Commun. math. Phys. 53, 155—184 (1977)

Communications in
Mathematical
Physics

Convergence of Operator Product Expansions on the Vacuum in Conformal Invariant Quantum Field Theory

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Abstract. In a conformal invariant quantum field theory (in 4 space time dimensions) Wilson operator product expansions converge on the vacuum, because they are closely related to conformal partial wave expansions.

1. Introduction

Let $\phi^i(x)$, $\phi^j(y)$ two local quantum fields. According to Wilson [1], their product should admit an asymptotic expansion at short distances of the form

$$\phi^i(\tfrac{1}{2}x)\phi^j(-\tfrac{1}{2}x)\Omega = \sum_k C^{ijk}(x)\phi^k(0)\Omega\,. \tag{1.1a}$$

Herein ϕ^k are local fields, and $C^{ijk}(x)$ are singular c-number functions. In a scale invariant theory they are homogeneous functions of x. The expansion is presumably valid for all states Ω in the field theoretic domain $\mathscr{D}$ which is created out of the vacuum by polynomials in smeared field operators. We shall however only consider the special case

$$\Omega = \text{vacuum}\,. \tag{1.1b}$$

Studies in perturbation theory [2] indicate that expansion (1.1) is then valid as an asymptotic expansion to arbitrary accuracy for matrix elements $(\Psi, \phi^i(x)\phi^j(y)\Omega)$, Ψ in $\mathscr{D}$. This means that the error in a truncated expansion can be made smaller than any given power of x at sufficiently small distances $\|x\|$ by taking into account sufficiently many terms. (For more precise formulation cp. e.g. Appendix A of [3].)

Asymptotic expansions need not converge. For instance the asymptotic expansion near $y=0$ of the function $f(y)=\exp(-1/y)$ of one positive real variable y in powers of y vanishes identically and does therefore not converge to the function f.

Among the fields ϕ^k there are derivatives of other local fields. In general there appears $\partial^\mu\phi$ etc. together with any nonderivative field ϕ. In a conformal invariant theory, non-derivative fields ϕ can be recognized by their conformal transformation law [4], viz. $[\phi(0), K^\mu]=0$, K^μ = generators of special conformal transformations.

From the work of Ferrara et al. one knows [4] that conformal symmetry imposes strong restrictions on the coefficients C^{ijk} in (1.1): The terms involving nonderivative fields determine all the others. Using this, the terms involving derivatives of one and the same nonderivative local field can be formally summed. Here we will prove more:

Theorem 1. *Consider conformal invariant quantum field theory (in four space time dimensions) and suppose that vacuum expansions* (1.1) *are valid as asymptotic expansions in homogeneous functions of* x *to arbitrary accuracy for* $(\Psi, \phi^i(\frac{1}{2}x)\phi^j(-\frac{1}{2}x)\Omega)$, Ψ *in* $\mathscr{D}$. *Then* $\phi^i\phi^j\Omega$ *admits a convergent expansion,*

$$\phi^i(x)\phi^j(y)\Omega = \sum_k{}' \int dz \phi^k(z)\Omega\mathscr{B}^{kij}(z;xy). \tag{1.2}$$

$\mathscr{B}^{kij}$ *are generalized c-number functions. Summation is over nonderivative fields* ϕ^k *only and integration is over Minkowski space. Convergence is strong convergence in Hilbert space after smearing with test functions* $f(xy)$.

The result is valid for non-derivative fields ϕ^i, ϕ^j of any dimensions d_i, d_j transforming according to arbitrary finite dimensional irreducible representations l_i, l_j of the Lorentz group $M \simeq \mathrm{SL}(2\mathbb{C})$. Multispinor-indices have been suppressed.

The functions $\mathscr{B}^{kij}$ are to a large extent determined by conformal symmetry. Let $U \approx \mathrm{SU}(2)$ the rotation subgroup of M and denote by $M^{\vee}$, $U^{\wedge}$ the sets of all finite dimensional irreducible representations of M resp. U. Write $\chi^i = [l_i, d_i]$ etc. We shall show that functions $\mathscr{B}^{ijk}$ are linear combinations of a finite number of kinematically determined kernels $\mathscr{B}^{ls}(z\chi_k; x\chi_i y\chi_j)$. Given χ_i, χ_j, and χ_k they are labelled by

$$l \in M^{\vee}, s \in U^{\wedge} \quad \text{such that} \quad s \subset l \quad \text{and} \quad l \subset l_i \otimes l_j, s \subset l_k. \tag{1.3}$$

$\otimes$ stands for the Kronecker product; and $\subset$ means "is contained in". If no pair (s, l) satisfying (1.3) exists, then ϕ^k cannot appear[1] in the operator product expansion of $\phi^i\phi^j$.

Example. ϕ^i, ϕ^j scalar. Then $l_i = l_j = \mathrm{id}$, the trivial 1-dimensional representation. So $l = \mathrm{id}$, $s = \mathrm{id}$ and l_k must be a completely symmetric tensor representation; $\mathscr{B}^{kij}$ is then unique up to normalization.

The proof of the theorem has two ingredients:

1. The Hilbert space of physical states carries a unitary representation U of the conformal group G^* = universal covering of SO(4, 2). It was shown by Lüscher and the author that this is true even if one only assumes weak conformal invariance, i.e. invariance of Euclidean Green functions under $\mathrm{SO}_e(5, 1)$ or its 2-fold spin covering [5].

2. All unitary irreducible representations of G^* with positive energy are finite component field representations in the terminology of [6]. This result was proven by the author in [7].

Using these facts one can derive partial wave expansions on G^*, i.e. decompose $\int dxdy f(xy)\phi^i(x)\phi^j(y)\Omega$ into states which transform irreducibly. Because of the

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[1] For massless free fields $\phi^k(x)$ there are further restrictions beyond this, cp. end of Section 7

Plancherel theorem, partial wave expansions are strongly convergent. They are here at the same time asymptotic expansions. Comparing with (1.1a) one finds that they can be rewritten in the form (1.2).

An independent proof of the theorem for theories in 2 space time dimensions was given by Lüscher [8]. He uses different methods employing a semigroup. Interesting further results on 2-dimensional models were obtained by Rühl and Yunn [9].

We conjectured in [10] *that the assertion of Theorem 1 would also hold true in realistic theories with mass and without conformal symmetry.*

Let us mention that one can also give a dynamical derivation of the vacuum expansions (1.1) themselves in conformal invariant quantum field theory (QFT). This is discussed elsewhere [11]. It is not, however, a derivation from QFT axioms and conformal symmetry alone: One also needs Lagrangean integral equations to identify composite fields, and meromorphy of Euclidean conformal partial waves in dimension must be assumed to get a discrete expansion in the first place.

Finally, the following corollaries of Theorem 1 may be of interest.

Let P^μ, K^μ the generators of translations and special conformal transformations, respectively, and

$$H=\tfrac{1}{2}(P^0+K^0) \quad \text{the “conformal Hamiltonian”}.$$

Assume that the hypothesis of Theorem 1 hold for arbitrary products of fields ϕ^i, ϕ^j. Let f test functions and $\phi^k(f)=\int dx f(x)\phi^k(x)$ smeared fields.

Then we have

Corollary 2. *The Hilbert space $\mathscr{H}$ of physical states is spanned by states of the form $\phi^k(f)\Omega$, $\phi^k(f)$ smeared fields, $\Omega=$vacuum.*

Corollary 3. *The conformal Hamiltonian H has a purely discrete spectrum with eigenvalues $\omega=0$ (vacuum) and*

$$\omega=d_k+m, \qquad m=0,1,2,\ldots,$$

d_k dimensions of nonderivative fields in the theory.

Corollary 2 is obtained by recalling that finite products $\phi^{i_1}(f_1)\ldots\phi^{i_N}(f_N)$ of fields generate a dense set of states out of the vacuum according to the principles of QFT. Then one applies Theorem 1 repeatedly.

Corollary 3 follows from Corollary 2 because states $\phi^k(f)\Omega$ for given k span an irreducible representation space of G^*, with spectrum of H determined in Ref. [5] to be of the form $\omega=d_k+m$, $d_k=\dim\phi^k$. Because only a denumerable number of fields appears in the operator product expansions by hypothesis, the corollary follows.

2. Harmonic Analysis

We wish to decompose $\phi^i(x)\phi^j(y)\Omega$ into states which transform irreducibly under G^*. It will suffice to consider scalar products $(\Psi, \phi^i(x)\phi^j(y)\Omega)$ with states Ψ in the dense domain $\mathscr{D}$.

For simplicity of writing consider first a theory of one hermitean scalar field $\phi(x)$, and $\phi^i = \phi^j = \phi$. The Wightman functions are

$$W(x_1 \dots x_N) = (\Omega, \phi(x_1) \dots \phi(x_N)\Omega). \tag{2.1}$$

Let $\mathcal{S} = \sum_N \mathcal{S}_N$ the space of finite sequences of Schwartz test functions f_0, $f_1(x_1) \dots f_N(x_1 \dots x_N)$. The subspace $\mathcal{S}_2$ consists of a sequence with only one nonvanishing term $f_2(x_1, x_2)$. The field theoretic domain $\mathcal{D}$ consists of vectors

$$\Psi(f) = \sum_k \int dx_1 \dots dx_k f_k(x_1 \dots x_k) \phi(x_1) \dots \phi(x_k)\Omega\,; \quad f \in \mathcal{S}. \tag{2.2}$$

According to the reconstruction theorem, the dense domain $\mathcal{D}$ in the Hilbert space $\mathcal{H}$ of physical states may be identified with a space of continuous linear functionals $F: f \mapsto \langle F, f \rangle$ on $\mathcal{S}$, i.e. sequences $F = (F_n)_{n=0,1,\dots}$ of generalized functions $F_n \in \mathcal{S}'_n$. We shall write $F(x_1 \dots x_n)$ in place of $F_n(x_1 \dots x_n)$ and use functional notation,

$$\langle F, f \rangle \equiv \sum_n \int dx_1 \dots dx_n \bar{F}(x_1 \dots x_n) f_n(x_1 \dots x_n)\,.$$

The identification is such that $F \in \mathcal{D}$ if and only if $F = Wf$ for an f in $\mathcal{S}$, with

$$Wf(x_1 \dots x_n) \equiv \sum_k \int dy_1 \dots dy_k f(y_k \dots y_1) \bar{W}(y_1 \dots y_k x_1 \dots x_n)\,. \tag{2.3a}$$

The scalar product on $\mathcal{D}$ becomes

$$(\Psi(f_1), \Psi(f_2)) = \langle Wf_1, f_2 \rangle\,. \tag{2.3b}$$

Since the Hilbert space $\mathcal{H}$ carries a unitary representation U of G^* it can be decomposed

$$\mathcal{H} = \int d\mu(\chi) \mathcal{H}^\chi = \int d\mu(\chi) \int d\nu(\varrho) \mathcal{H}^{\chi\varrho}\,. \tag{2.4a}$$

μ a measure on the set $G^{*\wedge} = \{\chi\}$ of all unitary irreducible representations (UIR's) of G^*. $\mathcal{H}^\chi$ consists of a direct sum or integral[2] of irreducible representation spaces $\mathcal{H}^{\chi\varrho}$ which carry equivalent UIR's χ.

In particular, states Wf in $\mathcal{D}$ may be so decomposed

$$Wf = \int d\mu(\chi) F^\chi = \int d\mu(\chi) \int d\nu(\varrho) F^{\chi\varrho}\,; \quad F^\chi \text{ in } \mathcal{H}^\chi \text{ etc.} \tag{2.4b}$$

Since an irreducible representation space of G^* must be contained in $\mathcal{H}$ as a whole, the spectrum condition allows only UIR's with positive energy. All such have been classified in [7]. First there is of course the trivial 1-dimensional representation. The others can be labelled by $\chi = [l, \delta]$, $l \in M^\vee$ a finite dimensional irreducible representation of $M \approx SL(2\mathbb{C})$ ("Lorentz spin") and $\delta \geq \delta_{\min}(l)$ real ("dimension"), cp. Proposition 6 below. The UIR χ may be realized in a space $\mathcal{F}_\chi$ of

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[2] In actual fact the measure ν is discrete, cp. Corollary 3 and its proof

(generalized) functions on Minkowski space with values in the finite dimensional representation space V^l of the Lorentz group M. Functions φ in $\mathscr{F}_\chi$ satisfy a spectrum condition, i.e. their Fourier transform is supported in sptr.$(\chi)\subseteqq \bar{V}_+$, the closed forward cone. The action of $T_\chi(g)$ of $g\in G^*$ on functions φ in $\mathscr{F}_\chi$ will be reviewed later on. Consider "intertwining maps"

$$\mathscr{B}^\chi : \mathscr{F}_\chi \mapsto \mathscr{H}^\chi \quad \text{such that} \quad U(g)\mathscr{B}^\chi = \mathscr{B}^\chi T_\chi(g) \quad \text{for} \quad g \quad \text{in} \quad G^*. \tag{2.5a}$$

They are linear combinations of isometric intertwining maps

$$\mathscr{B}^{\chi\varrho} : \mathscr{F}_\chi \mapsto \mathscr{H}^{\chi\varrho}$$
$$\mathscr{B}^\chi = \int d\nu(\varrho) a(\varrho) \mathscr{B}^{\chi\varrho} \quad \text{with} \quad \int |a(\varrho)|^2 d\nu(\varrho) < \infty. \tag{2.5b}$$

$\mathscr{B}^{\chi\varrho}$ are maps from $\mathscr{F}_\chi$ to $\mathscr{H}^{\chi\varrho}$ which preserve the norm and commute with the action of the group.

Every vector $F^{\chi\varrho}$ in $\mathscr{H}^{\chi\varrho}$ may be written in the form $F^{\chi\varrho} = \mathscr{B}^{\chi\varrho}\varphi^{\chi\varrho}$ where $\mathscr{B}^{\chi\varrho}$ is an intertwining map as were just introduced, and $\varphi^{\chi\varrho}\in\mathscr{F}_\chi$. $\mathscr{B}^{\chi\varrho}$ is unique and $\varphi^{\chi\varrho}$ is uniquely determined by $F^{\chi\varrho}$.

It suffices to consider states $\Psi(f)$ such that $(\Psi(f), \Omega) = 0$. The trivial 1-dimensional representation of G^* will then not appear in the decomposition, because the vacuum Ω is the only Lorentz invariant state.

