CONVERGENCE OF POLYNOMIAL ERGODIC AVERAGES

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Abstract. We prove the L^2 convergence for an ergodic average of a product of functions evaluated along polynomial times in a totally ergodic system. For each set of polynomials, we show that there is a particular factor, which is an inverse limit of nilsystems, that controls the limit behavior of the average. For a general system, we prove the convergence for certain families of polynomials.

1. Introduction

Bergelson and Leibman generalized Furstenberg's celebrated proof [F77] of Szemerédi's Theorem:

Theorem (Bergelson and Leibman [BL96])**.** *Let $(X, \mathcal{X}, \mu, T)$ be an invertible probability measure preserving system, let $\ell \geq 1$ be an integer and let $p_1(n)$, $p_2(n)$,..., $p_\ell(n)$ be polynomials taking integer values on the integers with $p_j(0) = 0$ for $j = 1, 2, \ldots, \ell$. If $A \in \mathcal{X}$ with $\mu(A) > 0$, then*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu\big(T^{p_1(n)}A \cap T^{p_2(n)}A \cap \ldots \cap T^{p_\ell(n)}A\big) > 0\ .$$

Furstenberg's Theorem corresponds to the case that all polynomials are degree one. Recently in [HK02], we proved that lim inf in Furstenberg's Theorem is actually a limit. Here we show that the same result holds for the polynomial version in a totally ergodic system and in an arbitrary system under some restrictions on the polynomials.

1.1. Statement of the result.

By *integer polynomial* we mean a polynomial in one variable taking integer values on the integers.

We prove a result of convergence in L^2 for a product of bounded measurable functions evaluated along polynomial times:

Theorem 1. *Let $(X, \mathcal{X}, \mu, T)$ be an invertible measure preserving system, let $\ell \geq 1$ be an integer and let $p_1(n)$, $p_2(n)$,..., $p_\ell(n)$ be integer*

The second author was partially supported by NSF grant DMS-0244994.

polynomials. If $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$ and $\{M_i\}$, $\{N_i\}$ are two sequences of integers with $N_i \to +\infty$, then:
(a) *The averages*

$$\frac{1}{N_i}\sum_{n=M_i}^{M_i+N_i-1}\int f_1(T^{p_1(n)}x)f_2(T^{p_2(n)}x)\dots f_\ell(T^{p_\ell(n)}x)\,d\mu(x) \tag{1}$$

converge as $i \to +\infty$.
(b) *Assume additionally that at least one of the following conditions holds:*

(i) *The system $(X, \mathcal{X}, \mu, T)$ is totally ergodic.*
(ii) *The polynomials $\{p_1, p_2, \dots, p_\ell\}$ are all of degree > 1.*
(iii) *The polynomials $\{p_1, p_2, \dots, p_\ell\}$ are all of degree 1.*

Then the averages

$$\frac{1}{N_i}\sum_{n=M_i}^{M_i+N_i-1} f_1(T^{p_1(n)}x)f_2(T^{p_2(n)}x)\dots f_\ell(T^{p_\ell(n)}x) \tag{2}$$

converge in $L^2(\mu)$ as $i \to +\infty$.

The case that the system is not totally ergodic and at least one polynomial is of degree 1 and at least one other is of higher degree remains open.

If one assumes that T is weakly mixing, V. Bergelson [Be87] showed that the limit in (2) exists and is constant. However, without the assumption of weak mixing one can easily show that the limit need not be constant, even for linear polynomials. Recently, N. Frantzikinakis and the second author [FK03] have shown that the limit is constant when the system is totally ergodic and the polynomials have no non-trivial relations over $\mathbb{Q}$.

In [FW96], H. Furstenberg and B. Weiss proved the existence of the limit for the pair of polynomials n and n^2. For the family of linear polynomials $\{n, 2n, \dots, \ell n\}$, the existence of the limit in (2) is proven by the authors in [HK02].

1.2. **Sketch of the proof.** We can clearly assume that the functions are real valued. Moreover we can assume:

($\mathcal{H}$) *The polynomials $p_i(n)$ in Theorem 1 are not constant and the polynomials $p_i(n) - p_j(n)$ are not constant for all $i \neq j$.*

By using the ergodic decomposition of μ when needed, we can assume that the system is ergodic.

