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CONVERGENCE OF COMMON FIXED POINT OF FINITE STEP ITERATION PROCESS FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we give a sufficient condition to converge to common fixed point of a finite step iteration process with errors for two finite families of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces. Also, we establish some weak and strong convergence theorems of the above said scheme and mappings using additional assumptions to the space in the framework of uniformly convex Banach spaces. The results presented in this paper improve and extend some results of Chen and Guo (2011) [1], Sitthikul and Saejung (2009) [19] and many others.

1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

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for all $x, y \in K$.

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \tag{1.2}$$

for all $x \in K$ and $p \in F(T)$.

(3) asymptotically nonexpansive [6] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.3}$$

for all $x, y \in K$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| \tag{1.4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(5) generalized asymptotically quasi-nonexpansive [7] if $F(T) \neq \emptyset$ and there exist sequences $\{k_n\}$ in $[1, \infty)$ and $\{s_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| + s_n \tag{1.5}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(6) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \tag{1.6}$$

for all $x, y \in K$ and $n \geq 1$.

If in definition (5), $s_n = 0$ for all $n \geq 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

Remark 1.1. It is easy to see that if $F(T)$ is nonempty, then nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings are the special cases of generalized asymptotically quasi-nonexpansive mappings.

The class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [6] in 1972 as an important generalization of the class of nonexpansive self-mappings, and proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Since then, iteration processes for asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in Banach spaces have studied extensively by many authors (see [2],[5],[8]-[10],[12]-[18]). In 2002, Xu and

Noor [21] introduced and studied a three-step iteration scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. Cho et al. [3] extended the work of Xu and Noor to a three-step iterative scheme with errors in Banach space and proved the weak and strong convergence theorems for asymptotically nonexpansive mappings. In 2009, Sitthikul and Saejung [19] introduced and studied a finite-step iteration scheme for a finite family of nonexpansive and asymptotically nonexpansive mappings and proved some weak and strong convergence theorems in the setting of Banach spaces. In 2009, Imnang and Suantai [7] introduced and studied a multi-step iteration iteration scheme for a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces. Recently, Chen and Guo [1] introduced and studied a new finite-step iteration scheme with errors for two finite families of asymptotically nonexpansive mappings as follows:

Let K be a nonempty convex subset of a Banach space E with $K+K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N : K \to K$ be $2N$ asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

$$\begin{aligned}
x_1 &= x \in K, \\
x_n^{(0)} &= x_n, \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1-\alpha_n^{(1)}) S_1^n x_n + u_n^{(1)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1-\alpha_n^{(2)}) S_2^n x_n + u_n^{(2)}, \\
&\vdots \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1-\alpha_n^{(N-1)}) S_{N-1}^n x_n + u_n^{(N-1)}, \\
x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1-\alpha_n^{(N)}) S_N^n x_n + u_n^{(N)}, \\
x_{n+1} &= x_n^{(N)}, \quad \forall\, n \geq 1, \qquad (1.7)
\end{aligned}$$

where $\{\alpha_n^{(i)}\} \subset [0,1]$ and $\{u_n^{(i)}\}$ are bounded sequences in K for all $i \in I = \{1, 2, \ldots, N\}$, and the weak and strong convergence theorems are proved, which improve and generalize some results in [19]. Letting $u_n^{(i)} = 0$ for all $n \geq 1$, $i \in I$ in (1.7). We have the following:

$$\begin{aligned}
x_1 &= x \in K, \\
x_n^{(0)} &= x_n, \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1-\alpha_n^{(1)}) S_1^n x_n, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1-\alpha_n^{(2)}) S_2^n x_n, \\
&\vdots
\end{aligned}$$

$$\begin{aligned} x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1-\alpha_n^{(N-1)}) S_{N-1}^n x_n, \\ x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1-\alpha_n^{(N)}) S_N^n x_n, \\ x_{n+1} &= x_n^{(N)}, \quad \forall\, n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $i \in I$ and the author [1] proved weak convergence theorem of iteration scheme (1.8).

The purpose of this paper is to study the weak and strong convergence of the iteration scheme (1.7) and (1.8) to converge to common fixed points for two finite families of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings in the framework of uniformly convex Banach spaces. The results presented in this paper improve and extend the corresponding results of Chen and Guo (2011) [1], Sitthikul and Saejung (2009) [19] and many others.

