#### **Purdue University**

## Purdue e-Pubs

# Department of Computer Science Technical Reports

Department of Computer Science

1986

## Convergence of O(h\*\*4) Cubic Spline Collocation Methods for Elliptic Partial Differential Equations

Elias N. Houstis Purdue University, enh@cs.purdue.edu

E. A. Vavalis

John R. Rice Purdue University, jrr@cs.purdue.edu

Report Number: 86-596

Houstis, Elias N.; Vavalis, E. A.; and Rice, John R., "Convergence of O(h\*\*4) Cubic Spline Collocation Methods for Elliptic Partial Differential Equations" (1986). *Department of Computer Science Technical Reports.* Paper 515.

https://docs.lib.purdue.edu/cstech/515

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries. Please contact epubs@purdue.edu for additional information.

#### CONVERGENCE OF AN O(b<sup>4</sup>) CUBIC SPLINE COLLOCATION METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

E.N. Houstis, E.A. Vavalis and J.R. Rice Purdue University Department of Computer Science West Lafayette, IN 47907

> CSD TR 596 May 15, 1986

#### ABSTRACT

This paper presents a new class of collocation methods using cubic splines for solving elliptic partial differential equations (PDEs). The error bounds obtained for these methods are optimal. The methods are formulated and a convergence analysis is carried out for a broad class of elliptic PDEs. Experimental results confirm the optimal convergence and indicate that these methods are computationally more efficient than methods based on either collocation with Hermite cubics or on Galerkin with cubic splines.

#### 1. INTRODUCTION

We consider the formulation and analysis of a method for approximating the solution u(x,y) of the elliptic linear partial differential equation

$$Lu = \alpha D_x^2 u + \beta D_x D_y u + \gamma D_y^2 u + \delta D_x u + \varepsilon D_y u + \zeta u = -f \quad \text{in} \quad \Omega = [a,b] \times [c,d] \tag{1.1}$$

subject to homogeneous Dirichlet or Neumann boundary conditions

$$Bu \equiv 0 \text{ on } \partial\Omega \equiv \text{ boundary of } \Omega$$
 (1.2)

where Bu is  $u, D_x u$  or  $D_y u$ . Throughout it is assumed that the coefficients  $\alpha, \beta, \gamma$  satisfy the ellipticity condition  $\beta^2 - 4\alpha\gamma < 0$ .

The method considered in this paper involves the determination of the bicubic spline piecewise polynomial  $u_{\Delta}(x,y)$  over the partition  $\Delta$  of  $\Omega$ . The spline  $u_{\Delta}$  is chosen to satisfy exactly the boundary conditions and an operator equation  $L'u_{\Delta} = -f$  at the interior grid points of  $\Delta$ , where L' is a high order perturbation of L plus additional spline end conditions. An implementation of the method exists in ELLPACK [Rice 85] for equation (1.1) and mixed homogeneous boundary conditions. The method of collocation at nodal points based on tensor product of cubic splines was first analyzed independently in [Cave 72] and [Ito 72] for Helmholtz elliptic PDEs with Dirichlet boundary conditions on a square. Second order convergence of the method was proved. The formulation of this method for more general elliptic PDEs was considered in [Ito 72] without deriving any error estimates. In [Arno 84] the nodal collocation method using the tensor product of smoothest splines of arbitrary odd order is analyzed for a certain class of elliptic PDEs. The results indicate the failure of this method to produce an optimal order approximation for the solution of (1.1),(1.2). Optimal order of convergence is obtained in various Sobolev spaces with order greater or equal to the order of the operator. In [Fyfe 69],[Arch 73],[Cave 72],[Dani 75],[Rubi 76],[Arch 77] variants of the nodal cubic spline collocation are studied yielding  $O(h^4)$  discrete and semidiscrete approximations to one dimensional elliptic and parabolic equations. In these studies the collocation approximation is required to satisfy a perturbed differential equation and boundary conditions within  $O(h^4)$ . These high order perturbations are derived through an accurate spline interpolant of the true solution and its derivatives. This idea was first introduced in [Fyfe 69] and the method was formulated as a

deferred correction type. Applying the same idea, an  $O(h^4)$  line cubic spline collocation method was introduced in [Rome 79] for the Poisson equation and in [Hous 84], [Vava 85] for larger classes of problems. The iterative solution of the resulting linear equations is studied extensively in [Hous 84] and [Vava 85].

#### 2. HIGH ORDER INTERPOLATION RELATIONS

In this section we derive a high order perturbation L' of L based on various interpolation results for cubic splines. Throughout we denote by  $\Delta_x = \{x_k = a + kh_x; k = 0 \text{ to } N \text{ with } h_x = (b-a)/N\}$  and  $\Delta_y = \{y_i = c + lh_y; l = 0 \text{ to } M \text{ with } h_y = (d-c)/M\}$  the uniform partitions of [a,b] and [c,d]. Then  $\Delta = \Delta_x \times \Delta_y$  is the induced uniform partition of  $\Omega$ . The nodal points  $(x_i, y_i)$  of  $\Delta$  are broken into three sets, for later use as follows:  $\Omega_i$  = points interior to  $\Omega$ ,  $\Omega_B$  = points of  $\partial\Omega$  but not corner points,  $\Omega_C$  = corner points of  $\Delta$ . Throughout, we denote by  $S_{3,\mathbf{x}_a} = \mathbf{P}_{3,\mathbf{x}_a} \cap C^2$  ([a,b]) the space of one-dimensional cubic splines with respect to a partition  $\pi_n$  of [a,b] and by  $S_{3,\mathbf{x}_a}^{(0)}$ , the subspace of  $S_{3,\mathbf{x}_a}$  whose elements satisfy the boundary conditions (1.2).

We define  $S_{3,A}^{(0)}$  to be the space of the *two-dimensional splines* associated with  $\Delta$  and which satisfy exactly the boundary conditions (1.2). We can construct a basis for  $S_{3,\Delta}^{(0)}$  by forming the tensor product of basis elements of the one-dimensional splines  $S_{3,\Delta x}^{(0)}$  and  $S_{3,\Delta y}^{(0)}$ . If  $\pi_n \equiv \{t_i = a + ih, i = -1 \text{ to } n+1, h = (b-a)/n\}$  is a uniform partition of [a,b] then the basis functions  $\{\hat{B}_i\}$  for  $S_{3,\kappa}$  can be chosen so that

$$\hat{B}_i(t_{i\pm 1}) = 1/6, \ \hat{B}_i(t_i) = 2/3, \ \hat{B}_i'(t_{i\pm 1}) = 1/h^2, \ \hat{B}_i''(t_i) = -2/h^2,$$
  
 $\hat{B}_i(t_{i-1}) = 1/(2h) \text{ and } \hat{B}_i(t_{i+1}) = -1/(2h).$ 

In the case of Dirichlet boundary conditions the basis functions  $B_i$  of the subspace  $S_{3/2}^{\infty}$  can be defined in terms of  $\hat{B}_i$ 's by

$$B_{0}(x) = \hat{B}_{0}(x) - 4\hat{B}_{-1}(x), \quad B_{1}(x) = \hat{B}_{1}(x) - \hat{B}_{-1}(x),$$
  

$$B_{i}(x) = \hat{B}_{i}(x), \quad i = 2, \dots, N-2$$
  

$$B_{N-1}(x) = \hat{B}_{N-1}(x) - \hat{B}_{N+1}(x) \text{ and } B_{N} = \hat{B}_{N}(x) - 4\hat{B}_{N+1}(x).$$
(2.1)

Similarly, for Neumann boundary conditions, the basis functions for  $S_{3\Delta}^{(0)}$  can be defined by

$$B_{0}(x) = \hat{B}_{0}(x), \quad B_{1}(x) = \hat{B}_{-1}(x) + \hat{B}_{1}(x),$$
  

$$B_{i}(x) = \hat{B}_{i}(x), \quad i = 2, \dots, N-2, \quad B_{N-1}(x) = \hat{B}_{N+1}(x) + \hat{B}_{N-1}(x) \text{ and} \qquad (2.2)$$
  

$$B_{N}(x) = \hat{B}_{N}(x) \quad .$$

Throughout we adopt the following representation of  $u_{\Delta}$ 

$$u_{\Delta}(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{M} U_{i,j} B_{i}(x) B_{j}(y).$$
(2.3)

In order to formulate the spline-collocation method and prove its convergence, we need to derive some results on bicubic spline interpolation. Throughout we denote by S(x,y) the bicubic spline interpolant of u in  $S_{3\Delta}^{(0)}$  such that

$$S_{i,j} = u_{i,j}, 1 \le i \le N, 1 \le j \le M,$$
 (2.4)

$$D_{x}^{2}S_{i,j} = D_{x}^{2}u_{i,j}, \ 0 \le j \le M \ , \ i = 0, N,$$
(2.5)

$$D_{y}^{2}S_{i,j} = D_{y}^{2}u_{i,j}, \ 0 \le i \le N \ , \ j = 0, \mathcal{M} \ ,$$
(2.6)

$$D_x^2 D_y^2 S_{ij} \approx D_x^2 D_y^2 u_{ij}, \quad i = 0, N \quad j = 0, M,$$
(2.7)

where we use the notation

$$S_{i,j} = S(x_i, y_j)$$
 and  $u_{i,j} = u(x_i, y_j)$ .

Next, we list several useful identities that S satisfies.

