

CONVERGENCE OF EMPIRICAL PROCESSES OF  
 MIXING rv's ON  $[0, 1]$

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Conditions are given for the weak convergence of weighted empirical cumulative processes of three types of mixing random variables (rv's) on  $[0, 1]$ .

**1. Introduction.** In this section we define three types of mixing conditions and derive a basic lemma concerning them. Section 2 gives a central limit theorem for sums of uniformly bounded strongly-mixing rv's with some examples. Section 3 applies Section 1 and Section 2 to obtain the main results—the convergence of empirical processes to Gaussian processes. This work generalises a theorem of Koul [4] for independent rv's.

Let  $\{X_{iN}, i = 1, 2, \dots, n_N\}$  be a sequence of rv's defined on some probability space,  $N = 1, 2, \dots$ . For  $1 \leq a \leq b \leq n_N$  let  $M_{a,b}^N$  be the  $\sigma$ -algebra generated by  $X_{aN}, \dots, X_{bN}$ . Let  $\phi_N, \phi_N, \alpha_N$  be functions on  $\{0, 1, \dots, n_N - 1\}$  such that  $\phi_N(0) = \infty, \phi_N(0) = \alpha_N(0) = 1$ . Suppose that for  $1 \leq k \leq k + i \leq n_N, A \in M_{1,k}^N, B \in M_{k+i, n_N}^N$

$$(1) \quad |P(AB) - P(A)P(B)| \leq \phi_N(i)P(A)P(B)$$

then we call  $\{X_{iN}\}$   $\phi_N$ -mixing. If we replace (1) by

$$(2) \quad |P(AB) - P(A)P(B)| \leq \phi_N(i)P(A)$$

or

$$(3) \quad |P(AB) - P(A)P(B)| \leq \alpha_N(i)$$

we call  $\{X_{iN}\}$   $\phi_N$ -mixing or  $\alpha_N$ -mixing respectively.

This extension of the usual notions (e.g. see Phillip [5], [6] for  $\phi_N(i) = \phi(i\gamma_N), \phi_N(i) = \phi(i\gamma_N), \alpha_N(i) = \alpha(i)$ ) allows us to obtain C.L.T. results even when  $\phi_N(1) \rightarrow \infty$  or  $\sum_i \alpha_N(i) \rightarrow \infty$  as  $N \rightarrow \infty$ .

**LEMMA 1.** Suppose  $1 \leq k \leq k + i \leq n_N$ . Let  $X, Y$  be real rv's measurable  $M_{1,k}^N$  and  $M_{k+i, n_N}^N$ , respectively. Then each of the following is an upper bound (when finite) for  $|EXY - EXEY|$ , for  $\{X_{iN}\}$   $\phi_N$ -mixing, ( $\phi_N$ -mixing,  $\alpha_N$ -mixing respectively).

- (a)  $\phi_N(i)E|X|E|Y|$
- (b)  $2\phi_N^{1/p}(i)E^{1/p}|X|^pE^{1/q}|Y|^q$ , for  $p^{-1} + q^{-1} = 1, 1 < p < \infty$
- (c)  $2\phi_N(i)C_2E|X|$ , for  $|Y| \leq C_2$
- (d)  $4\alpha_N(i)C_1C_2$ , for  $|X| \leq C_1, |Y| \leq C_2$

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- (e)  $10\alpha_N(i)^{1-1/p} C_2 2^{-1/p} E^{1/p} |X|^p$ , for  $|Y| \leq C_2$ ,  $1 \leq p < \infty$
- (f)  $k\alpha_N(i)^{1-1/p-1/q}$ , for  $1 \leq p < \infty$ ,  $1 \leq q < \infty$ ,

where  $k = K(E|X|^p, E|Y|^q)$  and  $K(x, y) = 8((x + y + x^2y^2)/2)^{1/p+1/q}$ .

PROOF. For (a) to (d) see Lemmas 1, 2 of Phillip [5] and page 171 of Billingsley [1]. (f) is proved by an easy generalisation of Lemma 1.3 of Ibragimov [3], who gives  $K(x, y) = 4 + 2(x + y + x^2y^2)$  for  $p = q$ . An alternative value,  $K(x, y) = 10x^{1/p}y^{1/q}$ , is given by Lemma 1 of Deo [2], and thus implies (e). (These authors all consider the stationary case).

