

Convergence of Multi-Grid Iterations Applied to Difference Equations

By Wolfgang Hackbusch

Abstract. Convergence proofs for the multi-grid iteration are known for the case of finite element equations and for the case of some difference schemes discretizing boundary value problems in a rectangular region. In the present paper we give criteria of convergence that apply to general difference schemes for boundary value problems in Lipschitzian regions. Furthermore, convergence is proved for the multi-grid algorithm with Gauss-Seidel's iteration as smoothing procedure.

1. Introduction. Systems of linear equations arising from boundary value problems can be solved very fast by the multi-grid iteration (cf. [1]–[6], [9], [11]). Although, the multi-grid algorithms are applied successfully to a general class of problems, the proofs of convergence are restricted to a very special class of problems. In the case of special finite element equations for boundary value problems with smooth boundaries proofs of convergence are given by Astrachancev [1] and Nicolaides [9]. In [6] the author established general criteria and proved the convergence for general finite element problems.

The second important class of problems are systems of difference equations discretizing boundary value problems. The model problem of Poisson's equation in a rectangle (and similar problems) can be analyzed easily by means of Fourier transformation (cf. Fedorenko [4]). In the case of certain difference schemes for problems with variable coefficients and a rectangular region, Bachvalov [2] and Wesseling [11] proved the convergence of the multi-grid iteration. But two gaps are still to be filled. Convergence proofs are missing for the case of nonrectangular regions. Moreover, all proofs cited above require a special smoothing procedure (cf. Section 4) related to the Jacobi iteration. In practice smoothing by the Gauss-Seidel iteration is preferred (cf. [3], [5]). This paper contains general criteria that apply to difference schemes in general regions and to smoothing by Gauss-Seidel.

In Section 2 we describe the multi-grid algorithm very briefly. For further comments we refer, for instance, to [6]. As pointed out in [6] the convergence can be concluded from an 'approximation property' and a 'smoothing property'. The first one is studied in Section 3. A criterion is proved and its assumptions are verified in the case of a very general difference scheme. It turns out that the crux of the assumptions is a certain regularity condition (3.6b) that is proved in [7] for the case of Dirichlet

Received April 13, 1979.

AMS (MOS) subject classifications (1970). Primary 65F10, 65N20.

Key words and phrases. Multi-grid method, difference equations, boundary value problems.

© 1980 American Mathematical Society
0025-5718/80/0000-0054/\$05.00

boundary values. The smoothing property is investigated in Section 4, in particular, for the case of Gauss-Seidel's iteration as smoothing procedure.

2. Multi-Grid Iteration. Let

$$(2.1) \quad h_0 > h_1 > \cdots > h_l > \cdots > 0$$

be a sequence of grid sizes. l is called the 'level number'. The discretization of the continuous problem (boundary value problem)

$$(2.2) \quad Lu = f$$

with step size h_l is denoted by

$$(2.3) \quad L_l u_l = f_l \quad (l \geq 0).$$

The solution u_l of (2.3), as well as the right-hand side f_l , belongs to a finite-dimensional vector space V_l .

The system (2.3) of linear equations is to be solved by the multi-grid algorithm described below. It uses auxiliary equations of the form $L_m u_m = g_m$ for $m = 0, 1, \dots, l-1$. The connection of grid functions of different levels is given by a prolongation

$$p_{l,l-1}: V_{l-1} \rightarrow V_l$$

and a restriction

$$r_{l-1,l}: V_l \rightarrow V_{l-1}.$$

Since a detailed explanation of the multi-grid algorithm is contained in [5], [6], we give only a brief description by means of a program.

```

procedure mgm( $l, u, f$ ): integer  $l$ ; array  $u, f$ ;
if  $l = 0$  then  $u := L_0^{-1} * f_0$  else
  begin integer  $j$ ; array  $v, d$ ;
    for  $j := 1$  step 1 until  $\nu$  do  $u := G_l(u, f)$ ;
     $d := r_{l-1,l} * (L_l * u - f)$ ;  $v := 0$ ;
    for  $j := 1$  step 1 until  $\gamma$  do mgm( $l-1, v, d$ );
     $u := u - p_{l,l-1} * v$ 
  end;

```

The meaning of the parameters is the following. $l \geq 0$ is the actual level number. $f \in V_l$ is the right-hand side to the problem in consideration (e.g., $f = f_l$ in case of (2.3)). u has an arbitrary input value $u_l^{(i)} \in V_l$ (i : number of iterations). The procedure *mgm* computes the next iterate $u = u_l^{(i+1)}$ as output. The procedure depends on the positive numbers ν (number of iterations of the smoothing procedure G_l) and γ (number of *mgm* iterations per level). The smoothing procedure is of the form

$$(2.4) \quad G_l(v_l, f_l) = G_l v_l + H_l f_l \quad (v_l, f_l \in V_l) \text{ with } G_l + H_l L_l = I.$$

The convergence of the multi-grid algorithm depends on the choice of ν, γ , on the coarsest step size h_0 and on the maximal ratio $\sup\{h_{l-1}/h_l: l \geq 1\} < \infty$. Usually, the last ratio is constant, e.g. equal to 2. In the following $\gamma = 2$ is fixed (for $\gamma = 1$ compare [6, Corollary 3.8]).

We say that the multi-grid iteration ‘converges’ if it converges for a suitable choice of h_0 and ν ; more precisely if the iteration matrix $M_l = M_l(\nu, h_0, h_1, \dots, h_l)$ [defined by $u_l^{(i+1)} - u_l = M_l(u_l^{(i)} - u_l)$, $u_l = L_l^{-1}f_l$] satisfies

$$(2.5) \quad \|M_l\| \leq C(\nu) < 1 \quad \text{for } \nu_{\min} \leq \nu \leq \nu_{\max}(h_1), l \geq 1,$$

where $C(\nu) \rightarrow 0$ as $\nu \rightarrow \infty$ and $\nu_{\max}(h) \rightarrow \infty$ as $h \rightarrow 0$. The matrix norm $\|\cdot\|$ is associated with some suitable vector norm on V_l .

We recall the following result of [6]. Here and in the sequel C denotes a generic constant independent of l .

PROPOSITION 1. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two suitable (not necessarily different) norms on V_l ($l \geq 0$) and define the matrix norms $\|A\|_{i,j}$ ($i, j = 1, 2$) of $A: V_l \rightarrow V_m$ by $\sup\{\|Av\|_j/\|v\|_i: 0 \neq v \in V_l\}$. Assume the smoothing property*

$$(2.6) \quad \|L_l G_l^\nu\|_{2,1} \leq C_0(\nu) h_l^{-\alpha} \quad \text{for all } l \geq 0, 1 \leq \nu \leq \nu_{\max}(h_1)$$

with $C_0(\nu) \rightarrow 0$ ($\nu \rightarrow \infty$), $\nu_{\max}(h) \rightarrow \infty$ ($h \rightarrow 0$), and G_l from (2.4) for suitable $\alpha \geq 0$. Assume the approximation property

$$(2.7) \quad \|L_l^{-1} - p_{l,l-1} L_{l-1}^{-1} r_{l-1,l}\|_{1,2} \leq C h_{l-1}^\alpha \quad \text{for all } l \geq 1$$

with α from (2.6). Furthermore, the estimates

$$(2.8) \quad \frac{1}{C} \|v_{l-1}\|_2 \leq \|p_{l,l-1} v_{l-1}\|_2 \leq C \|v_{l-1}\|_2 \quad \text{for all } v_{l-1} \in V_{l-1}, l \geq 1,$$

$$(2.9) \quad \|G_l^\nu\|_{2,2} \leq C \quad \text{for all } 1 \leq \nu \leq \nu_{\max}(h_1), l \geq 0,$$

$$(2.10) \quad h_l < h_{l-1} \leq C h_l \quad \text{for all } l \geq 1$$

are required. Then the multi-grid iteration with $\gamma = 2$ converges: (2.5) holds with $\|\cdot\| = \|\cdot\|_{2,2}$.