The decomposition (2.4b) becomes

$$Wf = \int d\mu(\chi) \int d\nu(\varrho) \mathscr{B}^{\chi\varrho}\varphi^{\chi\varrho} \quad \text{with} \quad \varphi^{\chi\varrho}\in\mathscr{F}_\chi. \tag{2.4c}$$

Since $\mathscr{F}_\chi$ is a function space, $\mathscr{B}^{\chi\varrho}\varphi^{\chi\varrho} = \int dx \mathscr{B}^{\chi\varrho}(x)\,\varphi^{\chi\varrho}(x)$, and $\mathscr{B}^{\chi\varrho}(x)$ takes values in $\mathscr{H}^{\chi\varrho}\subset\mathscr{H}$. We restrict Wf to a continuous linear functional $Wf(x_1x_2)$ on $\mathscr{S}_2$. Let us introduce kernels $\mathscr{B}^\varrho(x\chi; x_1x_2) = (\mathscr{B}^{\chi\varrho}(x), \phi(x_1)\phi(x_2)\Omega)$. Decomposition (2.4c) gives then

$$(\Psi(f), \phi(x_1)\phi(x_2)\Omega) \equiv Wf(x_1x_2) = \int d\mu(\chi) \int d\nu(\varrho) \int dx \varphi^{\chi\varrho}(x)^* \mathscr{B}^\varrho(x\chi; x_1x_2). \tag{2.6'}$$

The kernels are singular functions with values in V^l. Often, physicists write indices: $\bar{\varphi}^{\chi\varrho}_\alpha \mathscr{B}^\varrho_\alpha$ (sum over multispin or indices α).

One may associate kernels $\mathscr{B}(x\chi; x_1x_2)$ with arbitrary intertwining maps $\mathscr{B}^\chi : \mathscr{F}_\chi \mapsto \mathscr{H}^\chi$ in the same way as for $\mathscr{B}^{\chi\varrho}$. They are related by (2.5b) viz.

$$\mathscr{B}(x\chi; x_1x_2) = \int d\nu(\varrho) a(\varrho) \mathscr{B}^\varrho(x\chi; x_1x_2), \tag{2.5c}$$

where $a(\varrho)$ is an arbitrary ν-square-integrable function of ϱ. Correspondingly, we shall use $\mathscr{B}$ as a generic name for arbitrary linear combinations (2.5c) of kernels $\mathscr{B}^\varrho$.

Since functions φ in $\mathscr{F}_\chi$ satisfy a spectrum condition, kernels $\mathscr{B}(x\chi; ...)$ are nonunique as functions of x. In particular, the Fourier transform

$$\mathscr{B}^\sim(p\chi; x_1x_2) = \int dx e^{ipx} \mathscr{B}(x\chi; x_1x_2) \tag{2.7}$$

is only relevant for $p\in$ sptr.$(\chi)\subseteqq \bar{V}_+$.

We shall count kernels $\mathscr{B}(x\chi; ...)$ only as distinct if $\varphi^\sim(p)^* \mathscr{B}^\sim(p\chi; x_1x_2)$ differ for some φ in $\mathscr{F}_\chi$.

The intertwining property (2.5) imposes strong covariance condition on kernels $\mathscr{B}$. Further restrictions come from the spectrum condition for states $\phi(x_1)\Omega$. We write $y_2 > y_1$ if $y_2 - y_1 \in V_+$. Spectrum condition and covariance imply

Proposition 4. *Let $\varphi^{\sim}(p)$ the Fourier transform of an arbitrary element of $\mathscr{F}_\chi$. Then $\tilde{\varphi}(p)^* \tilde{\mathscr{B}}(p\chi; x_1 x_2)$ is boundary value of a holomorphic function of $z_j = x_j + iy_j$ $(j=1,2)$ in the tube $y_2 > y_1 > 0$. Kernels $\tilde{\mathscr{B}}(p\chi; x_1 x_2)$ are linear combinations of a finite number of kinematically determined functions $\mathscr{B}^{ls}(p\chi; x_2 x_2)$ (at most one for scalar $\phi = \phi^i = \phi^j$) which can be labelled as in (1.3). Moreover, they can be analytically continued in p to entire analytic functions of p, viz. $\int dx dy f(xy) \mathscr{B}^{\sim}(p\chi; xy)$ is holomorphic in p for all test functions f with compact support.*

Proof of Proposition 4 will be given in the following sections; explicit expressions for $\mathscr{B}^{\sim}$ will be given in Section 8.

Let us return to expansion (2.6′). We retain the assumption that $\phi(x_1)$, $\phi(x_2)$ are scalar fields so that only one linearly independent kernel exists (cf. Proposition 4). Thus all kernels are proportional and the integration over ϱ may be performed with the result

$$(\Psi(f), \phi(x_1)\phi(x_2)\Omega) = Wf(x_1 x_2) = \int d\mu(\chi) \int dx \varphi^\chi(x)^* \mathscr{B}(x\chi; x_1 x_2) \tag{2.6}$$

with $\varphi^\chi(x) \propto \int d\nu(\varrho) \varphi^{\chi\varrho}(x) \in \mathscr{F}_\chi$.

It remains to be shown that

i) the measure $\mu(\chi)$ is discrete so that

$$(\Psi(f), \phi(x_1)\phi(x_2)\Omega) = \sum_k \int dx \varphi^{\chi_k}(x)^* \mathscr{B}(x\chi_k; x_1 x_2); \tag{2.8}$$

ii)

$$\varphi^{\chi_k}(x)^* = (\Psi(f), \phi^k(x)\Omega),$$

where ϕ^k is a nonderivative field appearing in the Wilson expansion (1.1) with dimension d_k and Lorentz spin l_k if $\chi_k = [l_k, d_k]$. Later on we write $\mathscr{B}^k$ for $\mathscr{B}(\cdot\chi_k, \ldots)$.

We shall use the hypothesis that Wilson expansion (1.1) is valid as an asymptotic expansion at $x=0$. We will derive from (2.6) an asymptotic expansion at $x=0$ in homogeneous functions of x. As asymptotic expansions in homogeneous functions are unique, (2.8) can then be deduced by comparison.

By Proposition 4, kernels $\mathscr{B}^{\sim}(p\chi; x_1 x_2)$ are entire functions of p. They may therefore be expanded in an everywhere convergent power series

$$\mathscr{B}^{\sim}(p\chi; \tfrac{1}{2}x - \tfrac{1}{2}x) = \sum_{r=0}^{\infty} C_\beta^{\chi r}(x) p_{\beta_1} \ldots p_{\beta_r}; \quad \beta = (\beta_1 \ldots \beta_r). \tag{2.9}$$

For reasons of dilatational invariance one has for real $\lambda > 0$,

$$C_\beta^{\chi r}(\lambda x) = \lambda^{-2d+\delta+r} C_\beta^{\chi r}(x) \quad \text{for } \chi = [l, \delta], d = \dim \phi \tag{2.10}$$

(or, more generally $2d \equiv d_i + d_j$, $d_{i,j} = \dim \phi^{i,j}$).

We will insert power series expansion (2.9) in (2.6).

$$(\Psi(f), \phi(\tfrac{1}{2}x)\phi(-\tfrac{1}{2}x)\Omega) = \int d\mu(\chi) \int dp \sum_r C_\beta^{\chi r}(x) p_{\beta_1} \ldots p_{\beta_r} \varphi^{\sim\chi}(p)^*. \tag{2.11}$$

Suppose that in (2.6) f is a sequence of test functions whose Fourier transforms have compact support. Vectors $\Psi(f)$ with such f are still dense in the Hilbert space

$\mathscr{H}$ of physical states. Because of momentum conservation, the Fourier transforms $\varphi^{\sim\chi}(p)$ of $\varphi^{\chi}(x)$ will then also have compact support, and so $\varphi^{\chi}(x)$ are infinitely differentiable at $x=0$ (even entire in x). Because of homogeneity (2.10), expansion (2.11) implies the following *asymptotic* expansion

$$(\Psi(f), \phi(\tfrac{1}{2}x)\phi(-\tfrac{1}{2}x)\Omega) = \sum_r \int d\mu(\chi) \int dp C_\beta^{\chi r}(x) p_{\beta_1} \dots p_{\beta_r} \varphi^{\sim\chi}(p)^*$$

$$= \sum_r \int d\mu(\chi) i^r C_\beta^{\chi r}(x) V_{\beta_1} \dots V_{\beta_r} \varphi^{\chi}(0)^*, \tag{2.11'}$$

where it is understood that summation and integrations $\sum_r \int d\mu(\chi)$ are rearranged in order of increasing $\delta + r$. ($\chi = [l, \delta]$).

Expansion (2.11′) can reproduce Wilson expansion (1.1) only if (2.8) holds true. It follows from (2.6) and (2.8) that expansion (1.2) is true on a dense set of vectors $\Psi(f)$ as described before (2.11′). Being a partial wave expansion it is then generally true and strongly convergent as stated in our theorem.

At the same time we see from (2.9), (2.11′) how the coefficients in the Wilson expansion (1.1) are obtained by power series expansion from the kernels $\mathscr{B}^{\sim}(p\chi; x_1 x_2)$.

We have written our formulae for a scalar field $\phi = \phi^i = \phi^j$. They remain true generally when interpreted correctly; i.e. appropriate indices should be supplied and attention must be paid to the existence of several linearly independent kernels $\mathscr{B}^{ls}(x\chi; x_1 x_2)$. In particular, Equation (2.8i) should be read as

$$(\Psi(f), \phi^i(x_1)\phi^j(x_2)\Omega) = \sum_k \int dx \sum_{l,s} \varphi^{\chi_k ls}(x)^* \mathscr{B}^{ls}(x\chi_k; x_1 x_2)$$

while (2.8ii) becomes

$$\varphi^{\chi_k ls} \in \mathscr{F}_\chi; \qquad \varphi^{\chi_k ls}(x)^* = \sum_j a_j^{ls}(\Psi(f), \phi^j(x)\Omega)$$

with complex coefficients a_j^{ls} and $\sum_j$ running over fields ϕ^j with spin l_k and dimension d_k only. If no degeneracies in spin and dimension occur then the sum is redundant.

The analysis of the kernels $\mathscr{B}$ will be done in full generality in the following sections.

It only remains to *prove Proposition 4. The sequel of this paper will be devoted to this problem.* At the same time, we will obtain explicit expressions for the kernels $\mathscr{B}^{\sim}(p\chi; x_1 x_2)$, cp. Section 8.

One could try to determine the kernels $\mathscr{B}^{\sim}$ by imposing infinitesimal conformal invariance. In fact this program was already carried out be Ferrara et al. [4] for the scalar case even before global conformal invariance was understood, and Proposition 4 is implicit in their work for this case. For general spin the infinitesimal method becomes too complicated. We shall therefore resort to global methods which are more powerful. [In applications one wants to apply Theorem 1 repeatedly (as e.g. in Corollary 2) and fields of arbitrary Lorentz spin may then appear.]

3. The Conformal Group G^*

The group G^* is an infinite sheeted covering of $SO_e(4,2)$. Its geometry was examined in [7]. The following picture emerges.

G^* contains the quantum mechanical (q.m.) Lorentz group $M \approx SL(2\mathbb{C})$ and therefore also its two-element-center Γ_1 whose representation distinguishes between bosons and fermions. Γ_1 is also contained in the center of G^* but does not exhaust it. The group G^*/Γ_1 may be pictured as a group of transformations of superworld $\boldsymbol{M}$. That is, G^* can act on $\boldsymbol{M}$, but the action of Γ_1 is trivial. Points η of $\boldsymbol{M}$ may be parametrized

$$\eta=(\tau,\varepsilon),\quad -\infty<\tau<\infty\,;\quad \varepsilon=(\varepsilon^1\varepsilon^2\varepsilon^3,\varepsilon^5)\quad \text{a unit 4-vector}$$
$$\text{viz. } (\varepsilon^1)^2+(\varepsilon^2)^2+(\varepsilon^3)^2+(\varepsilon^5)^2=1\,. \tag{3.1}$$

The action of G^* on $\boldsymbol{M}$ is specified by the action of various subgroups.

A subgroup K^* of G^* acts on $\boldsymbol{M}$ by rotations of ε and translations b_σ of τ to $\sigma+\tau$. $K^* \simeq \mathbb{R}\times SU(2)\times SU(2)$, also K^* contains the center $\Gamma=\Gamma_1\Gamma_2 \simeq \mathbb{Z}_2\times\mathbb{Z}$ of G^*. Γ_2 is generated by an element $\gamma^\wedge$ which acts on $\boldsymbol{M}$ as

$$\gamma^\wedge(\tau,\varepsilon)=(\tau+\pi,-\varepsilon)\quad \text{viz.}\quad \gamma^\wedge=\mathscr{R}\exp i\pi H=\mathscr{R}b_\pi\,. \tag{3.2}$$

H is the generator of τ-translation, and $\mathscr{R}$ rotation (sic) of ε into $-\varepsilon$. Its square $\mathscr{R}^2=e$. K^*/Γ is the maximal compact subgroup of G^*/Γ.

A fundamental domain F in $\boldsymbol{M}$ with respect to the discrete subgroup Γ_2 is a submanifold such that

$$F\cap\gamma F=\emptyset \quad \text{for}\quad e\neq\gamma\in\Gamma_2\,,\quad \bigcup_{\gamma\in\Gamma_2}\gamma F=\boldsymbol{M}\,.$$

A fundamental domain $F=\boldsymbol{M}_c^4$ may be chosen as

$$\boldsymbol{M}_c^4=\{(\tau,\varepsilon)\in\boldsymbol{M}\,;\,-\pi<\tau\leqq\pi,\ \varepsilon^5\geqq-\cos\tau\}\,. \tag{3.3a}$$

Its interior may be identified with Minkowski space $M^4=\{x^\mu\}$ through the reparametrization

$$x^0=\frac{\sin\tau}{\cos\tau+\varepsilon^5}\,;\qquad x^i=\frac{\varepsilon^i}{\cos\tau+\varepsilon^5}\qquad (i=1,2,3)\,. \tag{3.3b}$$

Translations $n^\sim$ in $N^\sim$, Lorentz transformation m in M and dilatations a in A act in the costumary way on points of $\boldsymbol{M}^4\subset\boldsymbol{M}_c^4$ parametrized by x^μ (see below). Their action on translates $\gamma\boldsymbol{M}^4$ of $\boldsymbol{M}^4$ is then also determined because $n^\sim ma\gamma=\gamma n^\sim ma$, γ being in the center of G^*. It extends by continuity to all of $\boldsymbol{M}=\cup\gamma\boldsymbol{M}^4$ (union over Γ_2).

The action of G^* on $\boldsymbol{M}$ is completely specified by the action of its subgroups $K, N^\sim, M$, and A, for every g in G^* may be written in the form

$$g=kman^\sim,\qquad k\in K^*\quad \text{etc.} \tag{3.4}$$

[This decomposition is nonunique. Let $U=K^*\cap M\approx SU(2)$ the rotation subgroup of M. Then $kman^\sim=k'm'a'n^{\sim\prime}$ if and only if $k'=ku$, $m'=u^{-1}m$ with u in U, and $a=a'$, $n^\sim=n^{\sim\prime}$.]