**2. A C.L.T. for uniformly bounded rv's.**

THEOREM 1. Let  $\{X_{iN}\}$  be real rv's satisfying (3). Suppose  $EX_{iN} = 0$ ,  $|X_{iN}| \leq C < \infty$ ,  $1 \leq i \leq n_N$ , and  $t_N^2 = E(\sum_{j=1}^n X_{jN})^2$ ,  $N \geq 1$ , where  $n = n_N \rightarrow \infty$  as  $N \rightarrow \infty$ . Let  $k = k_N$ ,  $p = p_N$ ,  $q = q_N$  be sequences of positive integers such that  $k(p + q) \leq n$ , and

(4)  $t_N^{-2} q^2 k \sum_1^{k-1} \alpha_N(jp) \rightarrow 0$

(5)  $t_N^{-2} k q \sum_0^{q-1} \alpha_N(j) \rightarrow 0$

(6)  $t_N^{-2} q(p + q) \sum_1^{k-1} \alpha_N(jp) \rightarrow 0$

(7)  $t_N^{-2} (n - k(p + q)) \sum_1^{n-k(p+q)} \alpha_N(j) \rightarrow 0$

$k\alpha_N(q) \rightarrow 0$ ,  $t_N^{-4} k \{ p \sum_0^{p-1} j^2 \alpha_N(j) + p^2 (\sum_0^{p-1} \alpha_N(j))^2 \} \rightarrow 0$ , as  $N \rightarrow \infty$ . Then

(8)  $t_N^{-1} \sum_{j=1}^n X_{jN} \rightarrow_{\mathcal{L}} N(0, 1)$ .

PROOF. This follows from the proof of Theorem 1.6 of Ibragimov, [2]: define  $S_N''$  analogously to  $S_N''$  on line 4, page 359. Equations (4) and (5) deal with the cases  $i \neq j$ ,  $i = j$  respectively for the first term on the R.H.S. of line 4. Equations (6) and (7) ensure that the second and third terms on the R.H.S. of line 4 tend to zero. Hence  $ES_N''^2 \rightarrow 0$ .

Finally, in place of  $E|\sum_1^p x_i|^4 \leq MC^4 p^3 / 1np$  we have used

$$E|\sum_{1+j}^{p+j} X_{iN}|^4 \leq 4! 4C^4 \{ 3p \sum_0^{p-1} (i + 1)^2 \alpha_N(i) + 4p^2 (\sum_0^{p-1} \alpha_N(i))^2 \}$$

(cf. Lemma 4, page 172 of [1].)

COROLLARY 1. Let  $\{X_{iN}\}$  be real uniformly bounded rv's with mean zero satisfying (3). Suppose  $E(\sum_1^n X_{jN})^2/n \rightarrow T < \infty$  and

(9)  $n = n_N \rightarrow \infty$  as  $N \rightarrow \infty$ .

Then any of the following sets of conditions are sufficient to ensure that  $n^{-1/2} \sum_{j=1}^n X_{jN} \rightarrow_{\mathcal{L}} N(0, T)$ .

(a)  $\max_{1 \leq i \leq n} \alpha_N(i) = o(n^{-1/2})$

(b)  $\max_{1 \leq i < n} i \log i \alpha_N(i) = o(e^{1/2} / \log^{1/2} e)$ , where  $e = \log n_N$ .

(c)  $\sum_1^{n-1} \alpha_N(i) \leq k < \infty$  and  $n_N^{1-a} \alpha_N([n_N^b]) \rightarrow 0$ , where  $0 < 2b < a < 1 - b$ ,  $n = n_N$ .

- (d)  $\max_{1 \leq i < n} i^2 \alpha_N(i) = o(n^{\frac{3}{2}})$ .
- (e)  $\sum_1^{n-1} i^2 \alpha_N(i) = o(n_N^{\frac{3}{2}})$ , and  $\max_1^{n-1} \alpha_N(i) \leq K < \infty$ .
- (f)  $\sum_1^j i^2 \alpha_N(i) \leq K j^r$ ,  $1 \leq j < n_N$ , where  $r < \frac{3}{2}$ .
- (g)  $\sum_1^{n-1} i^2 \alpha_N(i) \leq K n^r$  and  $\sum_1^{n-1} \alpha_N(i) < K n^d$ ,

where either  $0 \leq d \leq \frac{1}{2}$ ,  $r < \frac{4}{3} - 2d$  or  $\frac{1}{2} \leq d \leq \frac{3}{10}$ ,  $r < \frac{3}{2} - 4d$ .

