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CONVERGENCE OF CERTAIN COSINE SUMS IN THE METRIC SPACE L

BABU RAM

ABSTRACT. We consider here the L^1 convergence of Rees-Stanojević cosine sums to a cosine trigonometric series belonging to the class S defined by Sidon and deduce as corollaries some previously known results from our result.

1. **Introduction.** Sidon [6] introduced the following class of cosine trigonometric series: Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

be a cosine series satisfying $a_k = o(1)$, $k \to \infty$. If there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, \qquad k \to \infty, \tag{1.2}$$

$$\sum_{k=0}^{\infty} A_k < \infty, \tag{1.3}$$

$$|\Delta a_k| \leqslant A_k, \quad \forall k, \tag{1.4}$$

we say that (1.1) belongs to the class S.

Let the partial sums of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

Recently, Garrett and Stanojević [3] proved that the partial Rees-Stanojević sums [5]

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \tag{1.5}$$

converge in the L^1 metric to (1.1) if and only if given $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon \tag{1.6}$$

for all $n \geqslant 0$. It has been shown in the same paper that the classical Young-Kolmogorov-Stanojević sufficient conditions for integrability of (1.1) imply (1.6).

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Generalising a classical result [1, p. 204], Teljakovskiĭ [7] proved the following.

THEOREM A. *If* (1.1) *belongs to the class* S, *then a necessary and sufficient condition for* L^1 *convergence of* (1.1) *is* $a_n \log n = o(1)$, $n \to \infty$.

2. **Lemmas.** The proofs of our results are based upon the following lemmas.

LEMMA 1 (Fomin [2]). *If* $|c_k| \leqslant 1$, *then*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \frac{\sin(k+1/2)x}{2\sin x/2} \right| dx \leqslant C(n+1),$$

where C *is a positive absolute constant.*

LEMMA 2. *If* (1.1) *belongs to the class* S, *then*

$$g_n(x) = S_n(x) - a_{n+1} D_n(x),$$

where $D_n(x)$ *denotes the Dirichlet kernel.*

PROOF. Since (1.1) belongs to the class S, we have

$$a_k \to 0 \quad \text{and} \quad \sum_{k=0}^{\infty} |\Delta a_k| < \infty.$$

The conditions of Lemma 1 of Garrett and Stanojević [3] are thus satisfied and the result follows.

3. **Main result.** The main result of this paper reads:

THEOREM. *If* (1.1) *belongs to the class* S, *then* (1.6) *holds. Hence*

$$\|f - g_n\|_{L^1} = o(1), \qquad n \to \infty.$$

PROOF. Making use of Abel's transformation and Lemma 1, we have

$$\begin{aligned}
\int_0^{\pi} |f(x) - g_n(x)|\, dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\
&\leqslant \sum_{k=n+1}^{\infty} \Delta A_k \int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx \\
&\leqslant C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k.
\end{aligned}$$

(1.2) and (1.3) now imply the conclusion of the Theorem.

4. **Corollaries.** (i) Using Lemma 2, we notice that

$$\begin{aligned}\int_0^\pi |f(x) - S_n(x)|\,dx &= \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)|\,dx \\ &\leqslant \int_0^\pi |f(x) - g_n(x)|\,dx + \int_0^\pi |g_n(x) - S_n(x)|\,dx \\ &\leqslant \int_0^\pi |f(x) - g_n(x)|\,dx + \int_0^\pi |a_{n+1}D_n(x)|\,dx\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi |a_{n+1}D_n(x)|\,dx &= \int_0^\pi |g_n(x) - S_n(x)|\,dx \\ &\leqslant \int_0^\pi |f(x) - S_n(x)|\,dx + \int_0^\pi |f(x) - g_n(x)|\,dx.\end{aligned}$$

Since $\lim_{n\to\infty} \int_0^\pi |f(x) - g_n(x)|\,dx = 0$ by our Theorem and $\int_0^\pi |a_{n+1}D_n(x)|\,dx$ behaves like $a_{n+1}\log n$ for large values of n, Theorem A of Teljakovskiĭ follows.

(ii) Let $a_k \to 0$ and $\sum_{k=1}^\infty (k+1)|\Delta^2 a_k| < \infty$. Then g_n converges to f in the metric space L since the trigonometric cosine series (1.1) with quasi-convex coefficients belongs to the class S if we choose $A_k = \sum_{m=k}^\infty |\Delta^2 a_m|$. This is Example 1 of [3].

5. **Remark.** In [4], Garrett and Stanojević proved (Corollary B, p. 70) that their Theorem B extends the Teljakovskiĭ result.

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Department of Mathematics, Rohtak University, Rohtak-124001, India