Let $N = \mathscr{R} N^{\sim} \mathscr{R}^{-1}$, it is called subgroup of special conformal transformations. The point $\eta_0 = (0, \varepsilon^{\wedge})$, $\varepsilon^{\wedge} = (\mathbf{0}, 1)$ is left invariant by MAN, and $\boldsymbol{M}$ is a homogeneous space $\boldsymbol{M} \simeq G^*/MAN$. MAN is isomorphic to a Poincaré group since

$$\mathscr{R} m \mathscr{R}^{-1} \equiv m^{\sim} \in M\,, \qquad \mathscr{R} a \mathscr{R}^{-1} = a^{-1} \quad \text{for} \quad ma \in MA\,. \tag{3.4'}$$

The fundamental domain $\boldsymbol{M}_c^4$ may also be made into a homogeneous space $\boldsymbol{M}_c^4 \simeq \boldsymbol{M}/\Gamma_2 \simeq G^*/P$ with $P = \Gamma_2 MAN$. The action of subgroups Γ_2, N, M, A, N on cosets $x = n_x^{\sim} P \in \boldsymbol{M}^4$ is the usual one: The center $\Gamma = \Gamma_2 \Gamma_1$ acts trivially, and[3]

M: Lorentz transformations m: $x^\mu \to \Lambda(m)^\mu_\nu x^\nu \equiv (mx)^\mu$.

A: dilatations a: $x^\mu \mapsto |a| x^\mu$, $|a| > 0$.

$N^{\sim}$: translations $n_y^{\sim}$: $x^\mu \mapsto x^\mu + y^\mu$, y^μ real.

N: spec. conf. transf. $n_{\theta y}$: $x^\mu \mapsto \sigma(x, y)^{-1}(x^\mu - y^\mu x^2)$

$\qquad y^\mu$ real, $\sigma(x, y) = 1 - 2x \cdot y + x^2 y^2$.

$$\mathscr{R}: \qquad x^\mu \mapsto \frac{\theta x^\mu}{x^2}\,, \quad \theta = \text{time reflection}\,. \tag{3.5}$$

If $n_y^{\sim} \in N^{\sim}$ then $\mathscr{R} n_y^{\sim} \mathscr{R}^{-1} = n_y \in N$.

Elements $m \in M \simeq \mathrm{SL}(2\mathbb{C})$ may be identified with unimodular two by two matrices; $\Lambda(m)$ is then given by the fundamental formula of spinor calculus. Let $\boldsymbol{x} = x^0 \mathbb{1} + \sum_1^3 x^k \sigma^k$ then $\Lambda(m)$ is determined by m through

$$m \boldsymbol{x} m^* = \boldsymbol{x}' \quad \text{with} \quad x'^\mu = \Lambda(m)^\mu_\nu x^\nu\,. \tag{3.6}$$

Translations act transitively on $\boldsymbol{M}^4$ and $\boldsymbol{M}^4$ is almost all of $\boldsymbol{M}_c^4 \simeq G^*/P$. Therefore the set $N^{\sim}P$ fills up all but a lower dimensional submanifold of G^*. Elements in $N^{\sim}P$ will be called regular. Every regular element g of G^* may be written in a unique way as

$$g = n^{\sim} \gamma man \quad \text{with} \quad n^{\sim} \in N^{\sim},\ \gamma \in \Gamma_2,\ m \in M,\ a \in A \quad \text{and} \quad n \in N\,. \tag{3.7}$$

Haar measure of G^* factorizes as $dg = dn^{\sim} dm\, da\, dn$ in this parametrization. In the following it is understood that restriction to regular elements of G^* is made whenever this is necessary in order that the formulae make sense.

Let x' and $p(x, g) \in P = \Gamma_2 MAN$ determined by x, g through the unique decomposition

$$g^{-1} n_x^{\sim} = n_{x'}^{\sim}\, p(x, g)^{-1}\,. \text{ Then } x' = g^{-1} x \tag{3.8}$$

viz. x' is determined by the action (3.5) of G^* on cosets $x \in G^*/P$. From (3.8) one deduces the cocycle condition

$$p(x, g_1 g_2) = p(x, g_1)\, p(g_1^{-1} x, g_2)\,. \tag{3.9}$$

Special cases: For $n^{\sim} \gamma ma \in N^{\sim} \Gamma_2 MA$ one has

[3] Our metric is $g_{\mu\nu} = \mathrm{diag.}\,(+---)$; $x \cdot y = g_{\mu\nu} x^\mu y^\nu$, $x^2 = x \cdot x$ etc.

$$p(x, n^{\check{}}\gamma ma) = \gamma ma \quad \text{independent of} \quad x\,. \tag{3.10}$$

The next lemma gives an explicit expression for $p(x,g)$.

Lemma 5. *Let $p(x,g)$ as defined in* (3.8), *and $\mathscr{R}\in G^*$ the reciprocal radius transformation defined after* (3.2). *Then $p(x,\mathscr{R})$ is MA-covariant in the sense that*

$$bp(x,\mathscr{R}) = p(bx,\mathscr{R})b^{\check{}} \quad \textit{for} \quad b\in MA\,, \quad b^{\check{}} = \mathscr{R}b\mathscr{R}^{-1}\in MA\,.$$

It is explicitly given by

$$\begin{aligned} &p(x,\mathscr{R}) = \gamma m_x a_x n_z \quad \textit{with} \quad \gamma = \gamma^{\wedge N},\ N = \operatorname{sign} x\,,\\ &|a_x| = |x^2|,\ m_x = i^{N-1}\mathbf{x}|x^2|^{-1/2},\ z^\mu = -x^\mu/x^2\,. \end{aligned} \tag{3.11}$$

Herein $\mathbf{x} = x^0\mathbb{1} + \sum x^k\sigma^k$, and $\operatorname{sign} x = \pm 1$ *for $x\in V_\pm$ and 0 otherwise.*

The quantity $p(x,g)$ for general regular $g\in G^*$ is expressible in terms of $p(x,\mathscr{R})$. Write $g = n^{\check{}}\gamma man$, $n^{\check{}}\in N^{\check{}}$, $\gamma\in\Gamma_2$ etc. Then

$$p(x,g^{-1}) = p(x,n^{-1})(\gamma ma)^{-1} \quad \textit{and} \quad p(x,n^{-1}) = p(x,\mathscr{R})p(\mathscr{R}nx,\mathscr{R})\,.$$

The quantity $p(x,\mathscr{R})$ was computed in [7, Eqs. (6.20) and (6.21)]. The other assertions of Lemma 5 follow from the cocycle condition (3.9) and (3.10), noting that $\mathscr{R}n_x\mathscr{R} = n_x^{\check{}}$, $bn_x^{\check{}} = n_{bx}^{\check{}}b$ for $b\in MA$. In particular, it follows from the last relation that $b^{\check{}}\mathscr{R}n_x^{\check{}} = \mathscr{R}n_{bx}^{\check{}}b$, and so by Definition (3.8), $b^{\check{}}n_{\mathscr{R}x}^{\check{}}p(x,\mathscr{R})^{-1} \equiv n_{\mathscr{R}bx}^{\check{}}b^{\check{}}p(x,\mathscr{R})^{-1}$ $= n_{\mathscr{R}bx}^{\check{}}p(bx,\mathscr{R})^{-1}b$. This shows MA-covariance of $p(x,\mathscr{R})$. □

Having completed the outline of the group G^*'s geometry, we now turn to its unitary irreducible representations with positive energy.

Let $\chi = [l,\delta]$, δ real and $l\in M^{\check{}}$ a finite dimensional irreducible representation of M by matrices $D^l(m)$ in a vector space V^l. We equip V^l with a scalar product, written u^*v, of vectors u, v in V^l which is such that

$$D^l(m^{\check{}})^{-1} = D^l(m)^* \quad \text{for} \quad m^{\check{}} = \mathscr{R}m\mathscr{R}^{-1} = \theta m\theta^{-1}\,, \quad m\in M\,.$$

We define a finite dimensional representation of $P = \Gamma_2 MAN$ in V^l by

$$D^\chi(\gamma man) = |a|^{-c}e^{i\pi Nc}D^l(m) \text{ with } c = \delta - 2, \text{ for } \gamma = \gamma^{\wedge N}\,. \tag{3.12}$$

As usual, $a\in A$ is dilatation by $|a|$, cp. (3.5), etc.

Let $\mathscr{E}_\chi$ the space of infinitely differentiable functions on G^* with values in V^l and having covariance property

$$f(gp) = |a|^2 D^\chi(p)^* f(g) \quad \text{for} \quad p = \gamma man\in\Gamma_2 MAN\,. \tag{3.13}$$

$\mathscr{E}_\chi$ becomes a representation space for G^* by imposing the transformation law

$$(T_\chi(g)f)(g') = f(g^{-1}g')\,. \tag{3.14}$$

Because of covariance property (3.13) and decomposition (3.7) of group elements, functions f in $\mathscr{E}_\chi$ are uniquely specified by their restriction $f(x) \equiv f(n_x^{\check{}})$ to $N^{\check{}}$. Transformation law (3.14) becomes in this language

$$(T_\chi(g)f)(x) = |a|^2 D^\chi(p(x,g))^* f(g^{-1}x) \tag{3.15}$$

with $|a|$ from $p(x,g) = \gamma man$. We are dealing with an induced representation on G^*/P. (P is called a parabolic subgroup, it is not the minimal one.)

A scalar product on $\mathscr{E}_\chi$ is constructed with the help of an intertwining map (or operator)

$$\Delta_+^\chi : \mathscr{E}_\chi \mapsto \mathscr{F}_\chi , \tag{3.16}$$

where $\mathscr{F}_\chi$ is a space of generalized functions on G^* with values in V^l having covariance property

$$\varphi(gp) = |a|^2 D^\chi(p)^{-1}\varphi(g) \quad \text{for} \quad g \in G^*,\ p = \gamma m a n \in \Gamma_2 MAN . \tag{3.17}$$

$\mathscr{F}_\chi$ is made into a representation space for G^* by the transformation law

$$(T_\chi(g)\varphi)(g') = \varphi(g^{-1}g') . \tag{3.18}$$

Generalized functions φ in $\mathscr{F}_\chi$ are determined by their restriction $\varphi(x) \equiv \varphi(n_x^\sim)$ to $N^\sim$. The transformation law becomes

$$(T_\chi(g)\varphi)(x) = |a|^2 D^\chi(p(\chi, g)^{-1})\varphi(g^{-1}\chi) \tag{3.19}$$

with notation as in (3.15). The intertwining map Δ_+^χ is required to commute with the action of the group

$$\Delta_+^\chi T_\chi(g) f = T_\chi(g)\Delta_+^\chi f \qquad \text{for} \qquad f \text{ in } \mathscr{E}_\chi .$$

It is given explicitly by

$$\begin{aligned}\varphi(x) = (\Delta_+^\chi f)(x) &= n_+(\chi) \int_{\tilde N} dn^{\sim\prime} f(n_x^\sim \mathscr{R} n^{\sim\prime}) \\ &= \int dx' \Delta_+^\chi(x-x') f(x')\end{aligned}$$

with

$$\Delta_+^\chi(x) = n_+(\chi)(-x^2 + i\varepsilon x^0)^{-\delta-j_1-j_2} D^l(i\mathbf{x}) . \tag{3.20}$$

Here (j_1, j_2) is the highest weight of the representation $l \in M^\vee$ of M, and D^l is the extension to GL(2$\mathbb{C}$) of l through $D^l(\varrho m) = \varrho^{2j_1+2j_2} D^l(m)$, $\varrho \in \mathbb{C}$. Equations (3.20) were derived in [7].

The Fourier transform of the intertwining kernel (= conformal invariant 2-point function) (3.20) is

$$\Delta_+^\chi(p) = n'_+(\chi) D^l\left(-\frac{\partial}{\partial p}\right)\theta(p)(p^2)^{-2+\delta+j_1+j_2} \tag{3.20'}$$

with a new normalization factor $n'_+(\chi)$. $\Delta_+^\chi(p)$ vanishes for momenta p outside the closed forward cone. The massless scalar 2-point function is obtained as a limit, $j_1 = j_2 = 0$, $\delta \to 1$, viz. $\Gamma(c+1)^{-1}\theta(p)(p^2)^c \to \theta(p)\delta(p^2)$ as $c \to -1$.

In Ref. [7] a complete classification of all UIR's of G^* with positive energy was given. The result will be quoted as our

Proposition 6. *The* UIR's *of G^* with positive energy can be labelled by $\chi = [l, \delta]$, l a finite dimensional irreducible representation of $M \simeq$ SL(2$\mathbb{C}$) and $\delta \geqq \delta_{\min}(l)$ real. If (j_1, j_2) is the highest weight of l (viz. $2j_1, 2j_2$ nonnegative integers) then $\delta_{\min}(l) = j_1 + j_2 + 2$ if $j_1 \neq 0$, $j_2 \neq 0$, and $\delta_{\min}(l) = j_1 + j_2 + 1$ otherwise, except for the trivial* 1-*dimensional representation which has $\delta = j_1 = j_2 = 0$. The nontrivial* UIR's *χ can be* *realized in the representation spaces $\mathscr{E}_\chi$ equipped with scalar product*

$$(f_1, f_2) = \int dx_1 dx_2 f_1(x_1)^* \cdot \Delta_+^\chi(x_1 - x_2) f_2(x_2)$$

with intertwining kernel (3.20).

Representations with $j_1 = 0$ *or* $j_2 = 0$ *and* $\delta = \delta_{\min}$ *are zero mass representations, the others have continuous mass spectrum,* sptr.$(\chi) = \bar{V}_+$.

Remark. An equivalent UIR. χ is realized in the space $\mathscr{F}_\chi = \Delta_+^\chi \mathscr{E}_\chi$. If $\varphi_1 = \Delta_+^\chi f_1$, $f_1 \in \mathscr{E}_\chi$ then the scalar product $(\varphi_1, \varphi_2) = \int dx f_1(x)^* \cdot \varphi_2(x)$. Generalized functions φ in $\mathscr{F}_\chi$ satisfy a spectrum condition since the intertwining kernel $\Delta_+^\chi(p)$ does, cp. (3.20′).

In the following, we shall often not distinguish in notation between the test function space $\mathscr{E}_\chi$ and the Hilbert space constructed from it. If we use functional notation for the elements of this Hilbert space, it is always understood that an arbitrary representative out of the equivalence class of functions modulo zero norm vectors is to be chosen.

4. Implications of the Spectrum Condition

Let us use the intertwining map Δ_+^χ to introduce

$$V(x\chi; x_1 x_2) = \int dx' \Delta_+^\chi(x - x')^* \mathscr{B}(x'\chi; x_1 x_2). \tag{4.1}$$

Because UIR's of G^* acting in $\mathscr{F}_\chi$ and $\mathscr{E}_\chi$ are equivalent and intertwined by Δ_+^χ, the conformal partial wave expansion (2.6′) may be written in the equivalent form

$$(\Psi(f), \phi(x_1)\phi(x_2)\Omega) = Wf(x_1 x_2) = \int d\mu(\chi) \int d\nu(\varrho) \int dx \xi^{\chi\varrho}(x)^* V^\varrho(x\chi; x_1 x_2) \tag{4.2}$$

with $\xi^{\chi\varrho} \in \mathscr{E}_\chi$.

