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CONVERGENCE OF NONCONFORMING MULTIGRID METHODS WITHOUT FULL ELLIPTIC REGULARITY

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Abstract. We consider nonconforming multigrid methods for symmetric positive definite second and fourth order elliptic boundary value problems which do not have full elliptic regularity. We prove that there is a bound (< 1) for the contraction number of the W-cycle algorithm which is independent of mesh level, provided that the number of smoothing steps is sufficiently large. We also show that the symmetric variable V-cycle algorithm is an optimal preconditioner.

1. Introduction

The multigrid theory for conforming finite element methods where the finite element spaces on successive grids are nested is now well understood (cf., for example, the books [42], [46], [10] and the references therein).

However, for certain problems the simplest finite element methods are nonconforming or conforming but nonnested. For example, the simplest method for the stationary Stokes equations uses the Crouzeix-Raviart element (nonconforming), and the simplest finite element methods for the plate bending problem use the Morley finite element (nonconforming) or the (reduced) Hsieh-Clough-Tocher macro-element (conforming but nonnested). Also, some simple nonconforming methods can overcome the phenomenon of locking in elasticity problems and plate problems (cf., [5], [37], [28], [67]).

The convergence of multigrid methods for nonconforming elements was studied in [14]–[18], [20], [21], [23], [49], [9], [43], [50], [62], [63], [52], [55], [64], [65], [68] and [60]. The convergence of the multigrid method for macro-elements was studied in [66]. The results for the nonconforming or conforming but nonnested multigrid methods can also be obtained from the more abstract theory of Bramble, Pasciak and Xu (cf., [13]) once their “regularity and approximation” assumption is verified for each concrete problem. The results in all the papers (except [65]; see below) cited above for nonconforming and macro elements have been obtained under the condition that the underlying boundary value problem has full elliptic regularity.

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In this paper we study the convergence of multigrid methods for nonconforming finite elements without assuming full elliptic regularity. We follow the methodology of Bank and Dupont in [6], where the convergence of conforming, nested W-cycle multigrid methods is established without full elliptic regularity. The two key ingredients in their approach are: (i) the equivalence between mesh-dependent norms and fractional order Sobolev norms on the finite element space, and (ii) a duality argument involving fractional order Sobolev spaces. Since the nonconforming finite element space may not be a subspace of the fractional order Sobolev space, there are no straightforward generalizations of (i) and (ii) to the nonconforming case. We overcome this difficulty by relating the nonconforming finite element to a conforming finite element.

The idea of using conforming "relatives" in the treatment of nonconforming finite elements was first used in the context of additive Schwarz preconditioners for nonconforming finite elements (cf., [22], [24], [25]). Let $(K, \mathcal{P}, \mathcal{N})$ and $(K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ be two finite elements (cf., [30], [27]), where K is the shared element domain, $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are the spaces of shape functions, and $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are the sets of nodal variables. We say that $(K, \mathcal{P}, \mathcal{N}) \preceq (K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ if $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ and $\mathcal{N} \subseteq \tilde{\mathcal{N}}$, and refer to $(K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ as a "relative" of $(K, \mathcal{P}, \mathcal{N})$. Let V and $\tilde{V}$ be the finite element spaces on the same triangulation associated with $(K, \mathcal{P}, \mathcal{N})$ and $(K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ respectively. Then we say that $V \preceq \tilde{V}$ if $(K, \mathcal{P}, \mathcal{N}) \preceq (K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$. Our idea is to find a conforming finite element space $\tilde{V}_h$ for a given nonconforming finite element space V_h such that $V_h \preceq \tilde{V}_h$. Then we obtain multigrid convergence results for V_h by exploiting its connection with $\tilde{V}_h$. In the theory we do not require that the $\tilde{V}_h$ on successive grids be nested. Therefore, by applying the theory to $V_h = \tilde{V}_h$, we also have multigrid convergence results for conforming but nonnested finite element methods.

After the completion of the first draft of this paper, we learned that W-cycle convergence in the (nonconforming) energy norm without full elliptic regularity was obtained in [65] by a different technique. However, one of the assumptions (Assumption A.4) in [65] concerns a discretization error estimate for nonconforming finite elements which is not in the literature and was not proved in [65]. It turns out that this estimate follows from our theory (cf., the remark after Theorem 3.8). Thus the estimate from our approach combined with the theory in [65] would give another complete proof of the W-cycle convergence in the (nonconforming) energy norm.

The rest of the paper is organized as follows. In Section 2 we set up the notation and assumptions of an abstract framework for our finite element multigrid analysis which is applicable to both second and fourth order problems. Preliminary estimates are established in Section 3. In Section 4 we obtain the convergence of the k-th level W-cycle algorithm and the full multigrid W-cycle method in both the (nonconforming) energy norm and a lower order norm. In particular we show that the contraction number of the k-th level W-cycle algorithm is bounded away from 1 uniformly when the number of smoothing steps is sufficiently large. For fourth order problems, the convergence in the lower order norm and the connection to the conforming relative result in a better pointwise convergence rate for the nonconforming method. We also prove that the symmetric variable V-cycle multigrid algorithm is an optimal preconditioner. Applications of our theory to second and fourth order problems are given in Sections 5 and 6.

For future reference, we state the W–cycle and variable V–cycle algorithms here. Let $V_1, V_2, \ldots$ be finite-dimensional vector spaces, and let $A_k : V_k \longrightarrow V_k$, $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ and $I_k^{k-1} : V_k \longrightarrow V_{k-1}$. The equation to be solved is

$$A_k z = g. \tag{1.1}$$

The W-cycle multigrid algorithm. Let m_1 and m_2 be two nonnegative integers. The W-cycle multigrid algorithm with initial guess z_0 yields $\mathit{WMG}(k, z_0, g)$ as an approximate solution to the equation (1.1).

For $k = 1$, $\mathit{WMG}(1, z_0, g)$ is the solution obtained from a direct method. In other words,

$$\mathit{WMG}(1, z_0, g) = A_1^{-1} g.$$

For $k > 1$, $\mathit{WMG}(k, z_0, g)$ is defined recursively in three steps.

- *Pre-smoothing.* Let $z_l \in V_k$ $(1 \le l \le m_1)$ be defined recursively by the equations

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}\Big(g - A_k z_{l-1}\Big), \qquad 1 \le l \le m_1, \tag{1.2}$$

  where Λ_k dominates the spectral radius of A_k.
- *Correction.* Let $\overline{g} := I_k^{k-1}(g - A_k z_{m_1})$. Let $q_i \in V_{k-1}$ $(0 \le i \le 2)$ be defined recursively by

$$\begin{aligned} &q_0 = 0, \qquad \text{and} \\ &q_i = \mathit{WMG}(k-1, q_{i-1}, \overline{g}), \quad i = 1, 2. \end{aligned} \tag{1.3}$$

  Let $z_{m_1+1} = z_{m_1} + I_{k-1}^k q_2$.
- *Post-smoothing.* Let $z_l \in V_k$ $(m_1+2 \le l \le m_1+m_2+1)$ be defined recursively by the equations

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}\Big(g - A_k z_{l-1}\Big), \qquad m_1 + 2 \le l \le m_1 + m_2 + 1. \tag{1.4}$$

  Then $\mathit{WMG}(k, z_0, g) = z_{m_1+m_2+1}$.

The symmetric variable V-cycle algorithm. Let m_j $(j = 2, \ldots, k)$ be positive integers which are chosen so that $\beta_0 m_j \le m_{j-1} \le \beta_1 m_j$ for $j = 3, \ldots, k$, and $1 < \beta_0 \le \beta_1$. The symmetric variable V-cycle multigrid algorithm with initial guess z_0 yields $\mathcal{V}\mathit{MG}(k, z_0, g)$ as an approximate solution to the equation (1.1).

For $k = 1$, $\mathcal{V}\mathit{MG}(1, z_0, g)$ is the solution obtained from a direct method. In other words,

$$\mathcal{V}\mathit{MG}(1, z_0, g) = A_1^{-1} g.$$

For $k > 1$, $\mathcal{V}\mathit{MG}(k, z_0, g)$ is defined recursively in three steps.

- *Pre-smoothing.* Let $z_l \in V_k$ $(1 \le l \le m_k)$ be defined recursively by the equations

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}\Big(g - A_k z_{l-1}\Big), \qquad 1 \le l \le m_k, \tag{1.5}$$

  where Λ_k dominates the spectral radius of A_k.
- *Correction.* Let $\overline{g} := I_k^{k-1}(g - A_k z_{m_k})$, and

$$q = \mathcal{V}\mathit{MG}(k-1, 0, \overline{g}). \tag{1.6}$$

  Let $z_{m_k+1} = z_{m_k} + I_{k-1}^k q$.

- *Post-smoothing.* Let $z_l \in V_k$ $(m_k + 2 \le l \le 2m_k + 1)$ be defined recursively by the equations

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}\Big(g - A_k z_{l-1}\Big), \qquad m_k + 2 \le l \le 2m_k + 1, \tag{1.7}$$

Then $\mathcal{V}MG(k, z_0, g) = z_{2m_k+1}$.

2. An abstract framework

In this section we set up an abstract framework for our finite element multigrid analysis, which will be carried out in Sections 3 and 4 under the assumptions stated here. Throughout this paper, $\ell = 1$ (second order problems) or 2 (fourth order problems), and $\alpha \in (0, 1]$. The case $\alpha = 1$ corresponds to the case of full elliptic regularity.

We begin with the continuous problem. Let V be a Hilbert space and $a(\cdot, \cdot)$ be a symmetric bilinear form on V which is bounded and coercive:

$$|a(v_1, v_2)| \lesssim \|v_1\|_V \|v_2\|_V \quad \forall\, v_1, v_2 \in V, \tag{B}$$

$$a(v, v) \gtrsim \|v\|_V^2 \quad \forall\, v \in V. \tag{C}$$

In order to avoid the proliferation of constants, we adopt the notation $\lesssim$, $\gtrsim$ and $\approx$. The statement $F \lesssim G$ (or $G \gtrsim F$) means that F is bounded by G multiplied by a constant which is independent of mesh sizes. The statement $F \approx G$ means $F \lesssim G$ and $G \lesssim F$.

Let $F \in V'$. The continuous problem is to find $u \in V$ such that

$$a(u, v) = F(v) \quad \forall\, v \in V. \tag{2.1}$$

There exists a unique solution of (2.1) by (B), (C) and the Riesz Representation Theorem.

We assume that there exist two other Hilbert spaces Z and W such that

$$Z \hookrightarrow V \hookrightarrow W, \tag{R-1}$$

$$\|u\|_Z \lesssim \|F\|_{W'}, \tag{R-2}$$

where $F \in W'$ and u is the solution to (2.1).

Moreover, we assume the spaces Z and W are related by the following duality estimate.

$$|a(\zeta, v)| \lesssim \|\zeta\|_Z \|v\|_W \quad \forall\, \zeta \in Z,\, v \in V. \tag{D}$$

Remark. In applications V is a subspace of $H^{\ell}(\Omega)$, W is a subspace of $H^{\ell-\alpha}(\Omega)$, and Z is a subspace of $H^{\ell+\alpha}(\Omega)$. The elliptic regularity for (2.1) is then given by (R-1) and (R-2).

Next we describe the finite element spaces. Let $V_1, V_2, \dots$ and $\tilde{V}_1, \tilde{V}_2, \dots$ be two sequences of finite-dimensional vector spaces with corresponding mesh parameters $h_1, h_2, \dots$. We assume that there exist positive constants C_1 and C_2, independent of the mesh sizes, such that

$$C_1 h_{k-1} \le h_k \le C_2 h_{k-1} \quad \text{and} \quad 0 < C_1 \le C_2 < 1. \tag{M}$$

We assume that the spaces V_k and $\tilde{V}_k$ are connected to the spaces of the continuous problem through the following relations:

$$\tilde{V}_k \subseteq V \quad \text{(i.e., } \tilde{V}_k \text{ is conforming)}, \tag{C-1}$$

and there exists a Hilbert space X such that

$$W \hookrightarrow X \quad \text{and} \quad V_k, \tilde{V}_k \subseteq X \quad \text{for} \quad k \geq 1, \tag{C-2}$$

$$W = [X, V]_{1-(\alpha/\ell)}, \tag{C-3}$$

where $[X, V]_{1-(\alpha/\ell)}$ denotes the interpolation space obtained from X and V by the complex method of interpolation (cf., [7], [61], [44]). When $\alpha = 1 = \ell$, we interpret $[X, V]_0$ to be the space X.

Remark. Note that we do *not* assume $\tilde{V}_{k-1} \subseteq \tilde{V}_k$, i.e., $\tilde{V}_k$ is conforming but not necessarily nested, and the space V_k can be nonconforming and hence nonnested. However, they are all inside the space X, which is just $L^2(\Omega)$ in applications.