PROOF. If  $T = 0$  conditions (a)—(g) are redundant. If  $T > 0$  this is just a matter of checking the conditions of Theorem 1 with  $k = [n/(p + q)]$ , and  $p = [n^a]$ ,  $q = [n^b]$  where

- (a)  $a = \frac{1}{2}$ ,  $b = \frac{1}{4}$
- (d)  $a = \frac{5}{9}$ ,  $b = \frac{1}{3}$
- (e)  $a = \frac{1}{2}$ ,  $b = \frac{1}{4}$
- (f)  $a = \frac{2}{3}$ ,  $6b = (4 + \epsilon)/(1 + \epsilon)$  where  $\epsilon = 3/r - 2$
- (g)  $a = 1 - \epsilon - 2d$ ,  $b = \min(\frac{2}{3}, \frac{3}{4} - d)$  where  $\epsilon > 0$  is small.

For (b), use  $p = [n^{\frac{1}{2}} \cdot e^{\frac{1}{2}} \cdot \log^{\frac{1}{2}} e]$ ,  $q = [n^{\frac{1}{2}} \cdot e^{-\frac{1}{2}} \cdot \log^{-\frac{1}{2}} e]$ , and in (e)—(g) apply inequalities such as  $\sum_1^{n-1} \alpha_N(i) \leq K$ ,  $K$  an integer  $\Rightarrow \sum_1^p i^2 \alpha_N(i) \leq K \sum_{p-K+1}^p i^2 = O(p)$ .

For (f) one uses  $\sum_1^k \alpha_N(jp) \leq \sup \sum_1^k \beta_j = \sum_1^k K j^{-2} p^{r-2} (j^r - (j-1)^r)$ , where the sup is taken over  $\{\sum_1^j i^2 \beta_i \leq K j^r p^{r-2}, \beta_j \geq 0, j = 1, \dots, k\}$ .

NOTE. Of course there is no loss in assuming  $K = 1$  in (e). By (b),  $M$  in Theorem 1.6 of [2] can be improved to  $o(e^2/\ln e)^{\frac{1}{2}}$  where  $e = \ln n$ .

As an example we show that Theorem 3.2 of Serfling [8] holds with the assumption of strict stationarity removed provided that  $A^2$  given by (3.8) of [8] is well defined. This follows from

COROLLARY 2. Suppose for  $j = 1, 2$ ,  $\{X_{iN}^{(j)}, i = 1, \dots, n^{(j)}\}$  are  $\alpha_N$ -mixing sequences of real rv's with  $\alpha_N = \alpha_N^{(j)}$ ,  $n^{(j)} = n_N^{(j)}$  and  $n^{(1)}/n^{(2)} \rightarrow C < \infty$  as  $N \rightarrow \infty$ . Suppose that the two sequences are independent, that for each  $i$ ,  $X_{iN}^{(j)}$  has continuous cdf  $F_N^{(j)}$ , that

$$\sum_1^{n^{(j)-1}} \alpha_N^{(j)} \leq K < \infty,$$

and  $\alpha_N^{(j)}$  satisfies one of the sets of conditions (a)—(e) of Corollary 1,  $j = 1, 2$ . Let  $U_N$  be the two-sample Wilcoxon statistic

$$U_N = (n^{(1)}n^{(2)})^{-1} \sum_{i=1}^{n^{(1)}} \sum_{j=1}^{n^{(2)}} s(X_{jN}^{(2)} - X_{iN}^{(1)})$$

where  $s(u) = -1, 0, 1$  according as  $u < 0, = 0, > 0$ . Let  $\gamma_N = 2 \int F_N^{(1)} dF_N^{(2)} - 1$ . Then

$$n^{(1)}(U_N - \gamma_N) \rightarrow_{\mathcal{D}} N(0, 4V)$$

where

$$V = \lim_{N \rightarrow \infty} n^{(1)-1} \text{Var} \sum_1^{n^{(1)}} x_i + C \lim_{N \rightarrow \infty} n^{(2)-1} \text{Var} \sum_1^{n^{(2)}} y_i,$$

$$x_i = F_N^{(2)}(X_{iN}^{(1)}),$$

$$y_i = F_N^{(1)}(X_{iN}^{(2)})$$

provided  $V$  exists. Further, if  $\{x_i\}, \{y_i\}$  do not depend on  $N$  and for all  $i, j$

$$\text{Cov}(x_i, x_{i+j}) = \text{Cov}(x_1, x_{1+j})$$

$$\text{Cov}(y_i, y_{i+j}) = \text{Cov}(y_1, y_{1+j})$$

then  $V$  exists and equals

$$\text{Var } x_1 + 2 \sum_{i=1}^{\infty} \text{Cov}(x_1, x_{1+i}) + C(\text{Var } y_1 + 2 \sum_{i=1}^{\infty} \text{Cov}(y_1, y_{1+i})) .$$