As in Section 2 we write $\mathscr{B}$, V for arbitrary linear combinations of kernels $\mathscr{B}^\varrho$, V^ϱ [cp. Eq. (2.5c)] so that $V(x\chi; x_1 x_2)$ is the kernel of an arbitrary intertwining map

$$V^\chi : \mathscr{E}_\chi \mapsto \mathscr{H}^\chi.$$

The kernels $\mathscr{B}$ are determined by V through (4.1) to within the arbitrariness discussed in Section 2. We shall first determine V and then recover $\mathscr{B}$ from the result.

Let us first state implications of the spectrum condition for V.

Lemma 7. *The kernels $V(x\chi; x_1 x_2)$ are limits of generalized functions $V(x\chi; z_1 z_2)$ of x which are holomorphic in the complex parameters $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in the tube $y_2 > y_1 > 0$. The limit is taken by letting $y_1, y_2 \to 0$ through the tube.*

Proof. It is well known that $\phi(x_1)\phi(x_2)\Omega$ is boundary value of states $\Psi(z_1 z_2) \in \mathscr{H}$ which are holomorphic in $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ in the tube. (This result is reviewed in [5].) Let $E^{\chi\varrho}$ the projection operator on the subspace $\mathscr{H}^{\chi\varrho}$ in decomposition (2.4a) of $\mathscr{H}$. By construction of expansion (4.2) we have

$$(E^{\chi\varrho}\Psi(f), \Psi(z_1 z_2)) = \int dx \xi^{\chi\varrho}(x)^* V^\varrho(x\chi; z_1 z_2)$$

in which $\int dx \xi^{\chi\varrho}(x)^* V^\varrho(x\chi; z_1 z_2)$ is a holomorphic function of z_1, z_2 in the tube and has $\int dx \xi^{\chi\varrho}(x)^* V^\varrho(x\chi; x_1 x_2)$ as a limit. If $\xi_0 \in \mathscr{E}_\chi$ is an arbitrary vector, then, because $\mathscr{E}_\chi$ carries an irreducible unitary representation, vectors of the form $\int dg f(g) T(g) \xi_0$

with f an infinitely differentiable function with compact support on G^* form a dense set of vectors in the UIR-Hilbert space $\mathscr{E}_\chi$. Therefore $\xi^\chi(x)$ may be considered as an *arbitrary* element of $\mathscr{E}_\chi$. But $\mathscr{E}_\chi$ contains all Schwartz test functions with values in V^l; therefore $V(x\chi; z_1 z_2)$ is a generalized function of x and has the indicated holomorphy property. □

It follows from Lemma 7 that it will suffice to determine $V(x\chi; x_1 x_2)$ for relatively spacelike points x_1, x_2 on Minkowski space.

Lemma 7 cannot be carried over without further ado to $\mathscr{B}(x\chi; x_1 x_2)$ because these kernels are nonunique as functions of x. However, it does imply the first assertion of Proposition 4 because every element of $\mathscr{F}_\chi$ is of the form $\varphi = \Delta_+^\chi \xi$ with $\xi \in \mathscr{E}_\chi$.

5. Relatively Spacelike Pairs of Points

Our further analysis is based on the fact that the conformal group G^* acts transitively on pairs of relatively spacelike points on superworld $\boldsymbol{M}$. This will now be explained.

The manifold $\boldsymbol{M}$ admits a G^*-invariant causal ordering [5]. Two points $\eta_1 = (\tau_1, \varepsilon_1)$ and $\eta_2 = (\tau_2, \varepsilon_2)$ are relatively spacelike if and only if

$$|\tau_2 - \tau_1| < \operatorname{Arccos} \varepsilon_1 \cdot \varepsilon_2 . \tag{5.1}$$

Arccos$x$ is the principal value of arccos x which lies between $0 \dots \pi$.

Lemma 8. a) *G^* acts transitively on relatively spacelike pairs of points on $\boldsymbol{M}$.*

b) *η_1, η_2 in $\boldsymbol{M}$ are relatively spacelike if and only if there exists $k \in K^* \subset G^*$ such that $k\eta_1$, $k\eta_2$ are relatively spacelike points on Minkowski space $\boldsymbol{M}^4 \subset \boldsymbol{M}$ (cp. Eqs. (3.3)).*

c) *The little group*[4] *in G^* of a pair of relatively spacelike points on $\boldsymbol{M}$ is isomorphic to MA. The manifold of relatively spacelike pairs of points on $\boldsymbol{M}$ may therefore be identified with the homogeneous space G^*/MA.*

Proof. Let η_0 the origin of $\boldsymbol{M}^4 \subset \boldsymbol{M}$ and $\eta_\infty = \mathscr{R}\eta_0$. We call η_∞ the unique point at spatial infinity of Minkowski space. Explicitly $\eta_\infty = (0, \varepsilon^\vee)$, $\varepsilon^\vee = (\boldsymbol{0}, -1)$.

The little group of η_0 is MAN and the little group of η_∞ therefore $MAN^\sim = \mathscr{R} MAN \mathscr{R}^{-1}$.

a) Let (η_1, η_2) relatively spacelike. Since G^* acts transitively on $\boldsymbol{M}$ there is g such that $\eta_2 = g\eta_\infty$. By G^*-invariance of causal ordering, $\eta_1' = g^{-1}\eta_1$ is then relatively spacelike to η_∞. By (5.1) and (3.3a) this means that η_1' must belong to Minkowski space $\boldsymbol{M}^4$. The little group $MAN^\sim$ of η_∞ acts transitively on $\boldsymbol{M}^4$. There is therefore p in $MAN^\sim$ such that $\eta_1' = p\eta_0$. Since p leaves η_∞ invariant we have then $(\eta_1, \eta_2) = (gp\eta_0, gp\eta_\infty)$. Since every pair of relatively spacelike points may be written in this way, with $gp \in G^*$, we have proven transitivity.

b) The if part follows from G^*-invariance of the causal ordering. Conversely, choose η_3 in $\boldsymbol{M}^4$ and relatively spacelike to η_0. By transitivity a) there is g in G^* such that $(\eta_1, \eta_2) = (g\eta_0, g\eta_3)$. Decompose $g = kman^\sim$ as in (3.4). Then $(\eta_4, \eta_5) = (man^\sim\eta_0, man^\sim\eta_3)$ are relatively spacelike points in $\boldsymbol{M}^4$ since the Poincaré group carries $\boldsymbol{M}^4$ into itself, and $(\eta_1, \eta_2) = (k\eta_4, k\eta_5)$.

[4] Little group = subgroup of stability

c) The little group of the pair (η_0, η_∞) is $MAN \cap MAN^{\sim} = MA$. The assertion of c) follows from this and a). □

6. Global Transformation Law

According to the discussion in Section 2, physical states in the dense domain $\mathscr{D}$ may be thought of as continuous linear functionals $F = Wf$ on the test function space $\mathscr{S}$. They can be restricted to the subspace $\mathscr{S}_2^{\sim}$ which consists of Schwartz test functions $h(x_1 x_2)$ with support containing only relatively spacelike pairs of points on Minkowski space. These pairs may at the same time be thought of as relatively spacelike pairs of points in the fundamental domain M_c^4 in superworld M.

The space $\mathscr{S}_2^{\sim}$ is not globally G^*-invariant. We shall imbed it in a space $\mathscr{I}^{\sim}$ of test functions on $M \times M$ with compact support containing only relatively spacelike pairs of points. The space $\mathscr{I}^{\sim}$ is G^*-invariant, i.e. it admits an action

$$T(g): \mathscr{I}^{\sim} \mapsto \mathscr{I}^{\sim} \quad (g \text{ in } G^*)$$

of the group G^*. Afterwards we will extend functionals $F = Wf$ from $\mathscr{S}_2^{\sim}$ to $\mathscr{I}^{\sim}$ by a process of analytic continuation (cp. Sec. 8 of [5]). In doing so a physical state $\Psi(h)$ is associated to every h in $\mathscr{I}^{\sim}$. The global G^*-transformation law of these special states can be stated explicitly, so that we may thereafter deal with an explicitly known action of G^* in a concrete function space in place of an abstract unitary representation of G^* in an abstract Hilbert space of physical states.

Let us deal with general spin right away. Let $\chi_i = [l_i, d_i]$ and $\chi_j = [l_j, d_j]$ specified by Lorentz transformation law and dimension of the fields ϕ^i and ϕ^j whose operator product (1.1a) we want to expand. We denote by V^i, V^j the finite dimensional vector spaces in which act the representations l_i and l_j of M. It is understood that they are equipped with a scalar product which is such that $D^l(\theta m \theta)^{-1} = D^l(m)^*$.

The space $\mathscr{S}_2^{\sim}$ consists of test functions $h(x_1 x_2)$ with values in the tensor product $V^i \otimes V^j$ and

$$\Psi(h) = \int dx_1 dx_2 h_{\alpha\beta}(x_1 x_2) \phi_\alpha^i(x_1) \phi_\beta^j(x_2) \Omega \quad \text{for} \quad h \in \mathscr{S}_2^{\sim}. \tag{6.1}$$

Indices α, β label an orthonormal basis in V^i resp. V^j; summation over repeated indices α, β is understood.

Let $P^0 = MAN$ so that superworld $M = G^*/P^0$. Let us restrict the representations D^χ (3.12) of $P = P^0 \Gamma_2$ to P^0. Consider the finite dimensional representation π of $P^0 \times P^0$ in $V^i \otimes V^j$ by matrices

$$\pi(p_1, p_2) = [D^{\chi_i}(p_1^{-1})^* \otimes D^{\chi_j}(p_2^{-1})^*] \delta_P(p_1)^{-1/2} \delta_P(p_2)^{-1/2}$$
$$\text{where} \quad \delta_P(man) = |a|^4, \quad p_1, p_2 \in P^0. \tag{6.2}$$

The space $\mathscr{I}^{\sim}$ consists of all infinitely differentiable cross sections on the homogeneous vector bundle $E = (M \times M) \underset{\pi}{\times} (V^i \otimes V^j)$ with compact support containing in its interior only relatively spacelike pairs of points on M (notation of

[13]). In other words, $\mathscr{I}^{\sim}$ consists of infinitely differentiable functions on $G^* \times G^*$ with values in $V^i \otimes V^j$ having covariance property

$$h(g_1 p_1, g_2 p_2) = \pi(p_1; p_2)^{-1} h(g_1, g_2) \quad \text{for} \quad p_i \in P^0, g_i \in G^* \quad (i=1,2)\,. \tag{6.3}$$

The action $T(g)$ of G^* is

$$(T(g)h)(g_1, g_2) = h(g^{-1} g_1, g^{-1} g_2)\,. \tag{6.4}$$

Evidently such functions h are completely specified if they are known for one representative (g_1, g_2) out of every coset $(\eta_1, \eta_2) \in \boldsymbol{M} \times \boldsymbol{M} = (G^* \times G^*)/(P^0 \times P^0)$. Therefore, if a representative of every coset is fixed in some way, cross sections h may also be considered as vector-valued functions on $\boldsymbol{M} \times \boldsymbol{M}$. The support of h is the closure of the (open) set of all pairs $(\eta_1, \eta_2) \in \boldsymbol{M} \times \boldsymbol{M}$ such that $h(g_1, g_2) \neq 0$ for $(g_1, g_2) \in (\eta_1, \eta_2)$. $\mathscr{I}^{\sim}$ is made up of cross sections h with support properties as stated above.

Consider the subspace of $\mathscr{I}^{\sim}$ which consists of cross sections which vanish outside Minkowski space $\boldsymbol{M}^4 \times \boldsymbol{M}^4 \subset \boldsymbol{M} \times \boldsymbol{M}$. It may be identified with the space $\mathscr{S}_2^{\sim}$ as follows: Every $g_i \in G^*$ with $g_i P^0 \in \boldsymbol{M}^4 \subset \boldsymbol{M}$ may be written as $g_i = n^{\sim}_{x_i} p_i$ with $p_i \in P^0$, $n^{\sim}_x$ = translation by x. Therefore by (6.3)

$$\begin{aligned} h(g_1, g_2) &= \pi(p_1, p_2)^{-1} h(n^{\sim}_{x_1}, n^{\sim}_{x_2}) \\ g_i &\equiv \pi(p_1, p_2)^{-1} \dot{h}(x_1, x_2) \end{aligned}$$

$$\text{for} \quad g_i = n^{\sim}_{x_i} p_i \in n^{\sim}_{x_i} P^0 \in \boldsymbol{M}^4\,. \tag{6.5}$$

and $\dot{h}(x_1, x_2)$ in in $\mathscr{S}^{\sim}{}_2$ and determines $h(g_1, g_2)$ everywhere on $G^* \times G^*$ so long as h is in the subspace.

Let $h \in \mathscr{S}_2^{\sim}$ and $\Psi(h)$ defined by (6.1). It follows from the results of [5] that the Hilbert space of physical states carries a unitary representation U of G^* whose action on states $\Psi(h)$ is such that

$$U(g)\Psi(h) = \Psi(T(g)h) \tag{6.6}$$

provided $g \in G^*$ and $h \in \mathscr{S}_2^{\sim}$ are such that also $T(g)h \in \mathscr{S}_2^{\sim}$, i.e. g does not carry any point in the support of h outside Minkowski space. It follows from Lemma 8b and compact support of h that every in h in $\mathscr{I}^{\sim}$ may be written as a finite sum of the form

$$h = \sum_i T(g_i) h_i \quad \text{with} \quad h_i \in \mathscr{S}_2^{\sim} \text{and} \quad g_i \in G^*\,. \tag{6.7}$$

We may then define

$$\Psi(h) = \sum U(g_i)\Psi(h_i) \tag{6.8}$$

with $\Psi(h_i)$ defined by (6.1). We must show that this is consistent, i.e. independent of the choice of g_i and h_i in (6.7). Suppose that $h = \Sigma\, T(g'_i) h_i$ is another decomposition of h with h_i in $\mathscr{S}_2^{\sim}$ and $g_i \in G^*$. By making finer splittings and reordering we may achieve that both sums have equally many terms, and

$$T(g'_i)h'_i = T(g_i)h_i \quad \text{for all} \quad i\,. \tag{6.9}$$

But

$$\begin{aligned} \Psi(T(g'_i)h'_i) &= U(g'_i)\Psi(h'_i) = U(g'_i)\Psi(T(g'^{-1}_i g_i)h_i) \\ &= U(g'_i)U(g'^{-1}_i g_i)\Psi(h_i) = U(g_i)\Psi(h_i) = \Psi(T(g_i)h_i)\,. \end{aligned}$$

We used in turn Definition (6.8), hypothesis (6.9), Equation (6.6), the group law, and Definition (6.8) again. By summing over i we have

$$\Psi(h)=\Sigma_i \Psi(T(g_i)h_i)=\Sigma_i \Psi(T(g_i')h_i')$$

which proves consistency.

We have shown that Definition (6.8) is meaningful. It is then automatically consistent with (6.1) and, moreover, transformation law (6.6) holds generally true for arbitrary h in $\mathscr{I}^{\sim}$ and g in G^*. Equation (6.6) is the promised explicit form of the global transformation law.

