

Aplikace matematiky

Teresa Regińska

Convergence of approximation methods for eigenvalue problem for two forms

Aplikace matematiky, Vol. 29 (1984), No. 5, 333–341

Persistent URL: <http://dml.cz/dmlcz/104103>

Terms of use:

© Institute of Mathematics AS CR, 1984

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

CONVERGENCE OF APPROXIMATION METHODS FOR EIGENVALUE PROBLEM FOR TWO FORMS

TERESA REGIŃSKA

(Received April 13, 1983)

INTRODUCTION

In [2] R. D. Brown investigated approximation methods for eigenvalues of a real quadratic form b relative to a positive definite quadratic form a , where a and b are defined on a vector space V . He considered a general procedure for approximation, outlined by Aronszajn in [1]. His investigations were carried out on the basis of the theory of discrete convergence in Banach spaces in the form developed by Stummel in [6]. In this paper we prove a general convergence theorem in a different way. Namely, it is shown how the theory of external approximation of eigenvalue problems described in [5] can be adopted to the study of the methods considered by Brown. The convergence criteria obtained are somewhat weaker than those presented in [2].

1. EXTERNAL APPROXIMATION OF EIGENVALUE PROBLEMS

In this section we present a brief summary of the results contained in [5] concerning external approximation of eigenproblems.

Let X be a Banach space and $T \in \mathcal{L}(X)$. Let F be a normed space such that there exists an isomorphism $\omega : X \xrightarrow{\text{in}} F$. Next, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces with norms denoted by $\|\cdot\|_n$ and let $\{r_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences of linear maps from X onto X_n and X_n into F ($n = 1, 2, \dots$), respectively.

Definition 1. An approximation $\{X_n, r_n, p_n\}$ of X is said to be an external approximation convergent in F if r_n and p_n are uniformly bounded and

$$\forall u \in X \lim_{n \rightarrow \infty} \|\omega u - p_n r_n u\|_F = 0.$$

Let us introduce a sequence $\{T_n\}_{n=1}^\infty$ of linear bounded operators $T_n \in \mathcal{L}(X_n)$, $n = 1, 2, \dots$. As usual, $\sigma(T)$, $\varrho(T)$ and $\sigma(T_n)$, $\varrho(T_n)$ denote the spectrum and the resolvent set of T and T_n , respectively.

Definition 2. The approximation $\{T_n\}_{n=1}^\infty$ is stable at a point $\lambda \in \varrho(T)$ iff $\exists N_\lambda$ and $\exists M_\lambda \forall n > N_\lambda \lambda \in \varrho(T_n)$ and $\|(\lambda - T_n)^{-1}\| \leq M_\lambda < \infty$.

Let $N(r_n)$ denote the null space of r_n . We assume that for any n , $N(r_n)$ has a complementary subspace in X . So, we can introduce the set \mathcal{F} of all sequences of complementary subspaces for $N(r_n)$:

$$\mathcal{F} = \{ \{V_n\}_{n=1}^\infty : V_n \subset X, V_n \oplus N(r_n) = X \}.$$

Theorem 1. If there exists $\{V_n\} \in \mathcal{F}$ such that

$$(1.1) \quad \delta_n = \sup_{\substack{v \in V_n \\ \|v\|=1}} \|\omega T v - p_n T_n r_n v\|_F \rightarrow 0,$$

$$(1.2) \quad \varepsilon_n = \sup_{\substack{v \in V_n \\ \|v\|=1}} \|\omega v - p_n r_n v\|_F \rightarrow 0,$$

then for any $\lambda \in \varrho(T)$ there exists a constant $M_\lambda < \infty$ such that

$$\|(\lambda - T_n)^{-1}\| \leq M_\lambda.$$

Remark 1. If the residual spectrum $\sigma_r(T_n)$ of T_n ($\sigma_r(T_n) = \{\lambda \in \sigma(T_n) : (\lambda - T_n)x = 0 \equiv x = 0, \text{ and } (\lambda - T_n)X_n \neq X_n\}$) does not contain the points of $\varrho(T)$, then Theorem 1 implies that $\{T_n\}$ is stable at any $\lambda \in \varrho(T)$.