The proof combines three ingredients. We start with an induction similar to the PET induction of Bergelson in [Be87] to show that that the limit behavior is determined by some factor (known as the *characteristic factor*). We show that these factors are of the form of the factors $Z_k(X)$ introduced in [HK02] and we use properties proven in [HK02] to describe them as inverse limits of nilsystems. Lastly, we apply a recent result of Leibman [L02] to obtain the convergence of polynomial averages on nilsystems.

Using our current method, unfortunately we are unable to eliminate the hypothesis of total ergodicity for the general case of convergence in norm. For a system which is not totally ergodic, the estimates we use to show that a factor is characteristic depend on the specific polynomial family, making the use of the Van der Corput Lemma in the PET induction impossible in the general case.

1.3. **Notation.** In general, we write (X, μ, T) for a measure preserving system, omitting explicit mention of the σ-algebra. We abbreviate 'measure preserving system' as 'system'.

Let (X, μ, T) be a system. Every subset of X is implicitly assumed to be measurable. Every function on X is implicitly assumed to be measurable and real valued. For a function f defined on a system X and an integer p, we use the standard shorthand of $T^p f$ instead of the more cumbersome $f \circ T^p$.

A *factor* of (X, μ, T) is a system (Y, ν, S), given with a measurable map $\pi : X \to Y$ so that $\pi\mu = \nu$ and $S \circ \pi = \pi \circ T$. For $f \in L^1(\mu)$, we consider $\mathbb{E}(f \mid Y)$ as a function on Y; it is defined by the relation

$$\int_Y \mathbb{E}(f \mid Y)(y)\, g(y)\, d\nu(y) = \int_X f(x)\, g(\pi(x))\, d\mu(x) \text{ for every } g \in L^\infty(\nu)\,.$$

For an integer $k \geq 0$, we write $X^{[k]}$ for X^{2^k} and the points of this space are written as $\mathbf{x} = (x_j : 0 \leq j < 2^k)$. We write $T^{[k]}$ for the transformation $T \times T \times \cdots \times T$ (2^k times) of $X^{[k]}$. For $k \geq 1$ we often identify $X^{[k]}$ with $X^{[k-1]} \times X^{[k-1]}$ in the natural way and write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ for a point of $X^{[k]}$, with $\mathbf{x}', \mathbf{x}'' \in X^{[k-1]}$; thus we have $T^{[k]} = T^{[k-1]} \times T^{[k-1]}$.

2. Preliminaries

2.1. **Nilsystems.** Let G be a Lie group. The lower central series

$$G^{(1)} \supset G^{(2)} \supset \cdots \supset G^{(i)} \supset G^{(i+1)} \supset \ldots$$

of G is defined by

$$G^{(1)} = G \text{ and } G^{(i+1)} = [G^{(i)}, G] \text{ for } i \geq 1\,.$$

This means that $G^{(i+1)}$ is the subgroup of G spanned by

$$\{g^{-1}h^{-1}gh : g \in G^{(i)},\ h \in G\}\ .$$

The group G is said to be a *k-step nilpotent* group if $G^{(k+1)} = \{1\}$.

Let G be a k-step nilpotent Lie group and let Λ be a discrete cocompact subgroup of G. The compact manifold $X = G/\Lambda$ is called a *k-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g, x) \mapsto g \cdot x$. The unique Borel probability measure μ on X invariant under this action is called the *Haar measure* of X. Let a be a fixed element of G and let $T : X \to X$ be given by $Tx = a \cdot x$. The system (X, μ, T) is called a *k-step nilsystem* or a *translation on a nilmanifold*.

Ergodic properties of nilsystems have been widely studied; see in particular [AGH63], [Pa69], [Pa70] and [Le91]. More recently, the following theorem was shown by Leibman:

Theorem (Leibman [L02]). *Let (X, μ, T) be a nilsystem, $p_1(n), p_2(n), \ldots,$ $p_\ell(n)$ be integer polynomials and $f_1, f_2, \ldots, f_\ell$ be continuous functions on X. Then, for all sequences of integers $\{M_i\}$ and $\{N_i\}$ with $N_i \to +\infty$, the averages*

$$\frac{1}{N_i} \sum_{n=M_i}^{M_i+N_i-1} f_1(T^{p_1(n)}x) f_2(T^{p_2(n)}x) \ldots f_\ell(T^{p_\ell(n)}x)$$

converge for every $x \in X$.