Let $V_0 = \{0\}$. We assume that, for each positive integer k, there exists a symmetric positive definite bilinear form $a_k(\cdot,\cdot)$ on $V_{k-1} + V_k + V$ such that $a_k(\cdot,\cdot)$ reduces to $a(\cdot,\cdot)$ on V and $a_{k-1}(\cdot,\cdot)$ on V_{k-1}. The (possibly nonconforming) energy norm $\|\cdot\|_k$ on $V_{k-1} + V_k + V$ is then defined to be

$$\|v\|_k = [a_k(v, v)]^{1/2}. \tag{2.2}$$

It follows from the boundedness and coercivity conditions (B) and (C) that

$$\|v\|_V \approx \|v\|_k \qquad \forall\, v \in V. \tag{2.3}$$

Furthermore, we assume the following *inverse estimate* holds for $k \geq 1$:

$$\|v\|_k \lesssim h_k^{-\ell} \|v\|_X \qquad \forall\, v \in V_{k-1} + V_k + \tilde{V}_k. \tag{I}$$

We assume there exists an interpolation operator $\Pi_k : V \longrightarrow V_k$ which satisfies the following *interpolation estimates* for $k \geq 1$:

$$\|\Pi_k v - v\|_X + h_k^\ell \|\Pi_k v\|_k \lesssim h_k^\ell \|v\|_V \qquad \forall\, v \in V, \tag{Π-1}$$

$$\|\zeta - \Pi_k \zeta\|_X + h_k^\ell \|\zeta - \Pi_k \zeta\|_k \lesssim h_k^{\ell+\alpha} \|\zeta\|_Z \qquad \forall\, \zeta \in Z. \tag{Π-2}$$

The spaces V_k and $\tilde{V}_k$ are connected by the operators $E_k : V_k \longrightarrow \tilde{V}_k$ and $F_k : \tilde{V}_k \longrightarrow V_k$ which satisfy the following:

$$\|E_k v - v\|_X \lesssim h_k^\ell \|v\|_k \qquad \forall\, v \in V_k, \tag{E}$$

$$\|E_k \Pi_k \zeta - \zeta\|_X + h_k^\ell \|E_k \Pi_k \zeta - \zeta\|_V \lesssim h_k^{\ell+\alpha} \|\zeta\|_Z \quad \forall\, \zeta \in Z, \tag{EΠ}$$

$$\|F_k \tilde{v} - \tilde{v}\|_X \lesssim h_k^\ell \|\tilde{v}\|_V \qquad \forall\, \tilde{v} \in \tilde{V}_k, \tag{F}$$

$$F_k E_k v = v \qquad \forall\, v \in V_k. \tag{FE}$$

Remark. In applications the constructions and analyses of E_k and F_k rely on the relation $V_k \preceq \tilde{V}_k$.

Let $\zeta \in Z$ and $\zeta_k \in V_k$ be related by

$$a(\zeta, E_k v) = a_k(\zeta_k, v) \quad \forall\, v \in V_k. \tag{2.4}$$

We assume that

$$|a_k(\zeta - \zeta_k, v)| \lesssim h_k^\alpha \, \|\zeta\|_Z \, \|v\|_k \quad \forall\, v \in V_k, \zeta \in Z, \tag{N-1}$$

$$|a_k(\zeta - \zeta_k, \Pi_k \xi)| \lesssim h_k^{2\alpha} \, \|\zeta\|_Z \, \|\xi\|_Z \quad \forall\, \zeta, \xi \in Z. \tag{N-2}$$

Remark. In applications the estimates (N-1) and (N-2) are obtained by modifying standard estimates for nonconforming finite element methods.

So far the relations between the spaces V_k have not been specified. Now we connect V_{k-1} and V_k by the coarse-to-fine intergrid transfer operator $I^k_{k-1} : V_{k-1} \longrightarrow V_k$. We assume that the following estimates on I^k_{k-1} hold:

$$\|I^k_{k-1}v - v\|_X \lesssim h^\ell_k \|v\|_{k-1} \quad \forall\, v \in V_{k-1}, \tag{I-1}$$

$$\|I^k_{k-1}\Pi_{k-1}\zeta - \Pi_k\zeta\|_X \lesssim h^{\ell+\alpha}_k \|\zeta\|_Z \quad \forall\, \zeta \in Z. \tag{I-2}$$

We also assume that V_k is equipped with the inner product $(\cdot,\cdot)_k$ such that

$$(v,v)_k \approx (v,v)_X \quad \forall\, v \in V_k. \tag{P}$$

We can then define $A_k : V_k \longrightarrow V_k$ by

$$(A_k v_1, v_2)_k = a_k(v_1, v_2) \quad \forall\, v_1, v_2 \in V_k. \tag{2.5}$$

By our assumptions on $a_k(\cdot,\cdot)$, A_k is a linear symmetric positive definite operator.

It follows from (I), (P) and (2.5) that

$$\rho(A_k) \le C_* h_k^{-2\ell}, \tag{2.6}$$

where $C_* > 0$ is independent of k. The number Λ_k in (1.2), (1.4), (1.5) and (1.7) is then defined by $\Lambda_k = C_* h_k^{-2\ell}$.

Finally, the fine-to-coarse intergrid transfer operator $I^{k-1}_k : V_k \longrightarrow V_{k-1}$ is defined by

$$(I^{k-1}_k v_1, v_2)_{k-1} = (v_1, I^k_{k-1} v_2)_k \quad \forall\, v_1 \in V_k \quad \text{and} \quad v_2 \in V_{k-1}. \tag{2.7}$$

For the convergence analysis we also need the operator $P^{k-1}_k : V_k \longrightarrow V_{k-1}$ defined by

$$a_{k-1}(P^{k-1}_k v_1, v_2) = a_k(v_1, I^k_{k-1} v_2) \quad \forall\, v_1 \in V_k \quad \text{and} \quad v_2 \in V_{k-1}. \tag{2.8}$$

It is easy to see from (2.5), (2.7) and (2.8) that the operators A_k, A_{k-1}, I^{k-1}_k and P^{k-1}_k are related by

$$A_{k-1} P^{k-1}_k = I^{k-1}_k A_k. \tag{2.9}$$

3. PRELIMINARY ESTIMATES

In this section we derive some estimates in preparation for the convergence analysis in the next section.

Lemma 3.1. *The following estimates hold:*

$$\|E_k v\|_V \lesssim \|v\|_k \quad \text{and} \quad \|E_k v\|_X \lesssim \|v\|_X \qquad \forall\, v \in V_k, \tag{3.1}$$

$$\|F_k \tilde{v}\|_k \lesssim \|\tilde{v}\|_V \quad \text{and} \quad \|F_k \tilde{v}\|_X \lesssim \|\tilde{v}\|_X \qquad \forall\, v \in \tilde{V}_k, \tag{3.2}$$

$$\|I^k_{k-1} v\|_k \lesssim \|v\|_{k-1} \quad \text{and} \quad \|I^k_{k-1} v\|_X \lesssim \|v\|_X \qquad \forall\, v \in V_{k-1}. \tag{3.3}$$

Proof. Using (I), (2.3) and (E) we have

$$\|E_k v\|_V \lesssim \|E_k v - v\|_k + \|v\|_k \lesssim h_k^{-\ell}\, \|E_k v - v\|_X + \|v\|_k \lesssim \|v\|_k,$$
$$\|E_k v\|_X \lesssim \|E_k v - v\|_X + \|v\|_X \lesssim h^\ell_k \|v\|_k + \|v\|_X \lesssim \|v\|_X.$$

The estimates (3.2)–(3.3) are similarly established by using (I), (F) and (I-1). $\square$

Lemma 3.2. *The following estimates hold:*

$$\|E_k\Pi_k\zeta - \zeta\|_W \lesssim h_k^{2\alpha}\|\zeta\|_Z \quad \forall\,\zeta \in Z, \tag{3.4}$$

$$\|E_k\Pi_k v - v\|_X + h_k^{\ell}\|E_k\Pi_k v\|_V \lesssim h_k^{\ell}\|v\|_V \quad \forall\, v \in V. \tag{3.5}$$

Proof. The estimate (3.4) follows immediately from (C-3), (EΠ) and interpolation (cf., [7], [61]). The estimate (3.5) is obtained from (Π-1), (E) and (3.1) as follows:

$$\begin{aligned} &\|E_k\Pi_k v - v\|_X + h_k^{\ell}\|E_k\Pi_k v\|_V \\ &\quad\lesssim \|E_k(\Pi_k v - v)\|_X + \|E_k v - v\|_X + h_k^{\ell}\|v\|_V \\ &\quad\lesssim \|\Pi_k v - v\|_X + h_k^{\ell}\|v\|_V \lesssim h_k^{\ell}\|v\|_V. \end{aligned} \tag{3.6}$$

□

For the convergence analysis, we need the following mesh-dependent norms on V_k:

$$|\!|\!|v|\!|\!|_{s,k}^2 = (A_k^{s/\ell}v, v)_k \quad \forall\, v \in V_k. \tag{3.7}$$

The spaces $(V_k, |\!|\!|\cdot|\!|\!|_{k,s})$ form a Hilbert scale (cf., [44]).

From (2.2), (2.5), (2.6), (3.7), (P) and the Cauchy-Schwarz inequality, we have

$$|\!|\!|v|\!|\!|_{0,k} = \sqrt{(v,v)_k} \approx \|v\|_X \qquad \forall\, v \in V_k, \tag{3.8}$$

$$|\!|\!|v|\!|\!|_{\ell,k} = \|v\|_k \qquad \forall\, v \in V_k, \tag{3.9}$$

$$|\!|\!|v|\!|\!|_{s,k} \lesssim h_k^{t-s}|\!|\!|v|\!|\!|_{t,k} \qquad \forall\, v \in V_k \quad \text{and} \quad t < s, \tag{3.10}$$

$$|a_k(v_1, v_2)| \le |\!|\!|v_1|\!|\!|_{\ell+t,k}|\!|\!|v_2|\!|\!|_{\ell-t,k} \quad \forall\, v_1, v_2 \in V_k,\ t \in \mathbb{R}. \tag{3.11}$$

Lemma 3.3. *The following estimates hold:*

$$|\!|\!|I_{k-1}^k v|\!|\!|_{s,k} \lesssim |\!|\!|v|\!|\!|_{s,k-1} \quad \forall\, 0 \le s \le \ell, v \in V_{k-1}, \tag{3.12}$$

$$|\!|\!|P_k^{k-1}v|\!|\!|_{t,k-1} \lesssim |\!|\!|v|\!|\!|_{t,k} \qquad \forall\, \ell \le t \le 2\ell, v \in V_k, \tag{3.13}$$

$$|\!|\!|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta|\!|\!|_{\ell-\alpha,k} \lesssim h_k^{2\alpha}\|\zeta\|_Z \quad \forall\,\zeta \in Z. \tag{3.14}$$

Proof. The estimate (3.12) follows from (3.3), (3.8), (3.9) and interpolation (cf., [44]). The estimate (3.13) then follows from (2.8), (3.12) and duality.

From (I-2), (3.8), and (3.10) we have

$$|\!|\!|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta|\!|\!|_{0,k} \lesssim h_k^{\ell+\alpha}\|\zeta\|_Z \quad \forall\,\zeta \in Z, \tag{3.15}$$

$$|\!|\!|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta|\!|\!|_{\ell,k} \lesssim h_k^{\alpha}\|\zeta\|_Z \qquad \forall\,\zeta \in Z. \tag{3.16}$$

The estimate (3.14) follows from (3.15), (3.16) and interpolation. □

Lemma 3.4. *We have the following equivalence of norms:*

$$|\!|\!|v|\!|\!|_{\ell-\alpha,k} \approx \|E_k v\|_W \quad \forall\, v \in V_k. \tag{3.17}$$

Proof. From (C-3), (3.1), (3.8) and (3.9), we obtain by interpolation that

$$\|E_k v\|_W \lesssim |\!|\!|v|\!|\!|_{\ell-\alpha,k} \quad \forall\, v \in V_k. \tag{3.18}$$

Let $Q_k : X \longrightarrow \tilde{V}_k$ be the orthogonal projection with respect to the inner product of X. Then from (I), (FE), (Π-1), (3.2), (3.5), (3.8) and (3.9) we have

$$|\!|\!|F_kQ_kv|\!|\!|_{0,k} \lesssim \|F_kQ_kv\|_X \lesssim \|Q_kv\|_X \lesssim \|v\|_X \qquad \forall\, v \in X, \tag{3.19}$$

$$\begin{aligned} |\!|\!|F_kQ_kv|\!|\!|_{\ell,k} &\lesssim |\!|\!|F_kQ_k(v - E_k\Pi_kv)|\!|\!|_{\ell,k} + |\!|\!|\Pi_kv|\!|\!|_{\ell,k} \\ &\lesssim h_k^{-\ell}\|v - E_k\Pi_kv\|_X + \|v\|_V \lesssim \|v\|_V \qquad \forall\, v \in V. \end{aligned} \tag{3.20}$$

We find by (C-3), (3.19), (3.20) and interpolation that

$$\|\!|F_k Q_k v|\!\|_{\ell-\alpha,k} \lesssim \|v\|_W \quad \forall\, v \in W. \tag{3.21}$$

In particular, we have by (FE) and (3.21) that

$$\|\!|v|\!\|_{\ell-\alpha,k} = \|\!|F_k Q_k E_k v|\!\|_{\ell-\alpha,k} \lesssim \|E_k v\|_W \quad \forall\, v \in V_k. \tag{3.22}$$

□

Theorem 3.5. *Let* $\zeta \in Z$ *and* $\zeta_k \in V_k$ *be related by* (2.4). *Then the following estimates hold:*

$$\|\zeta - \zeta_k\|_k \lesssim h_k^{\alpha}\, \|\zeta\|_Z, \tag{3.23}$$

$$\|\!|\Pi_k \zeta - \zeta_k|\!\|_{\ell-\alpha,k} \lesssim h_k^{2\alpha}\, \|\zeta\|_Z. \tag{3.24}$$

Proof. We have the following estimate for nonconforming methods (cf., [27]):

$$\|\zeta - \zeta_k\|_k \le \inf_{v \in V_k} \|\zeta - v\|_k + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(\zeta - \zeta_k, v)|}{\|v\|_k}. \tag{3.25}$$

The estimate (3.23) follows from (N-1), (Π-2) and (3.25).