Definition 3. We will say that $\sigma(T_n)$ approximates $\sigma(T)$ if the following three implications take place:

- i) if $\Omega \subset \mathbb{C}$ is open and $\Omega \cap \sigma(T) \neq \emptyset$, then $\Omega \cap \sigma(T_n) \neq \emptyset$ for sufficiently large n ;
- ii) if $\lambda \in \sigma(T)$ and there is $\delta_0 < 0$ such that $K(\lambda, \delta_0) \cap \sigma(T) = \{\lambda\}$, where $K(\lambda, \delta_0)$ is a circle with radius δ_0 and center λ , then for every δ such that $0 < \delta < \delta_0$: $\sigma(T_n) \cap K(\lambda, \delta_0) \subset K(\lambda, \delta)$ for sufficiently large n ;
- iii) if $\lambda_n \in \sigma(T_n)$ and $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$, then $\lambda_0 \in \sigma(T)$.

In the sequel we quote four theorems concerning the convergence of an approximation.

Theorem 2. Let $\{X_n, r_n, p_n\}$ be an external approximation of X , convergent in F , and let $\{T_n\}$ be stable in $\varrho(T)$. If for any $u \in X$

$$(1.3) \quad \lim_{n \rightarrow \infty} \|r_n T u - T_n r_n u\|_n = 0,$$

where $\|\cdot\|_n$ stands for the norm in X_n , then $\sigma(T_n)$ approximates $\sigma(T)$ in the sense of Definition 3.

Let Γ be a Jordan curve in the resolvent set $\rho(T)$. If $\{T_n\}$ is stable for all $\lambda \in \Gamma$, then $\Gamma \subset \rho(T_n)$ for $n > N_0$. So the spectral projectors associated with Γ , i.e. $E : X \rightarrow X$ and $E_n : X_n \rightarrow X_n$, are well defined and

$$E = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} d\lambda, \quad E_n = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T_n)^{-1} d\lambda.$$

Theorem 3. *If the assumptions of Theorem 2 are satisfied, then*

- i) if $\dim EX = \infty$, then $\dim E_n X_n \rightarrow \infty$ as $n \rightarrow \infty$,
- ii) if $\dim EX = n$, then $\dim p_n E_n X_n \geq n$ for $n > n_0$.

The preservation of algebraic multiplicities of isolated eigenvalues can be obtained under a certain stronger assumption on T_n . Namely, we have

Theorem 4. *Let the assumptions of Theorem 2 be satisfied. If $\dim EX < \infty$ and*

$$(1.4) \quad \|(T_n r_n - r_n T)(\lambda - T)^{-1}|_{V_n}\| \rightarrow 0 \quad \text{for } \lambda \in \Gamma,$$

then $\dim EX = \dim p_n E_n X_n$.

The eigensubspace EX of T is approximated by $E_n X_n$ in the following sense (cf. [5]):

Theorem 5. *If the assumptions of Theorem 2 are satisfied. then*

$$\forall v \in EX \quad \text{dist}(\omega v, p_n E_n X_n) \rightarrow 0.$$

If, moreover (1.4) is satisfied, then

$$\hat{\delta}(\omega EX, p_n E_n X_n) \rightarrow 0,$$

where $\hat{\delta}(Y, Z)$ is the gap between closed subspaces Y and Z of X ($\hat{\delta}(Y, Z) = \max(\delta(Y, Z), \delta(Z, Y))$ where $\delta(Y, Z) = \sup_{\substack{y \in Y \\ \|y\|=1}} \text{dist}(y, Z)$).