PROOF. This is immediate on examining that of Theorem 3.2; the requirement  $C \neq 0$  is unnecessary.

**3. Empirical processes.** Theorems 2, 3, 4, 5, respectively, concern independent,  $\alpha_N$ -dependent,  $\phi_N$ -dependent, and  $\psi_N$ -dependent samples. For definition of  $C, (D, \mathcal{D})$  see [1].

Let  $(C_{iN}, \dots, C_{n_N N})$  be constants and let

$$\sigma_N^2 = n_N^{-1} \sum_{i=1}^{n_N} C_{iN}^2 .$$

Suppose  $\{X_{iN}\}$  have cdfs  $\{F_{iN}\}$  on  $[0, 1]$ , and  $n_N \rightarrow \infty$  as  $N \rightarrow \infty$ . Let

$$g_{Nq}(t) = n_N^{-1} \sum_{i=1}^{n_N} \left| \frac{C_{iN}}{\sigma_N} \right|^q F_{iN}(t) , \quad 0 \leq t \leq 1 ,$$

and  $L_N(t) = \sigma_N^{-1} n_N^{-1/2} \sum_{i=1}^{n_N} C_{iN} (I(X_{iN} \leq t) - F_{iN}(t))$ . Let  $r, \{t_i\}$  be numbers depending on  $\delta > 0$  such that

$$(10) \quad \begin{aligned} 0 &= t_0 < t_1 \dots < t_r = 1 , \\ t_i - t_{i-1} &\geq \delta, i = 2, \dots, r - 1 . \end{aligned}$$

THEOREM 2. (Koul, Theorem 2.2 of [2]). Suppose for  $N \geq 1 \{X_{iN}\}$  are independent for  $N \geq 1$ ,

$$(11) \quad \max_{i=1}^{n_N} C_{iN}^2 / (n_N \sigma_N^2) \rightarrow 0 ,$$

$$(12) \quad EL_N(s)L_N(t) \rightarrow K(s, t) , \quad 0 \leq s, t \leq 1, \text{ as } N \rightarrow \infty ,$$

and

$$(13) \quad \limsup_{N \rightarrow \infty} \sup_{0 \leq t \leq 1 - \delta} (g_{N2}(t + \delta) - g_{N2}(t)) \rightarrow 0 \quad \text{as } \delta \rightarrow 0 .$$

Then

$$(14) \quad L_N \rightarrow_{\mathcal{D}} L \quad \text{in } (D, \mathcal{D})$$

where  $L$  is a zero-mean Gaussian process such that  $P(L \in C[0, 1]) = 1$  and  $EL(s)L(t) = K(s, t)$ .

NOTE 1. Koul gives (incorrectly)  $t_i - t_{i-1} \leq \delta$  in (10).

Koul required  $F_{iN}$  continuous (which is not necessary from his proof),  $n_N = N$  and replaced (13) by the stronger condition

$$\limsup_{N \rightarrow \infty} \max_{1 \leq i \leq n_N} \max_{1 \leq j \leq r} (F_{iN}(t_j) - F_{iN}(t_{j-1})) \rightarrow 0 \quad \text{as } \delta \rightarrow 0 ,$$

which is equivalent to

$$(15) \quad \limsup_{N \rightarrow \infty} \max_{1 \leq i \leq n} \sup_t (F_{iN}(t + \delta) - F_{iN}(t)) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

(For example if  $C_{iN} \equiv 1$ , and

$$F_{iN}(t) = 1 \quad t = 1 \\ = (1 - k_{iN})t, \quad 0 \leq t < 1,$$

where  $n_N^{-1} \sum_{i=1}^{n_N} k_{iN} \rightarrow 0$ ,  $0 \leq k_{iN} \leq 1$ , but  $\max_{1 \leq i \leq n_N} k_{iN} \rightarrow 0$  then (13) holds but not this stronger condition.)