3. The Approximation Property.

3.1. *A Criterion Implying the Approximation Property.* Assume

$$(3.1a) \quad r_{l-1,l} L_l p_{l,l-1} = L_{l-1} + \delta_{l-1} \quad (l \geq 1),$$

where δ_{l-1} is small enough in the following sense:

$$(3.1b) \quad \|L_{l-1}^{-1} \delta_{l-1} r'_{l-1,l} L_l^{-1}\|_{1,2} \leq C h_{l-1}^\alpha \quad (l \geq 1).$$

$r'_{l-1,l}$ is a suitable restriction involved in (3.3) given below. If L_l is the stiffness matrix of a finite element method, (3.1a) holds with $\delta_{l-1} = 0$ (cf. [6]). δ_{l-1} vanishes, too, if L_{l-1} is defined as in [5].

Moreover, we need the estimate

$$(3.2) \quad \|L_{l-1}^{-1} r_{l-1,l} L_l\|_{2,2} \leq C \quad (l \geq 1)$$

and the existence of some linear mapping $r'_{l-1,l}: V_l \rightarrow V_{l-1}$ ($l \geq 1$) with

$$(3.3) \quad \|I - p_{l,l-1} r'_{l-1,l} L_l^{-1}\|_{1,2} \leq C h_{l-1}^\alpha \quad (l \geq 1).$$

α involved in (3.1b) and (3.3) is the exponent from (2.6).

LEMMA 1. Assume that there are norms $\|\cdot\|_0$ and $\|\cdot\|_3$ on V_l such that

$$(3.4a) \quad \|L_l^{-1}\|_{0,2} \leq C, \quad \|L_l^{-1}\|_{1,3} \leq C, \quad \|L_l\|_{2,0} \leq C \quad (l \geq 0).$$

Then (3.1a, b), (3.2), and (3.3) follow from (3.4b, c, d):

$$(3.4b) \quad \|r_{l-1,l}\|_{0,0} \leq C, \quad \|r'_{l-1,l}\|_{3,3} \leq C \quad (l \geq 1),$$

$$(3.4c) \quad \|\delta_{l-1}\|_{3,0} \leq Ch_{l-1}^\alpha \quad (l \geq 1),$$

$$(3.4d) \quad \|I - p_{l,l-1}r'_{l-1,l}\|_{3,2} \leq Ch_{l-1}^\alpha \quad (l \geq 1).$$

(3.4d) describes the approximation of grid functions of V_l by $p_{l,l-1}V_{l-1}$: For all $v_l \in V_l$ there is $v_{l-1} \in V_{l-1}$ (namely $r'_{l-1,l}v_l$) with $\|v_l - p_{l,l-1}v_{l-1}\|_2 \leq Ch_{l-1}^\alpha \|v_l\|_3$. If $\alpha > 0$, $\|\cdot\|_3$ must define a finer topology than $\|\cdot\|_2$.

In Proposition 1 the approximation property (2.7) may be replaced by (3.1)–(3.3):

CRITERION 1. (2.8), (3.1a, b), (3.2), and (3.3) imply the approximation property (2.7). By Lemma 1 also (2.8) and (3.4a–d) are sufficient.

Proof. Since $[I - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}L_l]p_{l,l-1} = -p_{l,l-1}L_{l-1}^{-1}\delta_{l-1}$ by (3.1a), it follows that

$$\begin{aligned} \|L_l^{-1} - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}\|_{1,2} &= \|[I - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}L_l]L_l^{-1}\|_{1,2} \\ &= \|[I - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}L_l][I - p_{l,l-1}r'_{l-1,l}]L_l^{-1} \\ &\quad - p_{l,l-1}L_{l-1}^{-1}\delta_{l-1}r'_{l-1,l}L_l^{-1}\|_{1,2} \\ &\leq \{1 + \|p_{l,l-1}\|_{2,2}\|L_{l-1}^{-1}r_{l-1,l}L_l\|_{2,2}\}\|[I - p_{l,l-1}r'_{l-1,l}]L_l^{-1}\|_{1,2} \\ &\quad + \|p_{l,l-1}\|_{2,2}\|L_{l-1}^{-1}\delta_{l-1}r'_{l-1,l}L_l^{-1}\|_{1,2}. \end{aligned}$$

Hence, (2.8), (3.1b), and (3.3) yield (2.7). \square

Using $r_{l-1,l}[I - L_l p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}] = -\delta_{l-1}L_{l-1}^{-1}r_{l-1,l}$, we obtain a similar result:

CRITERION 2. Assume (3.1a),

$$(2.8^*) \quad \|r_{l-1,l}\|_{1,1} \leq C \quad (l \geq 1),$$

$$(3.1b^*) \quad \|L_l^{-1}p'_{l,l-1}\delta_{l-1}L_{l-1}^{-1}\|_{1,2} \leq Ch_{l-1}^\alpha \quad (l \geq 1),$$

$$(3.2^*) \quad \|L_l p_{l,l-1}L_{l-1}^{-1}\|_{1,1} \leq C \quad (l \geq 1),$$

$$(3.3^*) \quad \|L_l^{-1}(I - p'_{l,l-1}r_{l-1,l})\|_{1,2} \leq Ch_{l-1}^\alpha \quad (l \geq 1),$$

for a suitable linear mapping $p'_{l,l-1}: V_{l-1} \rightarrow V_l$. Then (2.7) follows.

3.2. *Application of the Criterion.* In the following we verify the conditions of Criterion 1 for the following example.

Example. Let L_l ($l \geq 0$) be an elliptic difference operator of order $2m$, i.e. the discretization of an elliptic differential operator of order $2m$. Let H_0^s be the space of

all complex-valued grid functions defined on the d -dimensional grid $\Omega(h_l) = \{x \in \Omega \subset \mathbf{R}^d: x/h_l \in \mathbf{Z}^d\}$ endowed with the norm

$$|u|_s = (h/2\pi)^{d/2} \left\| \left[1 + h^{-2} \sum_{j=1}^d \sin^2(\xi_j/2) \right]^{s/2} \sum_{x \in \Omega(h)} u(x) e^{ix\xi/h} \right\|_{L^2([- \pi, \pi]^d)} \quad (s \geq 0),$$

$$|u|_{-s} = \sup \left\{ (h/2\pi)^d \left| \sum_{x \in \Omega(h)} u(x) \bar{v}(x) \right| / |v|_s : 0 \neq v \in H_0^s \right\} \quad (s \geq 0, h = h_l)$$

(denoted by $|\cdot|_{s,0}$ in [7]) corresponding to the norm of the Sobolev space $H_0^s(\Omega)$ if $s + 1/2 \neq \text{integer}$. We define the associated matrix norms by

$$|A|_{s,t} = \sup \{ |Au|_t / |u|_s : 0 \neq u \in H_0^s \}.$$

Choose α , $\|\cdot\|_1$, and $\|\cdot\|_2$ (and $\|\cdot\|_0$, $\|\cdot\|_3$ of Lemma 1) by

$$(3.5) \quad \alpha = \theta + \theta', \quad \|\cdot\|_1 = |\cdot|_{\theta-m}, \quad \|\cdot\|_2 = |\cdot|_{m-\theta'},$$

$$\|\cdot\|_0 = |\cdot|_{-m-\theta'}, \quad \|\cdot\|_3 = |\cdot|_{m+\theta}$$

for some $\theta, \theta' \in [0, m]$ with $\alpha = \theta + \theta' > 0$. The condition $\alpha > 0$ will be important in Section 4.

The estimate

$$(3.6a) \quad |L_l|_{\vartheta+m, \vartheta-m} \leq C(\vartheta) \quad (l \geq 0, \vartheta \in \mathbf{R})$$

holds if the coefficients of the difference scheme L_l are sufficiently smooth (cf. Lemma 7). In [7] we proved

$$(3.6b) \quad |L_l^{-1}|_{\vartheta-m, \vartheta+m} \leq C$$

for all $l \geq 0, \vartheta \in [-\theta'_0, \theta_0]$ ($\theta_0, \theta'_0 \in [0, \frac{1}{2}], \theta_0 + \theta'_0 > 0$)

under very weak assumptions. The main requirements are stability of L_l with respect to $l_2 = H_0^0$ and ellipticity of L_l . It suffices that the underlying region Ω is Lipschitzian. The assumption on the smoothness of the coefficients is very weak, too. (3.6b) holds even for some schemes with irregular discretizations near the boundary. Symmetry of positive definiteness of L_l are not required.

At first we discuss the estimates (3.4a–d) of Lemma 1.

Note 1. (3.5) and (3.6a, b) imply the estimates (3.4a) of Lemma 1 if $0 \leq \theta \leq \theta_0$ and $0 \leq \theta' \leq \theta'_0$.