In order to prove the main results of this paper, we need the following concepts and lemmas:

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Recall that a Banach space E is said to satisfy Opial's condition [11] if, for any sequence $\{x_n\}$ in E, $x_n \to x$ weakly implies that

$$\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

A Banach space E has the Kadec-Klee property [19] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

A mapping $T\colon K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

Lemma 1.2. (See [20]) *Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$*, then* $\lim_{n\to\infty} \alpha_n$ *exists. In particular,* $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges to zero, then* $\lim_{n\to\infty} \alpha_n = 0$*.*

Lemma 1.3. (See [16]) *Let* E *be a uniformly convex Banach space and* $0 < \alpha \le t_n \le \beta < 1$ *for all* $n \in \mathbb{N}$*. Suppose further that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of* E *such that* $\limsup_{n\to\infty} \|x_n\| \le a$*,* $\limsup_{n\to\infty} \|y_n\| \le a$ *and* $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ *hold for some* $a \ge 0$*. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$*.*

Lemma 1.4. (See [19]) *Let* E *be a real reflexive Banach space with its dual* E^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* E *and* $p, q \in w_w(x_n)$ *(where* $w_w(x_n)$ *denotes the set of all weak subsequential limits of* $\{x_n\}$*). Suppose* $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ *exists for all* $t \in [0,1]$*. Then* $p = q$*.*

Proposition 1.5. *Let* K *be a nonempty subset of a Banach space* E *and* $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K$ *be* $2N$ *generalized asymptotically quasi-nonexpansive mappings. Then there exist sequences* $\{k_n\}$*,* $\{h_n\} \subset [1,\infty)$ *and* $\{s_n\}$*,* $\{t_n\} \subset [0,\infty)$ *with* $k_n \to 1$*,* $h_n \to 1$ *and* $s_n \to 0$*,* $t_n \to 0$ *as* $n \to \infty$ *such that*

$$\|S_i^n x - S_i^n y\| \quad \le \quad k_n \|x-y\| + s_n, \quad \forall\, n \ge 1,$$

and

$$\|T_i^n x - T_i^n y\| \quad \le \quad h_n \|x-y\| + t_n, \quad \forall\, n \ge 1,$$

for all $x, y \in K$ *and* $i = 1, 2, \ldots, N$*.*

Proof. Since for each $i = 1, 2, \ldots, N$, S_i, $T_i : K \to K$ are generalized asymptotically quasi-nonexpansive mappings, there exist sequences $\{k_n^{(i)}\}$, $\{h_n^{(i)}\} \subset [1,\infty)$ and $\{s_n^{(i)}\}$, $\{t_n^{(i)}\} \subset [0,\infty)$ with $k_n^{(i)} \to 1$, $h_n^{(i)} \to 1$ and $s_n^{(i)} \to 0$, $t_n^{(i)} \to 0$ as $n \to \infty$ such that

$$\|S_i^n x - S_i^n y\| \quad \le \quad k_n^{(i)} \|x-y\| + s_n^{(i)}, \quad \forall\, n \ge 1,$$

and

$$\|T_i^n x - T_i^n y\| \quad \le \quad h_n^{(i)} \|x-y\| + t_n^{(i)}, \quad \forall\, n \ge 1,$$

for all $x, y \in K$ and $i = 1, 2, \ldots, N$.

Letting

$$k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\}, \quad h_n = \max\{h_n^{(1)}, h_n^{(2)}, \ldots, h_n^{(N)}\},$$

and

$$s_n = \max\{s_n^{(1)}, s_n^{(2)}, \ldots, s_n^{(N)}\}, \quad t_n = \max\{t_n^{(1)}, t_n^{(2)}, \ldots, t_n^{(N)}\},$$

then we have that $\{k_n\}$, $\{h_n\} \subset [1, \infty)$ and $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ with $k_n \to 1$, $h_n \to 1$ and $s_n \to 0$, $t_n \to 0$ as $n \to \infty$ and

$$\|S_i^n x - S_i^n y\| \quad \leq \quad k_n^{(i)} \|x - y\| + s_n^{(i)} \leq k_n \|x - y\| + s_n, \quad \forall\, n \geq 1,$$

and

$$\|T_i^n x - T_i^n y\| \quad \leq \quad h_n^{(i)} \|x - y\| + t_n^{(i)} \leq h_n \|x - y\| + t_n, \quad \forall\, n \geq 1,$$

for all x, $y \in K$ and for each $i = 1, 2, \dots, N$. □

2. Strong convergence theorems

In this section, we first prove the following lemmas in order to prove our main theorems.

Lemma 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N \colon K \to K$ be $2N$ generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1, \infty)$ and $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ given in Proposition 1.1 and $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.7), where $\{\alpha_n^{(i)}\} \subset [0, 1]$ for all $i \in I$ with the following restrictions:*

(i) $\sum_{n=1}^\infty (k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^\infty s_n < \infty, \quad \sum_{n=1}^\infty t_n < \infty$;
(iii) $\sum_{n=1}^\infty \left\| u_n^{(i)} \right\| < \infty$ *for all* $i \in I$.

Then the limit $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in F$.