**Lemma 2.1** If  $u \in C^{6}(\Omega)$ , then the following relations hold at the nodes  $(x_i, y_j)$  of a uniform partition  $\Delta$ 

$$D_x^2 S_{ij} = D_x^2 u_{ij} - (h_x^2) D_x^4 u_{ij} / 12 + O(h_x^4),$$
(2.8a)

$$D_{y}^{2}S_{i,j} = D_{y}^{2}u_{i,j} - (h_{y}^{2}) D_{y}^{4}u_{i,j}/12 + O(h_{y}^{4}).$$
(2.8b)

The relations (2.8) are direct consequences of the discussion in [Luca 74].

Corollary 2.1 Under the hypotheses of Lemma 2.1, we have

$$(D_x^2 S_{i-1,j} - 2D_x^2 S_{i,j} + D_x^2 S_{i+1,j})/h_x^2 = D_x^4 u_{i,j} + O(h_x^2)$$
(2.9)

for  $1 \le i \le N-1$  and  $0 \le j \le M$ .

The relation (2.9) is established by a straightforward application of Lemma 2.1, while a similar relation holds for  $D_y^4 u$ . Similarly, using well known relationships valid for any smooth one-dimensional cubic

spline [Luca 74] for uniform meshes and Corollary 2.1, it is relatively simple to prove the following.

**Lemma 2.2** If  $u \in C^{6}(\Omega)$ , then at the interior nodes  $(\Omega_{1})$  of  $\Delta$  we have

$$(D_x^2 S_{i-1,j} + 10 D_x^2 S_{i,j} + D_x^2 S_{i+1,j})/12 = D_x^2 u_{i,j} + O(h_x^4),$$
(2.10a)

$$(D_{y}^{2}S_{i,j-1} + 10D_{y}^{2}S_{i,j} + D_{y}^{2}S_{i,j+1})/12 = D_{y}^{2}u_{i,j} + O(h_{y}^{4}),$$
(2.10b)

$$D_x S_{i,j} = D_x u_{i,j} + O(h_x^4), \quad D_y S_{i,j} \approx D_y u_{i,j} + O(h_y^4).$$
(2.10c)

By Taylor's expansion, it can be shown that  $D_x^4 u_{0,j} = 2D_x^4 u_{1,j} - D_x^4 u_{2,j} + O(h_x^2)$  and  $D_x^4 u_{N,j} = 2D_x^4 u_{N-1,j} - D_x^4 u_{N-2,j} + O(h_x^2)$  for  $1 \le j \le M$ . Similar relations hold for  $D_y^4 u_{i,0}$  and  $D_y^4 u_{i,M}$  for  $1 \le i \le N$ . These together with (2.9) imply the following result.

**Corollary 2.2** If  $u \in C^{6}(\Omega)$ , then at the boundary nodes  $(\Omega_{B})$  of  $\Delta$  we have

$$(14D_x^2S_{0,j} - 5D_x^2S_{1,j} + 4D_x^2S_{2,j} - D_x^2S_{3,j})/12 = D_x^2u_{0,j} + O(h_x^4)$$
(2.11a)

$$(14D_x^2 S_{N,j} - 5D_x^2 S_{N-1,j} + 4D_x^2 S_{N-2,j} - D_x^2 S_{N-3,j})/12 = D_x^2 u_{N,j} + O(h_x^4)$$
(2.11b)

$$(14D_y^2S_{i,0} - 5D_y^2S_{i,1} + 4D_y^2S_{i,2} - D_y^2S_{i,3})/12 = D_y^2u_{i,0} + O(h_y^4)$$
(2.11c)

$$(14D_{y}^{2}S_{i,M} - 5D_{y}^{2}S_{i,M-1} + 4D_{y}^{2}S_{i,M-2} - D_{y}^{2}S_{i,M-3})/12 = D_{y}^{2}u_{i,M} + O(h_{y}^{4})$$
(2.11d)

It is worth noticing that the relations (2.8) to (2.11) are independent of the spline end conditions. The following results will be used later to develop discretization error bounds for a class of two-dimensional elliptic boundary value problems.

Lemma 2.3 If  $u \in C^4(\Omega)$ , then we have

(i) 
$$D_x D_y S_{i,j} = D_x D_y u_{i,j} + O(h_x^4) + O(h_y^4)$$
 (2.12a)

(ii) 
$$D_x^k D_y^l S_{i,j} = D_x^k D_y^l u_{i,j} + O(h_x^{4-k}) + O(h_y^{4-l}), \quad 2 \le k+l \le 4$$
 (2.12b)

at each node of  $\Delta$ , and, with  $h = \max(h_x, h_y)$ ,

(iii) 
$$||D_{x}^{k}D_{y}^{l}(u-S)||_{\infty} = O(h^{4-(k+l)}), \quad 0 \le k, l \le 2.$$
 (2.12c)

The relations (2.12) can be derived from the discussion in [Carl 73] with minor modifications. Using the relations of Lemmas 2.1 to 2.3 and the corresponding corollaries we can prove the following result on which the formulation of the spline-collocation method is based.

**Theorem 2.1** Suppose  $u \in C^6[\Omega]$ ,  $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in C[\Omega]$ ,  $f \in C^4[\Omega]$  and  $h = \max(h_x, h_y)$ . Then

$$LS_{i,j} = -f_{i,j} + O(h^2), \text{ for } 0 \le i \le N, 0 \le j \le M,$$
(2.13)

and

$$L'S_{i,j} = -f_{i,j} + O(h^4), \text{ for } (x_i, y_j) \in \Omega_I,$$
 (2.14)

where the functions  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $\varepsilon$  and  $\zeta$  are evaluated at the associated points and the operator L' (a high order perturbation of L) is given symbolically at the point  $(x_i, y_j)$ , by the stencil S.1. Furthermore at boundary nodes  $\Omega_B$  the following relations hold for each type of boundary conditions:

Case of Dirichlet conditions at  $x = x_0$ 

$$L'S_{0,j} = \frac{\alpha}{12} (14D_x^2 S_{0,j} - 5D_x^2 S_{1,j} + 4D_x^2 S_{2,j} - D_x^2 S_{3,j}) + \beta D_x D_y S_{0,j} + \delta D_x S_{0,j} = -f_{0,j} + O(h^4), \quad (2.15a)$$

Case of Neumann conditions at  $x = x_0$ 

$$L'S_{0j} = \frac{\alpha}{12} (14D_x^2 S_{0j} - 5D_x^2 S_{1j} + 4D_x^2 S_{2j} - D_x^2 S_{3j}) + \varepsilon D_y S_{0j}$$

$$\frac{\gamma}{12} (D_y^2 S_{0j-1} + 10D_y^2 S_{0j} + D_y^2 S_{0j+1}) + \zeta S_0 = -f_{0j} + O(h^4).$$
(2.15b)

Similar relations hold for the other boundary sides.

#### 3. CUBIC SPLINE COLLOCATION METHOD FOR GENERAL ELLIPTIC PDES

In this section we define the *cubic spline collocation method* for the case of homogeneous boundary conditions (1.2). Based on the relation (2.14), we define an approximation  $u_{\Delta} \in S_{3,\Delta}^{(0)}$  to the solution u of (1.1), (1.2) by using the method of collocation such that it satisfies:

$$\begin{pmatrix} \frac{1}{12}D_y^2S(x_i, y_{j+1}) \\ \frac{10}{12}[\alpha D_x^2S(x_i, y_j) + \gamma D_y^2S(x_i, y_j)] \\ \frac{\alpha}{12}D_x^2S(x_{i-1}, y_j) & +\delta D_xS(x_i, y_j) + \epsilon D_yS(x_i, y_j) \\ +\beta D_x D_yS(x_i, y_j) + \varsigma S(x_i, y_j) \\ \frac{1}{12}D_y^2S(x_i, y_{j-1}) \end{pmatrix}$$

Stencil S.1. Definition of L'S at  $\Omega_I$  in terms of the cubic spline interpolant S and its derivatives at the grid points  $\Omega_I \cup \Omega_B$ . L'S is defined at boundary nodes by (2.15a) or (2.15b).

(i) the interior collocation equations

$$\left[L'u_{\Delta} - (-f)\right]_{(x_i,y_j)} = 0 \quad at \ the \ interior \ grid \ points \ \Omega_I$$
(3.1)

where L' is defined by S.1,

(ii) the boundary collocation equations

$$\left[L'u_{\Delta} - (-f)\right]_{(z_i,y_j)} = 0 \quad at the boundary knots \Omega_B$$
(3.2)

where L' is defined by (2.15a) or (2.15b) depending on the type of boundary conditions (1.2).

Notice that the cubic spline interpolant S satisfies the above equations (3.1) and (3.2) within an error of order  $O(h^4)$ . The convergence of this method for the case of Helmholtz equations with non constant coefficients is analyzed in Section 4. Here we examine the solvability of the collocation equations. We use the notations  $\sigma = h_x/h_y$ ,  $\Omega_i^0 = \{ (x_i, y_j) \in \Delta; 2 \le i \le N-2, 2 \le j \le M-2 \}$  and  $\Omega_i^1 = \Omega_i - \Omega_j^0$ . If we substitute  $u_{\Delta}$  in (3.1) with its representation (2.3) then the following result holds.