2. APPROXIMATION OF THE EIGENVALUE PROBLEM FOR TWO FORMS AND THE CONVERGENCE RESULTS

The eigenvalue problem for a pair of sesquilinear forms a and b on a complex vector space V is considered. It is assumed that a is symmetric and positive definite and, moreover, b is continuous with respect to a , i.e.: $\forall u, v \in V |b(u, v)| \leq c a^{1/2}(u, u) \cdot a^{1/2}(v, v)$, c a positive constant. Assume also that V is separable with respect to the norm $a^{1/2}$. Let X be the closure of V in the norm $a^{1/2}$. The form b can be continuously extended to X . So, our eigenvalue problems takes the form

(2.1) find $\lambda \in \mathbb{C}$ and $0 \neq u \in X$ such that

$$b(u, v) = \lambda a(u, v) \quad \forall v \in V,$$

which is equivalent to the eigenproblem for an operator $T \in \mathcal{L}(X)$ defined by a and b as follows:

$$(2.2) \quad \forall u \in X \quad b(u, v) = a(Tu, v) \quad \forall v \in V.$$

We will consider the approximate methods for the problem (2.1), which are generated by sequences of sesquilinear forms a_n and b_n defined on $V \times V$. It is assumed that a_n ($n = 0, 1, \dots$) are symmetric and positive definite and b_n are bounded with respect to a_n .

Let X_n be the closure of V in the norm $a_n^{1/2}$, $n = 0, 1, \dots$. The norms in X and X_n will be denoted by $\| \cdot \|$ and $\| \cdot \|_n$, respectively. The forms b_n have continuous extensions on X_n . The n -th approximate eigenvalue problem takes the form

$$(2.3) \quad \text{find } \lambda \in \mathbb{C} \text{ and } 0 \neq u \in X_n \text{ such that } b_n(u, v) = \lambda a_n(u, v) \quad \forall v \in V.$$

This problem is equivalent to the eigenproblem for an operator $T_n \in \mathcal{L}(X_n)$ which is defined by a_n and b_n as follows:

$$(2.4) \quad \forall u \in X_n \quad b_n(u, v) = a_n(T_n u, v) \quad \forall v \in V.$$

It will be assumed that the following conditions are satisfied:

$$C 1 \quad a_0 \leq a_n \leq a;$$

C 2 a is quasi-bounded with respect to a_0 , i.e.

$$\forall u \in V \quad \exists M_u < \infty \quad |a(u, v)| \leq M_u \|v\|_0 \quad \forall v \in V.$$

(a is quasi-bounded with respect to a_0 iff there exists a symmetric operator \hat{L} in X_0 such that $\forall u, v \in V \quad a(u, v) = a_0(\hat{L}u, v)$). The forms a_n generate a certain approximation of the space X . We will show that it is a special kind of the external approximation of X . We are going to construct suitable maps r_n and p_n .

Let us first remark that the assumptions C 1 and C 2 imply that a is quasi-bounded with respect to a_n , $n = 1, 2, \dots$. In fact, $a(u, v) = a_n(A_n \hat{L}u, v) \quad \forall v \in V$, where A_n is a bounded operator defined by $a_0(u, v) = a_n(A_n u, v) \quad \forall v \in V$. Denote $\hat{L}_n = A_n \hat{L}$. The operator \hat{L}_n considered in X_n is bounded from below ($a_n(\hat{L}_n u, u) \geq a_n(u, u) \quad \forall u \in V$), so \hat{L}_n is semi-bounded in X_n . Every semi-bounded symmetric operator with a dense domain has a semi-bounded selfadjoint extension with the same lower bound (cf. [3], XII. 5.1). Let L_n be the selfadjoint extension of \hat{L}_n on the space X_n . L_n is positive definite. Thus, there is a unique positive definite and selfadjoint square root $L_n^{1/2}$ of L_n and the domain $D(L_n)$ of L_n is dense in $D(L_n^{1/2})$ (cf. [4], V. § 3.11).