It follows that the functionals Wf on $\mathscr{S}_2^{\sim}$ extend to continuous linear functionals on $\mathscr{I}^{\sim}$ by virtue of the definition

$$\langle Wf, h\rangle=(\Psi(f),\Psi(h)) \quad \text{for} \quad f\in\mathscr{S},\ h\in\mathscr{I}^{\sim},$$

and

$$(U(g)\Psi(f),\Psi(h))\equiv\langle U(g)Wf, h\rangle=\langle Wf, T(g^{-1})h\rangle\ . \tag{6.10}$$

The conformal partial wave expansion of these expressions is obtained by decomposing the states $\Psi(h)$ as described in Section 2. We write it in terms of elements of $\mathscr{E}_\chi$ as in Section 4.

$$\langle Wf, h\rangle=\int d\mu(\chi)\int d\nu(\varrho)\int dx\,\xi^{\chi\varrho}(x)^* V^{\chi\varrho}(x;h)\equiv\int d\mu(\chi)\int d\nu(\varrho)V^{\chi\varrho}[\xi^{\chi\varrho}, h]$$
$$\text{with}\quad \xi^{\chi\varrho}\in\mathscr{E}_\chi,\ \text{for}\quad h\in\mathscr{I}^{\sim}. \tag{6.11a}$$

If h is in $\mathscr{S}_2^{\sim}\subset\mathscr{I}^{\sim}$ it is determined by a function $\dot{h}(x_1x_2)$ of relatively spacelike pairs (x_1, x_2) of points on Minkowski space through (6.5), and so

$$V^{\chi\varrho}[\xi, h]=\int dx_1dx_2\int dx\,\xi(x)^* V^{\varrho}(x\chi;x_1x_2)\dot{h}(x_1x_2)$$
$$\text{for}\quad h\in\mathscr{S}^{\sim}\subset\mathscr{I}^{\sim}. \tag{6.11b}$$

Here and everywhere we use vector notation: $h(x_1, x_2)$ takes values in $V^i\otimes V^j$, the kernel $V^{\varrho}(x\chi;x_1x_2)$ is a linear map from $V^i\otimes V^j$ to V^l, $\xi(x)$ takes values in V^l, and we write $v_2^* v_1$ for the scalar product of two vectors v_1, v_2 in V^l.

We write V^χ for an arbitrary linear combination of $V^{\chi\varrho}$ as usual. The intertwining property (G^*-invariance) of V^χ reads because of (6.10)

$$V^\chi[T_\chi(g)\xi, T(g)h]=V^\chi[\xi, h]\ . \tag{6.12}$$

This must hold for arbitrary ξ^χ in $\mathscr{E}_\chi$, because of irreducibility of the UIR-space $\mathscr{E}_\chi$, cp. the proof of Lemma 7. Thus $V^\chi[\cdot,\cdot]$ are G^*-invariant sesquilinear forms on $\mathscr{E}_\chi\times\mathscr{I}^{\sim}$. They determine the kernels $V(x\chi;x_1x_2)$ for relatively spacelike Minkowski space arguments x_1, x_2 through (6.11b).

7. G^*-Invariant Sesquilinear Forms on $\mathscr{E}_\chi\times\mathscr{I}^{\sim}$

We wish to determine the most general sesquilinear form $V^\chi[\cdot,\cdot]$ on $\mathscr{E}_\chi\times\mathscr{I}^{\sim}$ which is G^*-invariant in the sense of (6.12) and such that the kernel $V(x\chi;x_1x_2)$ determined by it admits analytic continuation as required by Lemma 7. This problem can be solved by a standard method of the theory of induced representations, viz. Bruhat theory of intertwining maps [14].

First we give an alternative description of the space $\mathcal{I}^{\sim}$. It will exhibit the representation acting in $\mathcal{I}^{\sim}$ as an induced representation on G^*/MA. Let us define a representation L of MA by operators $L(ma)$ acting in the vector space $V^i \otimes V^j$,

$$L(ma)^{-1} = D^{\chi_i}(ma)^* \otimes D^{\chi_j}(m^{\sim}a^{-1})^*, \qquad m^{\sim} = \mathcal{R} m \mathcal{R}^{-1}. \tag{7.1}$$

Lemma 9. *There is a bijective intertwining map Q from $\mathcal{I}^{\sim}$ to the space of all infinitely differentiable cross sections on the homogeneous vector bundle $(G^*/MA) \underset{L}{\times} (V^i \otimes V^j)$ with base G^*/MA and fibre $V^i \otimes V^j$.*

Explicitly, $Q\mathcal{I}^{\sim}$ consists of infinitely differentiable functions h_1 on G^* with values in $V^i \otimes V^j$ having covariance property

$$h_1(gma) = L(ma)^{-1} h_1(g) \quad \text{for} \quad ma \in MA\,. \tag{7.2}$$

It is made into a representation space for G^* by imposing the transformation law

$$(T(g)h_1)(g') = h_1(g^{-1}g') \qquad (h_1 \in Q\mathcal{I}^{\sim})\,. \tag{7.3}$$

The map Q is explicitly given by

$$(Qh)(g) = h(g, g\mathcal{R}) \quad \text{for} \quad h \in \mathcal{I}^{\sim}. \tag{7.4}$$

Evidently it commutes with the action of the group, $T(g)Q = QT(g)$ by (6.4). Covariance property (7.2) of Qh follows from (6.3) since $\mathcal{R} ma \mathcal{R}^{-1} = m^{\sim}a^{-1}$ by (3.4'). $\mathcal{R}$ was defined after (3.2). To prove the lemma it only remains to be shown that Qh determines h. This follows from the fact (Lemma 8) that G^* acts transitively on relatively spacelike pairs of points on superworld $\boldsymbol{M}$. As g ranges over G^*, the pair $(g\eta_0, g\mathcal{R}\eta_0) = (g\eta_0, g\eta_\infty)$ ranges over all relatively spacelike pairs of points on superworld $\boldsymbol{M}$, cp. the proof of Lemma 8. Therefore the set of pairs $(g, g\mathcal{R})$ contains a representative out of every coset $(\eta_1, \eta_2) \in \boldsymbol{M} \times \boldsymbol{M} = (G^* \times G^*)/(P^0 \times P^0)$. This suffices to determine h by the discussion following (6.4). □

Because of Lemma 9 we may consider $V^{\chi}[\cdot,\cdot]$ as a G^*-invariant sesquilinear form on $\mathcal{E}_\chi \times Q\mathcal{I}^{\sim}$. This will be helpful.

Elements $h \in Q\mathcal{I}^{\sim}$ admit an integral representation

$$h(g) = \int_{MA} dma L(ma) h'(gma) \quad \text{for} \quad h \in Q\mathcal{I}^{\sim} \tag{7.5}$$

with h' an infinitely differentiable vector valued function with compact support on G^*; dma is (right- and left) invariant Haar measure on MA. This integral representation makes covariance property (7.2) manifest.

According to (3.13), $f \in \mathcal{E}_\chi$ may also be considered as functions on G^* with values in V^l and admitting an integral representation

$$f(g) = \int_P dp\, \delta_P(p)^{-\frac{1}{2}} D^{\chi}(p^{-1})^* f'(gp)\,. \tag{7.6}$$

Here f' is an infinitely differentiable function on G^* with values in V^l and compact support, and dp is left-invariant Haar measure on $P=\Gamma_2 MAN$. Integration over P includes a summation over Γ_2. The measure dp is not right-invariant; instead $d(pp_1) =\delta_P(p_1)dp$ with modulus function [15]

$$\delta_P(p)=|a|^4 \quad \text{for} \quad p=\gamma man \in \Gamma_2 MAN. \tag{7.7}$$

Integral representation (7.6) fulfills the covariance condition (3.13) for arbitrary f'.

For the sesquilinear form $V^\chi[\cdot,\cdot]$ we may then make the general Ansatz

$$V^\chi[f, Q^{-1}h] = \int_{G^*\times G^*} dg dg' f'(g)^* t(g,g') h'(g') \quad \text{for} \quad h\in Q\mathscr{S}^\sim, f\in \mathscr{E}_\chi \tag{7.8}$$

and h', f' related to h, f by (7.5), (7.6), with a kernel $t(g,g')$ which maps $V^i\otimes V^j \mapsto V^l$. The kernel $t(g,g')$ is a generalized function on $G^*\times G^*$, but we will use functional notation as physicists always do.

Expression (7.8) must depend on h' only through h. If $h''(g)=L(b)h'(gb)$ with $b\in MA$ then h'' and h' determine the same h. Therefore we must require

$$t(g, g'b)L(b)^{-1} = t(g,g') \quad \text{for} \quad b\in MA. \tag{7.9a}$$

Similarly, $f''(g)=\delta_P(p)^{\frac{1}{2}} D^\chi(p^{-1})^* f'(gp)$ and $f'(g)$ determine the same f. Since $V^\chi[f,h]$ should depend on f' only through f we get the consistency condition

$$\delta_P(p)^{-\frac{1}{2}} D^\chi(p) t(gp, g') = t(g,g') \quad \text{for} \quad p\in P. \tag{7.9b}$$

From transformation law (7.3) and integral representation (7.5) we have

$$(T(g)h)(g') = h(g^{-1}g') = \int_{MA} db L(b) h'(g^{-1}g'b) \quad \text{for} \quad h\in Q\mathscr{S}^\sim.$$

Similarly from (3.14) and (7.6)

$$(T_\chi(g)f)(g') = f(g^{-1}g') = \int_P dp\, \delta_P(p)^{-\frac{1}{2}} D^\chi(p^{-1})^* f'(g^{-1}g'p) \quad \text{for} \quad f\in\mathscr{E}_\chi.$$

Therefore, G^*-invariance (6.12) reads

$$V^\chi[T_\chi(g_1)f, T(g_1)h] = \int dg dg' f'(g_1^{-1}g)^* t(g,g') h'(g_1^{-1}g) = V^\chi[f,h].$$

This requires

$$t(g_1 g, g_1 g') = t(g,g') \quad \text{for all} \quad g_1\in G^*. \tag{7.9c}$$

It remains to determine the general solution of Equations (7.9a)–(7.9c).

The general solution of (7.9c) is

$$t(g,g') = t^\#(g'^{-1}g) \tag{7.10}$$

with a (generalized) function $t^\#$ on G^* whose values are maps: $V^i\otimes V^j\mapsto V^l$. Covariance conditions (7.9a) and (7.9b) read then

$$t^\#(b^{-1}gp) = \delta_P(p)^{\frac{1}{2}} D^\chi(p)^{-1} t^\#(g) L(b) \quad \text{for} \quad b\in MA, p\in P. \tag{7.11}$$

Let us abbreviate $MA=H$ and let $P=\Gamma_2 MAN$ as before. We define a left action of $H\times P$ on G^* by

$$(b,p)\circ g = bgp^{-1} \quad \text{for} \quad p\in P, b\in H=MA.$$

Evidently this satisfies the group law $(p_1,b_1)\circ(p_2,b_2)\circ g=(p_1p_2,b_1b_2)\circ g$. The manifold G^* decomposes therefore into orbits under $H\times P$, and $H\times P$ acts transitively on each orbit. Let us determine the orbits.

Consider the action of $H\times P$ on cosets in $M_c^4=G^*/P$ and their elements. P acts transitively within each coset; therefore the problem reduces to determining the orbits in M_c^4 under H. Let us parametrize the finite points of M_c^4 by Minkowskian coordinates $x=(x^\mu)$ as in (3.3b). There are then three open orbits consisting respectively of positive timelike x, negative timelike x, and spacelike x. In addition there are several lower dimensional orbits. (They consist of the point $x=0$, pos. lightlike x, negative lightlike x, the unique point at spatial infinity of M^4, and the remaining points at infinity, respectively.)

Correspondingly, the open orbits on G^* consist of

$$G^*_+=\{g=n\tilde{}_x p \quad \text{with} \quad p\in P, \quad x \text{ pos. timelike}\}$$

$$G^*_-=\{g=n\tilde{}_x p \quad \text{with} \quad p\in P, \quad x \text{ neg. timelike}\}$$

$$G^*_\sim=\{g=n\tilde{}_x p \quad \text{with} \quad p\in P, \quad x \text{ spacelike}\}$$

and in addition there are several lower dimensional orbits.

Suppose that $t^\#(g)$ is known on one of the open orbits, say G^*_+. It is clear that $V(x\chi;x_1x_2)$ will then be determined on an open set of arguments. Analyticity properties (Lemma 7) can then be used to determine it everywhere.

Let us choose a standard $x^{\hat{}\mu}=(1,000)$. Correspondingly we select $n\tilde{}_{x^\wedge}$ as a standard point in G^*_+. Let us determine the little group of $n\tilde{}_{x^\wedge}$ in $H\times P$. The rotation group $U\subset M$ consists of $u\in M$ such that $ux^\wedge=x^\wedge$. Suppose $bn\tilde{}_{x^\wedge}p^{-1}=n\tilde{}_{x^\wedge}$. Consider this equation $\operatorname{mod}(P)$. It follows that $bx^\wedge=x^\wedge$. This requires $b\in U$. On the other hand $bn\tilde{}_{x^\wedge}p^{-1}=n\tilde{}_{bx^\wedge}bp^{-1}$. Therefore we must have $p=b$. In conclusion

$$bn\tilde{}_{x^\wedge}p^{-1}=n\tilde{}_{x^\wedge} \quad \text{for} \quad b\in H, p\in P \quad \text{if and only if} \quad (b,p)=(u,u), u\in U. \tag{7.12}$$

Thus the little group of $n\tilde{}_{x^\wedge}$ in $H\times P$ is isomorphic to the rotation group U.

Let $g\in G^*_+$. Then it can be written in the form

$$g=bn\tilde{}_{x^\wedge}p^{-1}=n\tilde{}_x bp^{-1} \quad \text{with} \quad b\in H, p\in P, x=bx^\wedge \quad \text{for} \quad g\in G^*_+. \tag{7.13}$$

Covariance condition (7.11) says that

$$t^\#(bn\tilde{}_{x^\wedge}p^{-1})=\delta_P(p)^{-\frac{1}{2}}D^\chi(p)t^\wedge L(b)^{-1} \quad \text{with} \quad t^\wedge=t^\#(n\tilde{}_{x^\wedge}). \tag{7.14}$$

For consistency, $t^\wedge$ must be U-invariant

$$t^\wedge=D^\chi(u)t^\wedge L(u)^{-1}\equiv D^l(u)t^\wedge[D^{l_i}(u)\otimes D^{l_j}(u)]^{-1} \qquad \text{for} \quad u\in U. \tag{7.15}$$

In other words, $t^\wedge$ is a U-invariant map $V^i\otimes V^j\mapsto V^l$. Next we will classify all such maps.

Finite dimensional irreducible representations of $M\simeq \mathrm{SL}(2\mathbb{C})\simeq \mathrm{Spin}(3,1)$ are constructed by analytic continuation (Weyls unitary trick) from UIR's of Spin(4), the twofold covering of SO(4). The Clebsch-Gordanology of both groups is therefore the same, and they contain U as a common subgroup.