Corollary 1. *The statement of Theorem 1 holds for nilsystems.*

Note that for nilsystems, the result holds without the assumption of total ergodicity and without restrictions on the polynomials.

2.2. The seminorms $\|\!|\cdot\|\!|_k$ and the factors $Z_k(X)$.

In this Section, (X, μ, T) is an ergodic system. We review a construction and some results of Section 3 of [HK02].

For every integer $k \geq 0$, we define a probability measure $\mu^{[k]}$ on $X^{[k]}$, invariant under $T^{[k]}$ by induction.

Set $\mu^{[0]} = \mu$. For $k \geq 0$, let $\mathcal{I}^{[k]}$ be the σ-algebra of $T^{[k]}$-invariant subsets of $X^{[k]}$. Then $\mu^{[k+1]}$ is the relatively independent square of $\mu^{[k]}$ over $\mathcal{I}^{[k]}$. This means that if F', F'' are bounded functions on $X^{[k]}$,

(3)
$$\int_{X^{[k+1]}} F'(\mathbf{x}')F''(\mathbf{x}'')\, d\mu^{[k+1]}(\mathbf{x}', \mathbf{x}'') := \int_{X^{[k]}} \mathbb{E}(F' \mid \mathcal{I}^{[k]})\, \mathbb{E}(F'' \mid \mathcal{I}^{[k]})\, d\mu^{[k]}\ .$$

For a bounded function f on X we define

$$\|\!|f|\!\|_k^{2^k} = \int_{X^{[k]}} \prod_{j=0}^{2^k-1} f(x_j)\, d\mu^{[k]}(\mathbf{x}) \tag{4}$$

and note that

$$\|\!|f|\!\|_{k+1} := \left(\int_{X^{[k]}} \mathbb{E}^2\Big(\prod_{j=0}^{2^k-1} f(x_j) \;\Big|\; \mathcal{I}^{[k]} \Big)\, d\mu^{[k]}(\mathbf{x}) \right)^{1/2^{k+1}} . \tag{5}$$

It is shown in [HK02] that for every $k \geq 1$, $\|\!|\cdot|\!\|_k$ is a seminorm on $L^\infty(\mu)$.

Let $f \in L^\infty(\mu)$; from Equation (5), we immediately have that

$$\|\!|f|\!\|_1 = \Big|\int f\, d\mu\Big| \ ; \quad \text{for every } k \geq 1,\ \|\!|f|\!\|_k \leq \|\!|f|\!\|_{k+1} \leq \|f\|_\infty \ .$$

For $k \geq 1$ and an integer n, we have

$$\|\!|f \cdot T^n f|\!\|_k^{2^k} = \int_{X^{[k]}} \Big(\prod_{j=0}^{2^k-1} f(x_j) \Big) \cdot (T^{[k]})^n \Big(\prod_{j=0}^{2^k-1} f(x_j) \Big)\, d\mu^{[k]}(\mathbf{x}) \ . \tag{6}$$

By using the Ergodic Theorem and definition (5), we have

$$\|\!|f|\!\|_{k+1}^{2^{k+1}} = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \|\!|f \cdot T^n f|\!\|_k^{2^k} \ . \tag{7}$$

An increasing sequence $\{Z_k(X) : k \geq 0\}$ of factors of X is built in Section 4 of [HK02]. These are characterized by the property:

(8) For $f \in L^\infty(\mu)$, $\mathbb{E}\big(f \mid Z_k(X)\big) = 0$ if and only if $\|\!|f|\!\|_{k+1} = 0$.

$Z_0(X)$ is the trivial factor of X and $Z_1(X)$ is its Kronecker factor.

2.3. **The case of a totally ergodic system.** We assume in this Section that the system (X, μ, T) is totally ergodic and prove a generalization of relation (7).

