

(Euler, see e.g. [6]), or if $a \in K_s$ (Shell [9]) or $a \in K_T$ (Thron [11]) where

$$K_s = \{a; a = te^{-t} \text{ for } |t| \leq \log 2\},$$

$$K_T = \{a; |a| \leq e^{-1}\}.$$

The set $K_c \cap \mathbf{R}$ is the segment $-e \leq a \leq 1/e$.

By applying results of the Fatou—Julia theory of iteration [3, 4, 5] one can settle most cases.

Theorem 1. *If $a = te^{-t}$, $|t| < 1$ or t a root of unity, then the sequence (1) converges to e^t .*

For almost all t such that $|t| = 1$, the sequence diverges.

Thron considered also the composition of functions $T_i(z) = e^{a_i z}$ with differing values a_i . He proved

Theorem A [11]. *If $a_i \in K_T$, $i = 1, 2, \dots$, then the sequence*

$$(3) \quad w_n = T_1 \circ T_2 \circ \dots \circ T_n(1), \quad T_i(z) = e^{a_i z},$$

converges to a limit u such that $|\log u| \leq 1$.

This may be regarded as expressing stability of infinite exponentiation with respect to changes of the exponents a_i within the region K_T . A result of this type remains true for other regions.

Theorem 2. *If $a \in K_c$, so that $a = te^{-t}$ for some t with $|t| < 1$, then for any neighbourhood N of e^t there is a corresponding neighbourhood U of a such that for any sequence μ_i of points in U the sequence (3) converges to a limit in N .*

Returning to the case of equal a_i we can show

Theorem 3. *For each $n = 1, 2, \dots$ there is a countable set of values a such that $w_n = T^n(1)$ in (1) satisfies $w_n = w_{n+k}$, $k \geq 1$, while w_n is different from w_i for $i < n$. One may find such that values a with arbitrarily large real part.*

A value of a in Theorem 3 leads to a sequence (1) with terminating convergence and there is a countable set of such values a which lie outside K_c . However a 's of this type obviously fail to have the stability property of Theorem 2.

2. Lemmas from iteration theory

If f is an entire or rational function the n -th iterate f^n (where $f^1 = f$, $f^{n+1} = f \circ f^n$, $n = 1, 2, \dots$) is a function of the same type. Iteration of rational functions was studied extensively by Fatou [3] and Julia [5] and the analogous theory for transcendental entire functions more briefly by Fatou [4].

Suppose that f is a non-linear entire function. In fact in our applications $f(z)$ will always have the form $f(z)=e^{az}$, a constant. Denote by $\mathcal{F}(f)$ the set of points of the complex plane in whose neighbourhood the sequence $f^n, n \geq 1$, fails to be a normal family. The complement of \mathcal{F} will be denoted by $\mathcal{C}(f)$.

A fixed point α of f is a solution of $f(\alpha)=\alpha$ and $f'(\alpha)$ is called the multiplier of α . If $|f'(\alpha)| < 1$ the fixed point is called attractive and $f^n(z) \rightarrow \alpha$ as $n \rightarrow \infty$, uniformly in a neighbourhood of α , so that $\alpha \in \mathcal{C}(f)$. If $|f'(\alpha)| > 1$ then α is called repulsive and clearly $\alpha \in \mathcal{F}(f)$. If $f'(\alpha)=1$ then $\alpha \in \mathcal{F}(f)$ since the expansion near α of f gives

$$f(z) = \alpha + (z - \alpha) + a_{m+1}(z - \alpha)^{m+1} + \dots, \quad a_{m+1} \neq 0, \quad m \geq 1$$

$$f^n(z) = \alpha + (z - \alpha) + na_{m+1}(z - \alpha)^{m+1} + \dots$$

If $\alpha \in \mathcal{C}(f)$ then for any limit function φ of a subsequence f^{n_k} in the component of $\mathcal{C}(f)$ which contains α we have $\varphi(\alpha)=\alpha$ so that φ is analytic and $\varphi^{(m+1)}(\alpha)$ is the limit of the $(m+1)$ st derivative of the f^{n_k} at α , which leads to a contradiction.