NOTE 2. If (16) holds (13) is equivalent to (13) with  $g_{Nq}$  replacing  $g_{N2}$  for any  $q > 0$ . The latter holds if  $g_{Nq}(t) \rightarrow g(t)$ ,  $0 \leq t \leq 1$ , where  $g$  is continuous.

THEOREM 3. Suppose for  $N \geq 1$   $\{X_{iN}\}$  are  $\alpha_N$ -mixing, (12), (13),

$$(16) \quad \max_{1 \leq i \leq n} |C_{iN}| \leq k_0 \sigma_N \quad \text{where } k_0 < \infty,$$

for some  $b$  in  $(\frac{1}{2}, 1)$  and some  $d < (1 + b)/2$

$$(17) \quad \sum_0^{n-1} (j + 1)^2 \alpha_N(j)^{1-b} \leq k_3 n^d, \quad N \geq 1 \text{ where } k_3 < \infty$$

$$(18) \quad \sum_0^{n-1} \alpha_N(j)^{1-b} < k_2, \quad N \geq 1 \text{ where } k_2 < \infty,$$

and

$$(19) \quad g_{N1} \text{ is a continuous, strictly increasing function,} \quad N \geq 1.$$

Then (14) holds.

NOTE 3. (19) can be removed if (13) is strengthened to  $\sup_t |H_N(t) - g(t)| \rightarrow 0$  as  $N \rightarrow \infty$  where  $H_N = g_{N1}$  or  $n^{-1} \sum_1^n F_{iN}$ , and  $g \in C$ .

Since our draft report [10], Deo [2] has published a special case of Theorem 3, viz  $\alpha_N = \alpha$ ,  $C_{iN} = 1$ ,  $F_{iN} = F$ ,  $d = 0$ .

PROOF. For convenience we suppress  $N$ . Using  $d = 0$  in Corollary 1(g), we conclude that the finite-dimensional distributions of  $L_N$  converge to those of  $L$ . (One can also prove this under stronger conditions by adapting Theorem 19.4 of [1], based on Rosen [7]). By Theorems 15.1, 15.5 of [1] it suffices to prove (19.51) of [1] for  $L_N$ . By Note 2 with  $q = 1$  this is so if (19.51) holds for  $\bar{L}_N = L_N(g_1^{-1})$ . For  $0 \leq s < t \leq 1$  set

$$\Delta_i = F_i(g_1^{-1}(t)) - F_i(g_1^{-1}(s)) \\ Z_i = I(s < g_1(X_i) \leq t) - \Delta_i, \quad 1 \leq i \leq n.$$

By Lemma 1(e) with  $i_1 \leq i_2 = i_1 + i \leq i_3 = i_2 + j \leq i_4 = i_3 + k$ ,

$$(20) \quad |EZ_{i_1} \cdots Z_{i_4}| / 10 \Delta_{i_1}^b \\ \leq \min \{ \alpha(i)^{1-b}, \alpha(k)^{1-b}, \alpha(j)^{1-b} + 10 \alpha(i)^{1-b} \alpha(k)^{1-b} \Delta_{i_3}^b \}.$$

Therefore

$$E|\bar{L}_N(t) - \bar{L}_N(s)|^4 \leq 10.4! (hk_0^3 \cdot 3k_3 n^{d-1} + 10k_0^2 k_2^2 h^2)$$

where  $h = k_0^{1-b}(t - s)^b$ . Let  $g = \min(2b, b + 2 - 2d)$ , and

$$R_\varepsilon = 10.4! (3k_0^3 k_3 (2\varepsilon^{-1})^{2-2d} + 10k_0^2 k_2^2) k_0^{1-b}.$$