Proposition 1. *Assume that (X, μ, T) is totally ergodic. Then for every integer $k \geq 1$, any $f \in L^\infty(\mu)$ and any non-zero integer a,*

$$\|\!|f|\!\|_{k+1}^{2^{k+1}} = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \|\!|f \cdot T^{an} f|\!\|_k^{2^k} \ .$$

The proof relies on the following two Lemmas.

Lemma 1. *Let a be a non-zero integer and let (Y, ν, S) be a totally ergodic system. Then the σ-algebra of $S^a \times S^a$-invariant subsets of $(Y \times Y, \nu \times \nu)$ coincides up to $\nu \times \nu$-null sets with the σ-algebra of $S \times S$ invariant subsets.*

Proof. We can clearly reduce to the case that a is positive. Let $f \in L^2(\nu \times \nu)$ be invariant under $S^a \times S^a$. We write σ_f for the correlation measure of this function for the $\mathbb{Z}^2$-action spanned by $S \times \mathbf{id}$ and $\mathbf{id} \times S$. By definition, σ_f is the positive finite measure on $\mathbb{T} \times \mathbb{T}$, defined by

$$\widehat{\sigma_f}(m, n) := \int_{\mathbb{T}\times\mathbb{T}} \exp\bigl(2\pi i(ms + nt)\bigr)\, d\sigma_f(s, t) \\ = \int_{Y\times Y} f(S^m x, S^n y)\, f(x, y)\, d\nu(x)\, d\nu(y)\ .$$

Since f is invariant under $S^a \times S^a$, we have $\widehat{\sigma_f}(m+qa, n+qa) = \widehat{\sigma_f}(m, n)$ for all integers m, n, q. It follows that the measure σ_f is concentrated on the union for $0 \leq j < a$ of the lines $D_j = \bigl\{(s, -s + j/a) : s \in \mathbb{T}\bigr\}$.

Assume that $\sigma_f(D_j) > 0$ for some $j \in \{0, 1, \ldots, a-1\}$. Let τ be the maximal spectral type of (Y, ν, S). It is classical that σ_f is absolutely continuous with respect to the measure $\tau\times\tau$ and thus $\tau\times\tau(D_j) > 0$. By Fubini's Theorem, there exists $s \in \mathbb{T}$ so that τ has atoms at the points s and $-s+j/a$. This means that $\exp(2\pi i s)$ and $\exp(2\pi i(-s+j/a))$ are eigenvalues of (Y, ν, S) and therefore so is $\exp(2\pi i j/a)$. By hypothesis, $j = 0$.

Therefore σ_f is concentrated on the line D_0 and it follows that f is invariant under $S \times S$. □

We note that the previous Lemma only needed that S^a be ergodic. Similarly, the next Lemma only needs that T^a be ergodic.

Lemma 2. *For every integer $a \neq 0$ and every integer $k \geq 0$, the σ-algebra of $(T^{[k]})^a$-invariant subsets of $X^{[k]}$ coincides up to $\mu^{[k]}$-null sets with the σ-algebra $\mathcal{I}^{[k]}$ of $T^{[k]}$ invariant subsets.*

Proof. For $k = 0$ the statement is a reformulation of the hypothesis of total ergodicity. For $k = 1$, this is a reformulation of Lemma 1 applied with $(Y, \nu, S) = (X, \mu, T)$. We procede by induction.

Let $k \geq 1$ be so that the statement holds for every non-zero integer a. Let

$$\mu^{[k]} = \int_\Omega \mu_\omega\, dP(\omega)$$

be the ergodic decomposition of $\mu^{[k]}$ for $T^{[k]}$. The induction hypothesis means that for P-almost every ω, the system $(X^{[k]}, \mu_\omega, T^{[k]})$ is totally ergodic. The invariant σ-algebra of $T^{[k+1]}$ is included in the invariant σ-algebra of $(T^{[k+1]})^a$ and so it suffices to prove the opposite inclusion.