For the discussion of (3.4b, d) we restrict our considerations to the case of $m = 1$. Let $h_{l-1}/h_l \in \mathbf{Z}$ and define $p_{l,l-1}^0$ by

$$(p_{l,l-1}^0 u)(x) = \prod_{j=1}^d \max\{0, 1 - |x_j - y_j|/h_{l-1}\} \quad (x \in \Omega(h_l), y \in \Omega(h_{l-1})),$$

where $u \in V_{l-1}$ is the unit vector with $u(y) = 1, u(z) = 0$ for $z \neq y$. $p_{l,l-1}^0$ is an example of an interpolation of order 2. Furthermore, define $r_{l-1,l}^0$ as the mapping adjoint to $p_{l,l-1}^0$: $(u, p_{l,l-1}^0 v) = (r_{l-1,l}^0 u, v)$, where $(v, w) = h^d \sum_{x \in \Omega(h)} v(x) \bar{w}(x)$ with $h = h_l$ or h_{l-1} , respectively. In the usual case of $d = 2$ and $h_{l-1} = 2h_l$, the mappings

$p_{l,l-1}^0$ and $r_{l-1,l}^0$ become

$$(p_{l,l-1}^0 u)(x) = \begin{cases} u(x) & \text{if } x \in \Omega(h_{l-1}), \\ \frac{1}{2} [u(x + e_j h_l) + u(x - e_j h_l)] & \text{if } x + e_j h_l \in \Omega(h_{l-1}) \\ & \text{or } x - e_j h_l \in \Omega(h_{l-1}), \\ \frac{1}{4} \sum_{j,k=1,2} u(x + (-1)^k e_1 h_l + (-1)^j e_2 h_l) & \text{otherwise,} \end{cases}$$

$$(r_{l-1,l}^0 u)(x) = \frac{1}{4} u(x) + \frac{1}{8} \sum_{j,k=1,2} u(x + (-1)^k e_j h_l) \\ + \frac{1}{16} \sum_{j,k=1,2} u(x + (-1)^j e_1 h_l + (-1)^k e_2 h_l),$$

where e_j ($j = 1, 2$) are the unit vectors $(1, 0)$, $(0, 1)$. Note that $u(y) = 0$ if $y \notin \Omega(h_l)$.

Note 2. Let $m = 1$. $p_{l,l-1}^0$ and $r_{l-1,l}^0$ defined above satisfy (3.7a, b):

$$(3.7a) \quad |p_{l,l-1}^0|_{s,s} \leq C, \quad |r_{l-1,l}^0|_{s,s} \leq C \quad \text{for all } l \geq 1, |s| \leq 2,$$

$$(3.7b) \quad |I - p_{l,l-1}^0 r_{l-1,l}^0|_{s,t} \leq C h_{l-1}^{s-t} \quad \text{for all } l \geq 1, -2 \leq t \leq s \leq 2, s - t \leq 2.$$

COROLLARY 1 TO NOTE 2. Assume $m = 1$ and (3.5) and set $r'_{l-1,l} = r_{l-1,l}^0$. Then (2.8) and the estimates (3.4b, d) of Lemma 1 hold for $p_{l,l-1} = p_{l,l-1}^0$ with $\alpha = \theta + \theta' > 0$. Moreover, (2.8) and (3.4d) remain valid if the coefficients of $p_{l,l-1}$ and $p_{l,l-1}^0$ differ by $O(h_l^{1+\theta})$ and/or if the coefficients of $p_{l,l-1}$ and $p_{l,l-1}^0$ differ by $O(1)$ at points near the boundary (i.e., $\text{distance}(x, \partial\Omega) \leq Ch_l$). Similarly, (3.4b) remains true if the coefficients of $r_{l-1,l}$ and $r_{l-1,l}^0$ differ by $O(h^{1+\max(\theta, \theta')})$ or by $O(1)$ near the boundary.

Example. $p_{l,l-1}$ and $r_{l-1,l}$ defined in [5, Eq. (3.4)] satisfy (3.7a, b).

COROLLARY 2 TO NOTE 2. Generalizations to $m > 1$ are obvious. $p_{l,l-1}^0$ must be defined by interpolation of order $> m$.

Note that this requirement is weaker than the requirement "order of interpolation $\geq 2m$ " of Brandt [3, p. 377].

Since L_l and L_{l-1} should be consistent discretizations of the same differential operator (2.2), the difference $\delta_{l-1} = r_{l-1,l} L_l p_{l,l-1} - L_{l-1}$ is expected to consist of terms of the following form:

$$\delta_{l-1} = \sum_{\beta, \beta' \in \mathbb{Z}^d, |\beta| + |\beta'| \leq 2m+1} \sum_{\gamma \in \mathbb{Z}^d} T^\gamma \partial^\beta d_{\gamma, \beta, \beta', l-1}(x, h) \partial^{\beta'},$$

$$\sup \{ |d_{\gamma, \beta, \beta', l-1}(x, h)| : x \in \Omega(h), l \geq 1 \} \leq C h^\vartheta,$$

$$\vartheta = \begin{cases} 1 & \text{if } |\beta| + |\beta'| = 2m + 1, |\beta|, |\beta'| \leq m + 1, \\ \theta + \theta' & \text{if } |\beta| + |\beta'| \leq 2m, |\beta|, |\beta'| \leq m, \end{cases}$$

where β, β' , and γ are multi-indices with $|\beta| = \beta_1 + \dots + \beta_d$, $\partial^\beta = \partial_1^{\beta_1} \dots \partial_d^{\beta_d}$, $(\partial_j u)(x) = [u(x) - u(x - e_j h)]/h$, $(T^\gamma u)(x) = u(x + \gamma h)$, $h = h_{l-1}$ (cf. [7]). The definition of ∂ and T makes sense since $u(x)$ is extended by zero outside $\Omega(h)$. (3.8b) may be replaced by other conditions involving Hölder continuity of $d_{\gamma, \beta, \beta', l-1}(\cdot, h)$.

Example. Consider the differential operator $L = -(a(x_1)u_{x_1})_{x_1} - (b(x_2)u_{x_2})_{x_2}$. Discretize $(au_{x_1})_{x_1}$ by $L_I^I u = -h^{-2}[-a^+ u^+ + (a^+ + a^-)u - a^- u^-]$ with $u = u(x)$, $u^\pm = u(x_1 \pm h, x_2)$, $a^\pm = a(x_1 \pm h/2)$. Similarly, L_I^{II} is the discretization of the second term of L . L_I is the sum $L_I^I + L_I^{II}$ with $h = h_l$. Let $p_{l, l-1} = p_{l, l-1}^0$ and $r_{l-1, l} = r_{l-1, l}^0$ as defined above. Then $\delta_{l-1}^I = r_{l-1, l} L_I^I p_{l, l-1} - L_{l-1}^I$ becomes

$$\begin{aligned} T_1 \partial_1 [a(x_1 - h/2) - \frac{1}{2}a(x_1 - 3h/4) - \frac{1}{2}a(x_1 - h/4)] \partial_1 \\ + T_1 T_2 \partial_2^2 [-h^2(a(x_1 - h/4) + a(x_1 - 3h/4))/16] \partial_1^2 \\ + T_2 \partial_2^2 [-h(a(x_1 + 3h/4) + a(x_1 + h/4) - a(x_1 - h/4) - a(x_1 - 3h/4))/16] \partial_1, \end{aligned}$$

where $(T_1 u)(x) = u(x_1 + h, x_2)$, $(T_2 u)(x) = u(x_1, x_2 + h)$ and $h = h_{l-1}$. The brackets contain the coefficients of (3.8a). Obviously, (3.8b) holds if $a(\cdot)$ is Hölder continuous with exponent $\theta + \theta' = \alpha < 1$. If $a(\cdot)$ has Lipschitz continuous derivatives, (3.8b) holds with $\theta = \theta' = 1$.

Note 3. Let α and the norms be chosen according to (3.5). (3.8a, b) implies the estimate (3.4c) of Lemma 1. (3.4c) holds even if δ_{l-1} contains a further term of order $O(h_{l-1}^{-2m})$ at points near the boundary.