We state some general properties of \mathcal{F} and \mathcal{C} .

I. $\mathcal{C}(f)$ is open. $\mathcal{F}(f)$ is perfect and non-empty [4].

In fact \mathcal{C} is open by definition so \mathcal{F} is at least closed. For $f(z)=e^{az}$ all large solutions of $e^{az}=z$ are repulsive fixed points so that in this case \mathcal{F} is clearly non-empty.

II. $\mathcal{C}(f)$ and $\mathcal{F}(f)$ are completely invariant under f in the sense that if $z \in \mathcal{C}$ then $f(z) \in \mathcal{C}$, and if further $f(w)=z$ then $w \in \mathcal{C}$ [4].

III. For any integer $p > 1$, $\mathcal{F}(f) = \mathcal{F}(f^p)$; [4].

IV. If in a component D of $\mathcal{C}(f)$ the sequence f^n converges to a finite limit function then D is simply-connected.

(This follows from applying the maximum principle to $f^n - f^m$ on any closed curve which lies in D .)

V. If α is an attractive fixed point of f then the component of $\mathcal{C}(f)$ which contains α is simply connected and contains a singular point of f^{-1} ; [3, 4].

If D is the component in question then $f^n \rightarrow \alpha$ in D , which is simply connected by IV. If D contains no singularity of f^{-1} then continuing the branch for which $f^{-1}(\alpha)=\alpha$ yields a function $g(=f^{-1})$ which is analytic and univalent in D and by II maps D into D with $g(\alpha)=\alpha$. If h is the conformal map of the unit disc Δ to D such that $h(0)=\alpha$ the application of Schwarz's Lemma to $h^{-1} \circ g \circ h = k$ shows that $|k'(0)| \leq 1$, which yields $|f'(\alpha)| = 1/|g'(\alpha)| \geq 1$, a contradiction.

VI. If α is a fixed point of f such that $f'(\alpha)$ is a root of unity, then $\alpha \in \mathcal{F}(f)$ but α lies on the boundary of one or more components D of $\mathcal{C}(f)$ in which $f^n \rightarrow \alpha$ as $n \rightarrow \infty$, and at least one such D contains a singularity of f^{-1} . (Proved in [3] for rational f .)

If α is a fixed point of f such that $f'(\alpha) = \lambda$ is a primitive p -th root of unity, then $f^p(\alpha) = \alpha$, $(f^p)'(\alpha) = \lambda^p = 1$ so that $\alpha \in \mathcal{F}(f^p) = \mathcal{F}(f)$.

Let us simplify the notation by putting $\alpha = 0$. As shown in [1, Theorem 2] the expansion of $F = f^p$ about 0 has the form

$$(4) \quad F(z) = z + a_{m+1}z^{m+1} + \dots, \quad a_{m+i} \neq 0,$$

where $m = kp$ for some positive integer k . It suffices to study the iteration of F near 0 (since $\mathcal{F}(F) = \mathcal{F}(f)$) and this has been worked out e.g. in [3] and in somewhat greater detail in [1].

Near $z = 0$ (see e.g. [1, Lemma 4]) the set $\mathcal{C}(F)$ contains a star of m equally spaced domains G_j , $1 \leq j \leq m$, where each G_j is bounded by a simple closed curve which lies in the region $\alpha_j < \arg z < \beta_j$ and approaches $z = 0$ in the directions $\arg z = \alpha_j, \beta_j$, where

$$(5) \quad \begin{aligned} \alpha_j &= -\gamma + \pi/(3m) - (2j-1)\pi/m, \\ \beta_j &= -\gamma - \pi/(3m) - (2j-3)\pi/m, \\ \gamma &= (\pi + \arg a_{m+1})/m. \end{aligned}$$

Thus $\beta_j - \alpha_j = (4\pi)/3m$. Moreover we have $F(G_j) \subset G_j$ and $F^n(z) \rightarrow 0$ uniformly as $n \rightarrow \infty$ for $z \in G_j$.