Let us decompose $V^i \otimes V^j$ into irreducibles under M, $V^i \otimes V^j = \sum V^{l'}$, with Clebsch Gordon maps [5]

$$C(l_i l_j; l'): V^i \otimes V^j \mapsto V^{l'} . \tag{7.16}$$

Let us decompose representations l and l' of M into irreducible representations $s \in U^\wedge$ of U,

$$V^l = \sum_{\substack{s \in U^\wedge \\ s \subset l}} V^{ls} \quad \text{etc.} \tag{7.17}$$

We identify $V^{ls} = V^{l's} \equiv W^s$. Consider the projection operators $\pi(ls)$ and their adjoints, viz. U-invariant imbeddings $\pi^*(ls)$,

$$\pi(ls): V^l \mapsto W^s ; \quad \pi^*(l's): W^s \mapsto V^{l'} . \tag{7.18}$$

The most general U-invariant map from $V^i \otimes V^j \mapsto V^l$ is a linear combination

$$t^\wedge = \sum_{l's} c_{l's} t^{\wedge l's}, \qquad t^{\wedge l's} = \pi^*(l's)\pi(ls)C(l_i l_j; l) \tag{7.19}$$

sum over $l' \in M^\vee$, $s \in U^\wedge$ such that $l' \subset l_i \otimes l_j$, $s \subset l$, $s \subset l'$ with complex coefficients $c_{l's}$. $M^\vee$ is the set of all finite dimensional irreducible representations of M.

With this we have found the most general form of $t(g, g')$ for $g'^{-1} g \in G_+^*$. The result is given by Equations (7.10), (7.14) with (7.13), and (7.19). The sesquilinear form $V^\chi[f, h]$ on $\mathscr{E}_\chi \times \mathscr{I}^\sim$ is then determined by (7.8) for f, h having suitable support properties. It remains to recover the corresponding kernels $V(x\chi; x_1 x_2)$ and continue them analytically.

Let $h \in \mathscr{S}_2^\sim \subset \mathscr{I}^\sim$. Then on the one hand h is determined by a function $\dot{h}(x_1 x_2)$ of Minkowski space arguments x_1, x_2 by (6.5) and, on the other hand, it is also determined by $Qh(g)$ according to Lemma 9. Let us find the connection.

First we observe that

$$n^\sim_{x_2} p_2 = n^\sim_{x_1} p_1 \mathscr{R} \quad \text{for} \quad p_1 = n_{\mathscr{R}y},\ p_2 = p(y, \mathscr{R}),\ y = x_2 - x_1 \tag{7.20}$$

in the notation of (3.8), and p_1, p_2 are in P^0 for spacelike y by Lemma 5. It follows from covariance (6.3)

$$\begin{aligned} \dot{h}(x_1 x_2) &\equiv h(n^\sim_{x_1}, n^\sim_{x_2}) = \pi(p_1, p_2) h(n^\sim_{x_1} p_1, n^\sim_{x_2} p_2) \\ &= \pi(p_1, p_2) h(n^\sim_{x_1} p_1, n^\sim_{x_1} p \mathscr{R}) = \pi(p_1, p_2) Qh(n^\sim_{x_1} n_{\mathscr{R}y}) \end{aligned}$$

and $Qh(g)$ vanishes unless $g = n^\sim_{x_1} n_{\mathscr{R}y} \operatorname{mod}(MA)$ for some x_1 and spacelike y. In particular it vanishes if $g = n^\sim n m a \gamma$ with $\gamma \neq e$, $\gamma \in \Gamma_2$ etc.

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[5] Remember that we write V^i for V^{l_i}, the vector space which carries the irreducible representation l_i of M

If we write $g = n^{\sim} n \gamma m a$, then Haar measure $dg = dn^{\sim} dn dm da$ and $dn^{\sim}_x = dx$, $dn_{\mathscr{R}y} = d\mathscr{R}y = (-y^2)^{-4} dy$. If $Qh(g) = \int_{MA} db L(b) h'(gb)$ then

$$\int dg' t(g, g') h'(g') = \sum_{\gamma} \int dn^{\sim} dn db \;\; t(g, n^{\sim} n \gamma b) L(b) h'(n^{\sim} n \gamma b)$$

$$= \int_{N^{\sim} \times N} dn^{\sim} dn \;\; t(g, n^{\sim} n) Qh(n^{\sim} n) ,$$

because of the above mentioned support property. Thus finally

$$\int dg' t(g, g') h'(g') = \int \int (-y^2)^{-4} dy dx t(g, n^{\sim}_x n_{\mathscr{R}y}) \pi(p_1, p_2)^{-1} \dot{h}(x_1 x_2)$$

with $x_1 = x, x_2 = x + y, p_1 = n_{\mathscr{R}y}, p_2 = p(y, \mathscr{R})$.

Similarly, let $f \in \mathscr{E}_\chi$ and write $f(n^{\sim}_x) \equiv f(x)$. Splitting $g = n^{\sim} p$ with $p \in P$, the measure factorizes, $dg = dn^{\sim} dp$, as we have just said. Some integrations in (7.8) can therefore be carried out with the help of (7.6) and covariance condition (7.9b). As a result

$$\begin{aligned} V^\chi[f, h] &= \int \int \int (-y^2)^{-4} dy dx dz f(z)^* t(n^{\sim}_z, n^{\sim}_x n_{\mathscr{R}y}) \pi(p_1, p_2)^{-1} \dot{h}(x, x + y) \\ &= \int \int \int (x_2 - x_1)^{-8} dx_1 dx_2 dz f(z)^* t^{\#}(n_{\mathscr{R}(x_2 - x_1)} n^{\sim}_{z - x_1}) \pi(p_1, p_2)^{-1} \dot{h}(x_1 x_2) \end{aligned}$$

for $f \in \mathscr{E}_\chi$, $h \in \mathscr{S}_2^{\sim}$. Thus by comparison with (6.11b)

$$V(x_3 \chi; x_1 x_2) = (-x_{21}^2)^{-4} t^{\#}(n_{\mathscr{R}x_{21}}^{-1} n^{\sim}_{x_{31}}) \pi(p_1, p_2)^{-1} \tag{7.21}$$

with $p_1 = n_{\mathscr{R}x_{21}}, p_2 = p(x_{21}, \mathscr{R}); x_{ij} = x_i - x_j$.

It only remains to insert the previously derived expression for $t^{\#}$.

Evidently, $V(\cdots)$ is translationally invariant, i.e. depends only on coordinate differences. We may therefore put $x_1 = 0$.

According to definition (3.8)

$$n_{\mathscr{R}y}^{-1} n^{\sim}_z = n^{\sim}_{z'} p(z, n_{\mathscr{R}y})^{-1} = b n^{\sim}_{\hat{x}} p^{-1} \tag{7.22}$$

with $z' = n_{\mathscr{R}y}^{-1} z = \mathscr{R}(\mathscr{R}z - \mathscr{R}y)$ and $p = p(z, n_{\mathscr{R}y}) b$ provided $b\hat{x} = z', b \in MA$.

A suitable b in MA exists if y, z are such that z' is positive timelike. We have $z'^2 = (\mathscr{R}z - \mathscr{R}y)^{-2} = (z - y)^2 / z^2 y^2$. Since y is spacelike by hypothesis, we may put $y^0 = 0$ without loss of generality. We see that z' will be positive timelike if

$$z \text{ pos. timelike, } z - y \text{ spacelike or vice versa; } y \text{ spacelike.} \tag{7.23}$$

We restrict our attention to this case. It corresponds with the previous assumption that the argument of $t^{\#}$ is in the orbit G_+^*.

According to Definitions (7.1), (6.2)

$L(b) = \pi(b, b^{\sim})$ with

$$b^{\sim} = \mathscr{R} b \mathscr{R}^{-1} = m^{\sim} a^{-1} \quad \text{for} \quad b = ma \in MA, m^{\sim} = \theta m \theta^{-1} . \tag{7.24}$$

Expression (7.14) for $t^{\#}(\cdot)$ yields then

$$V(z\chi; 0y) = (y^2)^{-4} \delta_P(p)^{-\frac{1}{2}} D^\chi(p) \hat{t} \pi(p_1 b, p_2 b^{\sim})^{-1} \tag{7.25}$$

with same b, p, p_1, p_2 as before in (7.21), (7.22). This is valid for z, y as described in (7.23).

Expression (7.22) for p can be simplified. We use Lemma 5 repeatedly. $p=p(z,n_{\mathscr{R}y})b=p(z,\mathscr{R})p(\mathscr{R}z',\mathscr{R})b=p(z,\mathscr{R})b^{\check{}}p(b^{\check{}-1}\mathscr{R}z',\mathscr{R})=p(z,\mathscr{R})b^{\check{}}p(-x^{\wedge},\mathscr{R})$.

But $p(-x^{\wedge},\mathscr{R})=\gamma^{\wedge-1}\,\mathrm{mod}(N)$; $\gamma^{\wedge}=$ generator of Γ_2. Writing $p(z,\mathscr{R})=\gamma m_z a_z n$ as in Lemma 5 we obtain

$$p=\gamma^{\wedge r-1}m_z a_z b^{\check{}}(\mathrm{mod}\,N)\equiv\gamma^{\wedge r-1}m_z m^{\check{}} a_z a^{-1}(\mathrm{mod}\,N)$$

with $r=\mathrm{sign}\,z$; $|a|^2=z'^2=(z-y)^2/z^2y^2$; $|a_z|=|z^2|$. Similarly

$$p_1=n_{\mathscr{R}y}\,;\, p_2=m_y a_y(\mathrm{mod}\,N) \quad \text{with} \quad |a_y|=-y^2\,. \tag{7.26}$$

We will now introduce an M-covariant version of $t^{\wedge}$ in order to switch $D^{\chi}(p)$ through $t^{\wedge}$ in (7.25). According to spinor calculus, irreducible representations l of N may be labelled by their highest weight (j_1,j_2); j_1, j_2 half-integer. Completely symmetric tensor representations of rank j are labelled $(\frac{1}{2}j,\frac{1}{2}j)$ in this way.

Lemma 10. *Given three finite dimensional irreducible representations l, l_i, l_j of M, define*

$\Sigma=$ max. rank *of any completely symmetric tensor representation of M contained in the tensor product $l\otimes l_i\otimes l_j$.*

Consider linear maps $t(x): V^{l_i}\otimes V^{l_j}\mapsto V^l$ such that

1) *$t(x)$ is a homogeneous polynomial of x of degree Σ.*

2) *$t(x)$ are M-covariant in the sense that*

$$D^l(m)t(x)[D^{l_i}(m)\otimes D^{l_j}(m)]^{-1}=t(mx) \quad \textit{for} \quad m\in M\,.$$

All such are obtained from U-invariant maps $t^{\wedge}$ as were classified in (7.19) *by setting*

$$t(x)=|x^2|^{\frac{1}{2}\Sigma}D^l(m)t^{\wedge}[D^{l_i}(m)\otimes D^{l_j}(m)]^{-1} \tag{$*$}$$

for positive timelike $x=|x^2|^{\frac{1}{2}}mx^{\wedge}$.

Conversely let $t(x)$ defined by $(*)$ for positive timelike x. Then it can be analytically continued to all x and satisfies 1) *and* 2).

We shall relegate the proof of this lemma to Appendix A.

Since representations D^{χ} of MAN are trivial on N we have

$$D^{\chi}(p)=D^{\chi}(m_z a_z b^{\check{}}) \quad \text{for} \quad z\in V_+, \quad \text{with} \quad \Lambda(m_z)^{\mu}{}_{\nu}|a_z|=2z^{\mu}z^{\nu}-g^{\mu\nu}z^2$$

by Lemma 5. Moreover $(m_z a_z b^{\check{}} x^{\wedge})^{\mu}=(z'^2)^{-1}\Lambda(m_z)^{\mu}{}_{\nu}|a_z|(\theta z')^{\nu}$, whence

$$m_z a_z b^{\check{}} x^{\wedge}=\frac{(z-y)^2z^2}{y^2}\left\{\frac{z}{z^2}-\frac{z-y}{(z-y)^2}\right\}.$$

Now we are ready to use Lemma 10 to switch $D^{\chi}(p)$ through $t^{\wedge}$ in (7.25). At the same time we insert the definitions of $\delta_P(p)$ and $\pi(\cdot,\cdot)$. They give $\delta_P(p)=|a_z a^{-1}|^4$, $\pi(p_1 b, p_2 b^{\check{}})^{-1}=D^{\chi_i}(ma)^*\otimes D^{\chi_j}(m_y a_y m^{\check{}} a^{-1})^*\cdot|a_y|^2$. Altogether

$$V(z\chi;0y)=|y|^{-2+c-c_i-c_j-\Sigma}(|z||z-y|)^{-2-c+\Sigma}\left(\frac{|z|}{|z-y|}\right)^{c_j-c_i}$$

$$\cdot t\left(\frac{z}{z^2}-\frac{z-y}{(z-y)^2}\right)[D^{l_i}(m_z)\otimes D^{l_j}(m_z m^{\check{}} m^{-1} m_y^{\check{}-1})] \tag{7.27}$$

with $|y|=|y^2|^{1/2}$ etc. We use Lemma 5 again to evaluate the argument of D^{l_j}. One has $m_{x'}=\mathbb{1}$ and so $m=mm_{x'}=m_{mx'}m^{\sim}=m_{z'}m^{\sim}=m^{\sim}_{\mathcal{R}z-\mathcal{R}y}m^{\sim}$. Thus $m_z m^{\sim} m^{-1} m_y^{\sim-1}$ $=m_z m^{\sim-1}_{\mathcal{R}z-\mathcal{R}y} m_y^{\sim-1}=m_{z-y}$.

Irreducible representations l of $M\simeq \mathrm{SL}(2\mathbb{C})$ are extended to $\mathrm{GL}(2\mathbb{C})$ in a standard way. Suppose l has highest weight (j_1,j_2), then one defines

$$|l|=j_1+j_2\,;\; D^l(\varrho m)=\varrho^{2|l|}D^l(m) \quad \text{for} \quad m\in \mathrm{SL}(2\mathbb{C}),\ \varrho\in\mathbb{C}\,. \tag{7.28}$$

With this notation, Equation (7.27) becomes

$$V(z\chi;0y)=|y|^{-2+c-c_i-c_j-\Sigma}(|z||z-y|)^{-2-c+\Sigma-2|l_i|-2|l_j|}\left(\frac{|z|}{|z-y|}\right)^{c_j-c_i}$$

$$\cdot t\left(\frac{z}{z^2}-\frac{z-y}{(z-y)^2}\right)[D^{l_i}(\boldsymbol{z})\otimes D^{l_j}(\boldsymbol{z}-\boldsymbol{y})]\,. \tag{7.29}$$

This is valid for z positive timelike, y and $z-y$ spacelike. An expression for arbitrary arguments is obtained by using the spectrum condition, viz. Lemma 7. We note that expression (7.29) is real analytic in its domain of validity. This guarantees uniqueness of analytic continuation to the whole domain of holomorphy given in Lemma 7. As a result we have the following Proposition 11. Of course kernels $V(x\chi;x_1x_2)$ depend also on spin and dimension $\chi_i=[l_i,d_i]$, $\chi_j=[l_j,d_j]]$ of the fields $\phi^i(x_1)$, $\phi^j(x_2)$ whose product we want to expand. We shall therefore indicate this dependence by writing $V(x\chi;x_1\chi_i x_2\chi_j)=V(x\chi;x_1x_2)$.