If

$$(21) \quad \varepsilon n^{-\frac{1}{2}} < 2(t - s)$$

then

$$(22) \quad E|\bar{L}_N(t) - \bar{L}_N(s)|^4 \leq R_\varepsilon(t - s)^g.$$

Therefore (21)  $\Rightarrow$  (22) with  $\bar{L}_N$  replaced by  $\bar{L}_{N_i} = L_{N_i}(g_1^{-1})$ ,  $i = 1, 2$  where  $L_{N_1}(t) = \sigma_N^{-1} n^{-\frac{1}{2}} \sum_{C_i \geq 0} C_i(I(X_i \leq t) - F_i(t))$ , and  $L_{N_2}(t) = L_N(t) - L_{N_1}(t)$ . Hence by Theorem 12.2 of [1], for  $m = 1, 2, \dots$

$$P(M_{m_j} \geq \varepsilon) \leq K_\varepsilon \delta^g$$

where  $K_\varepsilon = R_\varepsilon \cdot K_{1,g}^1 \cdot \varepsilon^{-4}$  and

$$M_{m_j} = \max_{i=1}^m |\bar{L}_{N_j}(s + i\delta/m) - \bar{L}_{N_j}(s)|, \quad j = 1, 2,$$

whenever

$$(23) \quad \varepsilon n^{-\frac{1}{2}} \leq 2\delta/m.$$

For  $s \leq t \leq s + p$ ,

$$(24) \quad |L_N(t) - L_N(s)| \leq |L_{N_1}(s + p) - L_{N_1}(s)| + |L_{N_2}(s + p) - L_{N_2}(s)| + n^{\frac{1}{2}}(g_1(s + p) - g_1(s)).$$

Hence if  $V_N = \sup_{s \leq t \leq s + \delta} |L_N(t) - L_N(s)|$  then  $V_N \leq 3M_{m_1} + 3M_{m_2} + n^{\frac{1}{2}}\delta/m$  (c.f. (22.17), (22.18) of [1].) Hence if (23) holds and  $n^{\frac{1}{2}}\delta < \varepsilon m$  then

$$(25) \quad P(V_N \geq 7\varepsilon) \leq 2K_\varepsilon \delta^g.$$

Choosing  $m$  satisfying  $r^{-1} < m\varepsilon n^{-\frac{1}{2}} \leq 2r^{-1}$  where  $r = [\delta]^{-1}$ , and using Corollary 8.3 of [1] with  $t_i = i/r$ , (19.51) for  $\bar{L}_N$  now follows.

**THEOREM 4.** *If for some  $d < 1$  the conditions of Theorem 3 hold with “ $\alpha_N$ -mixing” replaced by “ $\phi_N$ -mixing” and  $\alpha_N(j)^{1-b}$  is replaced by  $\phi_N(j)$  in (17) and (18), then (14) holds.*

**PROOF.** Instead of (20) one uses

$$|EZ_{i_1} \dots| / (4\Delta_{i_1}) \leq \min\{\phi(i), \phi(k), \phi(j) + 4\phi(i)\phi(k)\Delta_{i_3}\}.$$

**COROLLARY 3.** *The condition  $\sum_0^\infty i^2 \phi(i)^{\frac{1}{2}} < \infty$  of Theorems 22.1, 22.2 of [1] can be weakened to*

$$\sum_0^n i^2 \phi(i) = O(n^d) \quad \text{for some } d < 1.$$

(This improves Sen [8] who showed that  $\sum i \phi(i)^{\frac{1}{2}} < \infty$  was sufficient.)

**THEOREM 5.** *Suppose for  $N \geq 1$   $\{X_{iN}\}$  are  $\phi_N$ -mixing. Suppose for  $N \geq 1$  (12), (13), (16) and*

$$\sum_1^{n-1} i \phi_N(i) \leq k_2 < \infty \quad \text{for } N \geq 1.$$

*Then (14) holds.*

PROOF. Here we avoid assuming (19) by proving for  $s \leqq t \leqq u$

$$E|L_N(t) - L_N(s)|^2 |L_N(u) - L_N(t)|^2 \leqq k(g_{N1}(u) - g_{N1}(s))^2, \quad N \geqq 1$$

where  $k < \infty$ . This is done by breaking the L.H.S. into 29 separate sums and applying Lemma 1(a). The proof now proceeds as for Theorem 2.

NOTE 4. With obvious changes (e.g. replacing  $\sum_{i=1}^{n-1}$  by  $\int_0^1 di$ ) the results in this paper apply to processes  $\{X_{iN}, 1 \leqq i \leqq n_N\}$  where  $i, N, n$  vary continuously.