Let $t_n : X \rightarrow X_n$ be the unique bounded linear transformation such that $t_n v = v, \quad \forall v \in V$. We will show that $D(L_n^{1/2}) = t_n X$. To this end we apply the second representation theorem ([4], VI, § 2.6). The assumptions $x_k \in V, x_k \xrightarrow[k \rightarrow \infty]{} 0$

in X_n and $\|x_k - x_l\| \xrightarrow[k, l \rightarrow \infty]{} 0$ imply, by C 2, that for any $u \in V$, $|a(u, x_k)| \leq \|L_n u\|_n \cdot \|x_k\|_n \rightarrow 0$. Thus the form a is closable in X_n . So, let $\bar{a}^{(n)}$ be the closure of a in X_n . For $u, v \in X$ we have $\bar{a}^{(n)}(t_n u, t_n v) = a(u, v)$, and the selfadjoint operator associated with $\bar{a}^{(n)}$ in X_n is equal to L_n defined above. The second representation theorem for the densely defined, closed symmetric, and positive definite form $\bar{a}^{(n)}$ yields that $D(L_n^{1/2}) = t_n X$ and $\forall u, v \in X$

$$(2.5) \quad a(u, v) = \bar{a}^{(n)}(t_n u, t_n v) = a_n(L_n^{1/2} t_n u, L_n^{1/2} t_n v).$$

Finally, let us remark that the mapping t_n of X into X_n is injective. In fact, if $x_k \in V$ and $x_k \xrightarrow[k \rightarrow \infty]{} x$ in X then $t_n x_k \xrightarrow[k \rightarrow \infty]{} t_n x$ in X_n and $\forall u \in V |a(u, x)| = \lim |a_n(L_n u, x_k)| \leq \|L_n u\|_n \cdot \lim \|x_k\|_n = \|L_n u\|_n \cdot \|t_n x\|_n$. So, if $\|t_n x\| = 0$ then $\forall u \in V a(u, x) = 0$, i.e. $x = 0$.

Let us define $r_n = L_n^{1/2} t_n$.

Lemma 1. *If C 1 and C 2 are satisfied, then $r_n \in \mathcal{L}(X, X_n)$ and $r_n^{-1} \in \mathcal{L}(X_n, X)$ for $n = 0, 1, \dots$. Moreover, $\|r_n\|_{\mathcal{L}(X, X_n)} = \|r_n^{-1}\|_{\mathcal{L}(X_n, X)} = 1$.*

Proof. From (2.5) it follows that $\forall u \in X \|r_n u\|_n^2 = a_n(L_n^{1/2} t_n u, L_n^{1/2} t_n u) = \|u\|$. Next, let us remark that $\forall w \in D(L_n) a_n(L_n w, w) = \bar{a}^{(n)}(w, w) \geq a_n(w, w)$. In [4] (V, § 3.11) it is proved that under that condition $L_n^{-1/2}$ is a bounded operator on X_n . So, $\forall v \in X_n r_n^{-1}$ is well defined since t_n is injective, as has been shown above. Moreover, by (2.5) $\forall v \in X_n \|r_n^{-1} v\|^2 = \|t_n^{-1} L_n^{-1/2} v\|^2 = \bar{a}^{(n)}(L_n^{-1/2} v, L_n^{-1/2} v) = a_n(v, v)$ which completes the proof of Lemma 1.

So, we can put $p_n = r_n^{-1}$. We have $p_n r_n x = x$ for any $x \in X$. Thus we have

Corollary 1. $\{X_n, r_n, p_n\}$ is an external approximation of X , convergent in X in the sense of Definition 1.

Lemma 2. *If C 1 and C 2 are satisfied together with*

$$C 3 \quad \forall u \in V \sup_{\substack{v \in V \\ \|v\| = 1}} |a_n(u, v) - a(u, v)| \rightarrow 0,$$

then $\forall u \in V \|r_n u - u\|_n \rightarrow 0$.

Proof. Let us apply the integral expression for $L_n^{1/2}$ (cf. [4], V, § 3.11):

$$L_n^{1/2} u = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (L_n + \lambda)^{-1} L_n u \, d\lambda \quad \text{for } u \in D(L_n) \subset X_n.$$

Similarly, we can express the identity operator on X_n :

$$I u = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (I + \lambda)^{-1} u \, d\lambda.$$