Proposition 11. *Let $V(x_3\chi;x_1\chi_1x_2\chi_2)$ a 3-point function which satisfies the spectrum condition (Lemma 7) and which is conformal invariant in the sense explained earlier, with transformation law specified by $\chi_1=[l_1,2+c_1]$, $\chi_2=[l_2,2+c_2]$, $\chi=[l,2+c]$. [In this, c_1,c_2,c are real, l_1,l_2,l finitedimensional irreducible representations of $M\simeq\mathrm{SL}(2\mathbb{C})$ acting in vector spaces V^{l_1}, V^{l_2}, V^l.] Then*

$$V(x_3\chi;x_1\chi_1x_2\chi_2)=(-x_{12}^2)^{-\delta_{12}}(-x_{31}^2)^{-\delta_{31}}(-x_{32}^2)^{-\delta_{32}}$$

$$\cdot t\left(\frac{x_{31}}{(-x_{31}^2)}-\frac{x_{32}}{(-x_{32}^2)}\right)[D^{l_1}(\boldsymbol{x}_{31})\otimes D^{l_2}(\boldsymbol{x}_{32})] \tag{7.30}$$

with $\delta_{12}=\frac{1}{2}(2-c+c_1+c_2+\Sigma)$; $\delta_{31}+\delta_{32}=2+c-\Sigma+2|l_1|+2|l_2|$; $\delta_{31}-\delta_{32}=c_1-c_2$, and $t(x)$ are linear maps: $V^{l_1}\otimes V^{l_2}\mapsto V^l$ which satisfy the hypothesis of Lemma 10. $|l|$ etc. and Σ are defined in (7.28) *and Lemma 10; if $l_1\otimes l_2\otimes l$ does not contain a completely symmetric tensor-representation of M, then a conformal invariant 3-point function does not exist. An $i\varepsilon$-prescription is understood,*

$$(-x_{ij}^2)^\alpha=[-(x_i-x_j)^2+i\varepsilon(x_i^0-x_j^0)]^\alpha \tag{7.31}$$

$$\boldsymbol{x}=x^0\mathbb{1}+\sum_1^3 x^k\sigma^k,\ \sigma^k \text{ Pauli matrices.}$$

Expression (7.30) is a well defined distribution for arbitrary c, c_1, c_2.

Corollary 12. *Let $V(x\chi_k;x_1\chi_1x\chi_2)$ a conformal invariant 3-point function which satisfies the spectrum conditions for a 3-point Wightman function. Then it can be analytically continued to the permuted extended tube and satisfies all the Wightman* *axioms for a 3-point Wightman function* $(\Omega, \phi^k(x)^* \phi^i(x_1)\phi^j(x_2)\Omega)$ *of three possibly distinct local fields (with Lorentz spin and dimension* $\chi_k = [l_k, d_k]$ *etc.).*

When two of the fields are identical, the Wightman 3-point function has further symmetry properties. These are not automatically ensured by (7.30).

Remark. The kernels $V(x\chi; x_1\chi_1 x_2\chi_2)$ are *not* Clebsch Gordan kernels for the tensor product $\chi_1 \otimes \chi_2$ of UIR's of G^*. Indeed, states $\phi^1(x_1)\phi^2(x_2)\Omega$ transform in general according to a unitary representation of G^* which is not a Kronecker product, cp. epilogue of Ref. [5]. In particular it restricts to a nontrivial representation of the center of G^*, while for a Kronecker product of irreducible representations every element of the center would have to be represented by a multiple of the identity.

We add some remarks on zero mass representations. Most of the UIR's χ of G^* with positive energy have continuous mass spectrum, but there are also zero mass representations (cp. [7] and Proposition 6). A priori they could appear in the conformal partial wave expansion (4.2) and then also in the light cone expansion (1.2). We shall now argue that this only happens in exceptional cases[6].

Let us first discuss the meaning of this. Suppose $\phi(x)$ is a local field and $\Box\phi(x)\Omega = 0$. Then also $\Box\phi(x) = 0$ because a local field can never anihilate the vacuum. Therefore $\phi(x)$ is a free zero mass field. Appearance of zero mass representations in the conformal partial wave expansion would therefore mean that there appear massless free fields in the operator product expansion. This can happen. [Example: The expansion of the product of a massless free field $\phi(x)$ with its stress energy tensor must contain $\phi(x)$ again.] But it happens only in special cases. The reason lies in the nonexistence of a suitable 3-point function. Considered as functions of x, 3-point functions $V(x\chi; x_1\chi_1 x_2\chi_2)$ must be in the representation space $\mathscr{F}_\chi$. As such they must satisfy a spectrum condition. For continuous mass representations it says that the Fourier transform $V^\sim(p\chi; x_1\chi_1 x_2\chi_2)$ has support concentrated in the closed forward light cone, $p \in \bar{V}_+$. Because of the $i\varepsilon$-prescription, expression (7.30) satisfies this condition.

If χ is a zero mass representation, however, elements of $\mathscr{F}_\chi$ satisfy certain differential equations, in particular their Fourier transform is concentrated at $p^2 = 0$. Expression (7.30) does not meet this condition in general. Consider for instance the *scalar case* $l_1 = l_2 = \mathrm{id}$, $c = -1$: The Fourier transform $V^{0\sim}$ is given by Equation (8.4) below for this case [Caution: the limit $c \to -1$ must be taken with care in order not to lose contributions concentrated at $p^2 = 0$, cp. after (3.20′)]. We see that $V^{0\sim}$ cannot vanish identically for p in the interior of the forward light cone unless the argument of one of the Γ-functions in front is a nonpositive integer, i.e. $c_1 - c_2$ is an odd integer. More careful inspection reveals that $V^{0\sim}(p, -1; x_1 c_1 x_2 c_2)$ is concentrated at $p^2 = 0$ if and only if $c_1 - c_2 = \pm 1$.

8. Recovery of Kernels $\mathscr{B}^\sim(p\chi; x_1 x_2)$

We introduce the Fourier transform of 3-point functions with respect to the first argument

$$V^\sim(p\chi; x_1\chi_1 x_2\chi_2) = \int dx e^{ipx} V(x\chi; x_1\chi_1 x_2\chi_2) . \tag{8.1}$$

[6] This observation originates in a remark made by Castell some years ago [17]

The kernels $\mathscr{B}^{\sim}$ are obtained from them by Equation (4.1), viz.

$$V^{\sim}(p\chi;x_1\chi_1x_2\chi_2)=\Delta_+^{\chi}(p)\mathscr{B}^{\sim}(p\chi;x_1\chi_1x_2\chi_2)\,, \tag{8.2}$$

where $\Delta_+^{\chi}(p)$ is the Fourier transform of the 2-point function (intertwining kernel) (3.20). As we discussed earlier (in Sec. 2), kernels $\mathscr{B}$ are nonunique and determined only to the extent that (8.2) determines them.

We consider the scalar case first. We introduce a special notation for this case

$$\begin{aligned} &V^0(xc;x_1c_1x_2c_2)=V(x\chi;x_1\chi_1x_2\chi_2)\\ &\mathscr{B}^0(xc;x_1c_1x_2c_2)=\mathscr{B}(x\chi;x_1\chi_1x_2\chi_2) \quad \text{etc.}\end{aligned} \tag{8.3}$$

with

$$\chi_1=[\mathrm{id},2+c_1]\,,\qquad \chi_2=[\mathrm{id},2+c_2]\,,\qquad \chi=[\mathrm{id},2+c]\,,$$

where id stands for the trivial 1-dimensional representation of M.

From Proposition 11 we obtain [same Notation (7.31)]

$$\begin{aligned} &V^{0\sim}(pc;x_1c_1x_2c_2)\\ &=\int dx_3 e^{ipx_3}(-x_{12}^2)^{\frac{1}{2}(c-c_1-c_2-2)}(-x_{31}^2)^{\frac{1}{2}(-c-c_1+c_2-2)}(-x_{32}^2)^{\frac{1}{2}(-c+c_1-c_2-2)}\\ &=\Gamma(c+2)\Gamma\left(\frac{c+c_1-c_2+2}{2}\right)^{-1}\Gamma\left(\frac{c-c_1+c_2+2}{2}\right)^{-1}(-x_{12}^2)^{\frac{1}{2}(c-c_1-c_2-2)}\\ &\cdot\int_0^1 du\, u^{\frac{1}{2}(c+c_1-c_2)}(1-u)^{\frac{1}{2}(c-c_1+c_2)}\\ &\cdot\int dx e^{ipx}[-(z(u)-x)^2-u(1-u)x_{12}^2-i\varepsilon(z(u)^0-x^0)]^{-c-2}\end{aligned}$$

with $z(u)=ux_1+(1-u)x_2$. The second equation was obtained by inserting the standard integral representation

$$A^{-\nu}B^{-\mu}=\Gamma(\nu+\mu)\Gamma(\nu)^{-1}\Gamma(\mu)^{-1}\int_0^1 du\ u^{\nu-1}(1-u)^{\mu-1}[uA+(1-u)B]^{-\nu-\mu}\,.$$

The Fourier transform of the generalized function $[-x^2+a^2+i\varepsilon x^0]^{-\lambda}$ is well known for $a^2>0$, and so we obtain, for $x_{12}^2<0$

$$\begin{aligned} &V^{0\sim}(pc;x_1c_1x_2c_2)\\ &=2\pi^3\Gamma\left(\frac{c+c_1-c_2+2}{2}\right)^{-1}\Gamma\left(\frac{c-c_1+c_2+2}{2}\right)^{-1}(-x_{12}^2)^{-\frac{1}{2}(c_1+c_2+2)}\\ &\cdot\theta(p)\int_0^1 du\left(\frac{u}{1-u}\right)^{\frac{1}{2}(c_1-c_2)}e^{ip[ux_1+(1-u)x_2]}\\ &\cdot(\tfrac{1}{4}p^2)^{c/2}J_c([-u(1-u)x_{12}^2p^2]^{1/2})\,.\end{aligned} \tag{8.4}$$

J_c is the Bessel function; $\theta(p)=1$ for $p\in\bar{V}_+$ and 0 otherwise. The u-integral is regularized by analytic continuation in c [16]. Validity of (8.4) for arbitrary x_1, x_2

follows by uniqueness of analytic continuation. Dividing by $\Delta^{\chi}(p)$ we obtain finally

$$\begin{aligned}&\mathscr{B}^{0\sim}(pc;x_1c_1x_2c_2)\\&=n'_+(c)^{-1}2\pi^3\Gamma\left(\frac{c+c_1-c_2+2}{2}\right)^{-1}\Gamma\left(\frac{c-c_1+c_2+2}{2}\right)^{-1}(-x_{12}^2)^{-\frac{1}{2}(c_1+c_2+2)}\\&\cdot\int_0^1 du\left(\frac{u}{1-u}\right)^{\frac{1}{2}(c_1-c_2)}e^{ip[ux_1+(1-u)x_2]}\\&\cdot(4p^2)^{-c/2}J_c([-u(1-u)x_{12}^2p^2]^{1/2})\\&\text{for}\quad p\in\bar{V}_+,\quad\text{with } i\varepsilon\text{-prescription (7.31)}\end{aligned}\tag{8.5}$$

with a constant $n'_+(c)$ which is determined by the normalization of the scalar 2-point function, cp. Equation (3.20').

We see by inspection that $\mathscr{B}^{0\sim}$ has the holomorphy properties in p which were stated in Proposition 4. It is equal to $(-x_{12}^2+i\varepsilon x_{12}^0)^{\alpha}$ times an entire holomorphic function in x_1, x_2 and p, and so it is a generalized function in $\mathscr{D}'$ (notation of [16]) of x_1 and x_2 which is holomorphic in the parameter p.

Let us now turn to the general case. The first two assertions of Proposition 4 are clear from Equation (8.2), Lemma 7 and Proposition 11, viz. the classification of 3-point functions V. It remains to demonstrate holomorphy in p. This can be simplified very much by remembering once more the arguments of Section 7.

Let $h(x_1x_2)$ an arbitrary Schwartz test function with values in the dual of $V^{l_1}\otimes V^{l_2}$ and

$$\mathscr{B}_h^{\chi}(x)=\int dx_1dx_2\mathscr{B}(x\chi;x_1\chi_1x_2\chi_2)\mathrm{h}(x_1x_2)\,. \tag{8.6}$$

The kernels $\mathscr{B}$ have the following properties which define them [Eq. (8.2) is a consequence, cp. Sec. 4].

1. As functions of x_1 and x_2 kernels $\mathscr{B}(x\chi;x_1\chi_1x_2\chi_2)$ transform in the same way as $V(x\chi;x_1\chi_1x_2\chi_2)$. [I.e. they are both restrictions of cross sections on the same homogeneous vector bundle over $\boldsymbol{M}\times\boldsymbol{M}$, at least for $x_{12}^2<0$, cp. Sec. 6.]

2. As functions of x, kernels $\mathscr{B}(x\chi;x_1\chi_1x_2\chi_2)$ transform like elements of $\mathscr{E}_\chi$. The smeared kernels $\mathscr{B}_h^{\chi}(x)$ are in the Hilbert space $\mathscr{E}_\chi$, viz.

$$(\mathscr{B}_h^{\chi},\mathscr{B}_h^{\chi})=\int dp\mathscr{B}_h^{\chi\sim}(p)^*\cdot\Delta_+^{\chi}(p)\mathscr{B}_h^{\chi\sim}(p)<\infty\,. \tag{8.7}$$

3. Kernels $\mathscr{B}(x\chi;x_1\chi_1x_2\chi_2)$ are conformal invariant.

The statement of the transformation laws 1. and 2. gives meaning to 3.

Let f a function in the representation space $\mathscr{F}_{\chi^\sim}$, $\chi^\sim=[l^\sim,2-c]$, c real, and define $f'(x)=f(-x)$. Then f' transforms like an element of $\mathscr{E}_\chi$, $\chi=[l,2+c]$. This is seen by comparing Equations (3.15) and (3.19) and noting that the phase factor $e^{i\pi Nc}$ in definition (3.12) can be reverted by a space time reflection $\Pi\theta$. (It takes $\tau\to-\tau$, $\varepsilon\to-\varepsilon$ in the notation of Section 3), while

$$D^l(m)^*\equiv D^{l\sim}(m)^{-1}=(-)^{2|l|}D^{l\sim}(\Pi\theta m\theta\Pi)^{-1}\,.$$

It follows that $\mathscr{B}(x\chi;\,x_1\chi_1x_2\chi_2)$ transforms in the same way as a function of x as $V(-x\chi^\sim;\,-x_1\chi_1-x_2\chi_2)$. They are both conformal invariant and they also have the same transformation law as functions of x_1, x_2. This is so because

$V(-x\chi^{\sim}; -x_1\chi_1 - x_2\chi_2)$ transforms in the same way as a function of x_1, x_2 as $V(x\chi; x_1\chi_1 x_2\chi_2)$ [for $(x_1-x_2)^2<0$], since only the restriction of representation D^χ to $P^0=MAN$ enters now [cp. Eqs. (6.2) and (6.3)] for which the phase factor $e^{i\pi Nc}$ in (3.12) is absent.