Proof. Use $|\partial^\beta u|_0 \leq C|u|_{m+\theta}$ if $|\beta| \leq m + \theta$ and $h|\partial^\beta u|_0 \leq Ch^\theta |u|_{m+\theta}$ if $m + \theta \leq |\beta| = m + 1$. For perturbations near the boundary apply the following lemma (cf. [7]). \square

LEMMA 2. Let $\Omega(h)$ have 'property C' defined in [7]. Assume that the subset $\Gamma(h) \subset \Omega(h)$ satisfy distance $(x, h\mathbb{Z}^d \setminus \Omega(h)) \leq Ch$ for some $C \neq C(h)$ and all $x \in \Gamma(h)$, that means, all points of $\Gamma(h)$ have a distance less than Ch from the boundary. Define the restriction γ by $(\gamma u)(x) = u(x)$ if $x \in \Gamma(h)$, $(\gamma u)(x) = 0$ otherwise. Then $|\gamma|_{s,t} \leq C'h^{s-t}$ is valid.

A sufficient condition for 'property C' is that Ω is Lipschitzian.

From Notes 1–3, Lemma 1 and Criterion 1 one concludes that the approximation property (2.7) holds for a very general class of difference schemes L_l .

Example (Application to the Shortley-Weller scheme). Discretize $-\Delta u = f$ (in a Lipschitz region $\Omega \subset \mathbb{R}^2$), $u = g$ (on $\partial\Omega$) by the Shortley-Weller scheme L_l (cf. [5], [8, p. 203ff.]). In [7, Note 2.3] we proved (3.6b) with $\theta'_0 = 0$, $\theta_0 > 0$. But note that (3.6a) is not valid since the diagonal D_l of the matrix L_l can be arbitrarily large. Nevertheless, $(h_l^2 D_l)^{-1} L_l$ and $L_l (h_l^2 D_l)^{-1} : H_0^{\theta+m} \rightarrow H_0^{\theta-m}$ are uniformly bounded.

Define $\theta' = 0$, $p_{l, l-1} = p_{l, l-1}^0$, $r_{l-1, l} = r_{l-1, l}^0 (h_l^2 D_l)^{-1}$, $r'_{l-1, l} = (h_{l-1}^2 D_{l-1})^{-1} r_{l-1, l}^0$ (or define $p_{l, l-1}$ and $r_{l-1, l}$ as in [5]). Then (3.1a, b), (3.2), and (3.3) are fulfilled. For a proof modify Lemma 1: Split $r_{l-1, l} L_l$ into $r_{l-1, l}^0 \cdot [(h_l^2 D_l)^{-1} L_l]$ and $\delta_{l-1} r'_{l-1}$, into $[\delta_{l-1} (h_{l-1}^2 D_{l-1})^{-1}] \cdot r_{l-1, l}^0$. Thus, we have shown the approximation property (2.7) with $\alpha = \theta > 0$ by means of Criterion 1.

4. Criteria Implying the Smoothing Property.

4.1. *Preparing Lemmata.* The following lemma describes a norm equivalent to $|\cdot|_s$.

LEMMA 3. Let $\Omega(h)$ have 'property C' (cf. Lemma 2). Assume $L_{1,0}$ to be a positive definite and H_0^m -elliptic difference operator of order $2m$, i.e., $L_{1,0} = L_{1,0}^*$ and $|u|_m^2/C \leq (L_{1,0}u, u) \leq C|u|_m^2$, where $(u, v) = h^d \sum_{x \in \Omega(h)} u(x)\bar{v}(x)$. The fractional powers of $\Lambda := (L_{1,0})^{1/(2m)}$ are well defined. Then $|u|_s$ and $|\Lambda^s u|_0$ are equivalent: $(1/C')|u|_s \leq |\Lambda^s u|_0 \leq C'|u|_s$, for $-m \leq s \leq m$. C' does not depend on h_1 .

Proof. Use Lemma 2.1 of [7] and the following lemma. \square

LEMMA 4 (INTERPOLATION). Let H_1 and H_2 be two Hilbert spaces. $A: H_1 \rightarrow H_2$, $\Lambda_i: H_i \rightarrow H_i$ and $\Lambda_i^{-1}: H_i \rightarrow H_i$ ($i = 1, 2$) are assumed to be bounded. Furthermore, let Λ_1 and Λ_2 be positive definite. Then the inequality

$$\|\Lambda_2^\gamma A \Lambda_1^{-\gamma}\|_{H_1 \rightarrow H_2} \leq C_1^{(\gamma_2 - \gamma)/(\gamma_2 - \gamma_1)} C_2^{(\gamma - \gamma_1)/(\gamma_2 - \gamma_1)}$$

holds for all $\gamma \in [\gamma_1, \gamma_2]$ if it is valid for $\gamma = \gamma_1$ and $\gamma = \gamma_2$.

Proof. Set $\varphi(\gamma) = \|\Lambda_2^\gamma A \Lambda_1^{-\gamma}\|_{H_1 \rightarrow H_2}$ and note that

$$\varphi(\gamma)^2 = \|\Lambda_2 A \Lambda_1^{-2\gamma} A^* \Lambda_2^\gamma\|_{H_2 \rightarrow H_2} = \rho(\Lambda_2^\gamma A \Lambda_1^{-\gamma'} \Lambda_1^{-\gamma''} A^* \Lambda_2^{\gamma''}) \leq \varphi(\gamma')\varphi(\gamma'')$$

for all γ', γ'' with $\gamma' + \gamma'' = \gamma$ (ρ : spectral radius). Therefore, the estimate follows by bisection for all $\gamma = \gamma_1 + \nu 2^{-\mu}(\gamma_2 - \gamma_1)$ with $\nu, \mu \in \mathbb{Z}$, $\mu \geq 0$, $0 \leq \nu \leq 2^\mu$. The continuity of $\varphi(\gamma)$ concludes the proof. \square

The preceding lemmata yield the following estimates.

LEMMA 5. The estimates (4, 1a, b, c) hold with C independent of h_1 :

$$(4.1a) \quad |A|_{r,r} \leq C|A|_{s,s}^{(t-r)/(t-s)} |A|_{t,t}^{(r-s)/(t-s)} \quad (-m \leq s \leq r \leq t \leq m),$$

$$(4.1b) \quad |A|_{r,-r} \leq C|A|_{0,0}^{(t-r)/t} |A|_{t,-t}^{r/t} \quad (0 \leq r \leq t \leq m \text{ or } 0 \geq r \geq t \geq -m),$$

$$(4.1c) \quad |A|_{r,-s} \leq C|A|_{0,0}^{(2m-r-s)/(2m)} |A|_{2m,0}^{r/(2m)} |A|_{0,-2m}^{s/(2m)} \quad (r \geq 0, s \geq 0, r+s \leq 2m).$$

Proof. By virtue of Lemma 3, $|u|_s$ can be replaced with $|\Lambda^s u|_0$. Hence, $|A|_{r,s}$ becomes $|\Lambda^s A \Lambda^{-r}|_{0,0}$. Applying Lemma 4 with $\Lambda_1 = \Lambda_2 = \Lambda$ we obtain (4.1a). (4.1b) follows by choosing $\Lambda_1 = \Lambda$, $\Lambda_2 = \Lambda^{-1}$. For the proof of (4.1c) apply Lemma 3 and (4.1a) with $2m$ instead of m . We abbreviate $|A|_{p,q}$ by $a(p, q)$. Lemma 4 with $\Lambda_1 = I$, $\Lambda_2 = \Lambda$ yields $a(r+s, 0) \leq Ca(0, 0)^{1-\beta} a(2m, 0)^\beta$ with $\beta = (r+s)/(2m)$. Similarly, $a(0, -r-s) \leq Ca(0, 0)^{1-\beta} a(0, -2m)^\beta$ follows. Applying (4.1a) to $\Lambda^{-r-s} A$ instead of A , one obtains $a(r, -s) \leq Ca(r+s, 0)^{r/(r+s)} a(0, -r-s)^{s/(r+s)}$. Inserting the estimates of $a(r+s, 0)$ and $a(0, -r-s)$ we are led to (4.1c). \square

Smoothing by Gauss-Seidel's iteration is expressed by

$$(4.2a) \quad G_l(v_l, f_l) = (D_l - R_l)^{-1}(S_l v_l + f_l), \quad G_l = (D_l - R_l)^{-1} S_l,$$

where

$$(4.2b) \quad L_I = D_I - R_I - S_I.$$

Definition 1. The splitting (4.2b) is called 2-cyclic (cf. [10, p. 39]) if there are two distinct subsets $\Omega_1(h)$ and $\Omega_2(h)$ of $\Omega(h)$ with $\Omega_1(h) \cup \Omega_2(h) = \Omega(h)$ such that

$$D_I = \omega_1 L_I \omega_1 + \omega_2 L_I \omega_2, \quad R_I = -\omega_2 L_I \omega_1, \quad S_I = -\omega_1 L_I \omega_2,$$

where the restrictions ω_j are defined by $(\omega_j u)(x) = u(x)$ if $x \in \Omega_j(h)$ and $(\omega_j u)(x) = 0$ otherwise.