Since $(L_n + \lambda)^{-1} L_n u = u - \lambda(L_n + \lambda)^{-1} u$ for $u \in D(L_n)$ and $(I + \lambda)^{-1} u = u - \lambda(I + \lambda)^{-1} u$; we have

$$\begin{aligned} L_n^{1/2} u - u &= \frac{1}{\pi} \int_0^\infty \lambda^{1/2} [(I + \lambda)^{-1} - (L_n + \lambda)^{-1}] u \, d\lambda = \\ &= \frac{1}{\pi} \int_0^\infty \lambda^{1/2} (L_n + \lambda)^{-1} (L_n - I) (I + \lambda)^{-1} u \, d\lambda \text{ for } u \in D(L_n). \end{aligned}$$

The last term is obtained from the resolvent equation

$$(I + \lambda)^{-1} - (L_n + \lambda)^{-1} = (L_n + \lambda)^{-1} (L_n - I) (I + \lambda)^{-1}.$$

From the above it follows that

$$\|L_n^{1/2} u - u\|_n \leq \frac{1}{\pi} \int_0^\infty \lambda^{1/2} (1 + \lambda)^{-2} \, d\lambda \|L_n u - u\|_n,$$

since $(I + \lambda)^{-1} u = (1 + \lambda)^{-1} u$ and $\|(L_n + \lambda)^{-1}\| \leq [\text{dist}(-\lambda, \sigma(L_n))]^{-1} \leq (1 + \lambda)^{-1}$. Thus, for any $u \in V$

$$\begin{aligned} \|L_n^{1/2} u - u\|_n &\leq c \|L_n u - u\|_n = c \sup_{\substack{v \in V \\ \|v\| = 1}} |a_n(L_n u - u, v)| = \\ &= \sup_{\substack{v \in V \\ \|v\| = 1}} |a(u, v) - a_n(u, v)| \rightarrow 0 \end{aligned}$$

according to the assumption C 3.

Theorem 6. *If C 1, C 2 and C 3 are satisfied together with*

$$\text{C 4} \quad \sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} |b_n(u, v) - b(u, v)| \rightarrow 0 \text{ as } n \rightarrow \infty;$$

C 5 *if sequences $\{u_n\}$ and $\{v_n\}$ of elements of V satisfy $a_n(u_n, w) \rightarrow a(u, w)$ and $a_n(v_n, w) \rightarrow a(v, w) \forall w \in V$ and the norms $\|u_n\|_n, \|v_n\|_n$ are uniformly bounded then $b_n(u_n, v_n) \rightarrow b(u, v)$,*

then the family $\{T_n\}$ defined by (2.4) is stable.

Proof. We have to show that $\delta_n (\delta_n = \|T - r_n^{-1} T_n r_n\|)$ converges to zero as $n \rightarrow \infty$. Let us take u and v from the space V . Then

$$\begin{aligned} a(r_n^{-1} T_n r_n u, v) &= a(t_n^{-1} L_n^{-1/2} T_n L_n^{1/2} u, v) = \bar{a}^{(n)}(v, L_n^{-1/2} T_n L_n^{1/2} u) = \\ &= a_n(L_n v, L_n^{-1/2} T_n L_n^{1/2} u). \end{aligned}$$

Since $L_n^{1/2}$ is selfadjoint in X_n , by the definition of T_n

$$a_n(L_n v, L_n^{-1/2} T_n L_n^{1/2} u) = a_n(T_n L_n^{1/2} u, L_n^{1/2} v) = b_n(L_n^{1/2} u, L_n^{1/2} v).$$

Thus

$$\begin{aligned} \delta_n &= \sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} a((T - r_n^{-1} T_n r_n) u, v) = \sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} |b(u, v) - b_n(L_n^{1/2} u, L_n^{1/2} v)| \leq \\ &\leq \sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} |b(u, v) - b_n(u, v)| + \sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} |b_n(u, v) - b_n(L_n^{1/2} u, L_n^{1/2} v)|. \end{aligned}$$

The first term tends to zero according to the assumption C 4. Suppose that the second term does not converge to zero. Thus, there exist $\varepsilon < 0$ and sequences $\{u_n\}$ and $\{v_n\}$ from the unit sphere in $V \cap X$ such that

$$(2.6) \quad |b_n(u_n, v_n) - b_n(L_n^{1/2} u_n, L_n^{1/2} v_n)| \geq \varepsilon.$$