Proof. Let $q \in F$. Then from (1.7), we have

$$\begin{aligned}
\left\| x_n^{(1)} - q \right\| &= \left\| \alpha_n^{(1)} T_1^n x_n + (1 - \alpha_n^{(1)}) S_1^n x_n + u_n^{(1)} - q \right\| \\
&\leq \alpha_n^{(1)} \|T_1^n x_n - q\| + (1 - \alpha_n^{(1)}) \|S_1^n x_n - q\| + \left\| u_n^{(1)} \right\| \\
&\leq \alpha_n^{(1)} [h_n \|x_n - q\| + t_n] + (1 - \alpha_n^{(1)}) [k_n \|x_n - q\| + s_n] + \left\| u_n^{(1)} \right\| \\
&\leq \alpha_n^{(1)} k_n h_n \|x_n - q\| + \alpha_n^{(1)} t_n + (1 - \alpha_n^{(1)}) k_n h_n \|x_n - q\| \\
&\quad + (1 - \alpha_n^{(1)}) s_n + k_n h_n \left\| u_n^{(1)} \right\| \\
&\leq k_n h_n \|x_n - q\| + k_n h_n \Big\{ \left\| u_n^{(1)} \right\| + s_n + t_n \Big\}. \qquad (2.1)
\end{aligned}$$

Again using (1.7) and (2.1), we obtain

$$\begin{aligned}
\left\|x_n^{(2)}-q\right\| &= \left\|\alpha_n^{(2)}T_2^n x_n^{(1)}+(1-\alpha_n^{(2)})S_2^n x_n+u_n^{(2)}-q\right\| \\
&\le \alpha_n^{(2)}\left\|T_2^n x_n^{(1)}-q\right\|+(1-\alpha_n^{(2)})\left\|S_2^n x_n-q\right\|+\left\|u_n^{(2)}\right\| \\
&\le \alpha_n^{(2)}[h_n\left\|x_n^{(1)}-q\right\|+t_n]+(1-\alpha_n^{(2)})[k_n\left\|x_n-q\right\|+t_n] \\
&\quad +\left\|u_n^{(2)}\right\| \\
&\le \alpha_n^{(2)}k_nh_n\left\|x_n^{(1)}-q\right\|+\alpha_n^{(2)}t_n+(1-\alpha_n^{(2)})k_nh_n\left\|x_n-q\right\| \\
&\quad +(1-\alpha_n^{(2)})s_n+k_nh_n\left\|u_n^{(2)}\right\| \\
&\le \alpha_n^{(2)}k_nh_n\Big[k_nh_n\left\|x_n-q\right\|+k_nh_n\Big\{\left\|u_n^{(1)}\right\|+s_n+t_n\Big\}\Big] \\
&\quad +(1-\alpha_n^{(2)})k_nh_n\left\|x_n-q\right\|+(1-\alpha_n^{(2)})s_n+k_nh_n\left\|u_n^{(2)}\right\| \\
&\le k_n^2h_n^2\left\|x_n-q\right\|+k_n^2h_n^2\Big[\left\|u_n^{(1)}\right\|+\left\|u_n^{(2)}\right\|\Big] \\
&\quad +k_n^2h_n^2(s_n+t_n). \qquad (2.2)
\end{aligned}$$

Continuing the above process, we get that

$$\left\|x_n^{(i)}-q\right\| \le k_n^ih_n^i\left\|x_n-q\right\|+k_n^ih_n^i\sum_{k=1}^{i}\left\|u_n^{(k)}\right\|+k_n^ih_n^i(s_n+t_n) \qquad (2.3)$$

for all $n \ge 1$ and $i \in I$. In particular,

$$\begin{aligned}
&\left\|x_{n+1}-q\right\| = \left\|x_n^{(N)}-q\right\| \\
&\le k_n^Nh_n^N\left\|x_n-q\right\|+k_n^Nh_n^N\sum_{k=1}^{N}\left\|u_n^{(k)}\right\|+k_n^Nh_n^N(s_n+t_n) \\
&= [1+(k_n^Nh_n^N-1)]\left\|x_n-q\right\|+k_n^Nh_n^N\sum_{k=1}^{N}\left\|u_n^{(k)}\right\|+k_n^Nh_n^N(s_n+t_n) \\
&\le [1+(k_n^Nh_n^N-1)]\left\|x_n-q\right\|+R\sum_{k=1}^{N}\left\|u_n^{(k)}\right\|+R(s_n+t_n) \qquad (2.4)
\end{aligned}$$

for some $R > 0$ and all $n \ge 1$. It follows from the conditions (i), (ii) and (iii) that $\sum_{n=1}^{\infty}(k_n^Nh_n^N-1) < \infty$, $\sum_{n=1}^{\infty}(s_n+t_n) < \infty$ and $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N}\left\|u_n^{(k)}\right\|\right) < \infty$. By Lemma 1.2, we have that $\lim_{n\to\infty}\left\|x_n-q\right\|$ exists. This completes the proof. □

Lemma 2.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E with $K+K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N$: $K \to K$ be $2N$ uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1,\infty)$ and $\{s_n\}$, $\{t_n\} \subset [0,\infty)$ given in Proposition 1.1 and $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.7), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$ with the following restrictions:*

(i) $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} s_n < \infty, \quad \sum_{n=1}^{\infty} t_n < \infty$;
(iii) $\sum_{n=1}^{\infty} \left\| u_n^{(i)} \right\| < \infty$ *for all* $i \in I$.

