in [0, 1)$. Suppose the path $\{x_t\}$ is bounded. Then we have the following:

(a) $\lim_{t \to 0^+} x_t = \tilde{x}$ *exists,*

(b) $\tilde{x}$ *is a fixed point of T and it is the unique solution of the variational inequality:*

$$\langle (I - A)\tilde{x}, J(\tilde{x} - v)\rangle \leq 0 \text{ for all } v \in F(T).$$

Proof. (a) Since $\{x_t\}$ is bounded, it follows from reflexivity of X that there exists a subsequence $\{x_{t_n}\} \subseteq \{x_t\}$ such that $x_{t_n} \rightharpoonup z \in C$ as $t_n \to 0$, where $\{t_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \to \infty} t_n = 0$. Set $x_n := x_{t_n}$. As in Theorem 1, $g : C \to C$ a nonexpansive with $F(T) = F(g)$. Also $x_n - gx_n \to 0$ as $n \to \infty$. Since J is weakly continuous, it follows from Lemma 4 that $z \in F(g)$. By (5), we get

$$\|x_n - z\|^2 \leq \frac{1}{1 - k}\langle Az - z, J(x_n - z)\rangle.$$

Since J is weakly continuous duality mapping, it follows that $x_n \to z$ as $n \to \infty$.

We have already proved that there exists a subsequence $\{x_{t_n}\}$ of $\{x_t : t \in (0, 1)\}$ that converges strongly to a point $z \in F(T)$. Now it remains to prove that the entire net $\{x_t\}$ converges strongly to z. Suppose, for contradiction, that there exists another sequence $\{x_{t_{n'}}\} \subset \{x_t\}$ such that $x_{t_{n'}} \to z' \neq z$ as $t_{n'} \to 0$. Then, we have $z' \in F(T)$. From (9), we have

$$\langle z - Az, J(z - z')\rangle \leq 0 \text{ and } \langle z' - Az', J(z - z')\rangle \leq 0.$$

This gives that $z = z'$. Therefore, $\lim_{t \to 0^+} x_t$ exists and $\lim_{t \to 0^+} x_t = z \in F(T)$.

(b) Since $\lim_{t\to 0^+} x_t = z$, it follows Proposition 2(b)·and Lemma 7 that z is a unique point satisfying

$$\langle (I-A)z, J(z-v)\rangle \le 0 \text{ for all } v \in F(T).$$

□

Corollary 3 (Theorem 1.2, Schu [19]). *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed convex bounded subset of $X, u \in C$ and $T : C \to C$ a continuous pseudocontractive mapping. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Then*

(a) *for each $n \in \mathbb{N}$, there is exactly one $x_n \in C$ such that*

$$x_n = (1-\lambda_n)u + \lambda_n T x_n,$$

(b) *$\{x_n\}$ converges strongly to a fixed point of T.*

Remark 2. By putting $Ax = u$ for all $x \in C$ in Theorem 2 and Theorem 3, we can also obtain Theorem 2 and Theorem 3 of Morales and Jung [14] as Corollary 2.

4. Applications

In 1980, Reich [18] proved the following theorem.

Theorem R (Reich [18]). *Let X be a uniformly smooth Banach space and C a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive mapping with a fixed point and let $z \in C$. For each $t \in (0,1)$, let x_t be given by $x_t = tz + (1-t)Tx_t$. Then $\{x_t\}_{0<t<1}$ converges to a fixed point of T as $t \to 0^+$. Thus,*

$$Q(z) := s - \lim_{t\to 0^+} z_t$$

defines the unique sunny nonexpansive retraction form C onto $F(T)$.

O'Hara, Pillay and Xu [16] introduced the Reich's property.

Definition 1. A Banach space X is said to have *Reich property* if for any closed and convex subset C of X, any nonexpansive mapping $T : C \to C$ with a fixed point and any $z \in C$, $\{x_t\}$ defined by $x_t = tz + (1-t)Tx_t$ converges strongly to a fixed point of T as $t \to 0^+$.

Thus, every uniformly smooth Banach space has Reich's property. Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a pseudocontractive mapping. Let Σ_C denote the set of all strongly pseudocontractive mappings $A : C \to C$ with constant $k \in [0,1)$. We now introduce the following property:

Definition 2. We say that a Banach space X has *property* (S) if for any closed convex subset C of X, any pseudocontractive mapping $T : C \to C$ with $F(T) \neq \emptyset$ and any $A \in \Sigma_C$, the path $\{x_t\}$ defined by (4) converges strongly to a fixed point of T as $t \to 0^+$.

The following theorem shows that property (S) plays a key role in the existence of sunny nonexpansive retraction.

Theorem 4. *Let X be a smooth Banach space with property* (S)*. Let C be a nonempty closed convex subset of X and $T : C \to C$ a pseudocontractive mapping with $F(T) \neq \emptyset$. If we define $Q : \Sigma_C \to F(T)$ by*

$$Q(A) := \lim_{t \to 0^+} x_t, \quad A \in \Sigma_C,$$

then $\langle AQ(A) - BQ(B), J(Q(A) - Q(B)) \rangle \geq \|Q(A) - Q(B)\|^2$ for all $A, B \in \Sigma_C$. In particular, if $A = u \in C$ is a constant, then Q is the sunny nonexpansive retraction from C onto $F(T)$.