Throughout this section we shall assume

(4.3) $\Omega(h)$ have 'property C' (cf. Lemma 2); $\alpha, \|\cdot\|_1, \|\cdot\|_2$ be defined by (3.5).

LEMMA 6. Let the splitting (4.2b) be 2-cyclic and assume $L_I = L_I^*$ to be positive definite. Then $|L_I G_I^\nu|_{0,0} \leq |D_I|_{0,0}/(\nu + 1/2)$ holds for all $\nu \geq 1$.

Proof. Numbering first the grid points of $\Omega_1(h)$ yields the following block structure:

$$L_I = \begin{bmatrix} d_1 & -s \\ -r & d_2 \end{bmatrix}, \quad G_I = \begin{bmatrix} 0 & d_1^{-1}s \\ 0 & d_2^{-1}rd_1^{-1}s \end{bmatrix},$$

$$L_I G_I^\nu = \begin{bmatrix} 0 & s\{[d_2^{-1}rd_1^{-1}s]^{\nu-1} - [d_2^{-1}rd_1^{-1}s]^\nu\} \\ 0 & 0 \end{bmatrix}.$$

Hence, $|D_I^{-1/2} L_I G_I^\nu D_I^{-1/2}|_{0,0}^2 = \left| \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \right|_{0,0}$ follows from $s^* = r$ with $A = B^{2\nu-1}(I - B)^2$ and $B = B^* = d_2^{-1/2}rd_1^{-1}sd_2^{-1/2}$. It is well known that $\rho(G_I) = \rho(B) = \|B\| \leq 1$ (cf. Note 5), where $\|\cdot\|$ denotes the $|\cdot|_{0,0}$ -norm restricted to the last block.

$$\|A\| = \rho(A) = \sup\{|\lambda^{2\nu-1}(1 - \lambda)^2|: \lambda \in \text{spectrum of } B\}$$

$$\leq \sup\{\lambda^{2\nu-1}(1 - \lambda)^2: 0 \leq \lambda \leq 1\} \leq 1/(\nu + 1/2)^2$$

implies $|L_I G_I^\nu|_{0,0} \leq |D_I^{1/2}|_{0,0} \|A\|^{1/2} \leq |D_I|_{0,0}/(\nu + 1/2)$. \square

4.2. General Criteria.

CRITERION 3. Assume (4.3), $\theta = \theta'$, (3.6a) for $\vartheta = 0$, and

$$(4.4a) \quad |L_I G_I^\nu|_{0,0} \leq h_I^{-2m} C(\nu) \quad \text{for } 1 \leq \nu \leq \nu_{\max}(h_1), l \geq 0;$$

$$C(\nu) \rightarrow 0 \quad (\nu \rightarrow \infty), \quad \nu_{\max}(h) \rightarrow \infty \quad (h \rightarrow 0),$$

$$(4.4b) \quad |G_I^\nu|_{m,m} \leq C \quad \text{for all } 1 \leq \nu \leq \nu_{\max}(h_1), l \geq 0.$$

Then the smoothing property (2.6) holds with $C_0(\nu) = C'[C(\nu)]^{\theta/m}$.

Proof. (3.6a) ($\vartheta = 0$) and (4.4b) yield $|L_I G_I^\nu|_{m,-m} \leq |L_I|_{m,-m} |G_I^\nu|_{m,m} \leq C$. Hence, (4.1b) ($t = m, r = m - \theta$) implies (2.6). \square

The following criterion applies also to the case of $\theta \neq \theta'$:

CRITERION 3*. Assume (4.3), (4.4a) and

$$(4.4b^*) \quad |L_I|_{2m,0} \leq C, \quad |L_I^*|_{2m,0} \leq C, \quad |G_I^\nu|_{0,0} \leq C, \\ |\tilde{G}_I^\nu|_{0,0} \leq C \quad (0 \leq \nu \leq \nu_{\max}(h_1), l \geq 0),$$

where $\tilde{G}_I = L_I G_I L_I^{-1}$. Then the smoothing property (2.6) holds with $C_0(\nu) = C' \cdot C(\nu)^{\alpha/(2m)} [\alpha = \theta + \theta', \text{ cf. (3.5)}]$.

Proof. (4.4b*) implies $|L_I G_I^\nu|_{2m,0} = |\tilde{G}_I^\nu L_I|_{2m,0} \leq |\tilde{G}_I^\nu|_{0,0} |L_I|_{2m,0} \leq C$. Since $|L_I^*|_{2m,0} = |L_I|_{0,-2m}$, also $|L_I G_I^\nu|_{0,-2m} \leq |L_I|_{0,-2m} |G_I^\nu|_{0,0} \leq C$ holds. (4.1c) yields (2.6). \square

First we shall verify the conditions of Criteria 3 and 3* for positive definite schemes. In a second step it is shown that additional terms of lower order may be added. Hence, all difference schemes with a hermitian principle part satisfy the smoothing property. In a third step we treat perturbations of order $O(h_l^{-2m})$ located at points near the boundary. Such perturbations often arise from special discretizations at the boundary.

Usually, the function $C_0(\nu)$ of (2.6) is $C/(\nu + 1)^{\alpha/(2m)}$. Therefore, $C_0(\nu) \rightarrow 0$ requires $\alpha = \theta + \theta' > 0$. The choice of $\theta = \theta' = 0$ is excluded. The upper bound $\nu_{\max}(h)$ of ν in (2.6) may be omitted (i.e. $\nu_{\max} = \infty$) if L_I is positive definite. In the case of other schemes $\nu_{\max}(h)$ might become finite (but $\nu_{\max}(h) \rightarrow \infty$ as $h \rightarrow 0$).

4.3. *Case of Positive Definite Difference Schemes.* Throughout this subsection we assume

$$(4.5) \quad L_I = L_I^*, \quad \frac{1}{C} |u|_m \leq (L_I u, u) \leq C |u|_m$$

as in Lemma 3. The proofs of convergence in [1], [2], [4], [6], [9], [11] require smoothing by

$$(4.6) \quad G_I(v_l, f_l) = v_l - \omega_l h_l^{2m} (L_I v_l - f_l), \quad G_I = I - \omega_l h_l^{2m} L_I.$$

If the diagonal of L_I is a multiple of I , G_I corresponds to a damped Jacobi iteration.

Note 4 (Smoothing by Jacobi Iteration). Assume (4.3), (4.5), (4.6) and $0 \leq \omega_l \leq h_l^{-2m} / |L_I|_{0,0}$. Then the smoothing property (2.6) holds for all ν ($\nu_{\max} = \infty$) with

$$(4.7) \quad C_0(\nu) = C/(\nu + 1/2)^{\alpha/(2m)} \quad (\alpha = \theta + \theta' \text{ from (3.5)}).$$

Proof. One may choose $L_{I,0} = L_I$ in Lemma 3. Thus, it suffices to estimate $A = \Lambda^{\theta-m} L_I G_I^\nu \Lambda^{\theta'-m} = L_I^\beta (I - \omega_l h_l^{2m} L_I)^\nu$, $\beta = \alpha/(2m)$, with respect to $|\cdot|_{0,0}$. But this norm is equal to the spectral radius. Since the spectrum of L_I is contained in $[0, 1/(\omega_l h_l^{2m})]$,

$$\rho(A) = \sup \{ \lambda^\beta (1 - \omega_l h_l^{2m} \lambda)^\nu : 0 \leq \omega_l h_l^{2m} \lambda \leq 1 \} \leq C/(\nu + 1)^\beta$$

proves Note 4. \square

The techniques of the following subsections can be applied to smoothing by (4.6), too. But since we are mainly interested in smoothing by Gauss-Seidel's iteration, henceforward our considerations are restricted to this subject.