**Lemma 3.1** Let A be the coefficient matrix of interior collocation equations (3.1) and assume  $\zeta \leq 0$ . Then A is diagonally dominant for sufficiently small  $h_x$ ,  $h_y$  provided that

 $\sigma^{2}/p \in \left(\frac{18}{32}, \frac{32}{18}\right), at \Omega_{I}^{0} points,$   $\sigma^{2}/p \in \left(\frac{40}{132}, \frac{132}{40}\right), (Dirichlet \ case), \sigma^{2}/p \in \left(\frac{68}{128}, \frac{128}{68}\right), (Neumann \ case), (3.3)$ at the points  $(x_{i}, y_{j}), i=1, N-1, j=1, M-1,$ 

$$\sigma^2/p \in \left(\frac{40}{128}, \frac{128}{40}\right)$$
, (Dirichlet case),  $\sigma^2/p \in \left(\frac{70}{128}, \frac{128}{70}\right)$ , (Neumann case),

at the points  $(x_i, y_j)$ , i=1, N-1, j=2,..., M-2, or i=2,..., N-2, j=1, M-1

where 
$$\sigma = \frac{h_x}{h_y}$$
 and  $p = \frac{\alpha}{\gamma}$ .

**Proof.** First consider the equations that correspond to the collocation points  $(x_i, y_j) \in \Omega_j^0$ . Based on the stencil S.2 and those given in Appendix I, the diagonal dominance condition is written as

$$d_{i,j} = \frac{1}{72\sigma^2 h_y^2} \left\{ \left| -72(\alpha + \gamma \sigma^2) + O(h_y^2) \right| - \left\{ 4 \left| 8(\alpha + \gamma \sigma^2) + \beta \frac{\sigma}{4} + O(h_y) \right| + 2 \left| 32\alpha - 18\gamma \sigma^2 + O(h_y) \right| + 12 \left| \alpha + \gamma \sigma^2 \right| \right\}$$
(3.4)

Without loss of generality we can assume that  $\alpha$  and  $\gamma$  are positive. Since  $\beta^2 - 4\alpha\gamma < 0$ , we conclude that  $8(\alpha + \gamma\sigma^2) + \beta\frac{\sigma}{4}$  are always positive. The expressions  $32\alpha - 18\gamma\sigma^2$ ,  $-18\alpha + 32\gamma\sigma^2$  become positive for  $\sigma^2 \in (\frac{18\alpha}{32\gamma}, \frac{32\alpha}{18\gamma})$ . In this case it is easy to observe that  $d_{i,j} = -\frac{\zeta}{9\sigma^2}$ . Let c the function  $c = -8\zeta h_{\gamma}^2 \sigma^2$  and  $d'_{i,j} = 72h_{\gamma}^2 \sigma^2 d_{i,j}$ .

Now consider the point  $(x_1, y_1) \in \Omega_I^1$ . From equation (3.3) we have  $\sigma^2 \in \left(\frac{40}{132}p, \frac{132}{40}p\right)$  thus

$$\min_{\sigma} d'_{1,1} \equiv \frac{307}{8} \alpha + c \quad if \quad \sigma^2 > \frac{19}{32} p , \quad \frac{1050}{19} \alpha + c \quad if \quad \sigma^2 > \frac{19}{16} p ,$$

$$\frac{1228}{19} \alpha + c \quad if \quad \sigma^2 > \frac{32}{19} p \quad and \quad c \quad if \quad \sigma^2 > \frac{132}{40} p .$$

Similarly, for the points  $(x_1, y_j), 2 \le j \le M - 2$  and  $\sigma^2 \in \left(\frac{40}{12}p, \frac{19}{32}p\right)$  we have that

$$\min_{\sigma'} d'_{1,j} = c \quad if \quad \sigma^2 > \frac{19}{32} p, \quad 36 \alpha + c \quad if \quad \sigma^2 > \frac{16}{18} p, \\ \frac{36}{9} \alpha + c \quad if \quad \sigma^2 > \frac{32}{18} p, \quad c \quad if \quad \sigma^2 > \frac{132}{72} p.$$

The same analysis establishes identical lower bounds for the  $d_{i,j}$  associated with the rest of points in  $\Omega_j^1$ . In the case of Neumann boundary conditions and the stencils given in Appendix II, we obtain

min  $d'_{1,1} = c$  for  $\sigma^2 \in \left[\frac{68}{128}p, \frac{128}{68}p\right]$  and min  $d'_{i,j} = c$  for  $\sigma^2 \in \left[\frac{70}{128}p, \frac{128}{70}p\right]$ , for i=1,N-1, j=2,...,M-2 or i=2,...,N-2, j=1,M-1. Identical relations can be found for  $d_{i,j}$ 's associated with the rest of the points in  $\Omega_i^1$ . If  $\zeta$  is negative then the lower bounds for  $d_{i,j}$  are positive. Thus A is strictly diagonal dominant. This concludes the proof of the Lemma.

Notice that the boundary collocation equations (3.2) are not diagonally dominant with respect to the  $U_{i,j}$ 's. However, we can obtain diagonally dominant boundary equations by appropriate differentiation of the operator equation Lu=f.

The analysis and the implementation of the cubic spline collocation method for the general operator is more convenient if  $u_{\Delta}$  is defined in two phases and the method is viewed as a deferred correction type. Each phase involves the application of the standard second order spline collocation method with appropriate right sides. Specifically, we have the following second formulation of the spline collocation method:

Phase I

(i) Determine  $\hat{u}_{\Lambda}$  such that it satisfies

$$\left[L\hat{u}_{\Delta}-(-f)\right]_{(\mathbf{x},\mathbf{y}_{f})}=0 \quad at \ the \ node \ points \ \Omega_{f}\cup\Omega_{B}$$

(ii) Estimate  $D_x^4 u_{i,j}$ ,  $D_y^4 u_{i,j}$  as follows. Let

$$\Lambda_x g(x, y) = [g(x + h, y) - 2g(x, y) + g(x - h, y)]/h^2$$

and define  $\Lambda_{y}g(x, y)$  similarly as the second central difference in y. Then take

at the interior nodes  $\Omega_I$ :

$$D_x^4 u_{i,j} \equiv \Lambda_x \hat{u}_{\Delta}$$
,  $D_y^4 u_{i,j} \equiv \Lambda_y \hat{u}_{\Delta}$ 

at the boundary nodes  $\Omega_B$ :

$$D_{x}^{4}u_{0,j} \equiv 2\Lambda_{x}\hat{u}_{\Delta}(x_{1},y_{j}) - \Lambda_{x}\hat{u}_{\Delta}(x_{2},y_{j}), \quad D_{x}^{4}u_{N,j} \equiv 2\Lambda_{x}\hat{u}_{\Delta}(x_{N-1},y_{j}) - \Lambda_{x}\hat{u}_{\Delta}(x_{N-2},y_{j}), \\ D_{y}^{4}u_{i,0} \equiv 2\Lambda_{y}\hat{u}_{\Delta}(x_{i},y_{1}) - \Lambda_{y}\hat{u}_{\Delta}(x_{i},y_{2}), \quad D_{y}^{4}u_{i,M} \equiv 2\Lambda_{y}\hat{u}_{\Delta}(x_{i},y_{M-1}) - \Lambda_{y}\hat{u}_{\Delta}(x_{i},y_{M-2}).$$

Phase II

Determine  $u_{\Delta}$  such that it satisfies

$$\left[ L \mu_{\Delta} - (-f) \right]_{(x_i, y_j)} = 0 \quad \text{at the nodes } \Omega_j \cup \Omega_E$$

where

$$\hat{f}_{i,j} = -f_{i,j} - (\frac{h_x^2}{12})D_x^4 u_{i,j} - (\frac{h_y^2}{12})D_y^4 u_{i,j} \quad \text{for } i = 1, \dots, N-1, \quad j = 1, \dots, M-1.$$

Following the analysis of Lemma 3.1 we can prove diagonaly dominance for the second order coefficient collocation matrix.

Lemma 3.2. Let A be the coefficient matrix of the interior collocation matrix for the standard nodal cubic spline collocation method. Then for  $\zeta \leq 0$ , sufficiently small  $h_x$ ,  $h_y$ , and  $\sigma^2 lp$  restricted in the interval  $(\frac{1}{2}, 2)$  the matrix A is diagonally dominant.

$$\begin{pmatrix} \gamma h_x^2 & 4\gamma h_x^2 & \gamma h_x^2 \\ 2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y & 8\zeta h_x^2 h_y^2 & 2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y \\ \alpha h_y^2 & -6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & -24\epsilon h_x^2 h_y & -6(\epsilon h_x^2 h_y + \delta h_x h_y^2) \\ +8(\gamma h_x^2 + \alpha h_y^2) & +32\gamma h_x^2 - 18\alpha h_y^2 & +8(\gamma h_x^2 + \alpha h_y^2) \\ 4\alpha h_y^2 & \frac{8\zeta h_x^2 h_y^2}{+24\delta h_x h_y^2} & 32\zeta h_x^2 h_y^2 & 8\zeta h_x^2 h_y^2 \\ -18\gamma h_x^2 + 32\alpha h_y^2 & -72(\gamma h_x^2 + \alpha h_y^2) & -18\gamma h_x^2 + 32\alpha h_y^2 \\ \alpha h_y^2 & \frac{4\alpha h_y^2 + \frac{1}{4}\beta h_x h_y}{+6(\epsilon h_x^2 h_y + \delta h_x h_y^2)} & \frac{8\zeta h_x^2 h_y^2}{+32\gamma h_x^2 - 18\alpha h_y^2} & \frac{2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y}{+8(\gamma h_x^2 + \alpha h_y^2)} & \frac{4\alpha h_y^2}{+32\gamma h_x^2 - 18\alpha h_y^2} & \frac{4\alpha h_y^2}{+32\gamma h_x^2 - 4\alpha h_y^2} \\ \alpha h_y^2 & \frac{4\gamma h_x^2 + \alpha h_y^2}{+8(\gamma h_x^2 + \alpha h_y^2)} & \frac{4\gamma h_x^2}{+32\gamma h_x^2 - 18\alpha h_y^2} & \frac{\gamma h_x^2}{+8(\gamma h_x^2 + \alpha h_y^2)} & \alpha h_y^2 \\ \gamma h_x^2 & \frac{4\gamma h_x^2}{+32\gamma h_x^2 - 18\alpha h_y^2} & \gamma h_x^2 \end{pmatrix}$$

Stencil S.2 The coefficient stencil of the collocation equations at a point of  $\Omega_i^0$ multiplied by 72  $h_x^2 h_y^2$ .