Then $\lim_{n\to\infty} \left\| S_i^n x_n - T_i^n x_n^{(i-1)} \right\| = 0$ *for all* $i \in I$.

Proof. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. So we can assume that

$$\lim_{n\to\infty} \|x_n - q\| = d \tag{2.5}$$

for all $q \in F$, where $d \geq 0$ is nonnegative number. It follows from condition (iii), (2.3), (2.5) and $\lim_{n\to\infty} k_n h_n = 1$ that

$$\limsup_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| \leq d \tag{2.6}$$

and so

$$\limsup_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - q + u_n^{(N)} \right\| \leq d. \tag{2.7}$$

Also,

$$\limsup_{n\to\infty} \left\| S_N^n x_n - q + u_n^{(N)} \right\| \leq d. \tag{2.8}$$

Further, from (1.7) and (2.5), we have

$$\begin{aligned} d &= \lim_{n\to\infty} \|x_n^{(N)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N)}(T_N^n x_n^{(N-1)} - q + u_n^{(N)}) + (1-\alpha_n^{(N)})(S_N^n x_n - q + u_n^{(N)})\|. \end{aligned}$$

By Lemma 1.3, we get that

$$\lim_{n\to\infty} \left\| S_N^n x_n - T_N^n x_n^{(N-1)} \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - q + u_n^{(N)} \right\| = d.$$

From (2.6), we have

$$\begin{aligned} d &= \liminf_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - q + u_n^{(N)} \right\| \\ &\le \liminf_{n\to\infty} \left[h_n \left\| x_n^{(N-1)} - q \right\| + t_n \right] + \lim_{n\to\infty} \left\| u_n^{(N)} \right\| \\ &= \liminf_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| \le \limsup_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| \le d \end{aligned}$$

and so

$$\lim_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| = d. \tag{2.9}$$

It follows from the condition (iii), (2.3), (2.5) and $\lim_{n\to\infty} k_n h_n = 1$ that

$$\limsup_{n\to\infty} \left\| x_n^{(N-2)} - q \right\| \le d.$$

Further, we know that

$$\limsup_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - q + u_n^{(N-1)} \right\| \le d \tag{2.10}$$

and

$$\limsup_{n\to\infty} \left\| S_{N-1}^n x_n - q + u_n^{(N-1)} \right\| \le d. \tag{2.11}$$

From (1.7) and (2.9), we have

$$\begin{aligned} d &= \lim_{n\to\infty} \|x_n^{(N-1)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - q + u_n^{(N-1)}) \\ &\quad + (1-\alpha_n^{(N-1)})(S_{N-1}^n x_n - q + u_n^{(N-1)})\|. \end{aligned} \tag{2.12}$$

It follows from (2.9)-(2.11) and Lemma 1.3 that

$$\lim_{n\to\infty} \left\| S_{N-1}^n x_n - T_{N-1}^n x_n^{(N-2)} \right\| = 0.$$

Continuing the above process, we obtain the result of Lemma 2.2. This completes the proof. □

Lemma 2.3. *Under the assumptions of Lemma 2.2, if*

$$\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0 \tag{2.13}$$

for all $i \in I$. Then

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \forall\, i \in I.$$

Proof. Since $\lim_{n\to\infty}\left\|S_i^n x_n - T_i^n x_n^{(i-1)}\right\| = 0$ for all $i \in I$ by Lemma 2.2. It follows from (2.13) that

$$\lim_{n\to\infty}\left\|x_n - T_i^n x_n^{(i-1)}\right\| = 0 \tag{2.14}$$

for all $i \in I$. Next, from (1.7), we have

$$\|x_n - x_{n+1}\| \le \alpha_n^{(N)}\left\|x_n - T_N^n x_n^{(N-1)}\right\| + (1-\alpha_n^{(N)})\|x_n - S_N^n x_n\| + \left\|u_n^{(N)}\right\|.$$

Using (2.13), (2.14) and $\lim_{n\to\infty}\left\|u_n^{(N)}\right\| = 0$, we have