EXAMPLE 1. Let  $F_0$  be a cdf on  $[0, 1]$ , and  $\psi \geqq 0$ , a function on  $[0, 1]$  such that  $0 < x < 1 \Rightarrow \psi(x) < \infty$ . Suppose the conditions of Theorem 2 or 3 or 4 or 5 hold and

$$\int_0^1 (x - x^2)^b \psi(x) dx < \infty, \quad (\text{with } b = 1 \text{ for Theorems 2, 4, 5}),$$

$$\limsup_N \max_i \sup_t \frac{dF_{iN}}{dF_0}(t) < \infty,$$

and

$$\limsup_N \int \delta_N^2 \psi(F_0) dF_0 < \infty$$

where  $\delta_N = n^{\frac{1}{2}}(n^{-1}\sigma_N^{-1} \sum_1^n C_{iN} F_{iN} - F_0) \rightarrow \delta$  uniformly as  $N \rightarrow \infty$ . Let  $F_N(x) = n^{-1}\sigma_N^{-1} \sum_1^n C_{iN} 1(X_{iN} \leqq x)$ . Then  $A_N = n \int (F_N - F_0)^2 \psi(F_0) dF_0 \rightarrow_{\mathcal{L}} A = \int (L + \delta)^2 \psi(F_0) dF_0$  where  $L$  is given by (14).

PROOF. The condition on  $\{dF_{iN}/dF_0\}$  implies for some  $C < \infty$   $EL_N^2 \leqq C(F_0 - F_0^2)^b$  in  $[0, 1]$ ,  $N$  large (with  $b = 1$  if using Theorems 2, 4, 5.) Hence, for  $\varepsilon > 0$  there exists  $u \in (0, \frac{1}{2})$  and  $N_0$  such that

$$(\int_0^u + \int_{1-u}^1) E(L_N + \delta_N)^2 \psi(F_0) dF_0 < \varepsilon^2, \quad N_0 \leqq N \leqq \infty$$

where  $L_\infty = L, \delta_\infty = \delta$ . Therefore by Theorem 5.5 of [1],

$$B_n = \int_u^{1-u} (L_N + \delta_N)^2 \psi(F_0) dF_0 \rightarrow_{\mathcal{L}} B = \int_u^{1-u} (L + \delta)^2 \psi(F_0) dF_0.$$

Hence,

$$\begin{aligned} &|P(A_N \leqq x + \varepsilon) - P(A \leqq x + \varepsilon)| \\ &\leqq |P(A_N \leqq x + \varepsilon) - P(B_N \leqq x)| + |P(B_N \leqq x) - P(B \leqq x)| \\ &\quad + |P(B \leqq x) - P(A \leqq x + \varepsilon)| \\ &\leqq \varepsilon + P(x < A_N \leqq x + \varepsilon) + \varepsilon + P(x < A \leqq x + \varepsilon). \end{aligned}$$

Hence  $A_n \rightarrow_{\mathcal{L}} A$ .

EXAMPLE 2. Let  $a(\cdot)$  be continuous and nondecreasing on  $[0, 1]$  such that  $a(s) > 0$  for  $s > 0$ . Consider the cdf

$$\begin{aligned} F(s, u) &= a(s)/a(u), \quad 0 \leqq s < u \leqq 1 \\ &= 1, \quad 0 < u \leqq s \leqq 1. \end{aligned}$$

When testing  $H_{0N} : \{F_{iN}(s) \equiv F(s, i/n_N)\}$  for  $\{X_{iN}\}$  independent, an asymptotically  $\alpha$ -level test is to reject  $H_0 \Leftrightarrow$

$$\int_0^1 L_{0N}^2(s) \cdot \frac{(\int_s^1 a^{-1})}{(k^{-1}A(s) + ka(s))^4} da(s) > t_\alpha,$$

where  $L_{0N}$  denotes  $L_N$  with expectations under  $H_0$ , and  $P(\int_0^1 (W^0)^2 > t_\alpha) = \alpha$ .  $W^0$ ,  $W$  are the Brownian-Bridge and Wiener process,  $\{C_{iN}\}$  satisfy (16),  $A(s) = \int_s^1 a^{-1} - a(s) \int_s^1 a^{-2}$ , and  $k > 0$  is an arbitrary constant.

PROOF.  $n^{-1} \sum_1^n F(s, i/n) \rightarrow s + a(s) \int_s^1 a^{-1} \Rightarrow$  (13) and (12) holds with  $K(s, t) = a(s)A(t)$ ,  $s \leq t$  so that by Theorem 2 under  $H_0$ ,  $L_{0N} \rightarrow_{\mathcal{L}} A \cdot W(a/A)$ . Finally on uses Theorem 5.1 of [1] and expresses  $W$  in terms of  $W^0$ .

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