The definition (3) of $\mu^{[k+1]}$ can be rewritten as

$$\mu^{[k+1]} = \int_\Omega \mu_\omega \times \mu_\omega \, dP(\omega) \ .$$

Let A be a subset of $X^{[k+1]}$, invariant under $(T^{[k+1]})^a$ for some non-zero integer a. By Lemma 1 applied with $(Y, \nu, S) = (X^{[k]}, \mu_\omega, T^{[k]})$, we have that for almost every ω, the set A coincides $\mu_\omega \times \mu_\omega$-almost everywhere with a set invariant under $T^{[k+1]}$. Thus $\mu_\omega \times \mu_\omega(A \setminus T^{[k+1]}A) = 0$. We have that

$$\mu^{[k+1]}(A \setminus T^{[k+1]}A) = \int_\Omega \mu_\omega \times \mu_\omega(A \setminus T^{[k+1]}A) \, dP(\omega) = 0$$

and A coincides up to a $\mu^{[k+1]}$-null set with a set invariant under $T^{[k+1]}$, meaning that the statement holds for $k+1$. □

Proof of Proposition 1. By Equation (6) and the Ergodic Theorem, the limit in the Proposition exists and is equal to

$$\int_{X^{[k]}} \mathbb{E}^2\Big(\prod_{j=0}^{2^k-1} f(x_j) \ \Big| \ \mathcal{I}_a^{[k]} \Big) \, d\mu^{[k]}(\mathbf{x})$$

where $\mathcal{I}_a^{[k]}$ is the σ-algebra of $(T^{[k]})^a$ invariant sets. The Proposition follows immediately from Lemma 2 and formula (5). □

2.4. **Systems of level k.** For an integer $k \geq 0$, we say that an ergodic system (X, μ, T) is *a system of level k* if $X = Z_k(X)$. Thus, the unique system of level 0 is the trivial system and systems of level 1 are ergodic rotations. For every ergodic system (X, μ, T) and every integer $k \geq 0$ the system $Z_k(X)$ is a system of level k. We use:

Theorem 2 ([HK02], Theorems 10.2 and 10.4)**.** *For every integer $k \geq 1$, every system of level k is an inverse limit of a sequence of ergodic k-step nilsystems.*

From Corollary 1 we deduce immediately:

Corollary 2. *The statement of Theorem 1 holds for systems of level k for any integer $k \geq 1$.*

2.5. **Characteristic factors.** In the next sections, given a family of polynomials satisfying condition $(\mathcal{H})$, we produce an appropriate factor $Z_k(X)$ of the given ergodic system X (called the *characteristic factor*) so that the limit behavior of the averages (1) remains unchanged when each f_i is replaced by its conditional expectation on the factor $Z_k(X)$. Furthermore, assuming that the one of the three assumptions in the second part of Theorem 1 is satisfied, the limit behavior of the averages (2) also remains unchanged under the same change of functions. More precisely, we show:

Theorem 3. *Let (X, μ, T) be an ergodic system, $\{p_1(n), p_2(n), \dots, p_\ell(n)\}$ a family of integer polynomials satisfying property $(\mathcal{H})$ and $m \in \{1, \dots, \ell\}$.*
(a) *There exists an integer $k \geq 0$ so that for any functions $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$, if $\mathbb{E}\big(f_m \mid Z_k(X)\big) = 0$ then*

$$\sup_M \Big| \int \frac{1}{N} \sum_{n=M}^{M+N-1} T^{p_1(n)} f_1 \cdot T^{p_2(n)} f_2 \cdot \ \dots \ \cdot T^{p_\ell(n)} f_\ell \, d\mu \Big| \ \longrightarrow \ 0 \tag{9}$$

as $N \to +\infty$.
(b) *Assume that one of the three hypotheses of part (b) of Theorem 1 is satisfied. Then there exists an integer $k \geq 0$ so that for any functions $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$, if $\mathbb{E}\big(f_m \mid Z_k(X)\big) = 0$ then*

$$\sup_M \Big\| \frac{1}{N} \sum_{n=M}^{M+N-1} T^{p_1(n)} f_1 \cdot T^{p_2(n)} f_2 \cdot \ \dots \ \cdot T^{p_\ell(n)} f_\ell \Big\|_{L^2(\mu)} \ \longrightarrow \ 0 \tag{10}$$

It follows that under the corresponding hypotheses of Theorem 1, the difference between the averages (1) (or (2), respectively) and the same averages with $\mathbb{E}(f_j \mid Z_k(X))$ substituted for f_j for each j, converges to zero (or converges to zero in L^2-norm, respectively). As $Z_k(X)$ is a system of level k, Theorem 1 follows from Corollary 2.