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{2.15}$$

Since $\lim_{n\to\infty}\|x_n - T_1^n x_n\| = 0$ by (2.14) and

$$\begin{aligned}
\|x_n - T_i^n x_n\| &\le \left\|x_n - T_i^n x_n^{(i-1)}\right\| + \left\|T_i^n x_n^{(i-1)} - T_i^n x_n\right\| \\
&\le \left\|x_n - T_i^n x_n^{(i-1)}\right\| + L\left\|x_n^{(i-1)} - x_n\right\| \\
&\le \left\|x_n - T_i^n x_n^{(i-1)}\right\| + L\alpha_n^{(i-1)}\left\|T_{i-1}^n x_n^{(i-2)} - x_n\right\| \\
&\quad + L(1-\alpha_n^{(i-1)})\left\|S_{i-1}^n x_n - x_n\right\| + L\left\|u_n^{(i-1)}\right\| \qquad (2.16)
\end{aligned}$$

for all $i = 1, 2, \ldots, N$. From (2.13), (2.14), (2.16) and $\lim_{n\to\infty}\left\|u_n^{(i-1)}\right\| = 0$, we have

$$\lim_{n\to\infty}\|x_n - T_i^n x_n\| = 0 \tag{2.17}$$

for all $i \in I$. It follows from (2.15) and (2.17) that

$$\begin{aligned}
\|x_n - T_i x_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + \left\|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\right\| \\
&\quad + \left\|T_i^{n+1} x_n - T_i x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + L\|x_{n+1} - x_n\| \\
&\quad + L\|T_i^n x_n - x_n\| \\
&\le (1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\
&\quad + L\|T_i^n x_n - x_n\|. \qquad (2.18)
\end{aligned}$$

Using (2.15) and (2.17), we get that

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = 0.$$

for all $i \in I$. This completes the proof. □

Lemma 2.4. *Under the assumptions of Lemma* 2.2*, if*

$$\|x - T_i y\| \quad \leq \quad \|S_i x - T_i y\| \tag{2.19}$$

for all x, $y \in K$ *and* $i \in I$*. Then*

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \forall\, i \in I.$$

Proof. By (2.19), we obtain that

$$\begin{aligned} 0 \quad &\leq \quad \left\|x_n - T_i^n x_n^{(i-1)}\right\| \leq \left\|S_i x_n - T_i^n x_n^{(i-1)}\right\| \\ &\leq \quad \left\|S_i^n x_n - T_i^n x_n^{(i-1)}\right\| \end{aligned} \tag{2.20}$$

for all $i \in I$. It follows from (2.20) and Lemma 2.2 that

$$\lim_{n\to\infty} \left\|S_i x_n - T_i^n x_n^{(i-1)}\right\| = \lim_{n\to\infty} \left\|x_n - T_i^n x_n^{(i-1)}\right\| = 0. \tag{2.21}$$

Since

$$\|x_n - S_i x_n\| \quad \leq \quad \left\|x_n - T_i^n x_n^{(i-1)}\right\| + \left\|T_i^n x_n^{(i-1)} - S_i x_n\right\|. \tag{2.22}$$

Using (2.21) in (2.22), we obtain

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0 \tag{2.23}$$

for all $i \in I$. Also,

$$\|x_n - S_i^n x_n\| \quad \leq \quad \left\|x_n - T_i^n x_n^{(i-1)}\right\| + \left\|T_i^n x_n^{(i-1)} - S_i^n x_n\right\|. \tag{2.24}$$

Using (2.21) and Lemma 2.2 in (2.24), we obtain

$$\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0 \tag{2.25}$$

for all $i \in I$. Thus $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$ by Lemma 2.3. This completes the proof. □

Theorem 2.5. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N : K \to K$ be $2N$ uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1, \infty)$ and $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ given in Proposition* 1.1 *and $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by* (1.7)*, where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0, 1)$ and all $i \in I$ with the following restrictions:*

(i) $\sum_{n=1}^\infty (k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^\infty s_n < \infty, \quad \sum_{n=1}^\infty t_n < \infty$;
(iii) $\sum_{n=1}^\infty \left\|u_n^{(i)}\right\| < \infty$ *for all* $i \in I$.

Then $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$ *in* K *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf\{\|x - y\| : y \in F\}$.