From these sequences we can choose subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ weakly convergent in X . Let their weak limits be denoted by u and v , respectively. Thus $\forall u \in V$

$$|a_{n_k}(u_{n_k}, w) - a(u, w)| \leq \sup_{\substack{z \in V \\ \|z\| = 1}} |a_{n_k}(z, v) - a(z, v)| + |a(u_{n_k}, w) - a(u, w)|,$$

and the left-hand side converges to zero by the assumption C 3. So, C 5 implies that

$$b_{n_k}(u_{n_k}, v_{n_k}) \rightarrow b(u, v).$$

We have to show that the sequence $\{b_{n_k}(L_{n_k}^{1/2} u_{n_k}, L_{n_k}^{1/2} v_{n_k})\}$ has the same limit. Let us notice that $a_n(L_n^{1/2} u_n, w) = a_n(u_n, w) + a_n(u_n, L_n^{1/2} w - w)$ for any $w \in V$ since $L_n^{1/2}$ is selfadjoint in X_n . Thus, by Lemma 2,

$$\lim_{k \rightarrow \infty} a_{n_k}(L_{n_k}^{1/2} u_{n_k}, w) = \lim_{k \rightarrow \infty} a_{n_k}(u_{n_k}, w) = a(u, w).$$

Applying now C 5 to the sequences $\{L_{n_k}^{1/2} u_{n_k}\}$ and $\{L_{n_k}^{1/2} v_{n_k}\}$ we get $|b_{n_k}(u_{n_k}, v_{n_k}) - b_{n_k}(L_{n_k}^{1/2} u_{n_k}, L_{n_k}^{1/2} v_{n_k})| \rightarrow 0$ contrary to (2.6). Thus $\delta_n \rightarrow 0$. It is easy to show that if $\delta_n \rightarrow 0$, then $\varrho(T) \cap \sigma_r(T_n) = \emptyset$ for $n > n_0$. Thus $\{T_n\}$ is stable according to Remark 1.

Now, let us notice that, in our special case, the condition (1.1) of Theorem 1 implies the condition (1.3). Moreover, (1.1) implies the condition (1.4). Thus according to Corollary 1 and Theorem 6, all the assumptions of Theorems 2–5 are satisfied. Therefore, the final result concerning the convergence of the methods considered can be formulated in the form of the following theorem:

Theorem 7. *Let the conditions C 1–C 5 be satisfied. Then*

- i) $\sigma(T_n)$ approximates $\sigma(T)$ in the sense of Definition 3;
- ii) if Γ is a Jordan curve in $\varrho(T)$ and E and E_n are the spectral projectors associated with Γ and T and T_n , respectively, then
if $\dim EX = \infty$, then $\dim E_n X_n \rightarrow \infty$,
if $\dim EX = n$, then $\dim E_n X_n = n$ for sufficiently large n ;
- iii) $\hat{\delta}(EX, p_n E_n X_n) \rightarrow 0$.

The theorem on convergence of eigenelements presented in [2] (cf. Th. 1.2) is proved under the additional assumptions on b and b_n . Namely, it is assumed that b and b_n are symmetric forms on V completely continuous with respect to a and a_n , respectively ($n = 0, 1, \dots$).

3. APPLICATION TO ARONSZAJN'S METHOD

Aronszajn's method is a special case of the approximation (2.3) considered in Section 2. Aronszajn's method is defined for the selfadjoint problem, i.e. b is also symmetric (cf. [1], [2], [7]). Since our theorem admits nonselfadjoint case we will assume that b is nonsymmetric, but $b_n = b$, $n = 0, 1, \dots$

The initial approximate eigenproblem is chosen so as to be easily solvable and $a_0 \leq a$. To construct the intermediate forms a_n one defines $a' = a - a_0$ and a sequence $\{\varphi_j\}$ in V whose elements are linearly independent modulo the null space N of a' in V . Let π_n be the projection, orthogonal with respect to a' , of V onto $\text{span}(\varphi_1, \dots, \varphi_n)$. Define

$$a_n(u) = a_0(u) + a'(\pi_n u) \quad n = 1, 2, \dots$$

Then $a_0 \leq a_1 \leq \dots \leq a$. So a_n is a finite dimensional perturbation of a . In [2] Brown proved the following theorem (cf. Prop. 2.1 and Th. 5.1).