The reader can check that the constant k arising in the Theorem does not depend on the particular system, but only on the polynomials.

3. THE LINEAR CASE

Henceforth (X, μ, T) is an ergodic system.

We state more precisely a result of [HK02]; it implies that Theorem 3 holds when all polynomials are of degree 1.

Proposition 2. *Let $\ell \geq 1$ be an integer and let $a_1, a_2, \dots, a_\ell$ be pairwise distinct non-zero integers. There exists a constant $C = C(a_1, a_2, \dots, a_\ell)$ so that, for all functions $f_1, f_2, \dots, f_\ell$ with $|f_i| \leq 1$ and for every*

$m \in \{1, \dots, \ell\}$,

$$\limsup_{N\to\infty} \sup_M \Big\| \frac{1}{N} \sum_{n=M}^{M+N-1} T^{a_1 n} f_1 \,.\, T^{a_2 n} f_2 \ \cdot\ \dots\ \cdot\ T^{a_\ell n} f_\ell \Big\|_2 \le C \|\!| f_m |\!\|_{\ell+1} \ .$$

Furthermore, if (X, μ, T) *is totally ergodic, the constant* C *can be taken equal to* 1.

Proof. We procede by induction on ℓ.

Let a be a non-zero integer and let $f \in L^\infty(\mu)$. Let $\mathcal{I}_a$ denote the T^a-invariant σ-algebra of (X, μ).

Let $\{M_i\}$ and $\{N_i\}$ be two sequences of integers with $N_i \to +\infty$. Then when $i \to +\infty$,

$$\Big\| \frac{1}{N_i} \sum_{n=M_i}^{M_i+N_i-1} T^{an} f \Big\|_2 \to \|\mathbb{E}(f \mid \mathcal{I}_a)\|_2 \ .$$

If the system is totally ergodic this limit is equal to $|\int f\, d\mu| = \|\!| f |\!\|_1 \le \|\!| f |\!\|_2$ and the second part of the Proposition is proven for $\ell = 1$.

Returning to the general case, we have that

$$\begin{aligned}
\|\mathbb{E}(f \mid \mathcal{I}_a)\|_2^4 &= \lim_{N\to\infty} \Big(\frac{1}{N} \sum_{n=0}^{N-1} \int f \cdot T^{an} f \, d\mu \Big)^2 \\
&\le \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Big(\int f \cdot T^{an} f \, d\mu \Big)^2 \\
&\le |a| \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Big(\int f \cdot T^{n} f \, d\mu \Big)^2 = |a| \ \cdot\ \|\!| f |\!\|_2^4
\end{aligned}$$

by Relation (7). This proves the first part of the Proposition for $\ell = 1$.

Assume that the result of the Proposition holds for some $\ell \ge 1$. Let $a_1, a_2, \dots, a_{\ell+1}$ be distinct non-zero integers, let $m \in \{1, \dots, \ell+1\}$ and let $f_1, f_2, \dots, f_{\ell+1} \in L^\infty(\mu)$ with $|f_i| \le 1$. Choose $j \in \{1, 2, \dots, \ell+1\}$ with $j \ne m$ and let C be the constant associated to the family of integers $\{a_i - a_j : 1 \le i \le \ell+1,\ i \le j\}$.

For $n \in \mathbb{Z}$ define

$$u_n = T^{a_1} f_1 \cdot T^{a_2 n} f_2 \cdot \dots \cdot T^{a_{\ell+1}} f_{\ell+1} \ .$$

For all integers M, N with $N > 0$ and every integer h,

$$\Big|\frac{1}{N}\sum_{n=M}^{M+N-1}\int u_n\cdot u_{n+h}\,d\mu\Big| = \Big|\frac{1}{N}\sum_{n=M}^{M+N-1}\int\prod_{i=1}^{\ell+1}T^{a_in}(f\cdot T^{a_ih}f_i)\,d\mu\Big|$$

$$= \Big|\int f_j\cdot T^{a_jh}f_j\cdot\frac{1}{N}\sum_{n=M}^{M+N-1}\prod_{\substack{1\leq i\leq \ell+1\\ i\neq j}}T^{(a_i-a_j)n}(f_i\cdot T^{a_ih}f_i)\,d\mu\Big|$$

$$\leq \Big\|\frac{1}{N}\sum_{n=M}^{M+N-1}\prod_{\substack{1\leq i\leq \ell+1\\ i\neq j}}T^{(a_i-a_j)n}(f_i\cdot T^{a_ih}f_i)\Big\|_2 .$$