By (3.17) and duality we have

$$\|\!|\Pi_k \zeta - \zeta_k|\!\|_{\ell-\alpha,k} \approx \|E_k(\Pi_k \zeta - \zeta_k)\|_W = \sup_{\phi \in W' \setminus \{0\}} \frac{\big|\phi\big(E_k(\Pi_k \zeta - \zeta_k)\big)\big|}{\|\phi\|_{W'}}. \tag{3.26}$$

Let $\phi \in W'$ be arbitrary. We define $\xi \in Z$ and $\xi_k \in V_k$ by the following equations:

$$a(\xi, v) = \phi(v) \qquad \forall\, v \in V, \tag{3.27}$$

$$a_k(\xi_k, v) = \phi(E_k v) \qquad \forall\, v \in V_k. \tag{3.28}$$

From (R-2) we have

$$\|\xi\|_Z \lesssim \|\phi\|_{W'}. \tag{3.29}$$

Using (2.4), (3.27) and (3.28), we have

$$\phi\big(E_k(\Pi_k \zeta - \zeta_k)\big) = a(\xi, E_k \Pi_k \zeta - \zeta) + a_k(\xi_k, \zeta - \zeta_k) + a_k(\xi - \xi_k, \zeta). \tag{3.30}$$

The terms on the right-hand side of (3.30) can be estimated as follows.

Using (D) and (3.4), we have

$$|a(\xi, E_k \Pi_k \zeta - \zeta)| \lesssim \|\xi\|_Z\, \|E_k \Pi_k \zeta - \zeta\|_W \lesssim h_k^{2\alpha}\, \|\xi\|_Z\, \|\zeta\|_Z. \tag{3.31}$$

It follows from (Π-2), (N-2) and (3.23) that

$$\begin{aligned} |a_k(\xi_k, \zeta - \zeta_k)| &\lesssim |a_k(\xi_k - \Pi_k \xi, \zeta - \zeta_k)| + |a_k(\Pi_k \xi, \zeta - \zeta_k)| \\ &\lesssim h_k^{2\alpha}\, \|\xi\|_Z\, \|\zeta\|_Z. \end{aligned} \tag{3.32}$$

Similarly, we have

$$|a_k(\xi - \xi_k, \zeta)| \lesssim h_k^{2\alpha}\, \|\xi\|_Z\, \|\zeta\|_Z. \tag{3.33}$$

The estimate (3.24) now follows by combining (3.26) and (3.29)–(3.33). □

The following corollary is an immediate consequence of (Π-2), (EΠ), (3.1), (3.4), (3.17), (3.23) and (3.24).

Corollary 3.6. *Let* $\zeta \in Z$ *and* $\zeta_k \in V_k$ *be related by* (2.4). *Then the following estimates hold:*

$$\|\zeta - E_k \zeta_k\|_V \lesssim h_k^{\alpha} \|\zeta\|_Z \quad \textit{and} \quad \|\zeta - E_k \zeta_k\|_W \lesssim h_k^{2\alpha} \|\zeta\|_Z.$$

Remark. Let $\zeta \in Z$ and $\zeta_k \in V_k$ be the solutions of the continuous problem $a(\zeta, v) = \phi(v)\ \forall\, v \in V$ and the discrete problem $a_k(\zeta_k, v) = \phi(E_k v)\ \forall\, v \in V_k$. Because of the presence of the operator E_k, the discrete problem is well-posed for ϕ in some negative order Sobolev spaces even though V_k is nonconforming. Theorem 3.5 and Corollary 3.6 give the discretization error estimates for these new nonconforming finite element methods.

Lemma 3.7. *Let $\zeta \in Z$, and let $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ be defined by*

$$a_k(\zeta_k, v) = a(\zeta, E_k v) \qquad \forall\, v \in V_k, \tag{3.34}$$

$$a_{k-1}(\zeta_{k-1}, v) = a(\zeta, E_{k-1} v) \qquad \forall\, v \in V_{k-1}. \tag{3.35}$$

Then the following estimate holds:

$$\vert\!\vert\!\vert \zeta_{k-1} - P_k^{k-1}\zeta_k \vert\!\vert\!\vert_{\ell-\alpha,k-1} \lesssim h_k^{2\alpha}\, \|\zeta\|_Z. \tag{3.36}$$

Proof. Again, by (3.17) and duality we have

$$\begin{aligned} \vert\!\vert\!\vert \zeta_{k-1} - P_k^{k-1}\zeta_k \vert\!\vert\!\vert_{\ell-\alpha,k-1} &\approx \|E_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k)\|_W \\ &= \sup_{\phi \in W' \setminus \{0\}} \frac{\left|\phi\big(E_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k)\big)\right|}{\|\phi\|_{W'}}. \end{aligned} \tag{3.37}$$

Let $\xi \in Z$, $\xi_k \in V_k$ and $\xi_{k-1} \in V_{k-1}$ be defined by

$$a(\xi, v) = \phi(v) \qquad \forall\, v \in V, \tag{3.38}$$

$$a_k(\xi_k, v) = \phi(E_k v) \qquad \forall\, v \in V_k, \tag{3.39}$$

$$a_{k-1}(\xi_{k-1}, v) = \phi(E_{k-1} v) \qquad \forall\, v \in V_{k-1}. \tag{3.40}$$

Again, the estimate (3.29) holds. Using (D), (2.8), (3.34), (3.35) and (3.40), we find that

$$\begin{aligned} &\left|\phi\big(E_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k)\big)\right| \\ &\quad = \left|a_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k, \xi_{k-1})\right| \\ &\quad = \left|a_{k-1}(\zeta_{k-1}, \xi_{k-1}) - a_k(\zeta_k, I_{k-1}^k \xi_{k-1})\right| \\ &\quad = \left|a(\zeta, E_{k-1}\xi_{k-1} - E_k I_{k-1}^k \xi_{k-1})\right| \\ &\quad \lesssim \|\zeta\|_Z \|E_{k-1}\xi_{k-1} - E_k I_{k-1}^k \xi_{k-1}\|_W. \end{aligned} \tag{3.41}$$

On the other hand, by (M), (3.4), (3.12), (3.14), (3.17), (3.38)–(3.40) and Theorem 3.5, we have

$$\begin{aligned} &\|E_{k-1}\xi_{k-1} - E_k I_{k-1}^k \xi_{k-1}\|_W \\ &\quad \lesssim \|E_{k-1}(\xi_{k-1} - \Pi_{k-1}\xi)\|_W + \|E_{k-1}\Pi_{k-1}\xi - \xi\|_W \\ &\quad\quad + \|\xi - E_k \Pi_k \xi\|_W + \|E_k(\Pi_k \xi - I_{k-1}^k \Pi_{k-1}\xi)\|_W \\ &\quad\quad + \|E_k I_{k-1}^k (\Pi_{k-1}\xi - \xi_{k-1})\|_W \lesssim h_k^{2\alpha}\, \|\xi\|_Z. \end{aligned} \tag{3.42}$$

The lemma now follows from (3.29) and (3.37)–(3.42). $\square$

Finally we derive within our abstract framework the discretization error estimates for the standard (nonconforming) discretization.

Theorem 3.8. *Let $F \in X'$, $u \in Z$, and $u_k \in V_k$ be such that*

$$a(u, v) = F(v) \quad \forall\, v \in V, \tag{3.43}$$

$$a_k(u_k, v) = F(v) \quad \forall\, v \in V_k. \tag{3.44}$$

Then the following estimates hold:

$$\|u-u_k\|_k \lesssim h_k^{\alpha}\|u\|_Z + h_k^{\ell}\|F\|_{X'}\,, \tag{3.45}$$

$$|\!|\!|\Pi_k u - u_k|\!|\!|_{\ell-\alpha,k} \lesssim h_k^{2\alpha}\|u\|_Z + h_k^{\ell+\alpha}\|F\|_{X'}\,. \tag{3.46}$$

Proof. Let $u_k' \in V_k$ satisfy

$$a_k(u_k', v) = F(E_k v) \quad \forall\, v \in V_k. \tag{3.47}$$

Theorem 3.5 implies that

$$\|u-u_k'\|_k \lesssim h_k^{\alpha}\|u\|_Z, \tag{3.48}$$

$$|\!|\!|\Pi_k u - u_k'|\!|\!|_{\ell-\alpha,k} \lesssim h_k^{2\alpha}\|u\|_Z. \tag{3.49}$$

From duality we have

$$\|u_k-u_k'\|_k = \sup_{v\in V_k\setminus\{0\}} \frac{|a_k(u_k-u_k',v)|}{\|v\|_k}. \tag{3.50}$$

Using (E), (3.44) and (3.47), we obtain

$$|a_k(u_k-u_k',v)| = |F(v-E_k v)| \lesssim h_k^{\ell}\|F\|_{X'}\|v\|_k. \tag{3.51}$$

Combining (3.50) and (3.51), we have

$$\|u_k-u_k'\|_k \lesssim h_k^{\ell}\|F\|_{X'}. \tag{3.52}$$

The estimate (3.45) follows from (3.48) and (3.52).

By (3.17) and duality, we have

$$|\!|\!|u_k' - u_k|\!|\!|_{\ell-\alpha,k} \approx \|E_k(u_k'-u_k)\|_W = \sup_{\phi\in W'\setminus\{0\}} \frac{\big|\phi\big(E_k(u_k'-u_k)\big)\big|}{\|\phi\|_{W'}}. \tag{3.53}$$

Let $\xi \in Z$ and $\xi_k \in V_k$ satisfy (3.27), (3.28) and (3.29). It follows that

$$\phi\big(E_k(u_k'-u_k)\big) = a_k(\xi_k - \Pi_k\xi, u_k'-u_k) + a_k(\Pi_k\xi, u_k'-u_k). \tag{3.54}$$

From (Π-2) and (3.52) we have

$$|a_k(\xi_k-\Pi_k\xi, u_k'-u_k)| \lesssim h_k^{\ell+\alpha}\|F\|_{X'}\|\xi\|_Z. \tag{3.55}$$

On the other hand, from (Π-2), (EΠ), (3.44) and (3.47) we obtain

$$\begin{aligned} |a_k(\Pi_k\xi, u_k'-u_k)| &= |F(E_k\Pi_k\xi - \Pi_k\xi)| \\ &\le \|F\|_{X'}\big(\|E_k\Pi_k\xi-\xi\|_X + \|\xi-\Pi_k\xi\|_X\big) \\ &\lesssim h_k^{\ell+\alpha}\|F\|_{X'}\|\xi\|_Z. \end{aligned} \tag{3.56}$$

Combining (3.29) and (3.53)–(3.56), we have

$$|\!|\!|u_k'-u_k|\!|\!|_{\ell-\alpha,k} \lesssim h_k^{\ell+\alpha}\|F\|_{X'}. \tag{3.57}$$

The estimate (3.46) follows from (3.49) and (3.57). □

Remark. The estimate (3.45) is the Assumption A4 in [65].

The following corollary is an immediate consequence of (Π-2), (EΠ), (3.1), (3.4), (3.17), (3.45) and (3.46).