Proof. The necessity of Theorem 2.5 is obvious. So, we will prove the sufficiency. Assume that $\liminf_{n\to\infty} d(x_n, F) = 0$. Taking the infimum over all $q \in F$ in (2.4), we have

$$d(x_{n+1}, F) \leq [1 + (k_n^N h_n^N - 1)]d(x_n, F) + R\sum_{k=1}^{N} \left\|u_n^{(k)}\right\| + R(s_n + t_n).$$

By using the conditions (i) - (iii) and Lemma 1.2, we know that $\lim_{n\to\infty} d(x_n, F)$ exists and so $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. In fact, letting $b_n = (k_n^N h_n^N - 1)$, $c_n = R\Big\{\sum_{k=1}^{N} \left\|u_n^{(k)}\right\| + s_n + t_n\Big\}$ in (2.4). For any positive integers m, n, $m > n$, from $1 + x \leq e^x$ for all $x \geq 0$ and (2.4), we have

$$\begin{aligned}
\|x_m - q\| &\leq (1 + b_{m-1})\|x_{m-1} - q\| + c_{m-1} \\
&\leq e^{b_{m-1}}\|x_{m-1} - q\| + c_{m-1} \\
&\leq e^{b_{m-1}}[e^{b_{m-2}}\|x_{m-2} - q\| + c_{m-2}] + c_{m-1} \\
&\leq e^{(b_{m-1}+b_{m-2})}\|x_{m-2} - q\| + e^{b_{m-1}}c_{m-2} + c_{m-1} \\
&\leq e^{(b_{m-1}+b_{m-2})}\|x_{m-2} - q\| + e^{b_{m-1}}[c_{m-2} + c_{m-1}] \\
&\leq \dots \\
&\leq \Big(\sum_{k=n}^{m-1} e^{b_k}\Big)\|x_n - q\| + \Big(\sum_{k=n}^{m-2} e^{b_k}\Big)\sum_{k=n}^{m-1} c_k \\
&\leq \Big(\sum_{k=n}^{m-1} e^{b_k}\Big)\|x_n - q\| + \Big(\sum_{k=n}^{m-1} e^{b_k}\Big)\sum_{k=n}^{m-1} c_k \\
&\leq Q\|x_n - q\| + Q\sum_{k=n}^{\infty} c_k,
\end{aligned}$$

where $Q = \sum_{n=1}^{\infty} e^{b_k}$. Thus for any $q \in F$, we have

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - q\| + \|x_n - q\| \\
&\leq (1 + Q)\|x_n - q\| + Q\sum_{k=n}^{\infty} c_k.
\end{aligned}$$

Taking the infimum over all $q \in F$, we obtain that

$$\|x_m - x_n\| \leq (1 + Q)d(x_n, F) + Q\sum_{k=n}^{\infty} c_k.$$

It follows from $\sum_{n=1}^{\infty} c_n < \infty$ and $\lim_{n\to\infty} d(x_n, F) = 0$ that $\{x_n\}$ is a Cauchy sequence, K is a closed subset of E and so $\{x_n\}$ converges strongly to $q_0 \in K$. Further, $F(T_i)$ and $F(S_i)$ $(i = 1, 2, \ldots, N)$ are closed sets, and so F is a closed subset of K. Therefore, $q_0 \in F$, that is, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ in K. This completes the proof. □

A family $\{T_i : 1, 2, \ldots, m\}$ of m self-mappings of K with $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ is said to satisfy condition (B) [1] K if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ and such that $\max_{1\le i\le m} \{ \|x - T_i x\| \} \geq f(d(x, F))$ for all $x \in K$.

As an application of our Theorem 2.5, we establish another strong convergence result as follows.

Theorem 2.6. *Under the assumptions of Lemma* 2.4*, if the family* $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ *satisfies condition* (B)*. Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$*.*

Proof. By Lemma 2.4, we know that $\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$, and so $\max_{1\le i\le N} \{ \|x_n - S_i x_n\|, \|x_n - T_i x_n\| \} \to 0$ $(n \to \infty)$. It follows from the condition (B) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. By the proof of Theorem 2.5, we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function with $f(0) = 0$ and so $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.5, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$. This completes the proof. □

Remark 2.7. Since a nonexpansive and an asymptotically nonexpasive mappings with $F(T) \neq \emptyset$ are asymptotically quasi-nonexpasive mappings and hence generalized asymptotically quasi-nonexpasive mappings. Theorem 2.6 improves and generalizes Theorem 2.2 in [1] and Theorem 1 in [19].

Theorem 2.8. *Under the assumptions of Lemma* 2.4*, if there exists a* T_i *or* S_i*,* $i \in I$*, which is semi-compact. Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$*.*

Proof. Without loss of generality, we can assume that T_1 is semi-compact. From Lemma 2.1 we know that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - S_i x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$ by Lemma 2.4. Since T_1 is semi-compact and $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ as $i \to \infty$. Thus

$$\|x^* - T_i x^*\| = \lim_{i\to\infty} \|x_{n_i} - T_i x_{n_i}\| = 0$$

and

$$\|x^* - S_i x^*\| = \lim_{i\to\infty} \|x_{n_i} - S_i x_{n_i}\| = 0$$

for all $i \in I$. Which implies that $x^* \in F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i)$ and so $\liminf_{n\to\infty} d(x_n, F) \le \liminf_{i\to\infty} d(x_{n_i}, F) \le \lim_{i\to\infty} \|x_{n_i} - x^*\| = 0$. It follows from Theorem 2.5 that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$ in K. This completes the proof. □

Remark 2.9. Since an asymptotically nonexpasive mappings with $F(T) \neq \emptyset$ is an asymptotically quasi-nonexpasive mapping and hence generalized asymptotically quasi-nonexpasive mappings. Theorem 2.8 improves and generalizes Theorem 2.3 in [1].