#### 3.1 Convergence analysis

The convergence of a variation of the first formulation (3.1) and (3.2) of the method is studied in Section 4 for Helmholtz PDEs with Dirichlet or Neumann conditions. We believe that we can establish the optimal order of convergence for general PDEs using the second formulation, the one with two phases or a deffered correction. To do this, we need to prove optimal order of convergence of the standard nodal collocation method in a certain Sobolev space of order two. The numerical results of Table 11 indicates such convergence takes place. This would guarantee that the perturbation term  $\hat{f}$  used as a right side in phase II is  $O(h^4)$ . This result is already established in [Arno 84] for certain classes of problems. Then an analysis of phase II, similar to the one in [Cave 72] and [Ito 72], would lead to the proof of optimal order of convergence. The numerical results in Tables 6 to 12 strongly support the belief that optimal convergence holds for general PDEs. A detailed analysis along these lines is under way.

### 4. CONVERGENCE OF THE SPLINE COLLOCATION METHOD FOR HELMHOLTZ EQUA-TIONS

We first consider the simplifications that occur in the spline collocation method for the case of

Helmholtz equations. We then establish that it has optimal order of convergence. The Helmholtz type elliptic differential equation considered is

$$Lu = \alpha D_x^2 u + \gamma D_y^2 u + \zeta u = -f \quad \text{in} \quad \Omega = [a, b] \times [c, d], \tag{4.1}$$

subject to boundary conditions (1.2), where  $\alpha$ ,  $\gamma$ ,  $\zeta$  and  $f \in C^2[\Omega]$ . Without loss of generality the coefficient functions  $\alpha$  and  $\gamma$  are assumed to be strictly positive, because of the ellipticity condition.

#### 4.1. Dirichlet Boundary Conditions

First we consider the Helmholtz equation (4.1) with Dirichlet boundary conditions. Then the collocation approximation  $u_{\Delta}$  is defined by the equations (3.1) and (3.2) by setting  $\beta = \delta = \varepsilon = 0$ . Since  $Lu = D_x^2 u$  or  $D_y^2 u$  on the boundary sides of  $\Omega$ , one can define  $u_{\Delta}$  to satisfy the same spline end conditions as interpolant S at  $\Omega_B$  without loss of accuracy. Specifically, in the spline collocation method for (4.1) we determine a cubic spline approximation  $u_{\Delta}$  in  $S_{3\Delta}^{(0)}$  to u such that it satisfies :

(i) the interior collocation equations in  $\Omega_f$ 

$$\left[L' u_{\Delta} - (-f)\right]_{(x,y_j)} = 0 \tag{4.2}$$

where L' is defined by S.1 with  $\beta \equiv \delta \equiv \epsilon \equiv 0$ ,

(ii) the boundary collocation equations in  $\Omega_B$ 

 $D_x^2 u_{\Delta} = -f_{i,j} / \alpha_{i,j}, \qquad (4.3a)$ 

$$D_{y}^{2}u_{\Delta} = -f_{i,j}/\gamma_{i,j}, \qquad (4.3b)$$

and

(iii) the corner collocation equations in  $\Omega_C$ 

$$D_{x}^{2}D_{y}^{2}u_{\Delta} = -\bar{f}_{i,j} , \qquad (4.4)$$

where

$$\bar{f} = [\gamma D_x^2 f - 2D_x \gamma D_z f] / \gamma^2$$

---- .

If we adopt the representation (2.3) of  $u_{\Delta}$ , where the  $B_j$ 's are defined by (2.1), then the equation (4.4) can be explicitly solved to obtain

$$U_{0,0} = \mu \bar{f}_{0,0}, \quad U_{0,M} = \mu \bar{f}_{0,M}, \quad U_{N,0} = \mu \bar{f}_{N,0} \text{ and } U_{N,M} = \mu \bar{f}_{N,M}.$$

$$\mu = h_x^2 h_y^2 / 36.$$
(4.5)

with

In order to express the collocation equations (4.3) to (4.4) in a matrix form we consider the  $m \times m$ tridiagonal matrix  $Q_m = \text{trid}(1,4,1)$ . Using this notation the boundary equations (4.3) can be written as

$$Q_{M-1}\mathbf{v}_0^{(1)} = \mathbf{w}_0^{(1)}, \ Q_{M-1}\mathbf{v}_N^{(1)} = \mathbf{w}_N^{(1)}$$
 (4.6a)

and

$$Q_{N-1}\mathbf{v}_0^{(2)} = \mathbf{w}_0^{(2)}, \ Q_{N-1}\mathbf{v}_M^{(2)} = \mathbf{w}_M^{(2)}$$
 (4.6b)

where

$$\mathbf{v}_{0}^{(1)} = \begin{bmatrix} U_{0,1}, U_{0,2}, \dots, U_{0,M-2}, U_{0,M-1} \end{bmatrix}^{T},$$
  

$$\mathbf{v}_{0}^{(2)} = \begin{bmatrix} U_{1,0}, U_{2,0}, \dots, U_{N-2,0}, U_{N-1,0} \end{bmatrix}^{T},$$
  

$$\mathbf{v}_{N}^{(1)} = \begin{bmatrix} U_{N,1}, U_{N,2}, \dots, U_{N,M-2}, U_{N,M-1} \end{bmatrix}^{T},$$
  

$$\mathbf{v}_{M}^{(2)} = \begin{bmatrix} U_{1,M}, U_{2,M}, \dots, U_{N-2,M}, U_{N-1,M} \end{bmatrix}^{T},$$

and the right side  $w_0^{(1)}$  has components

$$h_x^2 f_{0,1} / \alpha_{0,1} - U_{0,0}, \ h_x^2 f_{0,j} / \alpha_{0,j}, \ \text{for} \ 2 \le j \le M-2, \ h_x^2 f_{0,M-1} / \alpha_{0,M-1} - U_{0,M}.$$

The rest of the right sides in (4.6) are defined by similar expressions.

Note that the boundary unknowns  $v_0^{(1)}$ ,  $v_N^{(1)}$ ,  $v_0^{(2)}$ ,  $v_M^{(2)}$ , in (4.6) can be explicitly determined and eliminated from the problem since  $Q_{N-1}$ ,  $Q_{M-1}$  are non-singular. Finally, the interior collocation equations (4.2) can be described in terms of the stencils given in the Appendix I. Figure 4.1 indicates the matrix structure of these equations while the right side of the  $(x_i, y_j)$  equation is

$$-72h_x^2 h_y^2 f(x_i, y_j) - p_{i,j}$$
(4.7)

where the  $p_{i,j}$ 's are zero except near the boundary and there they can be computed by multiplying the

boundary unknowns with the approximate coefficients in the stencils.

DXX	XXX	XX			
XDXX	XXXX	XXX			
XXDXX.	XXXXX.	.xxx			
.XXDXX	.XXXXX	XXX.			
XXDX	XXXX	XXX			
XXD	xxx	xx	. <i>.</i> <b>.</b> .	• - • • • •	••••
xxx	DXX	xxx	xx		
XXXX	XDXX,.	XXXX	xxx		
XXXXX.	XXDXX.	XXXXX.	.xxx	• • • • • •	
, XXXXX	, XXDXX	.xxxxx	xxx.		
XXXX	XXDX	XXXX	xxx		
xxx	XXD	xxx	xx	•••••	• • • • • •
xx	xxx	DXX	xxx	<b>x</b> x	
XXX	XXXX	XDXX	XXXX	<b>XXX</b>	• • • • • •
.XXX	XXXXX.	XXDXX.	XXXXX.	.xxx	
XXX.	. XXXXX	.XXDXX	. XXXXX	xxx.	
XXX	XXXX	XXDX	xxxx	XXX	
xx	xxx	XXD	xxx	xx	•••••
	xx	xxx	DXX	xxx	xx
	XXX	XXXX	XDXX	XXXX	XXX
	.xxx	XXXXX.	XXDXX.	XXXXX.	.xxx
	xxx.	,XXXXX	.XXDXX	.XXXXX	XXX.
	XXX		XXDX	XXXX	XXX
	xx	xxx	xxD	xxx	xx
		xx	xxx	D <b>XX</b>	xxx
		XXX	XXXX	XDXX	XXXX.
		.xxx	XXXXX.	XXDXX.	XXXXX.
		xxx.	XXXXX	.XXDXX	.XXXXX
		xxx	XXXX	XXDX	xxxx
• • • • • •		xx	xxx	XXD	xxx
			xx	xxx	DXX
			XXX.	XXXX.	XDXX.
			XXX	XXXXX	XXDXX
• • • • • • •			YYY	XXXXXX	XXDXX
• • • • • • •			· · · · · · · · · · · · · · · · · · ·	VAVAV.	YTY
•••••			YY		XYD

Figure 4.1 Structure of the "interior" collocation matrix for N = M = 7 where d = diagonal non-zero element, x = off diagonal non-zero element, . = zero element.