Corollary 3.9. *Let F, u and u_k be as in Theorem* 3.8. *Then the following estimates hold:*

$$\|u - E_k u_k\|_V \lesssim h_k^{\alpha}\|u\|_Z + h_k^{\ell}\|F\|_{X'},$$
$$\|u - E_k u_k\|_W \lesssim h_k^{2\alpha}\|u\|_Z + h_k^{\ell+\alpha}\|F\|_{X'}.$$

4. Convergence analysis

In this section we establish the convergence results for the multigrid algorithms. First we investigate the convergence of the W-cycle algorithm. Following the methodology in [6], we start with the convergence analysis of the two-grid algorithm, where we assume that the residual equation is solved exactly on the coarser grid, i.e., the q_2 in the correction step is replaced by

$$q = A_{k-1}^{-1}\overline{g}. \tag{4.1}$$

Let z be the exact solution of (1.1), and let $e_i = z - z_i$ for $i = 0, \dots, m$, where $m = m_1 + m_2 + 1$. In order to relate the final error e_m to the initial error e_0, we introduce the operator R_k defined by

$$R_k = I - \frac{1}{\Lambda_k}A_k. \tag{4.2}$$

From the pre-smoothing step (1.2) and the post-smoothing step (1.4), we have

$$e_j = R_k e_{j-1}, \quad j = 1, 2, \dots, m_1, m_1 + 2, \dots m. \tag{4.3}$$

Since Λ_k dominates the spectral radius of A_k, it is easy to see that

$$|\!|\!| R_k v |\!|\!|_{s,k} \leq |\!|\!| v |\!|\!|_{s,k} \quad \forall\, v \in V_k,\ s \in \mathbb{R}. \tag{4.4}$$

From (2.9), the correction step of the two-grid algorithm, and (4.1), we have

$$e_{m_1+1} = e_{m_1} - I_{k-1}^{k} q = e_{m_1} - I_{k-1}^{k} A_{k-1}^{-1} I_k^{k-1} A_k e_{m_1} = (I - I_{k-1}^{k} P_k^{k-1}) e_{m_1}. \tag{4.5}$$

It follows from (4.3) and (4.5) that

$$e_m = R_k^{m_2}(I - I_{k-1}^{k}P_k^{k-1})R_k^{m_1}e_0. \tag{4.6}$$

Lemma 4.1. *We have the following smoothing property:*

$$\begin{aligned} &|\!|\!| R_k^n v |\!|\!|_{s,k} \lesssim h_k^{-\beta}[\max(1,n)]^{-\beta/(2\ell)}\, |\!|\!| v |\!|\!|_{s-\beta,k} \\ &\qquad\qquad \textit{for any } s \in \mathbb{R},\ \beta \geq 0,\ n = 0, 1, 2, \dots . \end{aligned} \tag{4.7}$$

Proof. For $n \geq 1$, the proof of (4.7) is standard (cf., [6]). For $n = 0$, the estimate follows from (3.10). □

The following estimate on the operator $I - I_{k-1}^{k}P_k^{k-1}$ is the crux of the convergence analysis.

Lemma 4.2. *We have the following approximation property:*

$$|\!|\!| (I - I_{k-1}^{k}P_k^{k-1})v |\!|\!|_{\ell-\alpha,k} \lesssim h_k^{2\alpha} |\!|\!| v |\!|\!|_{\ell+\alpha,k} \quad \forall\, v \in V_k. \tag{4.8}$$

Proof. From (3.17) and duality we have

$$|\!|\!| (I - I_{k-1}^{k}P_k^{k-1})v |\!|\!|_{\ell-\alpha,k} \approx \sup_{\phi \in W'\setminus\{0\}} \frac{\big|\phi\big(E_k(I - I_{k-1}^{k}P_k^{k-1})v\big)\big|}{\|\phi\|_{W'}}. \tag{4.9}$$

Let $\phi \in W'$ be arbitrary. We define $\zeta \in Z$, $\zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ by

$$a(\zeta, v) = \phi(v) \qquad \forall\, v \in V, \tag{4.10}$$

$$a_k(\zeta_k, v) = \phi(E_k v) \qquad \forall\, v \in V_k\,, \tag{4.11}$$

$$a_{k-1}(\zeta_{k-1}, v) = \phi(E_{k-1} v) \qquad \forall\, v \in V_{k-1}\,. \tag{4.12}$$

From (R-2), we have

$$\|\zeta\|_Z \lesssim \|\phi\|_{W'}. \tag{4.13}$$

Using (2.8) and (4.11), we find

$$\begin{aligned} &\phi\big(E_k(I - I_{k-1}^k P_k^{k-1})v\big) \\ &\quad = a_k(\zeta_k, (I - I_{k-1}^k P_k^{k-1})v) \\ &\quad = a_k(\zeta_k, v) - a_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1}v) \\ &\quad = a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) + a_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k, P_k^{k-1}v). \end{aligned} \tag{4.14}$$

We can estimate the two terms on the last line of (4.14) by using (M), (3.11)–(3.14), (4.10)–(4.12), Theorem 3.5 and Lemma 3.7 as follows:

$$\begin{aligned} &|a_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v)| \\ &\quad \lesssim \|\zeta_k - I_{k-1}^k \zeta_{k-1}\|_{\ell-\alpha,k} \|v\|_{\ell+\alpha,k} \\ &\quad \lesssim \Big(\|\zeta_k - \Pi_k \zeta\|_{\ell-\alpha,k} + \|I_{k-1}^k \Pi_{k-1}\zeta - \Pi_k\zeta\|_{\ell-\alpha,k} \\ &\qquad\qquad + \|I_{k-1}^k(\Pi_{k-1}\zeta - \zeta_{k-1})\|_{\ell-\alpha,k} \Big) \|v\|_{\ell+\alpha,k} \\ &\quad \lesssim h_k^{2\alpha} \|\zeta\|_Z \, \|v\|_{\ell+\alpha,k}\,, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &|a_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k, P_k^{k-1}v)| \\ &\quad \lesssim \|\zeta_{k-1} - P_k^{k-1}\zeta_k\|_{\ell-\alpha,k-1} \|P_k^{k-1}v\|_{\ell+\alpha,k-1} \\ &\quad \lesssim h_k^{2\alpha} \|\zeta\|_Z \, \|v\|_{\ell+\alpha,k}\,. \end{aligned} \tag{4.16}$$

The estimate (4.8) now follows from (4.9) and (4.13)–(4.16). ☐

Theorem 4.3 (Convergence of the two-grid algorithm)**.** *For $m_1 + m_2$ sufficiently large, the two-grid algorithm is a contraction in the $\|\cdot\|_k$ norm, with contraction number uniformly bounded away from 1. For m_1 sufficiently large, the two-grid algorithm is also a contraction in the $\|\cdot\|_{\ell-\alpha,k}$ norm, and the contraction number is uniformly bounded away from 1.*

Proof. Since the final error e_m is related to the initial error e_0 by (4.6), we have by (3.9), (4.7) and (4.8) that

$$\begin{aligned} \|e_m\|_k &= \|R_k^{m_2}(I - I_{k-1}^k P_k^{k-1}) R_k^{m_1} e_0\|_{\ell,k} \\ &\lesssim h_k^{-\alpha} [\max(1, m_2)]^{-\alpha/(2\ell)} \, \|(I - I_{k-1}^k P_k^{k-1}) R_k^{m_1} e_0\|_{\ell-\alpha,k} \\ &\lesssim h_k^{\alpha} [\max(1, m_2)]^{-\alpha/(2\ell)} \, \|R_k^{m_1} e_0\|_{\ell+\alpha,k} \\ &\lesssim [\max(1, m_2)]^{-\alpha/(2\ell)} \, [\max(1, m_1)]^{-\alpha/(2\ell)} \, \|e_0\|_k\,. \end{aligned} \tag{4.17}$$

Similarly, we have by (4.4), (4.7) and (4.8) that

$$\begin{aligned}\|e_m\|_{\ell-\alpha,k} &= \|R_k^{m_2}(I - I_{k-1}^k P_k^{k-1})R_k^{m_1} e_0\|_{\ell-\alpha,k} \\ &\leq \|(I - I_{k-1}^k P_k^{k-1})R_k^{m_1} e_0\|_{\ell-\alpha,k} \\ &\lesssim h_k^{2\alpha} \|R_k^{m_1} e_0\|_{\ell+\alpha,k} \\ &\lesssim [\max(1, m_1)]^{-\alpha/\ell} \|e_m\|_{\ell-\alpha,k} \,.\end{aligned} \tag{4.18}$$

The theorem follows from (4.17) and (4.18). □

Remark. The estimate in (4.18) seems to indicate that a one-sided algorithm with only pre-smoothing may be more efficient for convergence in the $\|\cdot\|_{\ell-\alpha,k}$ norm.

The next theorem follows from (3.12), Theorem 4.3 and a standard perturbation argument (cf., [6]).

Theorem 4.4 (Convergence of the W-cycle multigrid algorithm)**.** *For $m_1 + m_2$ sufficiently large, the W-cycle multigrid algorithm is a contraction in the $\|\cdot\|_k$ norm, with contraction number uniformly bounded away from 1. For m_1 sufficiently large, the W-cycle algorithm is also a contraction in the $\|\cdot\|_{\ell-\alpha,k}$ norm, and the contraction number is uniformly bounded away from 1.*

Let $F \in X'$, $u \in Z$ and $u_k \in V_k$ be such that (3.43) and (3.44) hold. We can find an approximate solution for (3.44) by the following full multigrid method.

The full multigrid W-cycle algorithm. For $k = 1$, the approximate solution $\hat{u}_1 \in V_1$ is obtained by a direct method.

For $k > 1$, the approximate solution $\hat{u}_k \in V_k$ is obtained recursively from

$$\begin{aligned}u_{k,0} &= I_{k-1}^k \hat{u}_{k-1} \\ u_{k,j} &= WMG(k, u_{k,j-1}, f_k), \quad 1 \leq j \leq r, \\ \hat{u}_k &= u_{k,r},\end{aligned} \tag{4.19}$$

where r is a positive integer independent of k and $f_k \in V_k$ is defined by

$$(f_k, v)_k = F(v) \qquad \forall\, v \in V_k \,.$$

Theorem 4.5. *Let $F \in X'$, $u \in Z$ and $u_k \in V_k$ be such that* (3.43) *and* (3.44) *hold. Let $m_1 + m_2$ (resp., m_1) be sufficiently large so that the W-cycle algorithms are contractions in the $\|\cdot\|_k$ (resp., $\|\cdot\|_{\ell-\alpha,k}$) norms with contraction numbers uniformly bounded away from 1. Then, for r sufficiently large, the following estimates hold for the approximate solutions $\hat{u}_k$ $(k = 1, 2, \dots)$ obtained by the full multigrid algorithm:*

$$\|u_k - \hat{u}_k\|_k \lesssim h_k^{\alpha} \|u\|_Z + h_k^{\ell} \|F\|_{X'} \,, \tag{4.20}$$

$$\|u_k - \hat{u}_k\|_{\ell-\alpha,k} \lesssim h_k^{2\alpha} \|u\|_Z + h_k^{\ell+\alpha} \|F\|_{X'} \,. \tag{4.21}$$

Proof. By (M), (Π-2), (3.3), (3.9), (3.12), (3.14), (3.16), (3.45) and Theorem 3.8 we have

$$\begin{aligned} \|u_k - I_{k-1}^k u_{k-1}\|_k &\le \|u_k - \Pi_k u\|_k + \|\Pi_k u - I_{k-1}^k \Pi_{k-1} u\|_k \\ &\quad + \|I_{k-1}^k(\Pi_{k-1}u - u_{k-1})\|_k \\ &\lesssim h_k^\alpha \|u\|_Z + h_k^\ell \|F\|_{X'}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} |\!|\!| u_k - I_{k-1}^k u_{k-1} |\!|\!|_{\ell-\alpha,k} &\le |\!|\!| u_k - \Pi_k u |\!|\!|_{\ell-\alpha,k} + |\!|\!| \Pi_k u - I_{k-1}^k \Pi_{k-1} u |\!|\!|_{\ell-\alpha,k} \\ &\quad + |\!|\!| I_{k-1}^k(\Pi_{k-1}u - u_{k-1}) |\!|\!|_{\ell-\alpha,k} \\ &\lesssim h_k^{2\alpha} \|u\|_Z + h_k^{\ell+\alpha} \|F\|_{X'}. \end{aligned} \tag{4.23}$$

Let $k > 1$. By (4.19) and the assumption on the W-cycle algorithm, there exists a positive δ such that $\delta < 1$ and

$$\|u_k - \hat{u}_k\|_k \le \delta^r \|u_k - I_{k-1}^k \hat{u}_{k-1}\|_k \qquad \text{for } k = 1, 2, \ldots, \tag{4.24}$$

$$|\!|\!| u_k - \hat{u}_k |\!|\!|_{\ell-\alpha,k} \le \delta^r |\!|\!| u_k - I_{k-1}^k \hat{u}_{k-1} |\!|\!|_{\ell-\alpha,k} \quad \text{for } k = 1, 2, \ldots. \tag{4.25}$$

Combining (3.3), (4.22) and (4.24), we obtain

$$\begin{aligned} \|u_k - \hat{u}_k\|_k &\le \delta^r \left[\|u_k - I_{k-1}^k u_{k-1}\|_k + \|I_{k-1}^k(u_{k-1} - \hat{u}_{k-1})\|_k \right] \\ &\le \delta^r C' \left[\left(h_k^\alpha \|u\|_Z + h_k^\ell \|F\|_{X'} \right) + \|u_{k-1} - \hat{u}_{k-1}\|_k \right], \end{aligned} \tag{4.26}$$

where C' is independent of k.

Similarly, using (3.12), (4.23) and (4.25), we obtain

$$|\!|\!| u_k - \hat{u}_k |\!|\!|_{\ell-\alpha,k} \le \delta^r C'' \left[\left(h_k^{2\alpha} \|u\|_Z + h_k^{\ell+\alpha} \|F\|_{X'} \right) + |\!|\!| u_{k-1} - \hat{u}_{k-1} |\!|\!|_{\ell-\alpha,k} \right], \tag{4.27}$$

where C'' is independent of k.