3. Weak convergence theorems

In this section, we prove weak convergence theorems of the iteration scheme (1.7) and (1.8) in uniformly convex Banach spaces.

Theorem 3.1. *Under the assumptions of Lemma* 2.4*, if E satisfying Opial's condition. Assume that the mappings $I - S_i$ and $I - T_i$ for all $i \in I$, where I denotes the identity mapping, are demiclosed at zero. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings* $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$.

Proof. Let $q \in F$, from Lemma 2.1 the sequence $\{\|x_n - q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q^* \in K$. From Lemma 2.4, we get that

$$\lim_{n\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0 \text{ and } \lim_{n\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for all $i \in I$. Since the mappings $I - S_i$ and $I - T_i$ for all $i \in I$ are demiclosed at zero, therefore $S_i q^* = q^*$ and $T_i q^* = q^*$, which means $q^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to q^*. Suppose on contrary that there is another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p^* \in K$ and $q^* \neq p^*$. Then by the same method as given above, we can also prove that $p^* \in F$. From Lemma 2.1 the limits $\lim_{n\to\infty} \|x_n - q^*\|$ and $\lim_{n\to\infty} \|x_n - p^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - q^*\| = \lim_{n_k\to\infty} \|x_{n_k} - q^*\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - p^*\| = \lim_{n\to\infty} \|x_n - p^*\| = \lim_{n_j\to\infty} \left\|x_{n_j} - p^*\right\| \\
&< \lim_{n_j\to\infty} \left\|x_{n_j} - q^*\right\| = \lim_{n\to\infty} \|x_n - q^*\|
\end{aligned}$$

which is a contradiction so $q^* = p^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$. This completes the proof. □

Lemma 3.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N$ $: K \to K$ be $2N$ uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1, \infty)$ and $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ given in Proposition 1.5 and $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.8), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$ with the following restrictions:*

(i) $\sum_{n=1}^{\infty} (k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} s_n < \infty, \quad \sum_{n=1}^{\infty} t_n < \infty$.

Then $\lim_{n\to\infty} \|t x_n + (1-t)p - q\|$ exists for all p, $q \in F$ and $t \in [0,1]$.

Proof. By Lemma 2.1, we know that $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|t x_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 2.1. It, therefore, remains to prove the Lemma 3.2 for $t \in (0,1)$. For all $x \in K$, we define the mapping $T_n \colon K \to K$ by

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1 - \alpha_n^{(1)}) S_1^n x_n, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1 - \alpha_n^{(2)}) S_2^n x_n, \\
&\vdots \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - \alpha_n^{(N-1)}) S_{N-1}^n x_n, \\
T_n(x) &= \alpha_n^{(N)} T_N^n x^{(N-1)} + (1 - \alpha_n^{(N)}) S_N^n x.
\end{aligned}$$

It is easy to prove

$$\|T_n x - T_n y\| \leq u_n \|x - y\| + c_n', \tag{3.1}$$

for all $x, y \in K$, where $c_n' = R(s_n + t_n)$, $u_n = (1 + b_n)$ and $b_n = (k_n^N h_n^N - 1)$ with $\sum_{n=1}^{\infty} c_n' < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and $u_n \to 1$ as $n \to \infty$. Setting

$$S_{n,m} = T_{n+m-1} T_{n+m-2} \dots T_n, \quad m \geq 1 \tag{3.2}$$

and

$$b_{n,m} = \left\| S_{n,m}(t x_n + (1-t)p) - (t S_{n,m} x_n + (1-t) S_{n,m} q) \right\|. \tag{3.3}$$

From (3.1) and (3.2), we have

$$\begin{aligned} \|S_{n,m}x - S_{n,m}y\| &\leq u_n\, u_{n+1}\, \dots\, u_{n+m-1}\, \|x-y\| + \sum_{i=n}^{n+m-1} c_i' \\ &\leq \Big(\prod_{j=n}^{n+m-1} u_j\Big)\, \|x-y\| + \sum_{i=n}^{n+m-1} c_i' \\ &= L_n\, \|x-y\| + \sum_{i=n}^{n+m-1} c_i' \qquad (3.4) \end{aligned}$$

for all $x, y \in K$, where $L_n = \prod_{j=n}^{n+m-1} u_j$ with $L_n \to 1$ and $S_{n,m}x_n = x_{n+m}$, $S_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m} + L_n a_n(t) + \sum_{i=n}^{n+m-1} c_i'. \qquad (3.5) \end{aligned}$$