Note 5 (Smoothing by Gauss-Seidel). Assume (4.3), $\theta = \theta'$, (4.5). Let G_l be defined by (4.2a, b), where the splitting (4.2b) is required to be 2-cyclic. Then the smoothing property (2.6) holds with $C_0(\nu)$ from (4.7) for all ν ($\nu_{\max}(h) = \infty$).

Proof. Since (4.2b) is 2-cyclic and L_l is positive definite, D_l is positive definite, too. Thus, the theorem of Ostrowski (cf. [8, p. 297], [10, p. 77]) applies resulting in $|L_l^{1/2} G_l^\nu L_l^{-1/2}|_{0,0} \leq 1$ ($\nu \geq 0$). By Lemma 3 (4.4b) follows. Lemma 6 implies (4.4a) with $C(\nu) = C/(\nu + 1/2)$ since $|D_l|_{0,0} \leq |L_l|_{0,0} \leq Ch_l^{-2m}$ results from (4.5) and $|u|_m \leq Ch_l^{-m} |u|_0$. (3.6a) with $\vartheta = 0$ holds by virtue of (4.5). Hence, all conditions of Criterion 3 are satisfied. \square

Example. Let $L_l u = f$ be the discretization of $-\operatorname{div}[(a(x_1), b(x_2))^T \operatorname{grad} u] = \varphi$ in Ω and $u = 0$ on $\partial\Omega$ as in the example of Section 3.1. (4.5) holds if $a(x_1), b(x_2) \in [e, C] \subset (0, \infty)$. Use the 'red-black' ordering of the grid points: $\Omega_1(h) = \{x \in \Omega(h) : (x_1 + x_2)/h \text{ even}\}$. If, in addition, Ω is a Lipschitz region, all conditions of Note 5 are satisfied. The smoothing property holds for all $\theta = \theta' = \alpha/2 \in (0, m]$.

Note 5 illustrates the application of Criterion 3. In order to apply Criterion 3* the following lemmata give conditions implying (4.4b*).

LEMMA 7. The inequalities $|L_l|_{2m,0} \leq C$ and $|L_l^*|_{2m,0} \leq C$ hold if the coefficients are sufficiently smooth. More precisely, the estimates hold if L_l is a finite sum of terms of the form

$$T^\gamma \partial^\beta c(x, h_j) \partial^{\beta'} \quad (\gamma, \beta, \beta' \in \mathbb{Z}^d, \beta_j \geq 0, \beta'_j \geq 0, |\beta| + |\beta'| \leq 2m),$$

where all k th derivatives of $c(x, h_j)$ with respect to x are uniformly Lipschitz continuous on $\bar{\Omega}$ for $k = \max(|\beta|, |\beta'|) - 1$ [for T^γ and ∂^β compare Section 3, (3.8a)].

Proof. $|L_l|_{2m,0} \leq C$ requires $k \geq |\beta| - 1$. Since L_l^* contains $(-1)^{|\beta|+|\beta'|} T^{\beta'} \partial^{\beta'} c \partial^\beta T^{\gamma+\beta}$, also $k \geq |\beta'| - 1$ must hold. \square

LEMMA 8. The estimates $|G_l^\nu|_{0,0} \leq C$, $|\tilde{G}_l^\nu|_{0,0} \leq C$ are valid for all $\nu \geq 0$ and $l \geq 0$ if the splitting (4.2b) is 2-cyclic and if one of the following conditions holds:

$$(4.8a) \quad L_l \text{ satisfies (4.5), } |D_l^{-1}|_{0,0} \leq Ch_l^{2m},$$

$$(4.8b) \quad D_l = \omega_l h_l^{-2m} I, \quad L_l \text{ and } L_l^* \text{ are diagonally dominant (cf. [10, p. 23])},$$

$$(4.8c) \quad |D_l^{-1}(R_l + S_l)|_{0,0} \leq 1, \quad |(R_l + S_l)D_l^{-1}|_{0,0} \leq 1.$$

Note that $|\cdot|_{0,0}$ coincides with the usual spectral norm of matrices.

Proof. (a) One verifies that $\tilde{G}_l = S_l(D_l - R_l)^{-1}$. G_l^ν and \tilde{G}_l^ν have the representations

$$G_l^\nu = \begin{bmatrix} 0 & d_1^{-1} s [d_2^{-1} r d_1^{-1} s]^{\nu-1} \\ 0 & [d_2^{-1} r d_1^{-1} s]^\nu \end{bmatrix}, \quad \tilde{G}_l^\nu = \begin{bmatrix} [s d_2^{-1} r d_1^{-1}]^\nu & [s d_2^{-1} r d_1^{-1}]^{\nu-1} s d_2^{-1} \\ 0 & 0 \end{bmatrix} \quad (\nu \geq 1)$$

Assume (4.8a) and let B be as in the proof of Lemma 6. $|D_l^{1/2} G_l^\nu D_l^{-1/2}|_{0,0}^2 = |D_l^{-1/2} \tilde{G}_l^\nu D_l^{1/2}|_{0,0}^2 = \|B^{2\nu} + B^{2\nu-1}\| \leq 2$ shows $|G_l^\nu|_{0,0} \leq \sqrt{2} |D_l^{-1/2}|_{0,0} |D_l^{1/2}|_{0,0} \leq C$ and $|\tilde{G}_l^\nu|_{0,0} \leq C$.

(b) Let $\|\cdot\|_\infty$ be the matrix norm corresponding to the supremum norm. (4.8b) implies that the $\|\cdot\|_\infty$ norm of $D_l^{-1}(R_l + S_l) = (R_l + S_l)D_l^{-1}$ and of the adjoint matrix are bounded by 1. Hence, (4.8c) holds.

(c) From (4.8c) it follows that $\|d_1^{-1}s\|, \|d_2^{-1}r\|, \|sd_2^{-1}\|, \|rd_1^{-1}\| \leq 1$ ($\|\cdot\|$: spectral norm). Then the representations of G_l^ν and \tilde{G}_l^ν yield $|G_l^\nu|_{0,0}, |\tilde{G}_l^\nu|_{0,0} \leq \sqrt{2}$. \square

We summarize:

Note 6. Assume (4.3) and (4.5). Let the coefficients of L_l be sufficiently smooth (cf. Lemma 7). G_l is defined by (4.2a), where the splitting (4.2b) is 2-cyclic with $|D_l^{-1}|_{0,0} \leq Ch_l^{2m}$. Then the smoothing property (2.6) holds for all $\theta, \theta' \in [0, m]$, $\theta + \theta' = \alpha > 0$ with $C_0(\nu)$ from (4.7) and $\nu_{\max}(h) = \infty$.

Proof. (4.4a) follows as in Note 5. Thanks to Lemmata 7, 8 the Criterion 3* yields (2.6). \square

4.4. Perturbations by Lower Order Terms. In the following we shall assume that the difference scheme L_l is the sum $L_l' + L_l''$, where L_l' satisfies the smoothing property (2.6). We assume a 2-cyclic splitting of L_l and L_l' :

$$\begin{aligned} L_l &= D_l - R_l - S_l, \quad L_l' = D_l' - R_l' - S_l', \\ (4.9) \quad G_l &= (D_l - R_l)^{-1}S_l, \quad G_l' = (D_l' - R_l')^{-1}S_l', \end{aligned}$$

$$G_l'' = G_l - G_l', \quad D_l'' = D_l - D_l', \quad R_l'' = R_l - R_l', \quad S_l'' = S_l - S_l'.$$

L_l'' is called a lower order term if there is some $\beta > 0$ such that

$$(4.10) \quad |L_l''|_{0,0} \leq Ch_l^{\beta-2m} \quad (\beta > 0, l \geq 0).$$

The first criterion applies if $\beta > m - \max(\theta, \theta')$.

CRITERION 4. Let $L_l = L_l' + L_l''$ and L_l' have 2-cyclic splittings and define G_l and G_l' by (4.9). Choose the norms by (3.5) and assume

$$(4.10^*) \quad |L_l''|_{0,\theta-m} \leq Ch_l^{\beta-m-\theta} \quad [\text{or } |L_l''|_{m-\theta',0} \leq Ch_l^{\beta-m-\theta'}],$$

$$(4.11) \quad |D_l'^{-1}|_{0,0} \leq Ch_l^{2m}, \quad |L_l'|_{m,-m} \leq C,$$

and $\beta > m - \theta'$ [or $\beta > m - \theta$, respectively]. Then L_l has the smoothing property (2.6) if L_l' has.