Since $0 < \alpha \le \ell$, it follows from (M) and iterations of (4.26) and (4.27) that

$$\|u_k - \hat{u}_k\|_k \le \left[\sum_{j=1}^k \left(\frac{C'\delta^r}{C_1^\ell} \right)^j \right] \left(h_k^\alpha \|u\|_Z + h_k^\ell \|F\|_{X'} \right), \tag{4.28}$$

$$|\!|\!| u_k - \hat{u}_k |\!|\!|_{\ell-\alpha,k} \le \left[\sum_{j=1}^k \left(\frac{C''\delta^r}{C_1^{\ell+\alpha}} \right)^j \right] \left(h_k^{2\alpha} \|u\|_Z + h_k^{\ell+\alpha} \|F\|_{X'} \right). \tag{4.29}$$

The estimates (4.20) and (4.21) follow from (4.28) and (4.29) for r sufficiently large. □

The following corollary is an immediate consequence of (3.1), (3.17), Theorem 3.8, Corollary 3.9 and Theorem 4.5.

Corollary 4.6. *The following estimates hold under the assumptions of Theorem 4.5:*

$$\|u - \hat{u}_k\|_k + \|u - E_k \hat{u}_k\|_V \lesssim h_k^\alpha \|u\|_Z + h_k^\ell \|F\|_{X'}, \tag{4.30}$$

$$|\!|\!| \Pi_k u - \hat{u}_k |\!|\!|_{\ell-\alpha,k} + \|u - E_k \hat{u}_k\|_W \lesssim h_k^{2\alpha} \|u\|_Z + h_k^{\ell+\alpha} \|F\|_{X'}. \tag{4.31}$$

Remark. For fourth order problems, pointwise convergence of $\hat{u}_k$ follows from (4.31) and the Sobolev inequality (cf., the remark after Example 6.1).

Next we consider the symmetric variable V-cycle algorithm as a preconditioner. Let $B_k : V_k \longrightarrow V_k$ be defined by

$$B_k g = \mathcal{V}MG(k, 0, g).$$

It can be shown by mathematical induction that B_k is a linear symmetric positive definite operator with respect to $(\cdot, \cdot)_k$ (cf., Theorem 4.5 in [10]). Therefore $B_k A_k$ is symmetric positive definite with respect to $a_k(\cdot, \cdot)$. Our goal is to estimate the condition number of $B_k A_k$ with respect to the energy norm $\|\cdot\|_k$ induced by $a_k(\cdot, \cdot)$.

The following lemma furnishes the crucial "regularity and approximation" estimate in the Bramble-Pasciak-Xu theory for the symmetric variable V-cycle multigrid preconditioner.

Lemma 4.7. *The following estimate holds:*

$$|a_k((I - I_{k-1}^k P_k^{k-1})v, v)| \lesssim \left(\frac{(A_k v, A_k v)_k}{\lambda_k} \right)^{\alpha/\ell} (a_k(v, v))^{1-(\alpha/\ell)}, \tag{4.32}$$

for all $v \in V_k$, where λ_k $(= \rho(A_k))$ is the largest eigenvalue of A_k.

Proof. Let $v \in V_k$ be arbitrary. Using (3.7), (3.11) and (4.8), we have

$$\begin{aligned} |a_k((I - I_{k-1}^k P_k^{k-1})v, v)| &\lesssim \|(I - I_{k-1}^k P_k^{k-1})v\|_{\ell-\alpha,k} \, \|v\|_{\ell+\alpha,k} \\ &\lesssim h_k^{2\alpha} \|v\|_{\ell+\alpha,k}^2 = h_k^{2\alpha} (A_k^{1+(\alpha/\ell)} v, v)_k. \end{aligned} \tag{4.33}$$

Hölder's inequality implies that

$$(A_k^{1+(\alpha/\ell)} v, v)_k \le (A_k v, A_k v)_k^{\alpha/\ell} \, (A_k v, v)_k^{1-(\alpha/\ell)}. \tag{4.34}$$

The estimate (4.32) follows from (4.33), (4.34) and (2.6). □

We can now simply apply the Bramble-Pasciak-Xu theory ([10], [12], [13]) to obtain the following theorem.

Theorem 4.8. *The condition number of $B_k A_k$ with respect to the energy norm $\|\cdot\|_k$ is bounded by a positive constant which is independent of the mesh parameter k.*

5. Applications to a model second order problem

In this section we apply our theory to the Poisson equation with homogeneous Dirichlet boundary condition. Let Ω be a polygonal domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$. Consider the following boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

Let $V = H_0^1(\Omega)$, and let $a(\cdot, \cdot)$ on $V \times V$ be defined by $a(v_1, v_2) = \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx$. Conditions (B) and (C) follow from the Cauchy-Schwarz inequality and the Poincaré inequality (cf., [48]), respectively.

The weak formulation of (5.1) is to find $u \in V$ such that

$$a(u, v) = \int_\Omega f v \, dx \quad \forall \, v \in V. \tag{5.2}$$

By the elliptic regularity theory for non-smooth domains (cf., [36], [39], [40], [41]), there exists $\alpha \in (\frac{1}{2}, 1]$ such that for $f \in H^{-1+\alpha}(\Omega)$, the solution u of (5.2) belongs to $H^{1+\alpha}(\Omega)$ and

$$\|u\|_{H^{1+\alpha}(\Omega)} \lesssim \|f\|_{H^{-1+\alpha}(\Omega)}. \tag{5.3}$$

Let $Z = H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$ and $W = H_0^{1-\alpha}(\Omega)$ ($= H^{1-\alpha}(\Omega)$ since $1 - \alpha < 1/2$). Clearly, (R-1) holds and (R-2) follows from (5.3).

Let $X = L^2(\Omega)$. From interpolation of Sobolev spaces (cf., [61] and [57]) we have

$$Z = [H_0^1(\Omega), H^2(\Omega) \cap H_0^1(\Omega)]_\alpha, \tag{5.4}$$

and $W = [L^2(\Omega), H_0^1(\Omega)]_{1-\alpha}$. In particular, the condition (C-3) holds.

The Laplacian Δ is a bounded linear operator from $H^2(\Omega)$ to $L^2(\Omega)$, and from $H^1(\Omega)$ to $H^{-1}(\Omega)$. Therefore by interpolation (cf., [61]) we have

$$\|\Delta\zeta\|_{H^{-1+\alpha}(\Omega)} \lesssim \|\zeta\|_{H^{1+\alpha}(\Omega)} \quad \forall\, \zeta \in H^{1+\alpha}(\Omega). \tag{5.5}$$

Let $\zeta \in Z$ and $v \in V$. There exists a sequence $\phi_n \in C_0^\infty(\Omega)$ (the space of C^∞ functions with compact supports in Ω) which converges to $v \in V$. Since $H_0^1(\Omega) \hookrightarrow H_0^{1-\alpha}(\Omega)$, the sequence ϕ_n also converges to $v \in H_0^{1-\alpha}(\Omega)$. Therefore, we have

$$a(\zeta, v) = \lim_{n\to\infty} \int_\Omega \nabla\zeta \cdot \nabla\phi_n \, dx = \lim_{n\to\infty} (-\Delta\zeta, \phi_n) = (-\Delta\zeta, v), \tag{5.6}$$

where $(\cdot,\cdot)$ denotes the canonical duality bilinear form between $H^{-1+\alpha}(\Omega)$ and $H_0^{1-\alpha}(\Omega)$. The duality estimate (D) now follows from (5.5) and (5.6).

We now consider finite element multigrid methods for (5.2).

Example 5.1. Let $\{\mathcal{T}_k\}$ be a sequence of quasi-uniform triangulations (cf., [30], [27]) of Ω. For simplicity we may assume that $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. Therefore, (M) holds for $C_1 = C_2 = 1/2$.

Let $V_k^* = \{v \in L^2(\Omega) : v|_T$ is linear for all $T \in \mathcal{T}_k$, v is continuous at the midpoints of interelement boundaries$\}$ be the $\mathcal{P}_1$ nonconforming finite element space associated with $\mathcal{T}_k$ (cf., [35]), and let $\tilde{V}_k^* = \{v \in H^1(\Omega) : v|_T$ is quadratic for all $T \in \mathcal{T}_k\}$ be the $\mathcal{P}_2$ conforming finite element space associated with $\mathcal{T}_k$. The space V_k (resp., $\tilde{V}_k$) is the subspace of V_k^* (resp., $\tilde{V}_k^*$) whose members vanish at the boundary nodes. Note that $V_k \preceq \tilde{V}_k$ and the conditions (C-1) and (C-2) clearly hold. The finite element space V_k^* is equipped with the inner product $(\cdot,\cdot)_k$ defined by $(v_1, v_2)_k = h_k^2 \sum_m v_1(m)v_2(m)$, where the summation is taken over all the midpoints in the triangulation $\mathcal{T}_k$. The equivalence of $(v,v)_k$ and $(v,v)_{L^2(\Omega)}$ is standard. Hence (P) holds.

Let $a_k(\cdot,\cdot)$ be defined by $a_k(v_1, v_2) = \sum_{T\in\mathcal{T}_k} \int_T \nabla v_1 \cdot \nabla v_2 \, dx$. Then (I) is a standard inverse estimate (cf., [30], [27]). The discrete problem for (5.2) is to find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega f v \, dx \quad \forall\, v \in V_k. \tag{5.7}$$

The interpolation operator $\Pi_k : V \longrightarrow V_k$ is defined by

$$(\Pi_k v)(m) = \frac{1}{|e|} \int_e v \, ds, \tag{5.8}$$

where m is the midpoint of the edge e. Note that

$$(\Pi_k\zeta)\big|_T = \zeta\big|_T \quad \text{if } \zeta\big|_T \text{ is linear.} \tag{5.9}$$

The following estimate can be found in [35]:

$$\|\zeta - \Pi_k\zeta\|_{L^2(T)} + h_k|\zeta - \Pi_k\zeta|_{H^1(T)} \lesssim h_k^\beta|\zeta|_{H^\beta(T)}, \qquad \beta = 1, 2, \tag{5.10}$$

for all $T \in \mathcal{T}_k$ and $\zeta \in H^\beta(\Omega) \cap H_0^1(\Omega)$. The estimate ($\Pi$-1) follows from (5.10) with $\beta = 1$, and the estimate (Π-2) follows from (Π-1), (5.4), (5.10) with $\beta = 2$ and interpolation.

The operators $E_k : V_k \longrightarrow \tilde{V}_k$ and $F_k : \tilde{V}_k \longrightarrow V_k$ are defined by

$$\begin{cases} (E_k v)(m) = v(m) & \text{for all internal midpoints } m \in \mathcal{T}_k, \\ (E_k v)(p) = \text{average of } v_i(p) & \text{for all internal vertices } p \in \mathcal{T}_k, \end{cases}$$

where $v_i = v|_{T_i}$ and $T_i \in \mathcal{T}_k$ contains p as a vertex, and

$$(F_k\tilde{v})(m) = \tilde{v}(m) \quad \text{for all midpoints } m \in \mathcal{T}_k. \tag{5.12}$$

Note that F_k is well-defined because $V_k \preceq \tilde{V}_k$.

The relation (FE) is trivial, and the estimates (E) and (F) can be found in [24]. For the proof of (EΠ), it is convenient (because we can ignore the boundary conditions) to introduce the operator $E_k^* : V_k^* \longrightarrow \tilde{V}_k^*$ which is defined by the same formula in (5.11) for all midpoints and vertices of $\mathcal{T}_k$. Note that for $v \in V_k$, we have $E_k^* v = E_k v$ except at the vertices on $\partial\Omega$.