#### 4.2 Neumann Boundary Conditions

Next, we consider the spline collocation method for (4.1) with constant coefficients and Neumann

boundary conditions. Specifically, we determine  $u_{\Delta}$  in  $S_{3\Delta}^{(0)}$  such that it satisfies

(i) the interior collocation equations (4.2) at the interior grid points  $\Omega_{I}$ ,

and

(ii) the boundary collocation equations at the points  $\Omega_B \cup \Omega_C$ 

$$D_x^3 \mu_{\Delta} = -D_x f_{ij} / \alpha, \tag{4.8a}$$

$$D_{y}^{3}u_{\Delta} = -D_{y}f_{ij}\gamma, \qquad (4.8b)$$

In order to formulate (4.2) and (4.8) in a matrix form we consider the  $n \times n$  matrix  $T_n$  defined in terms of  $Q_n$  as

$$T_{1,1} = 2q_{1,1}, \quad T_{1,2} = 2q_{1,2}, \quad T_{2,1} = 2q_{2,1},$$
  
$$T_{n,n} = 2q_{n,n}, \quad T_{n-1,n} = 2q_{n-1,n}, \quad T_{n,n-1} = 2q_{n,n-1}$$

and  $T_{i,j} = q_{i,j}$ ,  $2 \le i, j \le n-2$ . Then the boundary equations (4.8) can be written as

$$(1/h_x^3)T_{M+1}\mathbf{v}_0^{(1)} = \mathbf{r}_0^{(1)}, \ (1/h_x^3)T_{M+1}\mathbf{v}_N^{(1)} = \mathbf{r}_N^{(1)},$$
 (4.9a)

and

$$(1/h_y^3)T_{N+1}\mathbf{v}_0^{(2)} = \mathbf{r}_0^{(2)}, \quad (1/h_y^3)T_{N+1}\mathbf{v}_N^{(2)} = \mathbf{r}_M^{(2)}, \quad (4.9b)$$

where the right side  $r_0^{(1)}$  components

$$-D_x f_{0,0}/\alpha_1 - D_x f_{0,j}/\alpha_1$$
 for  $1 \le j \le M$ ,  $-D_x f_{0,m+1}/\alpha_1$ 

The rest of the r's are defined by similar expressions on the other boundary sides. Notice that  $T_{M+1}$  and  $T_{N+1}$  are positive definite. Thus (4.8) can be explicitly solved for the unknowns associated with the boundary and corner grid points in  $\Omega_B \cup \Omega_C$ .

#### 4.3. Convergence Analysis

We now derive a priori error bounds for the error of the cubic spline collocation method applied to the Helmholtz equation (4.1) with Dirichlet or Neumann boundary conditions. If we represent the interpolant S of the solution u, as defined in Section 2, by  $S = \sum_{i=0}^{N} \sum_{j=0}^{M} U_{i,j}^{I} B_{i}(x)B_{j}(y)$  then the following result holds.

Lemma 4.1. If S in  $S_{3,\Delta}^{(0)}$  then for the coefficients  $U_{i,j}^{I}$  and  $U_{i,j}$  of S and  $u_{\Delta}$  that correspond to the grid points  $\Omega_{B} \cup \Omega_{C}$  we have

$$U_{ij}^{I} = U_{ij} + O(h^{4}) \tag{4.10}$$

Proof. In the case of Dirichlet boundary condition notice that both S and  $u_{\Delta}$  satisfy the same uncoupled, uniquely solvable spline end conditions. Thus, in this case we have that  $U_{i,j}^I = U_{i,j}$ . In the case of Neumann conditions we consider side  $x = x_0, x_N$ . From Lemma 2.3 and the definition of  $u_{\Delta}$  we have that  $D_x^3 S_{i,j} = D_x^3 u_{i,j} + O(h_x)$  and  $D_x^3 u_{\Delta} = D_x^3 u_{i,j}$  at  $x = x_0 x_N$ . After subtracting these relations and expressing the resulting relation in a matrix form, we obtain

$$T_{M+1}(\hat{\mathbf{v}}_0^{(1)} - \mathbf{v}_0^{(1)}) = O(h_x^4),$$

where  $\hat{\mathbf{v}}_0$  are the boundary coefficient of  $u_{\Delta}$ . Since  $||T_{M+1}^{-1}|| \le 1$  we conclude that (4.10) holds. We can similarly establish (4.10) for the boundary coefficients associated with the rest of the boundary sides. This concludes the proof of the Lemma.

According to the formulation of the method, the boundary unknowns that correspond to the grid points  $\Omega_B \cup \Omega_C$  are explicitly defined by the systems (4.6) or (4.9) and can be eliminated using these equations. The remaining unknowns  $U_{i,j}$  that correspond to interior grid points  $\Omega_I$  are determined by the interior collocation equations (4.2). The behavior of the coefficient matrix A of these equations has been studied in the Lemma 3.1. It is worth noticing that under the hypotheses of this lemma the collocation approximation  $u_{\Delta}$  for (4.1) is uniquely defined. Next, we prove two important results on which the convergence proof is based. Similar results hold for the standard nodal collocation method applied to the operator L [Cave 72].

Lemma 4.2. Let  $A_0$  be the coefficient matrix that corresponds to the Laplace operator,  $\sigma \equiv \frac{h_x}{h_y}$  and p

 $\equiv \frac{\alpha}{\gamma}$ . If  $\zeta \leq 0$ , then for sufficiently small  $h_x$  and  $h_y$  that they satisfy relations (3.3) the matrices A and  $A_0$  are monotone and

$$||A^{-1}|| \leq ||A_0^{-1}||.$$

**Proof.** The coefficient matrix  $A_0$  corresponds to A with  $\zeta \equiv 0$ . From stencils in Appendices, Lemma 3.1 and the hypotheses of this Lemma, we conclude easily that  $A_0$  and A are irreducibly diagonally dominant

with positive diagonal elements and non-positive off diagonal elements and  $A \ge A_0$ . Thus for sufficiently small h we have that  $0 \le A^{-1} \le A^0$ . This concludes the proof of the Lemma.

In order to derive an explicit bound for  $||A_0^{-1}||$  independent of h we apply a discrete maximum principle argument similar to the one used in [Cave 72].

Lemma 4.3. Let 
$$\sigma = \frac{h_x}{h_y}$$
,  $c = \min_{P \in \Omega_r} \alpha(P)$ , and  $p = \frac{\alpha}{\gamma}$ . Then for sufficiently small  $h_x$ ,  $h_y$ , Dirichlet boundary conditions and  $\sigma^2/p$  restricted in the interval  $(\frac{19}{32}, \frac{32}{18})$  we have  $||A_0^{-1}|| \le (.76 + 2.68 h_x^2)/c$ 

**Proof.** First, for any discrete function U defined in,  $\Omega_I^0$ , we consider the discrete operator  $L_h$  defined by the interior stensil S.2 at any point in  $\Omega_I^0$  and the boundary stencils given in Appendix I for the points in  $\Omega_I^1$ , with  $\beta = \delta = \varepsilon = \zeta = 0$ . From the discrete maximum principle [Varg 62] we conclude, by similar arguments as in [Cave 72], that for any discrete function V such that  $L_h V \ge 0$  on  $\Omega_I^0$  we have that

$$\max_{P \in \Omega_{i}} |V(P)| \leq \max_{P \in \Omega_{i} \cup \Omega_{o}} |V(P)| + \frac{1}{2c} \max_{P \in \Omega_{i}} |L_{h} V(P)|$$
(4.11)

Consider the discrete function W defined in  $\Omega_I \cup \Omega_B$  such that  $L_k W(P) = -1$  for all  $P \in \Omega_I$ , and W(P) = 0 on  $\Omega_B$ . Based on the definition of  $L_k$  for Dirichlet boundary conditions and the discrete equation  $L_k W = -1$  we obtain the relations

$$\begin{split} |w_{1,1}| &\leq \frac{26}{76} ||w|| + \frac{72\sigma^2 h_s^2}{76(\gamma + \alpha \sigma^2)} \\ |w_{1,j}| &\leq \frac{72\gamma + 16\alpha\sigma^2}{72\gamma + 76\alpha \sigma^2} ||w|| + \frac{72\sigma^2 h_s^2}{72\gamma + 76\alpha \sigma^2}, \quad j = 2, \dots, M-2, \\ |w_{i,1}| &\leq \frac{16\gamma + 72\alpha\sigma^2}{76\gamma + 72\alpha \sigma^2} ||w|| + \frac{72\sigma^2 h_s^2}{76\gamma + 72\alpha \sigma^2}, \quad i = 2, \dots, N-2. \end{split}$$

Similar inequalities hold for the rest of the  $w_{i,j}$ 's for (i,j) in  $\Omega_j^1$ . Thus, we conclude that

$$\max_{(i,j) \in \Omega_{i}} |w_{i,j}| \le (.34 + .35h_{x}^{2})/c \tag{4.12}$$

Application of inequality (4.11) for w and relation (4.12) gives  $||w|| \le (.76 + 2.68 h_x^2)/c$ . From the definiton of w we have that  $A_0 = \xi$  with  $\xi = (-1, -1, ..., -1)$ . Therefore we conclude that

$$||A_0^{-1}|| \le (.76+2.68h_x^2)/c$$

This concludes the proof of the Lemma.