By using $\Big[$[4], Theorem 2.3$\Big]$, we have

$$\begin{aligned} b_{n,m} &\leq \phi^{-1}(\|x_n - p\| - \|S_{n,m}x_n - S_{n,m}p\|) \\ &\leq \phi^{-1}(\|x_n - p\| - \|x_{n+m} - p + p - S_{n,m}p\|) \\ &\leq \phi^{-1}\Big(\|x_n - p\| - (\|x_{n+m} - p\| - \|S_{n,m}p - p\|)\Big), \qquad (3.6) \end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges to 0 as $n \to \infty$ for all $m \geq 1$. Thus, fixing n and letting $m \to \infty$ in (3.5), we have

$$\begin{aligned} \limsup_{m\to\infty} a_{n+m}(t) &\leq \phi^{-1}\Big(\|x_n - p\| - \Big(\lim_{m\to\infty} \|x_m - p\| - \|S_{n,m}p - p\| \Big)\Big) \\ &\quad + L_n a_n(t) + \sum_{i=n}^{n+m-1} c_i', \qquad (3.7) \end{aligned}$$

and again letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n(t) \leq \phi^{-1}(0) + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$. This completes the proof. □

Theorem 3.3. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N : K \to K$ be $2N$ uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1, \infty)$ and $\{s_n\}$, $\{t_n\} \subset [0, \infty)$ given in Proposition 1.5 and $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.8), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$ with the following restrictions:*

(i) $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$;
(ii) $\sum_{n=1}^{\infty} s_n < \infty, \quad \sum_{n=1}^{\infty} t_n < \infty$;
(iii) $\|x - T_i y\| \le \|S_i x - T_i y\|$ *for all* $x, y \in K$ *and* $i \in I$.

If the mappings $I - S_i$ and $I - T_i$ for all $i \in I$, where I denotes the identity mapping, are demiclosed at zero. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$.

Proof. By Lemma 2.1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. By Lemma 2.4, we get that

$$\lim_{n\to\infty} \left\|x_{n_j} - S_i x_{n_j}\right\| = 0 \text{ and } \lim_{n\to\infty} \left\|x_{n_j} - T_i x_{n_j}\right\| = 0$$

for all $i \in I$. Since the mappings $I - S_i$ and $I - T_i$ for all $i \in I$ are demiclosed at zero, therefore $S_i p = p$ and $T_i p = p$ for all $i \in I$ which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and $p,\, q \in w_w(x_n)$. By Lemma 3.2, the limit

$$\lim_{n\to\infty} \|t x_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 1.4. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

Remark 3.4. Since a nonexpansive and an asymptotically nonexpasive mappings with $F(T) \neq \emptyset$ are asymptotically quasi-nonexpasive mappings and hence generalized asymptotically quasi-nonexpansive mappings. Theorem 3.3 improves and generalizes Theorem 3.2 in [1] and Theorem 2 in [19].

Example 3.5. Let $E = [-\pi,\, \pi]$ and let T be defined by

$$Tx = x \, cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |x||cos\, x| \le |x| = \|x - z\|\,,$$

and hence T is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$\|Tx - Ty\| = \left\|\frac{\pi}{2}\, cos\, \frac{\pi}{2} - \pi\, cos\, \pi\right\| = \pi,$$

whereas

$$\|x - y\| = \left\|\frac{\pi}{2} - \pi\right\| = \frac{\pi}{2}.$$

Example 3.6. Let $E = \mathbb{R}$ and let T be defined by

$$T(x) = \begin{cases} \frac{x}{2}\, cos\, \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\, cos\, \frac{1}{x}$. Thus $2 = cos\, \frac{1}{x}$. This is impossssible. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$\|Tx - z\| = \|Tx - 0\| = |\frac{x}{2}||cos\, \frac{1}{x}| \leq \frac{|x|}{2} < |x| = \|x - z\|\,,$$

and hence T is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$\|Tx - Ty\| = \left\|\frac{1}{3\pi}\, cos\, \frac{3\pi}{2} - \frac{1}{2\pi}\, cos\, \pi\right\| = \frac{1}{2\pi},$$

whereas

$$\|x - y\| = \left\|\frac{2}{3\pi} - \frac{1}{\pi}\right\| = \frac{1}{3\pi}.$$

4. Conclusion

By Remark 1.1 it is clear that if $F(T)$ is nonempty, then nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings are the special cases of generalized asymptotically quasi-nonexpansive mappings, thus our results are good improvement and generalization of corresponding results of [1, 19] and many others from the existing literature.

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