Let $T \in \mathcal{T}_k$, and let S_T be the interior of the union of the closures of all the triangles in $\mathcal{T}_k$ neighboring T. We have the following estimate for E_k^* (cf., [24]).

$$\|E_k^* v - v\|_{L^2(T)}^2 \lesssim h_T^2 \sum_{K \in S_T} |v|_{H^1(K)}^2 \quad \forall\, v \in V_k^*. \tag{5.13}$$

It follows from (5.13) and a standard inverse estimate that

$$\|E^* v\|_{L^2(T)} \lesssim \|v\|_{L^2(S_T)} \quad \forall\, v \in V_k^*. \tag{5.14}$$

The definition of E_k^* also implies that

$$(E_k^*\eta)\big|_T = \eta \quad \text{if } \eta\big|_{S_T} \text{ is linear.} \tag{5.15}$$

Let $T \in \mathcal{T}_k$. Let ϕ be an arbitrary linear function on S_T and $\tilde{\phi} \in H^2(\Omega)$ be an extension of ϕ. For any $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$, it follows from (5.9), (5.10), (5.14) and (5.15) that

$$\|E_k^*\Pi_k\zeta - \zeta\|_{L^2(T)} \le \|E_k^*\Pi_k(\zeta - \tilde{\phi})\|_{L^2(T)} + \|\zeta - \tilde{\phi}\|_{L^2(T)} \lesssim \|\zeta - \phi\|_{L^2(S_T)}. \tag{5.16}$$

Since ϕ is an arbitrary linear function on S_T, it follows from (5.16) and the Bramble-Hilbert lemma (cf., [11]) that

$$\|E_k^*\Pi_k\zeta - \zeta\|_{L^2(T)} \lesssim h_T^2|\zeta|_{H^2(S_T)}. \tag{5.17}$$

Summing (5.17) over all the triangles $T \in \mathcal{T}_k$, we have

$$\|E_k^*\Pi_k\zeta - \zeta\|_{L^2(\Omega)} \lesssim h_k^2|\zeta|_{H^2(\Omega)} \quad \forall\, \zeta \in H^2(\Omega) \cap H_0^1(\Omega). \tag{5.18}$$

Since $E_k\Pi_k\zeta$ and $E_k^*\Pi_k\zeta$ differ only at the vertices along $\partial\Omega$, we have

$$\begin{aligned}\|E_k\Pi_k\zeta - E_k^*\Pi_k\zeta\|_{L^2(\Omega)}^2 &\lesssim h_k^2 \sum_{p\in\partial\Omega} \sum_{\substack{T\in\mathcal{T}_k\\ \bar{T}\ni p}} [(\Pi_k\zeta)\big|_T(p)]^2 \\ &\lesssim h_k^2 \sum_{p\in\partial\Omega} \sum_{\substack{T\in\mathcal{T}_k\\ \bar{T}\ni p}} [(\Pi_k\zeta)\big|_T(p) - \zeta(p)]^2 \\ &\lesssim h_k^4|\zeta|_{H^2(\Omega)}^2\end{aligned} \tag{5.19}$$

by (5.10). It follows from (5.18) and (5.19) that

$$\|E_k\Pi_k\zeta - \zeta\|_{L^2(\Omega)} \lesssim h_k^2|\zeta|_{H^2(\Omega)} \quad \forall\,\zeta\in H^2(\Omega)\cap H_0^1(\Omega). \tag{5.20}$$

By (5.10), (5.20) and standard inverse estimates, we have

$$\begin{aligned}\|E_k\Pi_k\zeta - \zeta\|_{H^1(\Omega)} &\lesssim h_k^{-1}\|E_k\Pi_k\zeta - \Pi_k\zeta\|_{L^2(\Omega)} + \|\Pi_k\zeta - \zeta\|_k \\ &\lesssim h_k|\zeta|_{H^2(\Omega)} \quad \forall\,\zeta\in H^2(\Omega)\cap H_0^1(\Omega).\end{aligned} \tag{5.21}$$

On the other hand, using (E), (I) and (Π-1), we have

$$\begin{aligned}&\|E_k\Pi_k\zeta - \zeta\|_{L^2(\Omega)} + h_k\|E_k\Pi_k\zeta - \zeta\|_{H^1(\Omega)} \\ &\quad\lesssim \|E_k\Pi_k\zeta - \Pi_k\zeta\|_{L^2(\Omega)} + \|\Pi_k\zeta - \zeta\|_{L^2(\Omega)} \\ &\qquad + h_k\big(\|E_k\Pi_k\zeta - \Pi_k\zeta\|_k + \|\Pi_k\zeta - \zeta\|_k\big) \\ &\quad\lesssim h_k\,\|\zeta\|_{H^1(\Omega)} \quad \forall\,\zeta\in H_0^1(\Omega).\end{aligned} \tag{5.22}$$

The estimate (EΠ) now follows from (5.4), (5.20), (5.21), (5.22) and interpolation.

Next we verify the assumptions (N-1) and (N-2). Let $\zeta\in H^2(\Omega)\cap H_0^1(\Omega)$, and let $\zeta_k\in V_k$ be related to ζ through (2.4). Let $v\in V_k+V$, then Green's formula implies that

$$a_k(\zeta,v) = -\sum_{T\in\mathcal{T}_k}\int_T(\Delta\zeta)v\,dx + \sum_e\int_e\frac{\partial\zeta}{\partial n}[v]\,ds, \tag{5.23}$$

where $[v]$ denotes the jump of v (in the direction of n) across the edge e, and the second summation is taken over all the edges of $\mathcal{T}_k$.

Since $E_kv\in H_0^1(\Omega)$, it follows from (2.4) and (5.23) that

$$a_k(\zeta_k,v) = -\sum_{T\in\mathcal{T}_k}\int_T(\Delta\zeta)E_kv\,dx \quad \forall\,v\in V_k. \tag{5.24}$$

By subtracting (5.24) from (5.23), we obtain

$$a_k(\zeta-\zeta_k,v) = -\sum_{T\in\mathcal{T}_k}\int_T(\Delta\zeta)(v-E_kv)\,dx + \sum_e\int_e\frac{\partial\zeta}{\partial n}[v]\,ds \quad \forall\,v\in V_k. \tag{5.25}$$

Using the Cauchy-Schwarz inequality and (E), we have

$$\left|\sum_{T\in\mathcal{T}_k}\int_T(\Delta\zeta)(v-E_kv)\,dx\right| \lesssim h_k\,\|\zeta\|_{H^2(\Omega)}\|v\|_k \quad \forall\,v\in V_k. \tag{5.26}$$

Since v is continuous at the midpoints, a standard argument (cf., [35]) shows that

$$\left|\sum_e \int_e \frac{\partial\zeta}{\partial n}[v]\,ds\right| \lesssim h_k\,\|\zeta\|_{H^2(\Omega)}\,\|v\|_k \quad \forall\, v \in V_k. \tag{5.27}$$

Combining (5.25)–(5.27), we have

$$|a_k(\zeta - \zeta_k, v)| \lesssim h_k\,\|\zeta\|_{H^2(\Omega)}\,\|v\|_k \quad \forall\, v \in V_k. \tag{5.28}$$

Assume now that $\zeta \in H_0^1(\Omega)$. By (2.4) and (3.1) we have

$$a_k(\zeta_k, \zeta_k) = a(\zeta, E_k\zeta_k) \le \|\zeta\|_{H^1(\Omega)}\|E_k\zeta_k\|_{H^1(\Omega)} \lesssim \|\zeta\|_{H^1(\Omega)}\|\zeta_k\|_k. \tag{5.29}$$

It follows from (5.29) that

$$\|\zeta_k\|_k \lesssim \|\zeta\|_{H^1(\Omega)}, \tag{5.30}$$

which then implies

$$|a_k(\zeta - \zeta_k, v)| \le \|\zeta - \zeta_k\|_k\,\|v\|_k \lesssim \|\zeta\|_{H^1(\Omega)}\,\|v\|_k \quad \forall\, v \in V_k. \tag{5.31}$$

The estimate (N-1) follows from (5.4), (5.28), (5.31) and interpolation.

From (5.25) we obtain

$$a_k(\zeta - \zeta_k, \Pi_k\xi) = -\sum_{T\in\mathcal{T}_k}\int_T (\Delta\zeta)(\Pi_k\xi - E_k\Pi_k\xi)\,dx + \sum_e \int_e \frac{\partial\zeta}{\partial n}[\Pi_k\xi]\,ds \tag{5.32}$$

for $\zeta, \xi \in H^2(\Omega) \cap H_0^1(\Omega)$.

It follows from the Cauchy-Schwarz inequality, (5.10) and (5.20) that

$$\left|\sum_{T\in\mathcal{T}_k}\int_T (\Delta\zeta)(\Pi_k\xi - E_k\Pi_k\xi)\,dx\right| \lesssim h_k^2\,\|\zeta\|_{H^2(\Omega)}\|\xi\|_{H^2(\Omega)}. \tag{5.33}$$

Since $\Pi_k\xi$ is continuous at the midpoints, we have by a standard argument (cf., [35])

$$\left|\sum_e \int_e \frac{\partial\zeta}{\partial n}[\Pi_k\xi]\,ds\right| = \left|\sum_e \int_e \frac{\partial\zeta}{\partial n}[\Pi_k\xi - \xi]\,ds\right| \lesssim h_k^2\,\|\zeta\|_{H^2(\Omega)}\|\xi\|_{H^2(\Omega)}. \tag{5.34}$$

Combining (5.32)–(5.34), we obtain

$$|a_k(\zeta - \zeta_k, \Pi_k\xi)| \lesssim h_k^2\,\|\zeta\|_{H^2(\Omega)}\|\xi\|_{H^2(\Omega)} \quad \forall\, \xi, \zeta \in H^2(\Omega) \cap H_0^1(\Omega). \tag{5.35}$$

On the other hand, for $\zeta, \xi \in H_0^1(\Omega)$, we get the following trivial estimate by using (5.10) and (5.30):

$$|a_k(\zeta - \zeta_k, \Pi_k\xi)| \le \|\zeta - \zeta_k\|_k\|\Pi_k\xi\|_k \lesssim \|\zeta\|_{H^1(\Omega)}\,\|\xi\|_{H^1(\Omega)}. \tag{5.36}$$

The estimate (N-2) follows from (5.4), (5.35), (5.36) and (bilinear) interpolation (cf., [7]).

Finally, we define the intergrid transfer operator I_{k-1}^k. Let m be a midpoint of an edge of a triangle in $\mathcal{T}_k$. If $m \in \partial\Omega$, then $(I_{k-1}^k v)(m) = 0$. If m lies in the interior of a triangle in $\mathcal{T}_{k-1}$, then $(I_{k-1}^k v)(m) = v(m)$. Otherwise if m lies on the common edge of two adjacent triangles T_1 and T_2 in $\mathcal{T}_{k-1}$, then $(I_{k-1}^k v)(m) = \frac{1}{2}\big[v|_{T_1}(m) + v|_{T_2}(m)\big]$. The proof of the estimate (I-1) can be found in [14] and [18].

Let $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$. A slight modification of the arguments in [14] and [18] (where the nodal interpolation operator was used) gives

$$\|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta\|_{L^2(\Omega)} \lesssim h_k^2\,\|\zeta\|_{H^2(\Omega)}. \tag{5.37}$$

For $\zeta \in H_0^1(\Omega)$, using (M), (Π-1) and (I-1) we have the estimate

$$\begin{aligned}
&\|I_{k-1}^k \Pi_{k-1}\zeta - \Pi_k \zeta\|_{L^2(\Omega)} \\
&\quad\lesssim \|I_{k-1}^k \Pi_{k-1}\zeta - \Pi_{k-1}\zeta\|_{L^2(\Omega)} \\
&\qquad + \|\Pi_{k-1}\zeta - \zeta\|_{L^2(\Omega)} + \|\zeta - \Pi_k\zeta\|_{L^2(\Omega)} \\
&\quad\lesssim h_k \|\zeta\|_{H^1(\Omega)}.
\end{aligned} \tag{5.38}$$

The estimate (I-2) now follows from (5.4), (5.37), (5.38) and interpolation.

We have verified all of the assumptions in Section 2 for this example. Therefore the results in Section 4 are applicable to the multigrid algorithms for (5.7).

In the next example, we omit the technical details since they can be carried out along the same lines as in Example 5.1.

Example 5.2. In this example, we assume that the sides of the polygonal domain Ω are parallel to the coordinate axes. Let $\{\mathcal{T}_k\}$ be a sequence of quasi-uniform "triangulations" of Ω consisting of rectangles. For simplicity we may assume that $\mathcal{T}_{k+1}$ is obtained by connecting midpoints of the opposite sides of the rectangles in $\mathcal{T}_k$.

Let $V_k = \{v : v\big|_{\mathrm{R}} \in \langle 1, x_1, x_2, x_1^2 - x_2^2\rangle \ \forall\, \mathrm{R} \in \mathcal{T}_k, v$ is continuous at the midpoints of the interelement boundaries and vanishes at the midpoints on $\partial\Omega\}$ be the nonconforming "rotated" bilinear element (cf., [52]), and $\tilde{V}_k = \{v \in H_0^1(\Omega) : \ v\big|_{\mathrm{R}}$ is biquadratic for all $\mathrm{R} \in \mathcal{T}_k\}$ be the conforming $\mathcal{Q}_2$ finite element space. Note that $V_k \preceq \tilde{V}_k$. The inner product for V_k is defined by $(v_1, v_2)_k = h_k^2 \sum_m v_1(m) v_2(m)$, where the summation is taken over all internal midpoints m of the triangulation $\mathcal{T}_k$.

Let $a_k(\cdot,\cdot)$ be defined by $a_k(v_1, v_2) = \sum_{\mathrm{R}\in\mathcal{T}_k} \int_{\mathrm{R}} \nabla v_1 \cdot \nabla v_2 \, dx$. The discrete problem is again given by (5.7).

The interpolation operator $\Pi_k : V \longrightarrow V_k$ is defined by the same formula in (5.8), and the estimate (5.10) remains valid (cf., [52]).

The operator $E_k : V_k \longrightarrow \tilde{V}_k$ is defined by $(E_k v)(m) = v(m)$ for all internal midpoints $m \in \mathcal{T}_k$, $(E_k v)(c) = v(c)$ for all centroids $c \in \mathcal{T}_k$, and $(E_k v)(p) =$ average of $v_i(p)$ for all internal vertices $p \in \mathcal{T}_k$, where $v_i = v\big|_{\mathrm{R}_i}$ and $\mathrm{R}_i \in \mathcal{T}_k$ contains p as a vertex.

The operator $F_k : \tilde{V}_k \longrightarrow V_k$ is defined by the same formula in (5.12), and the intergrid transfer operator is defined by averaging as in Example 5.1.