Proof. For any $A \in \Sigma_C$ and $t \in (0,1)$, let x_t be the unique point in C such that $x_t = tAx_t + (1-t)Tx_t$. By Property (S), $\lim_{t \to 0} x_t$ exists; hence $Q(A) = \lim_{t \to 0} x_t$. By Proposition 2(b), we have

$$\langle x_t - Ax_t, J(x_t - v) \rangle \leq 0 \text{ for all } v \in F(T).$$

Taking the limit as $t \to 0^+$ and using Lemma 1, we obtain

$$\langle Q(A) - AQ(A), J(Q(A) - v) \rangle \leq 0.$$

Thus, for $A, B \in \Sigma_C$, we have

$$\langle Q(A) - AQ(A), J(Q(A) - Q(B)) \rangle \leq 0$$

and

$$\langle Q(B) - BQ(B), J(Q(B) - Q(A)) \rangle \leq 0.$$

Adding these two inequalities, we get

$$\langle Q(A) - AQ(A) + BQ(B) - Q(B), J(Q(A) - Q(B)) \rangle \leq 0.$$

Therefore,

$$\|Q(A) - Q(B)\|^2 \leq \langle AQ(A) - BQ(B), J(Q(A) - Q(B)) \rangle.$$

If $A = u$ and $B = v$ then

$$\langle u - v, J(Qu - Qv) \rangle \geq \|Qu - Qv\|^2.$$

By Lemma 2(c), Q is a sunny nonexpansive retraction from C onto $F(T)$. □

The following theorem extends Theorem R to one for pseudocontractive mapping. This also improves Theorem 5 of Morales [13].

Theorem 5. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $T : C \to X$ a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. If T satisfies the weakly inward condition, then there exists a unique sunny nonexpansive retraction $Q : C \to F(T)$.*

Proof. For any $u \in C$ and $t \in (0,1)$, let x_t be the unique point in C such that $x_t = tu + (1-t)Tx_t$. By Theorem 1, X has property (S) and hence by Theorem 4, there exists a unique sunny nonexpansive retraction form C onto $F(T)$ which is given by $Q(u) = \lim_{t \to 0^+} x_t$. □

We now generalize Theorem 3.10 of O'Hara, Pillay and Xu [16] to pseudocontractive one.

Theorem 6. *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. C a nonempty closed convex subset of X and $T : C \to X$ a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. If T satisfies the weakly inward condition, then there exists a unique sunny nonexpansive retraction $Q : C \to F(T)$.*

Proof. The definition of the weak continuity of duality mapping J implies that X is smooth. For any $u \in C$ and $t \in (0,1)$, let x_t be the unique point in C such that $x_t = tu + (1-t)Tx_t$. By Corollary 4, X has property (S) and hence by Theorem 4, there exists a unique sunny nonexpansive retraction form C onto $F(T)$ which is given by $Q(u) = \lim_{t \to 0^+} x_t$. □

Finally, using Lemma 5, Theorem 1 and Theorem 3, we derive strong convergence theorems for firmly pseudocontractive mappings.

Theorem 7. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $A : X \to X$ a continuous strongly pseudocontractive mapping with constant $k \in [0,1)$ and $T : X \to X$ continuous firmly pseudocontractive mapping. Suppose that every closed convex bounded subset of X has fixed point property for nonexpansive self mappings. Suppose also that the set*

$$E' = \{x \in X : Tx = (2-\lambda)x + (\lambda - 1)Ax \text{ for some } \lambda > 1\}$$

is bounded. Then we have the following:

(a) *For each $t \in (0,1)$, there is a path $\{x_t\}$ in X defined by*

$$x_t = tAx_t + (1-t)(2I - T)x_t$$

such that $\lim_{t \to 0^+} x_t = \tilde{x}$ exists,

(b) *$\tilde{x}$ is a fixed point of T and it is the unique solution of variational inequality:*

$$\langle (I - A)\tilde{x}, J(\tilde{x} - v)\rangle \text{ for all } v \in F(T).$$

Theorem 8. *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let $A : X \to X$ be a continuous strongly pseudocontractive mapping with constant $k \in [0,1)$ and $T : X \to X$ a demicontinuous firmly pseudocontractive mapping such that the equation*

$$x = tAx + (1-t)(2I - T)x$$

has a solution x_t in C for each $t \in [0,1)$. Suppose the path $\{x_t\}$ is bounded. Then we have the following:

(a) $\lim_{t \to 0^+} x_t = \tilde{x}$ *exists,*

(b) $\tilde{x}$ *is a fixed point of T and it is the unique solution of the variational inequality:*

$$\langle (I - A)\tilde{x}, J(\tilde{x} - v)\rangle \leq 0 \text{ for all } v \in F(T).$$

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Jong Soo Jung
Department of Mathematics
Dong-A University
Busan 604-714, Korea
E-mail address: jungjs@mail.donga.ac.kr

Daya Ram Sahu
Department of Applied Mathematics
Shri Shankaracharya College of Engineering and Technology
Junwani, Bhilai - 490020, India
current address
Department of Mathematics
Banaras Hindu University
Varanasi - 221005, India
E-mail address: sahudr@yahoo.com