Lemma 4.4. For sufficiently small  $h_x$ ,  $h_y$ , Neumann boundary conditions and  $\sigma^2/p$  restricted in the interval  $(\frac{18}{32}, \frac{32}{17})$  we have  $||A_0^{-1}|| \le (.85 + .61 h_x^2)/c$ 

**Proof.** Following the analysis of the proof of Lemma 4.3 and the definition of  $L_h$  for Neumann conditions we have

$$\max_{\{i,j\} \in \Omega_i^1} |w_{i,j}| \le (.41 + .36h_x^2)/c .$$

Using inequality (4.11) and the above relation we obtain (.85 + .61  $h_x^2$ )/c as the upper bound for  $||A_0^{-1}||$  for Neumann boundary conditions. This concludes the proof of the lemma.

In the case of the standard nodal cubic spline collocation method the following result holds for Helmholtz equations with variable coefficients for the Dirichlet case and with constant coefficients for the Neumman case.

Lemma 4.5. For the standard nodal cubic spline collocation method, sufficiently small  $h_x$ ,  $h_y$ , and  $\sigma^2/p$  restricted in the interval  $(\frac{1}{2}, 2)$  we have  $||A_0^{-1}|| \le \frac{1}{3c}$ .

Based on the above results, we can now establish the optimal convergence of the spline collocation method for Helmholtz equations with Dirichlet or Neumann boundary conditions.

**Theorem 4.4.** Let  $u_{\Delta}$  be the spline collocation approximation of u for (4.1). Then we have the error bounds

$$\|D_{x}^{k}D_{y}^{l}(u-u_{\Delta})\|_{u} \leq c_{k,l}h^{4-(k+l)}$$
(4.13)

where  $c_{k,l}$  is independent of h.

**Proof.** From Theorem 2.1 and the definition of  $u_{\Delta}$  at the interior grid points  $\Omega_{I}$ , we have

$$L'(S - \mu_{\Delta})|_{(x,y)} = O(h^4).$$
(4.14)

Expressing (4.14) in matrix form we obtain

$$A[U_{i,j}^{I} - U_{i,j}] = O(h^{4}).$$
(4.15)

Lemma 4.1 implies that  $U_{i,j}^I - U_{i,j} = O(h^4)$  for  $(x_i, y_j)$  in  $U_B \cup U_c$  and from (4.15), we conclude that  $||U^I - U|| = O(h^4)$  since  $||A^{-1}||$  is bounded independently of h. This relation and the definition of the basis functions  $B_i$  of  $S_{3,0}^{(0)}$  imply that

$$\|D_{x}^{k}D_{y}^{l}(S-u_{\Delta})\|_{\infty} \leq c_{k,l}h^{4-(k+l)}.$$
(4.16)

The error bound (4.13) is then a direct consequence of (4.16) and (2.12c).

#### 5. NUMERICAL RESULTS

In this section we present some numerical results to confirm the convergence properties of the cubic spline collocation method. For this purpose, we have selected five problems with known solutions which are solved then for various uniform meshes. We used the first formulation for the data summarized in Tables 1 to 4, which indicate that the rate of convergence of the method is 3.9. This rate should be compared with the optimal fourth order convergence in the approximation with bicubic-splines. It is worth noticing that the coefficient of u in problem 3 is positive while the mesh ratio  $\sigma^2$  for Problem 4 is outside the interval  $(\frac{18}{32}, \frac{32}{18})$ . This indicates that the conditions we give under which the coefficient matrix of the method is invertable are only sufficient. We used the second formulation (phase version) for the data in Tables 5 to 12. The other methods used for these tables are from the ELLPACK system and described in [Rice 85]. The order is estimated at the grid points by

$$order = -\log \frac{||(u - u_{h_1})||_{max}}{||(u - u_{h_2})||_{max}} / \log(h_1 / h_2)$$

All computations were performed on a VAX 780 in double precision except those in Tables 5 and 6 that were done in single precision. In these experiments the systems of linear equations are solved by Gauss elimination using the LINPACK routines SGEFA and SGESL. Unlike general collocation based on Hermite bicubics [Hous 86], the cubic spline collocation equations can be solved by various iterative methods [Hous 84], [Vava 85] and their direct solution does not require pivoting. A systematic performance evaluation of the cubic spline collocation method is under way.

Grid	$\max_{\substack{0 \leq i, j \leq N, M}}  (u - u_{\Delta})(x_i, y_j) $	order of convergence
5×5 9×9 13×13 17×17 21×21	1.36 e-03 8.43 e-05 1.74 e-05 5.75 e-06 2.42 e-06	3.99 3.81 3.84 3.87
25×25 33×33	1.17 e-06 3.26 4-07	3.98 3.89

Table 1. The convergence of cubic-spline collocation method for a Poisson equation  $D_x^2 u + D_y^2 u = -f$ with Dirichlet boundary conditions (u=0) on the unit square. The function f is selected so that  $u = 3e^{x+y}(x^2-x)(y^2-y)$ .

Griđ	$\max_{\substack{0 \leq i,j \leq N,M}}  (u - u_{\Delta})(x_i, y_j) $	order of convergence	total time
6×6 8×8 11×11 17×17 21×21	7.38 e-02 1.56 e-02 4.11 e-03 6.70 e-04 2.81 e-05	4.61 3.74 3.86 3.89	.18 .40 .92 5.01 11.32

Table 2. The convergence of cubic-spline collocation method for a Helmholtz equation  $D_x^2 u + D_y^2 u - [100+\sin(3\pi y)+\cos(2\pi x)]u = -f$  with Dirichlet boundary conditions on the unit square. The function f is selected such that  $u=-.31[5.4-C(x)]S(x)(y^2-y)[5.4-C(y)][1+T(x,y)^{-1}-.5]$  where  $C(z) = \cos(4\pi z)$ ,  $S(x) = \sin(\pi x)$  and  $T(x,y) = 4(x-0.5)^2 + 4(y-0.5)^2$ .

Grid	$\max_{\substack{0 \leq i,j \leq N, \mathcal{H}}}  (u - u_{\Delta})(x_i, y_j) $	order of convergence
6×6 9×9 17×17 21×21	5.72 e-03 8.29 e-04 5.89 e-05 2.40 e-05	4.11 3.81 4.01

Table 3. The convergence of cubic-spline collocation method for a Helmholtz equation  $D_x^2 u + D_y^2 u + u = -f$  with Dirichlet boundary conditions (u=0) on the unit square. The function f is selected such that  $u = 3e^{x+y}(x^2-x)(y^2-y)$ .

	SPLI	VE COLLOCA	TION	SPI SPI	INE GALERI	KIN	HERMI	TE COLLOCA	ATION
GRID	Total time	Number of equations	band- width	total time	number of equations	band- width	total time	number of equations	band- width
4x4	.23	16	5	.73	36	21	.3	36	11
8×8	.77	64	9	3.48	100	33	3.92	196	1 <b>9</b>
1 <b>2×12</b>	2.1	144	13	9.21	196	45	16.32	484	27
16×16	4.9	256	17	19.91	324	57	46.48	900	35
20×20	8.9	400	21	34.40	484	69	105.00	1444	43 .
_ 24×24_	15.0	576	25	60.29	676	81	209.5	2116	51

 

 Table 4. Performance results are obtained by applying cubic spline collocation, cubic spline Galerkin and Hermite bicubic collocation to the PDE problem considered in Table 2. The first column in each method indicates the total time in seconds for discretization and solution.

. \_\_ . \_

	SPLINE C	OLLOCATION	SPLINE C	ALERKIN	HERMITI	E COLLOCATION
GRID	епог	order	error	order	error	order
4×4	3.22 e-1		2.13 e-1		6.51 e-2	
8×8	3.57 e-2	2.60	1.57 e-2	3.73	6.22 c-3	2.77
12×12	6.32 e-3	3.83	2.91 e-3	3.73	9.59 c-4	4.13
16×16	1.63 e-3	4.37	7.02 e-4	4.58	2.90 e-4	3.86
20×20	6.69 e-4	3.77	2.45 e-4	4.45	1.16 e-4	3.86
24×24	3.49 e-4	3.41	1.00 e-4	4.69	5.44 e-5	3.96

 Table 5. Errors and order of convergence of cubic spline collocation, cubic spline Galerkin and Hermite bicubic collocation applied to the PDE problem considered in Table 2.

.

.

	SPLINE COLLOCATION		SPLIN	TE GALE	RKIN	HODIE			
GRID	error	order	total time	error	order	total time	епог	order	total time
4×4	4.007 e-3		0.25	3.287 e-3		0.97	4.230 e-3		0.13
8×8	2.086 c-4	3.49	1.03	1.347 e-4	3.77	4.23	1.690 e-4	3.80	0.63
12×12	4.072 €-5	3.61	2.81	1.970 e-5	4.25	11.27	2.810 e-5	3.97	1.95
16×16	1,220 e-5	3.89	6.31	5.707 e-6	4.00	24.42	8.320 e-6	3.92	4.35
20×20	4.892 e-6	3.87	12.11	2.213 e-6	4.01	47.35	3.260 e-6	3.96	8.63

Table 6. Errors, order of convergence and total time (in seconds) for discretization and solution for the cubic spline collocation, cubic spline Galerkin and HODIE methods applied to the self adjoint elliptic operator  $D_x(e^{xy}D_xu) + D_y(e^{-xy}D_yu) - u/(1+x+y)=f(x,y)$  with Dirichlet boundary conditions (u=0) on the unit square. The function f is selected such that  $u = 0.75e^{xy} \sin(\pi x)\sin(\pi y)$ .