All the assumptions in Section 2 can be verified for this example by the same arguments used in Example 5.1. Hence the results in Section 4 are applicable to the multigrid methods for (5.7) using the "rotated" $\mathcal{Q}_1$ finite elements.

Remark. The nonconforming $\mathcal{P}_1$ and "rotated" $\mathcal{Q}_1$ finite elements are equivalent to the lowest order triangular and rectangular Raviart-Thomas mixed finite elements (cf., [53], [4], [2]). There are multigrid methods for (5.1) using the lowest order Raviart-Thomas elements (cf., [19], [2]) which are based on the multigrid methods for the nonconforming elements. The results in Section 4 are therefore applicable to these multigrid algorithms for the lowest order Raviart-Thomas finite elements.

6. APPLICATIONS TO A MODEL FOURTH ORDER PROBLEM

In this section we apply our theory to the biharmonic equation with homogeneous Dirichlet boundary conditions. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and

$f \in L^2(\Omega)$. Consider the following boundary value problem.

$$\Delta^2 u = f \quad \text{in } \Omega \quad \text{and} \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{6.1}$$

Let $V = H_0^2(\Omega)$, and let $a(\cdot,\cdot)$ on V be defined by either

$$a(v,w) = \sum_{i,j=1,2} \int_\Omega v_{x_i x_j} w_{x_i x_j}\, dx$$

or

$$a(v,w) = \int_\Omega \big[\Delta v \Delta w + (1-\sigma)\big(2v_{x_1x_2}w_{x_1x_2} - v_{x_1x_1}w_{x_2x_2} - v_{x_2x_2}w_{x_1x_1}\big)\big]\, dx,$$

where σ is the Poisson ratio and $0 < \sigma < \frac{1}{2}$. For either choice of the variational form $a(\cdot,\cdot)$, conditions (B) and (C) follow from the Cauchy-Schwarz inequality and the generalized Poincaré inequality (cf., [48]), respectively.

The weak formulation of (6.1) is to find $u \in V$ such that

$$a(u,v) = \int_\Omega f v\, dx \quad \forall\, v \in V. \tag{6.2}$$

By the elliptic regularity theory for non-smooth domains (cf., [36], [39], [40], [41]), there exists $\alpha \in (\frac{1}{2}, 1]$ such that for $f \in H^{-2+\alpha}(\Omega)$, the solution u of (6.2) belongs to $H^{2+\alpha}(\Omega)$ and

$$\|u\|_{H^{2+\alpha}(\Omega)} \lesssim \|f\|_{H^{-2+\alpha}(\Omega)}. \tag{6.3}$$

Let $Z = H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$ and $W = H_0^{2-\alpha}(\Omega)$. Clearly, (R-1) holds. Since $W' = H^{-2+\alpha}(\Omega)$, the estimate (R-2) follows from (6.3).

Let $X = L^2(\Omega)$. By the interpolation of Sobolev spaces (cf., [61], [57]), we have

$$Z = [H_0^2(\Omega), H^3(\Omega) \cap H_0^2(\Omega)]_\alpha, \tag{6.4}$$

and $W = [L^2(\Omega), H_0^2(\Omega)]_{1-\alpha/2}$. In particular, the condition (C-3) holds.

The biharmonic operator Δ^2 is a bounded linear operator from $H^3(\Omega)$ to $H^{-1}(\Omega)$, and from $H^2(\Omega)$ to $H^{-2}(\Omega)$. Therefore by interpolation we have

$$\|\Delta^2 \zeta\|_{H^{-2+\alpha}(\Omega)} \lesssim \|\zeta\|_{H^{2+\alpha}(\Omega)} \quad \forall\, \zeta \in H^{2+\alpha}(\Omega). \tag{6.5}$$

As in the case of the Poisson equation (cf., Section 5), the duality estimate (D) follows from (6.5) and a density argument.

We now consider finite element multigrid methods for (6.2). In the following examples, $\{\mathcal{T}_k\}_{k=1}^\infty$ is a sequence of quasi-uniform triangulations of Ω. For simplicity we assume that $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. Let $T \in \mathcal{T}_k$. We denote by S_T the interior of the union of the closures of the triangles in $\mathcal{T}_k$ neighboring T.

Example 6.1. Let $V_k^* = \{v \in L^2(\Omega) : v\big|_T$ is quadratic, v is continuous at the vertices and $\partial v/\partial n$ is continuous at the midpoints of interelement boundaries$\}$ be the Morley finite element space associated with $\mathcal{T}_k$ (cf., [47]), and let $\tilde{V}_k^*$ be the Hsieh-Clough-Tocher macro element space associated with $\mathcal{T}_k$ (cf., [34]). A function $\tilde{v} \in \tilde{V}_k^*$ is C^1 on $\bar{\Omega}$, and its restriction to each $T \in \mathcal{T}_k$ is piecewise cubic on the three triangles formed by the centroid and the vertices of T. The space V_k (resp., $\tilde{V}_k$) is the subspace of V_k^* (resp., $\tilde{V}_k^*$) whose members have zero nodal values along $\partial\Omega$. Note that $V_k \preceq \tilde{V}_k$. The inner product for V_k^* is defined by

$(v,w)_k = h_k^2\sum_p v(p)w(p) + h_k^4\sum_m \frac{\partial v}{\partial n}(m)\frac{\partial w}{\partial n}(m)$, where the summations are taken over all vertices p and midpoints m in $\mathcal{T}_k$.

The symmetric positive definite bilinear form $a_k(\cdot,\cdot)$ is defined by either

$$a_k(v,w) = \sum_{T\in\mathcal{T}_k}\sum_{i,j=1,2}\int_T v_{x_ix_j}w_{x_ix_j}\,dx$$

or

$$\begin{aligned} a_k(v,w) = \sum_{T\in\mathcal{T}_k}\int_T \Big[&\Delta v\Delta w + (1-\sigma) \\ &\times \big(2v_{x_1x_2}w_{x_1x_2} - v_{x_1x_1}w_{x_2x_2} - v_{x_2x_2}w_{x_1x_1}\big)\Big]\,dx. \end{aligned}$$

The discrete problem for (6.2) is to find $u_k \in V_k$ such that

$$a_k(u_k,v) = \int_\Omega f v\,dx \quad \forall v\in V_k. \tag{6.6}$$

Clearly, (M), (C-1), (C-2), (I) and (P) are satisfied.

The interpolation operator $\Pi_k : V \longrightarrow V_k$ is defined by

$$(\Pi_k v)(p) = v(p) \quad\text{and}\quad \frac{\partial(\Pi_k v)}{\partial n}(m) = \frac{1}{|e|}\int_e \frac{\partial v}{\partial n}\,ds, \tag{6.7}$$

where p and m range over the internal vertices and midpoints of $\mathcal{T}_k$, and m is the midpoint of the edge e. Note that

$$(\Pi_k\zeta)\big|_T = \zeta\big|_T \quad \text{if } \zeta\big|_T \text{ is quadratic.} \tag{6.8}$$

The following interpolation estimates are established by the standard techniques for almost affine family of finite elements (cf., [30]).

$$\begin{aligned} &\|\zeta-\Pi_k\zeta\|_{L^2(T)} + h_k|\zeta-\Pi_k\zeta|_{H^1(T)} \\ &\qquad + h_k^2|\zeta-\Pi_k\zeta|_{H^2(T)} \lesssim h_k^\beta|\zeta|_{H^\beta(T)}, \qquad \beta=2,3, \end{aligned} \tag{6.9}$$

for all $T\in\mathcal{T}_k$ and $\zeta\in H^\beta(\Omega)\cap H_0^2(\Omega)$. The estimate (Π-1) follows from (6.9) with $\beta=2$, and the estimate (Π-2) follows from (Π-1), (6.4), (6.9) with $\beta=3$ and interpolation. From (6.9) with $\beta=3$ we also have

$$\Big(\sum_{T\in\mathcal{T}_k}|\zeta-\Pi_k\zeta|_{H^1(T)}^2\Big)^{1/2} \lesssim h_k^2\|\zeta\|_{H^3(\Omega)} \quad \forall\,\zeta\in H^3(\Omega)\cap H_0^2(\Omega). \tag{6.10}$$

Let p and m be the internal vertices and midpoints of $\mathcal{T}_k$. The operators $E_k : V_k \longrightarrow \tilde{V}_k$ and $F_k : \tilde{V}_k \longrightarrow V_k$ are defined by

$$\begin{aligned} (E_k v)(p) &= v(p), \\ \frac{\partial(E_k v)}{\partial n}(m) &= \frac{\partial v}{\partial n}(m), \\ [\partial^\beta(E_k v)](p) &= \text{average of } (\partial^\beta v_i)(p), \qquad |\beta|=1, \end{aligned} \tag{6.11}$$

where $v_i = v|_{T_i}$ and T_i contains p as a vertex, and

$$(F_k\tilde{v})(p) = \tilde{v}(p) \quad\text{and}\quad \frac{\partial(F_k\tilde{v})}{\partial n}(m) = \frac{\partial\tilde{v}}{\partial n}(m)\,. \tag{6.12}$$

Note that F_k is well–defined because $V_k \preceq \tilde{V}_k$. Clearly the relation (FE) holds, and (F) follows from a simple element by element calculation.

Let $E_k^*: V_k^* \longrightarrow \tilde{V}_k^*$ be defined by the same formulas in (6.11) for all vertices and midpoints of $\mathcal{T}_k$. A straightforward computation (cf., the similar computation in [24] where the Argyris element was used instead of the Hsieh-Clough-Tocher element) yields

$$\|E_k^* v - v\|_{L^2(T)}^2 \lesssim h_k^4 \sum_{K \in S_T} |v|_{H^2(K)}^2 \quad \forall\, v \in V_k^*. \tag{6.13}$$

Moreover,

$$(E_k^* \eta)\big|_T = \eta \quad \text{if } \eta\big|_{S_T} \text{ is quadratic.} \tag{6.14}$$

It follows that

$$\|E_k^* v - v\|_{L^2(\Omega)}^2 \lesssim h_k^4 \sum_{T \in \mathcal{T}_k} |v|_{H^2(T)}^2 \quad \forall\, v \in V_k^*. \tag{6.15}$$

Let $v \in V_k$. Since $E_k v$ and $E_k^* v$ differ only by their first order derivatives at the vertices along $\partial\Omega$, we have

$$\|E_k v - E_k^* v\|_{L^2(\Omega)}^2 \lesssim h_k^6 {\sum}' \|\nabla^2 v\|_{L^\infty(T)}^2 \lesssim h_k^4 \|v\|_k^2 \quad \forall\, v \in V_k, \tag{6.16}$$

where the summation $\sum'$ in (6.16) is taken over the triangles in $\mathcal{T}_k$ neighboring $\partial\Omega$.

The estimate (E) follows from (6.15) and (6.16). A standard inverse estimate then yields

$$\Big(\sum_{T \in \mathcal{T}_k} |E_k v - v|_{H^1(T)}^2\Big)^{1/2} \lesssim h_k \|v\|_k \quad \forall\, v \in V_k. \tag{6.17}$$

Using (6.8), (6.9), (6.13), (6.14) and the Bramble-Hilbert lemma as in Example 5.1, we obtain

$$\|E_k^* \Pi_k \zeta - \zeta\|_{L^2(\Omega)} \lesssim h_k^3 |\zeta|_{H^3(\Omega)} \quad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \tag{6.18}$$

Since $E_k \Pi_k \zeta$ and $E_k^* \Pi_k \zeta$ differ only by their first order derivatives at the vertices along $\partial\Omega$, we have, by (6.9),

$$\begin{aligned} \|E_k \Pi_k \zeta - E_k^* \Pi_k \zeta\|_{L^2(\Omega)}^2 &\lesssim h_k^4 \sum_{p \in \partial\Omega} \sum_{\substack{T \in \mathcal{T}_k \\ \bar{T} \ni p}} \Big|\nabla(\Pi_k \zeta)\big|_T\Big|^2(p) \\ &\lesssim h_k^4 \sum_{p \in \partial\Omega} \sum_{\substack{T \in \mathcal{T}_k \\ \bar{T} \ni p}} \Big|\nabla(\Pi_k \zeta)\big|_T - \nabla\zeta\Big|^2(p) \\ &\lesssim h_k^6 |\zeta|_{H^3(\Omega)}^2 \qquad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \end{aligned} \tag{6.19}$$

It follows from (6.18) and (6.19) that

$$\|E_k \Pi_k \zeta - \zeta\|_{L^2(\Omega)} \lesssim h_k^3 |\zeta|_{H^3(\Omega)} \quad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \tag{6.20}$$

By (6.9), (6.20) and standard inverse estimates, we have

$$\begin{aligned} \|E_k \Pi_k \zeta - \zeta\|_{H^2(\Omega)} &\lesssim h_k^{-2} \|E_k \Pi_k \zeta - \Pi_k \zeta\|_{L^2(\Omega)} + \|\Pi_k \zeta - \zeta\|_{H^2(\Omega)} \\ &\lesssim h_k |\zeta|_{H^3(\Omega)} \quad \forall\, \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \end{aligned} \tag{6.21}$$

By (I), (E) and (Π-1) we also have the trivial estimate

$$\|E_k \Pi_k \zeta - \zeta\|_{L^2(\Omega)} + h_k^2 \|E_k \Pi_k \zeta - \zeta\|_{H^2(\Omega)} \lesssim h_k^2 \|\zeta\|_{H^2(\Omega)} \quad \forall\, \zeta \in H_0^2(\Omega). \tag{6.22}$$

The estimate (EΠ) follows from (6.4), (6.20), (6.21), (6.22) and interpolation.