For the usual case of $m = 1$ β takes the values 1 and 2. Hence, $\alpha = \theta + \theta' > 0$ implies $\beta > m - \theta'$ or $\beta > m - \theta$. (4.10*) holds if L_l'' is a difference scheme of order $\leq 2m - \beta$ with smooth coefficients (cf. Lemma 7). Note that (4.10*) implies (4.10).

Proof. By (4.10) and (4.11) the estimate $|D_l'^{-1}L_l''|_{0,0} \leq Ch_l^\beta$ holds. The same norm of $D_l'^{-1}D_l'', D_l'^{-1}R_l'',$ and $D_l'^{-1}S_l''$ is also of order $O(h_l^\beta)$ since the splitting is 2-cyclic. Hence, $|G_l''|_{0,0} \leq Ch_l^\beta$ is valid for sufficiently small h_l . The second estimate of (4.11) implies $|L_l'|_{0,0} \leq Ch_l^{-2m}$. Thus, $|G_l'|_{0,0} \leq C$ holds, too. $X(\nu) = G_l^\nu - G_l'^\nu$

can be estimated by

$$\begin{aligned} |X(\nu)|_{m-\theta',0} &\leq |X(\nu)|_{0,0} \leq c(\nu, h_l) := \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} |G_l'|_{0,0}^{\mu} |G_l''|_{0,0}^{\nu-\mu} \\ &\leq \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} C^{\mu} [Ch_l^{\beta}]^{\nu-\mu} = C^{\nu} [(1 + h_l^{\beta})^{\nu} - 1]. \end{aligned}$$

The further terms of

$$\|L_l G_l^{\nu}\|_{2,1} \leq \|L_l' G_l^{\nu}\|_{2,1} + |L_l''|_{0,\theta-m} |G_l^{\nu}|_{m-\theta',0} + |L_l'|_{0,\theta-m} |X(\nu)|_{m-\theta',0}$$

are bounded by

$$|L_l''|_{0,\theta-m} \leq Ch_l^{\beta-m-\theta}, \quad |L_l'|_{0,\theta-m} \leq Ch_l^{-m-\theta}, \quad |G_l^{\nu}|_{m-\theta',0} \leq C^{\nu}.$$

Since L_l' satisfies (2.6) with $C_0'(\nu)$ and $\nu'_{\max}(h)$ one obtains $\|L_l G_l^{\nu}\|_{2,1} \leq h_l^{-\alpha} c_0(\nu, h_l)$ with $c_0(\nu, h) = C_0'(\nu) + h^{\theta'-m+\beta} C^{\nu} + Ch_l^{\theta'-m} c(\nu, h)$. $\theta' - m + \beta > 0$ implies $c_0(\nu, 0) = C_0'(\nu)$. Thus, there exists $\nu_{\max}(h) \leq \nu'_{\max}(h)$ with $\nu_{\max}(0) = \infty$ such that $c_0(\nu, h) \leq C_0(\nu) := 2C_0'(\nu)$ for all $0 \leq \nu \leq \nu_{\max}(h)$. In the case of the second inequality of (4.10*) and $\beta - m < \theta$ apply the analogous estimates to $\tilde{G}_l^{\nu} L_l = L_l G_l^{\nu}$. \square

The following criterion is applicable for all $\beta > 0$. On the other hand L_l' must satisfy not only the smoothing property but also the sufficient conditions of Criterion 3*.

CRITERION 5. Let $L_l = L_l' + L_l''$ and L_l' have 2-cyclic splittings and define G_l and G_l' by (4.9). Assume (4.3), (4.10), (4.11), and $|L_l''|_{2m,0} \leq C$, $|L_l''|_{2m,0}^* \leq C$. Moreover, the estimates (4.4a) and (4.4b*) must be valid for L_l', G_l', \tilde{G}_l' (instead of L_l, G_l, \tilde{G}_l). Then the smoothing property (2.6) holds for L_l , too.

Proof. Repeating the proof of Criterion 4 for the special case of $\theta = \theta' = m$ one obtains (4.4a). The same proof shows (4.4b*) for a suitable choice of $\nu_{\max}(h)$. Hence Criterion 3* implies (2.6). \square

Note 6 and Criterion 5 establish the following result.

Note 7. Assume (4.3) and $C^{-1}|u|_m^2 \leq \operatorname{Re}(L_l u, u) + \lambda_0 |u|_0^2 \leq C|u|_m^2$ for some real λ_0 (H_0^m -coerciveness of L_l). L_l must consist of the terms $T^{\gamma} \partial^{\beta} c(x, h_l) \partial^{\beta}$ described in Lemma 7. G_l is defined by (4.2a), where the splitting (4.2b) is 2-cyclic with $|D_l^{-1}|_{0,0} \leq Ch_l^{2m}$. Then the smoothing property (2.6) holds with $C_0(\nu)$ from (4.7).

Proof. Define $L_l' = (L_l + L_l^*)/2 + \lambda_0 I$ and $L_l'' = L_l - L_l'$. $|L_l''|_{0,0} \leq Ch_l^{2m-1}$ and $|D_l^{-1}|_{0,0} \leq Ch_l^{2m}$ imply $|D_l'^{-1}|_{0,0} \leq C'h_l^{2m}$ for sufficiently small h_l . Hence, Note 6 shows that (4.4a) and (4.4b*) hold for L_l' and G_l' . (2.6) follows by Criterion 5. \square

4.5. Perturbation at the Boundary. In particular, if special discretizations are used at points near the boundary, the difference scheme L_l is a sum of a scheme L_l' with smooth coefficients as studied in the foregoing section and a further term L_l'' with $(L_l'' u)(x) \neq 0$ only at points near the boundary. The following note shows the smoothing property for an important class of discretizations.

Note 8. Let $L_l = L_l' + L_l''$ and L_l' have 2-cyclic splittings with diagonal matrices D_l, D_l' and define G_l and G_l' by (4.9). If $(L_l' u)(x) \neq 0$ for some u , $|x - x'| \leq Ch_l$ must hold for some $x' = \nu' h_l \notin \Omega(h_l)$ (cf. Lemma 2). Moreover, $(L_l \mu)(x)$ and

$(L'_I u)(x)$ must depend only on $u(x')$ with $|x' - x| \leq Ch_I(x, x' \in \Omega(h_I))$. Assume that (4.4a) and (4.4b*) hold for L'_I , G'_I , and \tilde{G}'_I with $C(\nu)$ from (4.7) (sufficient conditions are those of Note 7). Furthermore, (4.3) and (3.6b) with some $\vartheta \in (0, 1/2)$ are required for L'_I (instead of L_I). Let

$$(4.12a) \quad \begin{aligned} |D_I'^{-1}|_{0,0} &\leq Ch_I^{2m}, \quad |R_I + S_I|_{0,0} \leq Ch_I^{-2m}, \\ |D_I'^{-1}L'_I|_{0,0} &\leq C, \quad |L'_ID_I'^{-1}|_{0,0} \leq C. \end{aligned}$$

The inequalities

$$(4.12b) \quad 0 \leq D_I'^{-1}(R_I + S_I) \leq D_I'^{-1}(R'_I + S'_I), \quad 0 \leq (R_I + S_I)D_I'^{-1} \leq (R'_I + S'_I)D_I'^{-1}$$

must hold for all entries of the matrices. Then the smoothing property (2.6) is valid with the same $\nu_{\max}(h)$ as for L'_I .

It is to be emphasized that D_I is not required to be uniformly bounded.

Proof. (1) We abbreviate $|\cdot|_{0,0}$ by $\|\cdot\|$. There is γ as in Lemma 2 such that $L''_I = L'_I \gamma$. (4.4b*) implies $\|D'_I\| \leq Ch_I^{-2m}$ and $\|L'_I\| \leq Ch_I^{-2m}$. By virtue of the Perron-Frobenius theory (cf. [10, p. 26]) $\|D_I'^{-1}L_I\| \leq \|D_I'^{-1}L'_I\| \leq C$ can be concluded from (4.12a, b). Therefore,

$$\begin{aligned} |D'_ID_I'^{-1}L_I|_{2m,0} &\leq |D'_ID_I'^{-1}L_I - L'_I|_{2m,0} + |L'_I|_{2m,0} \leq |(D'_ID_I'^{-1}L_I - L'_I)\gamma|_{2m,0} + C \\ &\leq (\|D'_I\| \|D_I'^{-1}L_I\| + \|L'_I\|) |\gamma|_{2m,0} + C \leq C' \end{aligned}$$

yields the first inequality of (4.13a):

$$(4.13a) \quad |D'_ID_I'^{-1}L_I|_{2m,0} \leq C, \quad |D_I'^*D_I'^{-1}L_I^*|_{2m,0} \leq C.$$

Similarly, the second estimate is proved.