	SPLINE C	COLLOCATION	HODIE H	ELMHOLTZ	HERMITE	COLLOCATION	SPLINE GALERKIN	
GRID	error	order	епог	order	егтог	order	сгтог	order
5×5	5.89 e-4		5.55 e-4		3.18 e-4		7.28 e-4	
7×7	1.08 c-4	4.18	9.68 e-5	4.31	6.08 e-5	4.08	1.60 e-4	3.80
9×9	4.20 e-5	3.29	3.10 e-5	3.96	1.97 e-5	3.91	5.47 e-5	3.64
11×11	1.88 e-5	3.59	1.26 e-5	4.02	8.24 e-6	3.90	2.39 c-5	3.72
13×13	9.52 e-6	3.74	5.97 e-6	4.11	3.92 e-6	4.08	1.16 e-5	3.98
15×15	5.28 e-6	3.82	3.27 e-6	3.91	2.16 e-6	3.87	6.08 e-6	4.20
17×17	3.15 e-6	3.87	1.88 e-6	4.14	1,24 e-6	4.12	3.47 e-6	4.21
19×19	1.99 e-6	3.90	1.19 e-6	3.88	7.89 e-7	3.86	2.17 e-6	3.98

Table 7. Errors and order of convergence of cubic spline collocation, HODIE Helmholtz, Hermite bicubiccollocation and cubic spline Galerkin methods applied to the PDE problem considered in Table2.

	SPL.	INE-CO	LLOCATION	1	SPLINE-GALERKIN			
	a = 11	12	a = 9/2		a = 11/2		a = 9/2	
GRID	error	order	error	order	ertor	order	error	order
4×4	1.425 e-4		1.326 e-4		1.104 e-4		1.136 3-4	
8×8	5.808 e-6	3.78	5.276 e-6	3.81	8.872 e-6	2.98	4.650 e-6	3.77
12×12	1.040 e-6	3.81	8.366 e-7	4.07	1.620 e-6	3.76	8.505 e-7	3.76
16×16	3.320 e-7	3.68	2.422 e-7	4.00	4.970 e-7	3.81	2.284 e-7	4.24
20×20	1.354 e-7	3.80	9.4474 -8	3.97	1.868 e-7	4.11	8.994 e-8	3.94

**Table 8.** Errors and order of convergence for the spline collocation and spline Galerkin methods applied to the Laplace equation  $D_x^2 u + D_y^2 u = f$  with Dirichlet boundary conditions on the unit square for various values of a. The function f is selected such that  $u = (x^a - x^{a-1})(y^a - y^{a-1})$ .

	SPLINE CO	OLLOCATION	HERMITI	E COLLOCATION
GRID	error	total time	епог	total time
7×5	4.410 e-4	0.50	2.63 e-4	3.65
11×5	3.340 e-4	0.80	2.67 e-4	6.64
11×9	4.271 e-5	1.55	1.72 e-5	23.40
13×11	1.847 e-5	2.52	7.33 e-6	45.37
21×11	1.282 e-5	4.27	7.06 c-6	71.94
19×17	3.222 e-5	7.98	1.13 e-6	187.35

Table 9. Errors and total time (in seconds) for discretization and solution of cubic spline collocation and Hermite bicubic collocation applied to the general elliptic operator  $(x+y+1)D_x^2u + e^{x-y}D_y^2u$  $+ (x-1)(y+1)D_{xy}u + (x+1)D_xu + (y-1)D_yu + (xy+1)u$  such that  $u_y = 0$  on the segment ( $y = 0, 0 \le x \le .5$ ),  $u_x = 0$  on the segment ( $x = 0, 0 \le y \le .5$ ) and u = 0 on the rest of the boundary. The problem is on the unit square and has a non-uniform mesh. The function f is selected such that  $u = 3e^{x+y}(x^3-x^2)(y^3-y^2)$ . The cubic spline Galerkin and HODIE methods are not applicable to this problem.

	SPLINE	SPLINE COLLOCATION					
GRID	error	order	uotal time				
4×4	2.032 e-3		0.22				
8×8	9.500 e-5	3.62	1.03				
12×12	1.750 e-5	3.78	2.78				
16×16	5.090 e-6	3.96	6.08				
20×20	2.013 e-6	3.95	12.01				

 Table 10. Errors, order of convergence and total time of cubic spline collocation for the problem considered in Table 9, using a uniform mesh.

	U <sub>2</sub> –	S <sub>22</sub> ]]	U <sub>yy</sub> –	$S_{yy}$ []	U <sub>xy</sub> -	$S_{xy}[]$	<i>U<sub>x</sub></i> –	$S_{\mathbf{x}}[ $	<i>U</i> , –	S <sub>y</sub>
GRID	ептог	order	error	order	error	order	error	order	error	order
4×4	1.19 e-1		1.17 e-1		4.26 e-1		1.67 e-1		1.70 e-1	
8×8	3.80 e-2	1.91	2.91 e-2	1.65	1.67 e-1	1.15	4.62 e-2	1.52	4.61 e-2	1.54
12×12	1.61 e-2	1.90	1.16 e-2	2.03	9.54 e-2	1.35	2.20 e-2	1.64	2,22 e-2	1.62
16×16	8.95 e-3	1.90	6.17 e-3	2.05	5.95 e-2	1.52	1.26 e-2	1.79	1.30 e-2	1 <b>.73</b>
20×20	5.70 e-3	1.91	3.79 e-3	2.06	4.03 e-2	1.65	8.17 e-3	1.86	8.41 e-3	1.84

Table 11. Derivative errors and order of convergence for the problem considered in Table 6.

The order of convergence results in Tables 5-8 confirm the conclusion from Tables 1-3 that the order of convergence of the spline collocation method is close to 4.0. Table 10 gives data from a completely general operator with combined Dirichlet and Neumann boundary conditions. Again the order of convergence is close to 4.0 which suggests that the optimal order of convergence of the method is 4.0 in general. Table 11 gives results on the order of convergence of derivatives of the standard  $O(h^2)$  spline collocation method. Both first and second derivatives exhibit second order convergence. This behavior agrees with the one observed by [Arno 84].

Table 4 gives a comparison of the problem size and computational times for three finite element methods using piecewise cubic polynomials. Using Hermite cubics (the Hermite collocation method) for a given grid size, gives many more unknowns much more computational effort and, as seen in Table 5, higher accuracy. However, accuracy is not improved enough to compensate for the additional computer time. To achieve an accuracy of  $10^{-3}$  or better requires 8.9 seconds for spline collocation, 19.9 seconds for spline Galerkin and 16.3 seconds for Hermite collocation. Table 6 shows another example where spline collocation is more efficient than spline Galerkin and Table 9 shows another example where spline collocation is more efficient than Hermite collocation.

Table 6 gives a comparison with high order finite difference (HODIE) method which is of comparable accuracy with a given grid size and a little more efficient. However, the HODIE method is not as general in its applicability as spline collocation (it cannot handle  $u_{xy}$  terms). The HODIE Helmholtz method used in Table 7 is even faster, but this method is specifically designed for and restricted to Helmholtz problems with only a variable coefficient of u in the operator.

#### REFERENCES

- [Ahlb 73] J. H. Alhberg and T. Ito, A Collocation Method for Two-Point Boundary Value Problems, Math. Comp., 129-131, (1975), 761-776.
- [Arch 73] D. A. Archer, Some Collocation Methods for Differential Equations, Rice University, Houston, TX, 1973, Ph.D. Thesis.
- [Arch 77] D. A. Archer, An O (h<sup>4</sup>) Cubic Spline Collocation Method for Quasilinear Parabolic Equations, SIAM J. Numer. Anal., 14, (1977), 620–637.
- [Arno 84] D. N. Arnold and J. Saraven, On the Asymptotic Convergence of Spline Collocation Methods for Partial Differential Equations, SIAM J. Num. Anal., 21, (1984), 459–472.
- [Carl 72] R. E. Carlson and C. A. Hall, Error Bounds for Bicubic Spline Interpolation, J. Approx. Theory, (1972), 71-88.
- [Carl 73] R. E. Carlson and C. A. Hall, Bicubic Spline Interpolation in Rectangular Polygons, J. Approx. Theory, (1973), 366-377.
- [Cave 72] J. C. Cavendish, A Collocation Method for Elliptic and Parabolic Boundary Value Problems Using Cubic Splines, Univ. of Pittsburgh, Pittsburgh, PA., 1972, Ph.D. Thesis.

[DeBo 78] C. de Boor, Practical Guide to Splines, Springer-Verlag, New York, 1978.

- [Dani 75] J. W. Daniel and B. K. Swartz, Extrapolated Collocation for Two Point Boundary Value Problems Using Splines, J. Inst. Maths. Applics. 16, (1975), 161–174.
- [Fyfe 69] J. R. Fyfe, The use of cubic splines in the solution of two point boundary value problems, Computer Journal, 12 (1969), 188-192.
- [Hall 68] C. A. Hall, On Error Bounds for Spline Interpolation, J. Approx. Theory, (1968), 209-218.
- [Hous 84] E. N. Houstis, J. R. Rice and E. A. Vavalis, Spline-Collocation Methods for Elliptic Partial Differential Equations, in: Advances in Comp. Methods for PDE's, Vol. V, 1984, R. Vichnevetsky, R. S. Stepleman (Editors).
- [Hous 85] E. N. Houstis, J. R. Mitchell and J.R. Rice, Collocation Software for Second-Order Elliptic Partial Differential Equations, ACM Trans. Softwr., 11, No. 4 (1985), 379-412.
- [Luca 72] T. R. Lucas and G. W. Reddien, Jr., Some Collocation Methods for Non-linear Boundary Value Problems, SIAM J. Num. Anal., 9, (1972), 341-356.
- [Henr 68] P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, John Wiley, New York, 1962.
- [Ito 72] T. Ito, A Collocation Method for Boundary Value Problems Using Spline Functions, Brown University, Providence, R.I., 1972, Ph.D. Thesis.
- [Luca 74] B. K. Lucas, Error Bounds for Interpolating Cubic Splines Under Various End Conditions, SIAM J. Num. Anal., 11, (1974), 569-584.
- [Pren 75] P. M. Prenter, Spline and Variational Methods, John Wiley, New York, 1975.
- [Rice 85] J. R. Rice and R. F. Boisvert, Solving Elliptic Problems Using Ellipack, Springer-Verlag, New York, 1985.
- [Rome 79] V. P. Romenski, The Method of Spline-Collocation for the Poisson Equation, Akademia Nauk SSSR, 81, (1979), 81-86.
- [Rubi 74] S. G. Rubin and R. A. Graves, Viscous Flow Solution With a Cubic Spline Approximation, Computers and Fluids, 3, (1974), 1-36.