In conclusion, $\mathscr{B}(x\chi; x_1\chi_1 x_2\chi_2)$ has the same conformal covariance properties as $V(-x\chi^{\sim}; -x_1\chi_1 - x_2\chi_2)$. .

Moreover, we see from Proposition 11 that

$$V(x\chi; x_1\chi_1 x_2\chi_2) = t(x_{32}^2 x_{31} - x_{31}^2 x_{32}) V^0(xc', x_1c_1' x_2c_2')$$
$$c' = c + \Sigma + 2|l_1| + 2|l_2|; \quad c_k' = c_k + \Sigma + |l_1| + |l_2| \quad (k=1,2) \tag{8.8}$$

with $t(x)$ a matrix valued polynomial which satisfies the hypothesis of Lemma 10. This motivates the Ansatz

$$\mathscr{B}(x\chi; x_1\chi_1 x_2\chi_2) = t(x_{31}^2 x_{32} - x_{32}^2 x_{31}) \mathscr{B}^0(xc'; x_1c_1' x_2c_2')$$
$$-c' = -c + \Sigma + 2|l_1| + 2|l_2|; \tag{8.8a}$$
$$c_k' = c_k + \Sigma + |l_1| + |l_2| \quad (k=1,2); \ \chi = [l, 2+c] \text{ etc.}$$

with $t(x)$ a matrix valued polynomial which satisfies the hypothesis of Lemma 10 (with $l^{\sim}, l_1, l_2$ substituted for l, l_i, l_j). Correspondingly

$$\mathscr{B}^{\sim}(p\chi; x_1\chi_1 x_2\chi_2) = t(x_{31}^2 x_{32} - x_{32}^2 x_{31}) \mathscr{B}^{0\sim}(pc'; x_1c_1' x_2c_2') \tag{8.8b}$$

where now $x_{32} \equiv -i\dfrac{\partial}{\partial p} - x_2$; $x_{31} \equiv -i\dfrac{\partial}{\partial p} - x_1$.

It is clear that this defines an entire function of p because the same is true of $\mathscr{B}^{0\sim}$, and application of a differential operator of finite order cannot destroy holomorphy.

Therefore, Proposition 4 will be proven if we can show that Ansatz (8.8a) is general and satisfies the conformal covariance requirements 1 – 3 supra and (8.7). It suffices to do so for relatively spacelike x_1, x_2 because $\mathscr{B}^{\sim}(p\chi; x_1\chi_1 x_2\chi_2)$ shares the analyticity properties (Lemma 7) in x_1 and x_2 for $p \in \text{sptr.}\chi$.

Concerning generality, we only have to count. Given χ, χ_1, χ_2, there are according to Equation (8.2) as many linearly independent kernels $\mathscr{B}^{\sim}(p\chi; x_1\chi_1 x_2\chi_2)$ as 3-point functions $V(x\chi; x_1\chi_1 x_2\chi_2)$. In view of Proposition 11 it only remains to verify that the number of linearly independent polynomials $t(x)$ satisfying the hypothesis of Lemma 10 remains unchanged when $l^{\sim}$ is substituted for l. These polynomials $t(x)$ are in one to one correspondence with U-invariant maps $t^{\wedge}$. The vector spaces $V^{l^{\sim}}$ and V^l are the same, and representations $l^{\sim}$ and l agree on U. Therefore every U-invariant map $t^{\wedge}: V^{l_1} \otimes V^{l_2} \mapsto V^l$ is at the same time a U-invariant map from $V^{l_1} \otimes V^{l_2}$ to $V^{l^{\sim}}$ and vice versa. Therefore there are a fortiori equally many linearly independent ones.

Next we discuss finiteness condition (8.7). It follows from (7.30) that $V^{\sim}(p\chi; x_1\chi_1 x_2\chi_2)$ is a tempered distribution, and therefore, by (8.2) and (8.8b), $\mathscr{B}^{\sim}(p\chi; x_1\chi_1 x_2\chi_2)$ is polynomially bounded in p for $p \in \bar{V}_+$. The Fourier transform $h^{\sim}(p_1 p_2)$ of any Schwartz test function $h(x_1 x_2)$ falls off faster than any power of total momentum p_1+p_2. Because of momentum conservation (translation invariance), also $\mathscr{B}_h^{\check{\chi}}(p)$ falls then off faster than any power of p for $p\in\bar{V}_+$. Since it is also ∞ differentiable (even holomorphic) in p, it agrees with a test function on the support of $\Delta_+^\chi(p)$, and therefore $\int dp\mathscr{B}_h^{\check{\chi}}(p)^*\cdot\Delta_+^\chi(p)\mathscr{B}_h^{\check{\chi}}(p)<\infty$. This proves (8.7).

We turn to conformal covariance of Ansatz (8.8a). We need only consider the case of relatively spacelike x_1 and x_2.

According to the discussion of Section 7, conformal invariant 3-point functions are determined by matrix-valued functions $t^\#(g)$ on the group G^* which satisfy a covariance condition, viz. (7.11). G^* decomposes intro three open orbits G_+^*, G_-^* and $G_\sim^*$ plus some lower dimensional submanifolds. On each of the orbits $t^\#(g)$ is fixed once it is known at one point. Conformal invariance alone does not relate the values of $t^\#(g)$ on different orbits however. We showed that $t^\#(g)$ for $g\in G_+^*$ determines the 3-point function for arguments x_1, x_2, x such that $x-x_1$ is positive timelike and $x-x_2$ spacelike, or vice versa (x_1-x_2 is spacelike by hypothesis). G_-^* is obtained from G_+^* by space-time reflection, and $G_\sim^*$ is the open interior of what is left. Let us introduce step functions to match

$$\theta_\pm(x;x_1x_2)=\begin{cases}1 \text{ if } x-x_1\in\bar{V}_\pm,\ x-x_2 \text{ spacelike, or vice versa,}\\ 0 \text{ otherwise.}\end{cases}$$

$$\theta_\sim(x;x_1x_2)=\begin{cases}1 \text{ if } \operatorname{sign}(x-x_1)=\operatorname{sign}(x-x_2),\\ 0 \text{ otherwise.}\end{cases}$$

$\operatorname{sign}x$ is defined to be ± 1 if $x\in\bar{V}_\pm$, and 0 if x is spacelike. Note that $x-x_1\in V_+$, $x-x_2\in V_-$ is impossible if x_1-x_2 is spacelike, therefore $\theta_++\theta_-+\theta_\sim\equiv 1$ for all x.

It follows from the orbit structure that

$$[a_1\theta_+(x;x_1x_2)+a_2\theta_-(x;x_1x_2)+a_3\theta_\sim(x;x_1x_2)]V(-x\check{\chi};-x_1\chi_1-x_2\chi_2) \quad (8.9)$$

has the same conformal covariance properties as $V(-x\check{\chi},-x_1\chi_1-x_2\chi_2)$ for arbitrary constants a_1, a_2, a_3 (for x_1-x_2 spacelike). Moreover, in the scalar case $l_1=l_2=l=id$, expression (8.9) is the most general conformal invariant 3-point function because then $t^\#(g)$ at any point $g\in G^*$ is simply a number. It follows that the kernel $\mathscr{B}(x\chi, x_1\chi_1x_2\chi_2)$ is of the form (8.9) in the scalar case. But then the Ansatz (8.8a) ensures that the same is true in general, because of identity (8.8) for V, and so the Ansatz (8.8a) is indeed conformal invariant.

There is one technical subtlety involved here. Our discussion so far has been for singular functions of x_1, x_2, and x, that is functions which are defined everywhere exept on some lower dimensional submanifolds. What we need is distributions, though. So the question arises whether there exists a conformal invariant regularization. The regularization is unique (within the limits discussed in Sect. 2) if it exists, because $\mathscr{B}^{\check{}}(p\chi;x_1\chi_1x_2\chi_2)$ is boundary value of an analytic function of x_1 and x_2 for $p\in\operatorname{sptr.}\chi$. For some range of c, it is an integrable function of x_1 and x_2. Elsewhere it can be defined by analytic continuation in c. Explicit expressions (8.8b) and (8.5) show that this is possible[7], at least after a change of normalization has been effected through multiplication by $n'_+(c')$.

[7] This is consistent with the remark at the end of Section 7 since $\mathscr{B}^{\check{}}(p\chi;x_1\chi_1x_2\chi_2)$ may vanish at $p^2=0$

In conclusion, we have found the kernels $\mathscr{B}^{\hat{}}(p\chi_k; x_1\chi_i x_2\chi_j) = \mathscr{B}^{kij}(p; x_1 x_2)$ which enter into the conformal partial wave expansion of Section 2. They are given explicitly by Equations (8.8b), (8.5), and Lemma 10, and they have the properties listed in Proposition 4.

Acknowledgement. The author is indebted to M. Lüscher, B. Schroer, and I. Todorov for discussions.

Appendix A. Proof of Lemma 10

The first part is easy. Given $t(x)$, define $t^{\hat{}}$ by $t^{\hat{}} = t(x^{\hat{}})$. Then $t^{\hat{}}$ is U-invariant by covariance Condition ii), and formula (7.27) follows from (ii) and homogeneity. As for the converse, we note first that definition (7.27) of $t(x)$ for positive timelike x makes sense, i.e. $t(x)$ depends on m only through $mx^{\hat{}} = x/|x^2|^{1/2}$ because $t^{\hat{}}$ is invariant under the little group U of $x^{\hat{}}$. It remains to show that $t(x)$ is a polynomial.

Let $E = L(V^l, V^{l_i} \otimes V^{l_j})$ the vector space of all linear maps from V^l to $V^{l_i} \otimes V^{l_j}$. It carries a representation of M given by $D(m)v = D^l(m)v[D^{l_i}(m) \otimes D^{l_j}(m)]^{-1}$. This representation is isomorphic to the tensor product $l \otimes l_i \otimes l_j$. Because of Fermi-superselection rule, it is a 1-valued representation of $M/\Gamma_1 \simeq SO_e(3,1)$. It may therefore be decomposed into irreducibles which are all tensor representations of M. Thus $E = \oplus E^k$, sum over irreducible representations of M contained in $l \otimes l_i \otimes l_j$, with multiplicities. $t^{\hat{}}$ is a U-invariant vector in E, it decomposes as $t^{\hat{}} = \sum c_k v^k$ with complex coefficients c_k, and v^k a normalized U-invariant vector in E^k.
Such a vector exists only if E^k carries a completely symmetric tensor representation; let its rank also be denoted by k. The components of the vectors $\mathfrak{Y}_k(mx^{\hat{}}) = D(m)v^k$ are called spherical functions for M. It is well known that $\mathfrak{Y}_k(x) = |x^2|^{k/2}\mathfrak{Y}_k(x/\sqrt{x^2})$ are polynomials. So $t^{\hat{}} = \sum c_k |x^2|^{\frac{1}{2}(\Sigma - k)}\mathfrak{Y}_k(x)$. According to Weyl's unitary trick, representations of $M/\Gamma_1 \simeq SO_e(3,1)$ are obtained from representations of SO(4) by analytic continuation. SO(4) has nontrivial center, $SO(4)/\mathbb{Z}_2 \approx SO(3) \times SO(3)$. Now $l \otimes l_i \otimes l_j$ either comes from a one-valued or from a two-valued representation of $SO(3) \times SO(3)$. In the first (second) case it contains only completely symmetric tensor representations of even (odd) rank k. In any case $\Sigma - k$ is always even and $\geqq 0$, because Σ is the maximal value of k by definition. This shows that $t(x)$ is a polynomial.

References

1. Wilson, K.: Phys. Rev. **179**, 1499 (1969)
2. Zimmermann, W.: Commun. math. Phys. **15**, 208 (1969); Lectures on elementary particles and quantum field theory. Brandeis University, Vol. I. Cambridge, Mass.: MIT press 1970; Lowenstein, J. M.: Phys. Rev. **D4**, 2281 (1971)
3. Mack, G.: Nucl. Phys. **B5**, 592 (1971)
4. Ferrara, S., Gatto, R., Grillo, A. F.: Springer tracts in modern physics, Vol. 67. Berlin-Heidelberg-New York: Springer 1973, and references therein; Ann. Phys. (N. Y.) **76**, 116 (1973); see also Ferrara, S., Grillo, A. F., Parisi, G., Gatto, R.: Nucl. Phys. **B49**, 77 (1972); Lett. Nuovo Cimento **4**, 115 (1972)
5. Lüscher, M., Mack, G.: Commun. math. Phys. **41**, 203 (1975)
6. Abdus Salam, Mack, G.: Ann. Phys. (N. Y.) **53**, 174 (1969)
7. Mack, G.: All unitary ray representations of the conformal group SU(2, 2) with positive energy. DESY 75/50, submitted to Commun. math. Phys.

8. Lüscher, M.: Commun. math. Phys. **50**, 23—52 (1976)
9. Rühl, W., Yunn, B.C.: Commun. math. Phys. **48**, 215—233 (1976); Kupsch, J., Rühl, W., Yunn, B.C.: Ann. Phys. (N. Y.) **89**, 115 (1975)
10. Mack, G.: Duality in quantum field theory. Nucl. Phys. (in press)
11. Mack, G.: J. Phys. (Paris) **34**, 99 (1973); Caianello, E.R.: Renormalization and invariance in quantum field theory. New York: Plenum Press 1974; Dobrev, V., Petkova, V., Petrova, S., Todorov, I.: Dynamical derivation of vacuum operator product expansions in conformal quantum field theory. IAS preprint, Princeton 1976 (to be publ. in Phys. Rev.); Dobrev, V., Mack, G., Petkova, V., Petrova, S., Todorov, I.: Harmonic analysis on the n-dimensional Lorentz group and its application to conformal quantum field theory (to be publ. in Lecture Notes in Physics)
12. Rühl, W.: Commun. math. Phys. **30**, 287; **34**, 149 (1973)
13. Wallach, N.: Harmonic analysis on homogeneous spaces. New York: Marcel Dekker Inc. 1973
14. Warner, G.: Harmonic analysis on semisimple Lie groups, Vol. I, Chapter 5. Berlin-Heidelberg-New York: Springer 1972; Strathdee, J., Boyce, J.F., Delbourgo, R., Abdus Salam: Partial wave analysis, Part I. Trieste IC/67/9, Feb. 1967
15. Dieudonné, J.: Treatise on analysis, Vol. 2. New York: Academic Press 1970
16. Gelfand, I.M., Shilov, G.E.: Generalized functions, Vol. I. New York: Academic Press 1964
17. Castell, L.: Talks presented at the Symposium on De Sitter and conformal groups. University of Colorado, Summer 1970

Communicated by R. Haag

Received October 4, 1976