(2) Let d_1 , d_2 , r , and s be as in the proof of Lemma 6. (4.12b) yields $0 \leq d_1^{-1}s \leq d_1'^{-1}s'$, etc. Hence

$$(4.13b) \quad 0 \leq G_I^\nu \leq G_I'^\nu, \quad 0 \leq \tilde{G}_I^\nu \leq \tilde{G}_I'^\nu \quad (\nu \geq 0)$$

follows. The Perron-Frobenius theory shows $\|G_I^\nu\| \leq \|G_I'^\nu\|$. By $\|D_I G_I^\nu\| \leq \|s\| (1 + \|rd_1^{-1}\|) \|G_I^{\nu-1}\| \leq Ch_I^{-2m}$ [cf. (4.12a), (4.4b*)] and $\|D_I'^{-1}\| \leq Ch_I^{2m}$ we obtain the first estimate of (4.13c):

$$(4.13c) \quad \|D_I'^{-1}D_I G_I^\nu\| \leq C, \quad \|\tilde{G}_I^\nu D_I D_I'^{-1}\| \leq C \quad (1 \leq \nu \leq \nu_{\max}(h_1)).$$

The proof of the second one is similar.

(3) (4.13a) and (4.13b) imply

$$(4.13d) \quad |L_I G_I^\nu|_{0,-2m} \leq C, \quad |L_I G_I^\nu|_{2m,0} \leq C \quad (1 \leq \nu \leq \nu_{\max}(h_1)).$$

E.g., the first inequality follows from

$$\begin{aligned} |L_I G_I^\nu|_{0,-2m} &\leq |L_I D_I'^{-1} D'_I|_{0,-2m} |D_I'^{-1} D_I G_I^\nu|_{0,0} \\ &\leq |D_I'^* D_I'^{-1} L_I^*|_{2m,0} \|D_I'^{-1} D_I G_I^\nu\| \leq C. \end{aligned}$$

(4) Let γ be as in Lemma 2. By (3.6b) $|L_l'^{-1}|_{\vartheta-m, \vartheta+m} \leq C$ holds for some $0 < \vartheta < 1/2$. Lemma 2 proves $|\gamma|_{\vartheta+m, 0} \leq Ch_l^{\vartheta+m}$. Interpolation of (4.4a) and (4.4b*) yields $|L_l' G_l'^{\nu}|_{0, \vartheta-m} \leq Ch_l^{m-\vartheta}/(\nu+1)^{\beta}$ with $\beta = (m+\vartheta)/(2m)$. Therefore,

$$\|\gamma G_l'^{\nu}\| \leq |\gamma|_{\vartheta+m, 0} |L_l'^{-1}|_{\vartheta-m, \vartheta+m} |L_l' G_l'^{\nu}|_{0, \vartheta-m} \leq C(\nu+1)^{\beta}$$

is valid. Applying again the Perron-Frobenius theory, we obtain

$$(4.13e) \quad \|\gamma G_l'^{\nu}\| \leq C(\nu+1)^{\beta}, \quad \beta = (m+\vartheta)/(2m) \quad (0 \leq \nu \leq \nu_{\max}(h_1))$$

from (4.13b). Now,

$$(4.13f) \quad \|L_l'' G_l'^{\nu}\| \leq Ch_l^{-2m}/(\nu+1)^{\beta}, \quad \beta = (m+\vartheta)/(2m) \quad (1 \leq \nu \leq \nu_{\max}(h_1))$$

can be concluded from $\|L_l'' G_l'^{\nu}\| = \|L_l'' G_l\| \|\gamma G_l'^{\nu-1}\|$, since there is γ satisfying the conditions of Lemma 2 with $L_l' G_l = L_l'' G_l \gamma$. The second term is estimated in (4.13e). Split the first term into $L_l' G_l - L_l' G_l'$. $\|L_l' G_l\| \leq Ch_l^{-2m} |L_l' G_l|_{0, -2m} \leq Ch_l^{-2m}$ follows from (4.13d) ($\nu = 1$). (4.4b*) for L_l' and (4.13b) yield $\|L_l' G_l'\| \leq Ch_l^{-2m}$.

(5) Using $L_l'(G_l'^{\nu} - G_l'^{\mu}) = -\sum_{\mu=0}^{\nu-1} L_l' G_l'^{\mu} \mu G_l'' G_l'^{\nu-\mu-1}$ and $G_l'' = G_l'' \gamma$, one obtains

$$\begin{aligned} \|L_l'(G_l'^{\nu} - G_l'^{\mu})\| &\leq \sum_{\mu=0}^{\nu-1} \|L_l' G_l'^{\mu}\| \|G_l''\| \|\gamma G_l'^{\nu-\mu-1}\| \\ &\leq h_l^{-2m} C' \sum_{\mu=0}^{\nu-1} [(\mu+1)^{-1}(\nu-\mu)^{-\beta}] \leq Ch_l^{-2m}/(\nu+1)^{\vartheta/m}. \end{aligned}$$

This estimate, (4.4a) (for L_l'), and (4.13f) yield (4.4a) for L_l :

$$\|L_l G_l'^{\nu}\| \leq \|L_l'' G_l'^{\nu}\| + \|L_l'(G_l'^{\nu} - G_l'^{\mu})\| + \|L_l' G_l'^{\nu}\| \leq Ch_l^{-2m}/(\nu+1)^{\vartheta/m} \quad (1 \leq \nu \leq \nu_{\max}(h_1)).$$

Repeating the proof of Criterion 3* yields (2.6). \square

Example. Consider the Shortley-Weller discretization L_l (cf. last example of Section 3). L_l' is the usual five-point formula. Hence, (4.4a) and (4.4b*) are fulfilled with $\nu_{\max}(h) = \infty$. (3.6b) holds for all $\theta_0 = \theta'_0 < 1/2$. Also the conditions (4.12a, b) are satisfied. Thus, the smoothing property holds for all ν ($\nu_{\max} = \infty$).

Mathematisches Institut
Universität zu Köln
Weyertal 86-90
D-5000 Köln 41, Federal Republic of Germany

1. G. P. ASTRACHANCEV, "An iterative method of solving elliptic net problems," *Ž. Vyčisl. Mat. i Mat. Fiz.*, v. 11, 1971, pp. 439-448.
2. N. S. BACHVALOV, "On the convergence of a relaxation method with natural constraints on the elliptic operator," *Ž. Vyčisl. Mat. i Mat. Fiz.*, v. 6, 1966, pp. 861-885.
3. A. BRANDT, "Multi-level adaptive solutions to boundary-value problems," *Math. Comp.*, v. 31, 1977, pp. 333-390.
4. R. P. FEDORENKO, "The speed of convergence of one iterative process," *Ž. Vyčisl. Mat. i Mat. Fiz.*, v. 4, 1964, pp. 559-564.

5. W. HACKBUSCH, "On the multi-grid method applied to difference equations," *Computing*, v. 20, 1978, pp. 291–306.
6. W. HACKBUSCH, "On the convergence of multi-grid iterations," *Beiträge Numer. Math.*, v. 9. (To appear.)
7. W. HACKBUSCH, "On the regularity of difference schemes." (To appear.)
8. TH. MEIS & U. MARCOWITZ, *Numerische Behandlung partieller Differentialgleichungen*, Springer-Verlag, Berlin, Heidelberg, New York, 1978.
9. R. A. NICOLAIDES, "On the l^2 convergence of an algorithm for solving finite element equations," *Math. Comp.*, v. 31, 1977, pp. 892–906.
10. R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
11. P. WESSELING, *A Convergence Proof for a Multiple Grid Method*, Delft University of Technology, Report NA-21, 1978.
12. W. HACKBUSCH, "Bemerkungen zur iterierten Defektkorrektur und zu ihrer Kombination mit Mehrgitterverfahren," *Rev. Roumaine Math. Pures Appl.* (To appear.)