Next we turn to the assumptions (N-1) and (N-2). Let $\zeta \in H^3(\Omega) \cap H_0^2(\Omega)$ and $\zeta_k \in V_k$ be related to ζ through (2.4). Let $v \in V_k + V$; then by the Green's formula (cf., [59]) we have

$$a_k(\zeta, v) = -\sum_{T\in\mathcal{T}_k} \int_T \nabla(\Delta\zeta)\cdot\nabla v\,dx + \sum_e \int_e \big(G_1(\zeta)[v_{x_1}] + G_2(\zeta)[v_{x_2}]\big)\,ds, \tag{6.23}$$

where $G_1(\zeta)$ and $G_2(\zeta)$ are combinations of second order derivatives of ζ, $[v_{x_1}]$ and $[v_{x_2}]$ denote the jumps of v_{x_1} and v_{x_2} across the edge e, and the second summation is taken over all edges e of $\mathcal{T}_k$.

Since $E_k v \in H_0^2(\Omega)$, it follows from (2.4) and (6.23) that

$$a_k(\zeta_k, v) = -\sum_{T\in\mathcal{T}_k} \int_T \nabla(\Delta\zeta)\cdot\nabla(E_k v)\,dx. \tag{6.24}$$

By subtracting (6.24) from (6.23) we obtain

$$\begin{aligned} a_k(\zeta-\zeta_k, v) = &-\sum_{T\in\mathcal{T}_k} \int_T \nabla(\Delta\zeta)\cdot\nabla(v - E_k v)\,dx \\ &+ \sum_e \int_e \big(G_1(\zeta)[v_{x_1}] + G_2(\zeta)[v_{x_2}]\big)\,ds \quad \forall\, v \in V_k. \end{aligned} \tag{6.25}$$

By the Cauchy-Schwarz inequality and (6.17) we have

$$\Big|\sum_{T\in\mathcal{T}_k} \int_T \nabla(\Delta\zeta)\cdot\nabla(v - E_k v)\,dx\Big| \lesssim h_k \|\zeta\|_{H^3(\Omega)}\,\|v\|_k \quad \forall\, v \in V_k. \tag{6.26}$$

Since v_{x_1} and v_{x_2} are continuous at the midpoints, we have, by a standard argument for nonconforming finite elements,

$$\Big|\sum_e \int_e \big(G_1(\zeta)[v_{x_1}] + G_2(\zeta)[v_{x_2}]\big)\,ds\Big| \lesssim h_k \|\zeta\|_{H^3(\Omega)}\,\|v\|_k \quad \forall\, v \in V_k. \tag{6.27}$$

Combining (6.25)–(6.27), we obtain

$$|a_k(\zeta-\zeta_k, v)| \lesssim h_k \|\zeta\|_{H^3(\Omega)}\,\|v\|_k \quad \forall\, v \in V_k. \tag{6.28}$$

Let $\zeta \in H_0^2(\Omega)$. Then we have the obvious estimate

$$\|\zeta_k\|_k \lesssim \|\zeta\|_{H^2(\Omega)}, \tag{6.29}$$

which implies that

$$|a_k(\zeta-\zeta_k, v)| \lesssim \|\zeta-\zeta_k\|_k \|v\|_k \lesssim \|\zeta\|_{H^2(\Omega)}\|v\|_k \quad \forall\, v \in V_k. \tag{6.30}$$

The estimate (N-1) now follows from (6.4), (6.28), (6.30) and interpolation.

By (6.9) and (6.20) we have

$$\begin{aligned} \|E_k\Pi_k\xi - \Pi_k\xi\|_{L^2(\Omega)} &\lesssim \|E_k\Pi_k\xi - \xi\|_{L^2(\Omega)} + \|\xi - \Pi_k\xi\|_{L^2(\Omega)} \\ &\lesssim h_k^3 \|\xi\|_{H^3(\Omega)} \quad \forall\, \xi \in H^3(\Omega)\cap H_0^2(\Omega). \end{aligned} \tag{6.31}$$

A standard inverse estimate then implies that

$$\Big(\sum_{T\in\mathcal{T}_k} |E_k\Pi_k\xi - \Pi_k\xi|^2_{H^1(T)}\Big)^{1/2} \lesssim h_k^2\,\|\xi\|_{H^3(\Omega)} \quad \forall\, \xi \in H^3(\Omega)\cap H_0^2(\Omega). \tag{6.32}$$

It follows from (6.25) that

$$a_k(\zeta-\zeta_k,\Pi_k\xi) = -\sum_{T\in\mathcal{T}_k}\int_T \nabla(\Delta\zeta)\cdot\nabla(\Pi_k\xi - E_k\Pi_k\xi)\,dx + \sum_e \int_e \big(G_1(\zeta)[(\Pi_k\xi)_{x_1}] + G_2(\zeta)[(\Pi_k\xi)_{x_2}]\big)\,ds \tag{6.33}$$

for $\zeta,\xi\in H^3(\Omega)\cap H_0^2(\Omega)$.

By the Cauchy-Schwarz inequality and (6.32) we have

$$\left|\sum_{T\in\mathcal{T}_k}\int_T \nabla(\Delta\zeta)\cdot\nabla(\Pi_k\xi - E_k\Pi_k\xi)\,dx\right| \lesssim h_k^2\|\zeta\|_{H^3(\Omega)}\,\|\xi\|_{H^3(\Omega)} \tag{6.34}$$

for $\zeta,\xi\in H^3(\Omega)\cap H_0^2(\Omega)$.

A standard argument for nonconforming finite elements shows that

$$\begin{aligned}&\left|\sum_e \int_e \big(G_1(\zeta)[(\Pi_k\xi)_{x_1}] + G_2(\zeta)[(\Pi_k\xi)_{x_2}]\big)\,ds\right| \\ &\lesssim \left|\sum_e \int_e \big(G_1(\zeta)[(\Pi_k\xi)_{x_1}-\xi_{x_1}] + G_2(\zeta)[(\Pi_k\xi)_{x_2}-\xi_{x_2}]\big)\,ds\right| \\ &\lesssim h_k^2\|\zeta\|_{H^3(\Omega)}\,\|\xi\|_{H^3(\Omega)} \quad \forall\,\zeta,\xi\in H^3(\Omega)\cap H_0^2(\Omega).\end{aligned} \tag{6.35}$$

Combining (6.33)–(6.35), we obtain

$$|a_k(\zeta-\zeta_k,\Pi_k\xi)| \lesssim h_k^2\|\zeta\|_{H^3(\Omega)}\,\|\xi\|_{H^3(\Omega)} \quad \forall\,\zeta,\xi\in H^3(\Omega)\cap H_0^2(\Omega). \tag{6.36}$$

On the other hand, for $\zeta,\xi\in H_0^2(\Omega)$, the estimates (6.9) and (6.29) imply that

$$|a_k(\zeta-\zeta_k,\Pi_k\xi)| \lesssim \|\zeta\|_{H^2(\Omega)}\|\xi\|_{H^2(\Omega)}. \tag{6.37}$$

The estimate (N-2) now follows from (6.4), (6.36), (6.37) and (bilinear) interpolation.

The intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ is defined by averaging as follows. Let p be a vertex of $\mathcal{T}_k$ inside Ω. If p is also a vertex of $\mathcal{T}_{k-1}$, then $(I_{k-1}^k v)(p) = v(p)$. If p is the midpoint of the common edge of two triangles T_1 and $T_2\in\mathcal{T}_{k-1}$, then

$$(I_{k-1}^k v)(p) = \frac{1}{2}\big[v\big|_{T_1}(p) + v\big|_{T_2}(p)\big].$$

Let m be a midpoint of an edge e of $\mathcal{T}_k$ inside Ω and n be a unit normal of e. If m is in the interior of a triangle in $\mathcal{T}_{k-1}$, then

$$\frac{\partial(I_{k-1}^k v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m).$$

If m is on the common edge of two triangles T_1 and T_2 in $\mathcal{T}_{k-1}$, then

$$\frac{\partial(I_{k-1}^k v)}{\partial n}(m) = \frac{1}{2}\left[\frac{\partial v|_{T_1}}{\partial n}(m) + \frac{\partial v|_{T_2}}{\partial n}(m)\right].$$

The estimate (I-1) follows immediately from the estimates in [16], and the estimate (I-2) follows from the estimates in [16] and interpolation, as in Example 5.1.

Since all of the assumptions of our theory hold for this example, the results in Section 4 are applicable to the multigrid methods for (6.6).

Remark. For $1/2 < \alpha < 1$, the estimate (4.31) and the Sobolev inequality (cf., [61]) imply that

$$\sup_{x\in\bar{\Omega}} \big|u(x) - [E_k\hat{u}_k](x)\big| \lesssim h_k^{2\alpha}\|u\|_{H^{2+\alpha}(\Omega)} + h_k^{2+\alpha}\|f\|_{L^2(\Omega)}.$$

Since E_kv and v coincide at the vertices, we have

$$\max_p |u(p) - \hat{u}_k(p)| \lesssim h_k^{2\alpha}\|u\|_{H^{2+\alpha}(\Omega)} + h_k^{2+\alpha}\|f\|_{L^2(\Omega)},$$

where the summation is taken over all the vertices of $\mathcal{T}_k$.

In the case $\alpha = 1$, we have, for any $0 < \beta < 1$,

$$\max_p |u(p) - \hat{u}_k(p)| \le C_\beta \left[h_k^{2\beta}\|u\|_{H^{2+\beta}(\Omega)} + h_k^{2+\beta}\|f\|_{L^2(\Omega)}\right].$$

Remark. The symmetric variable V-cycle preconditioner for the Morley finite element method can also be used to precondition the Argyris finite element method (cf., [3], [26]).

Remark. The results in Example 6.1 are also valid for the Adini element (cf., [1], [29], [45]) and the incomplete biquadratic element (cf., [58]), which are connected to the Bogner-Fox-Schmit element (cf., [8]) and the Fraeijs de Veubeke-Sander quadrilateral element (cf., [54], [38], [32]), respectively.

Example 6.2. Let $V_k = \tilde{V}_k \subseteq H_0^2(\Omega)$ be the Hsieh-Clough-Tocher or the reduced Hsieh-Clough-Tocher macro finite element space associated with $\mathcal{T}_k$ (cf., [34], [31], [51]), and let $a_k(\cdot,\cdot) = a(\cdot,\cdot)$ on V_k. The discrete problem for (6.2) is to find $u_k \in V_k$ such that

$$a(u_k, v) = \int_\Omega fv\,dx \quad \forall\, v \in V_k. \tag{6.39}$$

There exists an interpolation operator $\Pi_k : V \longrightarrow V_k$ such that

$$\|\zeta - \Pi_k\zeta\|_{L^2(T)} + h_k^2|\zeta - \Pi_k\zeta|_{H^2(T)} \lesssim h_k^\beta|\zeta|_{H^\beta(S_T)}, \qquad \beta = 2, 3, \tag{6.40}$$

and

$$(\Pi_k\zeta)\big|_T = \zeta\big|_T \quad \text{if } \zeta\big|_{S_T} \text{ is quadratic.} \tag{6.41}$$

The estimates (Π-1) and (Π-2) follow from (6.40). The operator Π_k can be constructed by using the techniques in [33] and [56]. For the Hsieh-Clough-Tocher element, we can also take Π_k to be the composition of the interpolation operator for the Morley element defined in (6.7) and the connection operator defined in (6.11).

Let $E_k = F_k =$ identity map on V_k. The estimates (E), (EΠ), (F), (FE), (N-1) and (N-2) are then completely trivial. We can take $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be the nodal interpolation operator. Then (I-1) is a standard interpolation error estimate. Moreover, we have

$$(I_{k-1}^k v)\big|_T = v\big|_T \quad \text{if } v\big|_T \text{ is quadratic.} \tag{6.42}$$

The estimates (Π-1) and (I-1) imply that

$$\|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta\|_{L^2(\Omega)} \lesssim h_k^2\,\|\zeta\|_{H^2(\Omega)} \quad \forall\, \zeta \in H_0^2(\Omega). \tag{6.43}$$

Using (6.41), (6.42) and the Bramble-Hilbert lemma, we obtain (cf., the proof of (6.20))

$$\|I_{k-1}^{k}\Pi_{k-1}\zeta - \Pi_k\zeta\|_{L^2(\Omega)} \lesssim h_k^3\,\|\zeta\|_{H^3(\Omega)} \quad \forall\,\zeta \in H^3(\Omega) \cap H_0^2(\Omega). \tag{6.44}$$

The estimate (I-2) follows from (6.4), (6.43), (6.44) and interpolation.

Therefore the results from Section 4 can be applied to these macro element methods.

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