[Varg 62] R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ. 1962.

[Vava 85] E. A. Vavalis, High Order Spline Collocation Methods for Elliptic Partial Differential Equations, University of Thessaloniki, Thessaloniki, Greece, 1985, Ph.D. Thesis.

#### APPENDICES

In this appendix, we present stencils for the spline collocation method after the elimination of boundary unknowns. This is done both for the Dirichlet and Neumann boundary conditions. The equations are presented in the form of stencils to be used at each grid point which involves a boundary unknown. Notice that each grid point  $(x_i, y_i)$  is associated with the unknown  $U_{i,j}$ . The value of each entry is the coefficient of the corresponding unknown. In all stencils, the coefficients of the PDE are evaluated at the indicated collocation point. All stencils have been multiplied by the factor 72  $h_x^2 h_y^2$ . APPENDIX I : Spline Collocation Stencils for the Dirchlet problem.

$$at \ point \\ (x_1, y_1) : \begin{pmatrix} \gamma h_x^2 & 4\gamma h_x^2 & \gamma h_x^2 \\ 2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y & 8\zeta h_x^2 h_y^2 & 2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y \\ -6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & -24\epsilon h_x^2 h_y & -6(\epsilon h_x^2 h_y + \delta h_x h_y^2) & \alpha h_y^2 \\ +8\gamma h_x^2 + 4\alpha h_y^2 & +32\gamma h_x^2 - 19\alpha h_y^2 & +8(\gamma h_x^2 + \alpha h_y^2) \\ +8\gamma h_x^2 + 4\alpha h_y^2 & 32\zeta h_x^2 h_y^2 & -24\delta h_x h_y^2 & 4\alpha h_y^2 \\ +24\delta h_x h_y^2 & -76(\gamma h_x^2 + \alpha h_y^2) & -19\gamma h_x^2 + 32\alpha h_y^2 \\ -19\gamma h_x^2 + 16\alpha h_y^2 & -76(\gamma h_x^2 + \alpha h_y^2) & -19\gamma h_x^2 + 32\alpha h_y^2 \\ +6(\epsilon h_x^2 h_y + \delta h_x h_y^2) & +24\epsilon h_x^2 h_y & +6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & \alpha h_y^2 \\ +4(\gamma h_x^2 + \alpha h_y^2) & +16\gamma h_x^2 - 19\alpha h_y^2 & +4\gamma h_x^2 + 8\alpha h_y^2 \end{pmatrix}$$

$$at \ points \\ (x_1, y_j) \\ j = 2, ..., M-2: \begin{pmatrix} \gamma h_x^2 & 4\gamma h_x^2 & \gamma h_x^2 \\ 2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y & 8\zeta h_x^2 h_y^2 & 2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y \\ -6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & -24\epsilon h_x^2 h_y & -6(\epsilon h_x^2 h_y + \delta h_x h_y^2) & \alpha h_y^2 \\ +8\gamma h_x^2 + 4\alpha h_y^2 & +32\gamma h_x^2 - 19\alpha h_y^2 & +8(\gamma h_x^2 + \alpha h_y^2) \\ & \frac{8\zeta h_x^2 h_y^2}{+24\delta h_x h_y^2} & \frac{32\zeta h_x^2 h_y^2}{-72\gamma h_x^2 - 76\alpha h_y^2} & \frac{8\zeta h_x^2 h_y^2}{-18\gamma h_x^2 + 32\alpha h_y^2} & 4\alpha h_y^2 \\ & \frac{2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y}{+6(\epsilon h_x^2 h_y + \delta h_x h_y^2)} & \frac{8\zeta h_x^2 h_y^2}{+32\gamma h_x^2 - 19\alpha h_y^2} & \frac{2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y}{+6(\epsilon h_x^2 h_y + \delta h_x h_y^2)} & \frac{4\gamma h_x^2}{+32\gamma h_x^2 - 19\alpha h_y^2} & \frac{\gamma h_x^2}{+8(\gamma h_x^2 + \alpha h_y^2)} & \alpha h_y^2 \end{pmatrix}$$

APPENDIX II : Spline Collocation Stencils for the Neumann problem.

.

$$at \ point \\ (x_1, y_1) : \begin{pmatrix} \gamma h_x^2 & 4\gamma h_x^2 & \gamma h_x^2 \\ 2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y & 8\zeta h_x^2 h_y^2 & 2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y \\ -6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & -24\epsilon h_x^2 h_y & -6(\epsilon h_x^2 h_y + \delta h_x h_y^2) & \alpha h_y^2 \\ +8(\gamma h_x^2 + \alpha h_y^2) & +32\gamma h_x^2 - 17\alpha h_y^2 & +8(\gamma h_x^2 + \alpha h_y^2) \\ & 8\zeta h_x^2 h_y^2 & 32\zeta h_x^2 h_y^2 & -24\delta h_x h_y^2 & 4\alpha h_y^2 \\ +\delta h_x h_y^2 & -68(\gamma h_x^2 + \alpha h_y^2) & -17\gamma h_x^2 + 32\alpha h_y^2 & 4\alpha h_y^2 \\ & 2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y & 8\zeta h_x^2 h_y^2 & -17\gamma h_x^2 + 32\alpha h_y^2 \\ +6(\epsilon h_x^2 h_y + \delta h_x h_y^2) & +24\epsilon h_x^2 h_y & +6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & \alpha h_y^2 \end{pmatrix}$$

$$at \ points \\ (x_1, y_j) \\ j = 2, \dots, M-2 : \begin{pmatrix} \gamma h_x^2 & 4\gamma h_x^2 & \gamma h_x^2 \\ 2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y & 8\zeta h_x^2 h_y^2 & 2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y \\ -6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & -24\epsilon h_x^2 h_y & -6(\epsilon h_x^2 h_y + \delta h_x h_y^2) & \alpha h_y^2 \\ +8(\gamma h_x^2 + \alpha h_y^2) & +32\gamma h_x^2 - 17\alpha h_y^2 & +8(\gamma h_x^2 + \alpha h_y^2) \\ & \frac{8\zeta h_x^2 h_y^2}{+\delta h_x h_y^2} & \frac{32\zeta h_x^2 h_y^2}{-124\delta h_x h_y^2} & \frac{-24\delta h_x h_y^2}{-18\gamma h_x^2 + 32\alpha h_y^2} & 4\alpha h_y^2 \\ & \frac{2\zeta h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y}{+6(\epsilon h_x^2 h_y + \delta h_x h_y^2)} & \frac{8\zeta h_x^2 h_y^2}{-18\gamma h_x^2 - 17\alpha h_y^2} & \frac{2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y}{+6(\epsilon h_x^2 h_y + \delta h_x h_y^2)} & \frac{4\gamma h_x^2}{+32\gamma h_x^2 - 17\alpha h_y^2} & \frac{2\zeta h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y}{+8(\gamma h_x^2 + \alpha h_y^2)} & \alpha h_y^2 \\ & \gamma h_x^2 & 4\gamma h_x^2 & \gamma h_x^2 \end{pmatrix}$$

$$at \ point \\ (x_{1}, y_{M-1}): \begin{pmatrix} 2\varsigma h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y & 8\varsigma h_x^2 h_y^2 & 2\varsigma h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y \\ -6(\epsilon h_x^2 h_y - \delta h_x h_y^2) & -24\epsilon h_x^2 h_y & -6(\epsilon h_x^2 h_y + \delta h_x h_y^2) \\ +8(\gamma h_x^2 + \alpha h_y^2) & +32\gamma h_x^2 - 17\alpha h_y^2 & +8(\gamma h_x^2 + \alpha h_y^2) \\ & \frac{8\varsigma h_x^2 h_y^2}{+\delta h_x h_y^2} & \frac{32\varsigma h_x^2 h_y^2}{-17\gamma h_x^2 + 32\alpha h_y^2} & \frac{8\varsigma h_x^2 h_y^2}{-17\gamma h_x^2 + 32\alpha h_y^2} & 4\alpha h_y^2 \\ & \frac{2\varsigma h_x^2 h_y^2 + \frac{1}{4}\beta h_x h_y}{+6(\epsilon h_x^2 h_y + \delta h_x h_y^2)} & \frac{8\varsigma h_x^2 h_y^2}{-17\gamma h_x^2 + 32\gamma h_x^2 - 17\alpha h_y^2} & \frac{2\varsigma h_x^2 h_y^2 - \frac{1}{4}\beta h_x h_y}{+8(\gamma h_x^2 + \alpha h_y^2)} & \alpha h_y^2 \\ & \frac{\gamma h_x^2}{2} & \frac{4\gamma h_x^2}{2} & \frac{\gamma h_x^2}{2} \end{pmatrix}$$