Now f and the branches of f^{-1} which vanish at zero permute the components of \mathcal{C} of which the G_j form part. If f^{-1} has no singularity in any of these components then f^{-1} (and so $F^{-1} = f^{-p}$) is univalent in each such component D and F^{-1} maps D into itself. Here F^{-1} is understood to be the analytic continuation of

$$(6) \quad F^{-1}(z) = z - a_{m+1}z^{m+1} + \dots$$

throughout D . The iterates $(F^{-1})^n$ are normal in the components D and in particular in the star $\cup G_j$.

By applying the theory described above to the local iteration of F^{-1} (6) rather than F (4) near 0 we see that there is a star of domains G'_j , $1 \leq j \leq m$, of the same form as G_j , but rotated through an angle π/m (by (5)) such that $F^{-1}(G'_j) \subset G'_j$ and the iterates $(F^{-1})^n$ converge uniformly to 0 in $\cup G'_j$. Together the G_j and G'_j form a region H which includes a punctured neighbourhood $0 < |z| \leq \rho$ of 0. In H the sequence $(F^{-1})^n$ is normal, analytic and converges uniformly to 0. Hence $(F^{-1})^n \rightarrow 0$ uniformly in the whole neighbourhood $|z| < \rho$.

This is impossible by the same argument which was used to show that a fixed point of multiplier 1 is a member of \mathcal{F} . Thus one of the components D_j of \mathcal{C}

which contains a G_j will also contain a singularity of f^{-1} . In the components D_j we have $F^n = f^{pn} \rightarrow 0$ since this holds in each G_j . The function f permutes the D_j cyclically and we have $f^k(z) \rightarrow 0$ as $k \rightarrow \infty$ for z in each D_j .

The behaviour of iterates near a fixed point whose multiplier has the form $\lambda = e^{2\pi i \theta}$, θ irrational, may be approached via the centrum problem (assume the fixed point is at the origin): Find if possible a local change of variable

$$(7) \quad z = \varphi(t) = t + b_2 t^2 + \dots, \quad t \text{ near } 0,$$

which reduces the transformation

$$(8) \quad z_1 = f(z) = \lambda z + a_2 z^2 + \dots$$

to the rotation $t_1 = \lambda t$.

Such a φ must satisfy $\varphi(\lambda t) = f(\varphi(t))$ and the coefficients b_n of φ are uniquely determined by recursions which involve division by $\lambda^n - \lambda$. Thus we have a small divisor problem in which convergence of the series for φ depends on how well λ^{n-1} approximates 1 (or θ is approximated by rationals). Siegel [10] has proved the following result, which has been further refined by Rüssmann [8].

VII. *There is a subset E of the unit circumference which has Lebesgue measure 2π and is such that for any f which is analytic near 0 and has the form (8) with $\lambda \in E$, the corresponding series (7) for φ has positive radius of convergence.*

It follows that if f is entire then the fixed point 0 belongs to $\mathcal{C}(f)$ and that there is a neighbourhood N of 0 such that for any non-zero z_0 in N the images $f^n(z_0)$, $n=1, 2, \dots$, are dense in a simple closed curve which lies in N and has positive distance from 0.

3. Proof of Theorem 1

Suppose that $a = te^{-t}$, with $|t| \leq 1$, so that e^t is a fixed point of $T(z) = e^{az}$ with multiplier t . Since te^{-t} is univalent in $|t| \leq 1$ there is only one such t for a given a and e^t is the only possible limit for w_n in (1).

Suppose first that $|t| < 1$. By property V e^t belongs to a component D of $\mathcal{C}(T)$ which contains the only singular point of T^{-1} , namely the origin. But $T(D) \subset D$ by II so that $1 \in D$ and thus $w_n = T^n(1)$ converges to e^t . A similar argument applies if t is a root of unity, except that V is replaced by VI.