Theorem 8. *If*

- i) a is quasi-bounded with respect to a_0 (thus there exists a symmetric operator \hat{L} , $D(\hat{L}) = V$, such that $a(u, v) = a_0(\hat{L}u, v) \forall u, v \in V$),
- ii) b is completely continuous with respect to a_0 ,
- iii) $V' := N + \text{span}(\varphi_j)$ is dense in X ,
- iv) $\hat{L}(V')$ is dense in X_0 ,

then the condition C 5 is satisfied.

It is easy to see that the assumption C 3 is also satisfied. In fact, since $a(u, v) - a_n(u, v) = a'(u - \Pi_n u, v)$, for $u \in N$ we have $a(u, v) - a_n(u, v) = 0 \forall v \in V$. Moreover, for $u \in \text{span}(\varphi_j) \| \Pi_n u - u \|_X \rightarrow 0$. Thus, since $\|v\|_X \geq \|v\|_0$, we have

$$\begin{aligned} \sup_{\substack{v \in V \\ \|v\|_X = 1}} a'(u - \Pi_n u, v) &\leq \sup_{\substack{v \in V \\ \|v\|_X = 1}} a(u - \Pi_n u, v) + \sup_{\substack{v \in V \\ \|v\|_0 = 1}} a_0(u - \Pi_n u, v) = \\ &= \|u - \Pi_n u\|_X + \|u - \Pi_n u\|_{X_0} \leq 2\|u - \Pi_n u\|_X \rightarrow 0. \end{aligned}$$

So, Theorem 7 yields.

Corollary 2. *If the assumptions (i-iv) of Theorem 8 are satisfied, then the eigenelements of the intermediate problems*

find $\lambda \in \mathbb{C}$ and $0 \neq u \in X_n$ such that

$$b(u, v) = \lambda a_n(u, v) \quad \forall v \in V$$

approximate suitable eigenlements of (2.1) in the sense of the points i)–iii) of Theorem 7.

Similar results for Aronszajn's method are obtained in [2] in another way.

References

- [1] *N. Aronszajn*: Approximation methods for eigenvalues of completely continuous symmetric operator. Proc. of Symposium on Spectral Theory and Differential Equations, Stillwater, Oklahoma, 1951, 179–202.
- [2] *R. D. Brown*: Convergence of approximation methods for eigenvalues of completely continuous quadratic forms. Rocky Mt. J. of Math. 10, No. 1, 1980, 199–215.
- [3] *N. Dunford, J. T. Schwartz*: Linear Operators, Spectral Theory. New York, Interscience 1963.
- [4] *T. Kato*: Perturbation Theory for Linear Operators. Springer Verlag, Berlin 1966.
- [5] *T. Regińska*: External approximation of eigenvalue problems in Banach spaces, RAIRO Numerical Analysis, 1984.
- [6] *F. Stummel*: Diskrete Konvergenz linear Operatoren, I. Math. Ann. 190, 1970, 45–92; II. Math. Z. 120, 1971, 231–264.
- [7] *H. F. Weinberger*: Variational methods for eigenvalue approximation. Reg. Conf. Series in appl. math. 15, 1974.

Souhrn

KONVERGENCE APROXIMAČNÍ METODY PRO PROBLÉM VLASTNÍCH HODNOT DVOU FOREM

TERESA REGIŃSKA

Článek je věnován aproximaci problému vlastních hodnot dvou forem v Hilbertově prostoru X . Zkoumají se aproximační metody generované posloupnostmi forem a_n a b_n definovaných na hustém podprostoru X . Důkaz konvergence těchto metod je založen na teorii vnější aproximace problému vlastních hodnot. Obecné výsledky jsou aplikovány na Aronszajnovu metodu.

Author's address: Dr. *Teresa Regińska*, Institute of Mathematics Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, P. O. B. 137, Poland.