If $|t| = 1$ and t belongs to the subset E of VII, then the fixed point e^t belongs to $\mathcal{C}(T)$ and there is a neighbourhood N of e^t such that for any z_0 such that $z_0 \neq e^t$, $z_0 \in N$ the images $T^n(z_0)$ remain in N for $n=1, 2, \dots$ but fail to converge to e^t . Thus the only way in which $w_n = T^n(1)$ can converge to e^t is for w_n to be equal to e^t for $n \geq n_0$. Since this happens for at most a countable set of values of a (and hence of t), removing such a countable set of values from E leaves a set of measure 2π on $|t|=1$ for which (1) diverges.

4. Terminating convergence

Consider first the case when $w_1=w_2=w_3=\dots$, that is when $e^a=\exp(ae^a)$, so that $ae^a=a+2n\pi i$ for some integer n . Theorem 3 asserts that this last equation has solutions of arbitrarily large real part. This is easy enough to prove directly but it is convenient to quote the

Lemma 1 [Littlewood [7)]. *Suppose that $f(z)=a_0+a_1z+\dots$ is analytic in $D: |z|<1$ and that u_m is a sequence such that for some constant $K>1$ we have*

$$(9) \quad |u_m| \leq |u_{m+1}| \leq K|u_m|, \quad 1 \leq m < \infty,$$

$$(10) \quad u_m \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

Then if $f(z) \neq u_m, m=1, 2, \dots$, in D we have

$$\text{Max}_{|z|=r} |f(z)| \leq C_1(1-r)^{-C_2}, \quad 0 < r < 1,$$

where C_1 depends on u_1, a_0 and C_2 depends only on K .

From this follows

Lemma 2. *If f is an entire function and u_m is a sequence which satisfies (9) and (10), then if f omits the values $u_m, m=1, 2, \dots$, in a half-plane H it follows that f has at most polynomial growth as $z \rightarrow \infty$ in H .*

To prove Lemma 2 it suffices to consider H as $\text{Im } z > 0$. Then $z=\varphi(t)=i(1+t)/(1-t)$, $t=(z-i)/(z+i)$ maps $D: |t|<1$ onto H . Applying Lemma 1 to $f(\varphi(t))$ shows that

$$|f(\varphi(t))| \leq C_1(1-|t|)^{-C_2}, \quad |t| < 1.$$

If $z=re^{i\theta}$ we have

$$1-|t|^2 = 4r \sin \theta / (r^2 + 2r \sin \theta + 1)$$

whence

$$1-|t| > 2r \sin \theta / (r^2 + 2r \sin \theta + 1)$$

and

$$|f(z)| < C_1 \{(r+1)^2 / 2r \sin \theta\}^{C_2} < K(r/\sin \theta)^{C_2},$$

if $r > 1 \sin \theta > 0$.

Applying Lemma 2 to $f(z)=ze^z-z$, and the sequence $u_m=2m\pi i$ in a half-plane $\text{Re } x > A$ proves the claim made at the beginning of § 4.

To complete the proof of Theorem 3 we need to find a such that $w_{n-1} \neq w_n = w_{n+1}$ (for given $n > 1$). We have $w_n = w_{n+1}$ if $T_n(1) = \exp(aT_{n-1}(1)) = T_{n+1}(1) = \exp(aT_n(1))$, that is if $aT_{n-1}(1) = aT_n(1) + 2k\pi i$ for some integer $k \neq 0$ ($k=0$ is equivalent to $w_{n-1} = w_n$). We have only to note that $aT_n(1) - aT_{n-1}(1)$ is an entire function of a which has very large growth on the positive real axis. The application of Lemma 2 to this function and to the sequence $u_m = 2m\pi i$ shows that there are solutions a of our problem in any region $\text{Re } a > A$.

5. Proof of Theorem 2

Suppose that $a = te^{-t}$ where $|t| < 1$. Given a neighbourhood N of e^t and ϱ such that $|t| < \varrho < 1$ choose a disc $\Delta = \{z: |z - e^t| < d, d > 0\}$ such that $\bar{\Delta} \subset N$ and also $|T'_a(z)| < \varrho$ in Δ , $T_a(\Delta) \subset \Delta' = \{z: |z - e^t| < \varrho d\}$, where $T_a(z) = e^{az}$ (and $T'_a(e^t) = t$).

Now Δ belongs to the component D of $\mathcal{C}(e^{az})$ which contains e^t and in which $T_a^n \rightarrow e^t$. Thus there is a positive integer p such that $T_a^p(1) \in \Delta'$. By continuity there is a neighbourhood U of a such that

(i)
$$U \subset K_c,$$

(ii) for any a_1, \dots, a_p in U we have

$$T_{a_1} \circ T_{a_2} \circ \dots \circ T_{a_p}(1) \in \Delta,$$

(iii) for any b in U we have $T_b(\Delta) \subset \Delta$,

(iv) for all b in U we have $|T'_b(z)| < \lambda = 1/2(1 + \varrho) < 1$ for all z in Δ .

Suppose that a_i is any sequence of points in U and set $T_i = T_{a_i}$. For any n we have $T_{n+1} \circ \dots \circ T_{n+p}(1) \in \Delta$ by (ii), and $w_{n+p} = T_1 \circ \dots \circ T_n \circ T_{n+1} \circ \dots \circ T_{n+p}(1) \in T_1 \circ \dots \circ T_n(\Delta)$ which by (iv) has diameter at most $2\lambda^n d$. If $n > k$ by (iii) both w_{n+p} , w_{k+p} are in $T_1 \circ \dots \circ T_k(\Delta)$ so that $|w_{n+p} - w_{k+p}| < 2\lambda^k d$. Thus w_m is a Cauchy sequence which converges to a limit inside $\bar{\Delta} \subset N$. The proof is complete.

6. Periodic sequences of exponents

Suppose that for some natural number k and for all n we have $a_{n+k} = a_n$. As in Theorems A and 2 set $T_i(z) = e^{a_i z}$.

Theorem 4. *If the sequence of exponents is periodic with period k and if*

$$a_n = t_n \exp(-t_{n+1}), \quad t_{n+k} = t_n, \quad n = 1, 2, \dots,$$

where either $|t_1 t_2 \dots t_k| < 1$ or $t_1 t_2 \dots t_k$ is a root of unity, and $w_n = T_1 \circ T_2 \circ \dots \circ T_n(1)$, then for at least one p with $0 \leq p \leq k$ the sequence w_{mk+j} converges to e^t as $m \rightarrow \infty$.

In the case $k=1$ this reduces to Theorem 1. For $k > 1$ it has some similarity to Theorem 2.

Put $\varphi_i = T_i \circ T_{i+1} \circ \dots \circ T_{i+k-1}$. Then if t_i are as in Theorem 4 and $\lambda_i = e^{t_i}$ we have $T_n(\lambda_{n+1}) = \lambda_n$ so that λ_i is a fixed point of φ_i and further $\varphi'_i(\lambda_i) = t_1 \dots t_n$. Thus under the assumptions of the theorem λ_i belongs to a domain D in which the iterates $\varphi_i^N \rightarrow \lambda_i$ as $N \rightarrow \infty$. D contains at least one of the singularities of φ_i^{-1} ,

that is one of the k values

$$0, T_i(0) = 1, \dots, T_i \circ T_{i+1} \circ \dots \circ T_{i+k-2}(0).$$

Thus for such a value β we have $\varphi_i^m(\beta) \rightarrow 0$ as $m \rightarrow \infty$, that is $T_i \circ T_{i+1} \circ \dots \circ T_{i+mk-1+p}(0) \rightarrow \lambda_i$ as $m \rightarrow \infty$ for some $0 \leq p < k$. Choosing $i=1$ gives the result claimed.

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