By the inductive hypothesis,

$$\limsup_{N\to\infty}\sup_M\Big|\frac{1}{N}\sum_{n=M}^{M+N-1}\int u_n\cdot u_{n+h}\,d\mu\Big| \leq C|\!|\!| f_m\cdot T^{a_mh}f_m|\!|\!|_{\ell+1} .$$

By the Van der Corput Lemma,

$$\limsup_{N\to\infty}\sup_M\Big\|\frac{1}{N}\sum_{n=M}^{M+N-1}u_n\Big\|_2^2 \leq C\limsup_{H\to+\infty}\frac{1}{H}\sum_{h=0}^{H-1}|\!|\!| f_m\cdot T^{a_mh}f_m|\!|\!|_{\ell+1}$$

$$\leq C\limsup_{H\to+\infty}\Big(\frac{1}{H}\sum_{h=0}^{H-1}|\!|\!| f_m\cdot T^{a_mh}f_m|\!|\!|_{\ell+1}^{2^{\ell+1}}\Big)^{1/2^{\ell+1}} .$$

If the system is totally ergodic, $C = 1$ by the inductive hypothesis and the above lim sup equals $|\!|\!| f_m|\!|\!|_{\ell+2}^2$ by Proposition 1; this shows the second part of the Proposition for $\ell + 1$.

In the general case, the last expression is bounded by

$$C|a_m|^{1/2^{\ell+1}}\limsup_{H\to+\infty}\Big(\frac{1}{H}\sum_{h=0}^{H-1}|\!|\!| f_m\cdot T^hf_m|\!|\!|_{\ell+1}^{2^{\ell+1}}\Big)^{1/2^{\ell+1}} = C|a_m|^{1/2^{\ell+1}}|\!|\!| f_m|\!|\!|_{\ell+2}^2$$

by Equation (7) and the first part of the Proposition is proven. □

4. Polynomial families

4.1. Ordering polynomial families.

Definition 1. Let $r \geq 0$ be an integer. An *integer polynomial with r parameters* is an integer polynomial whose coefficients are polynomial functions of r integer parameters.

We abbreviate the expression "integer polynomial with r parameters" as I.P$_r$ or I.P. when the number of parameters is not important. An I.P$_0$ is simply an integer polynomial. We write an I.P$_r$ in the form

$p(h_1, \dots, h_r; n)$, where $h_1, \dots, h_r$ are the parameters and n is the variable.

The *degree* of a non-identically zero I.P. is its degree in the variable n, meaning it is the largest integer d so that the coefficient of n^d is not identically zero.

Definition 2. Let $r \geq 0$ be an integer. A *polynomial family with r parameters* is a finite non-empty sequence

$$\big\{p_1(h_1, \dots, h_r; n), \dots, p_\ell(h_1, \dots, h_r; n)\big\} \tag{11}$$

of integer polynomials in r parameters so that

(i) For $1 \leq i \leq \ell$, the polynomial p_i has a degree ≥ 1.
(ii) For $1 \leq i, j \leq \ell$ with $i \neq j$, the polynomial $p_i - p_j$ has a degree ≥ 1.

Moreover, a polynomial family with r parameters as in (11) is given with a *mark*, meaning an index $m \in \{1, 2, \dots, \ell\}$. The I.P. p_m is called the *marked polynomial*.

We abbreviate the expression "polynomial family with r parameters as P.F$_r$ or P.F.

The set of polynomial families is partitioned according to their *types*, which we now define:

Definition 3. Let $\mathcal{F}$ be a P.F$_r$, as in (11). ℓ is called the *length* of this P.F. The maximum degree d of the polynomials is called the *degree* of the P.F.

For $1 \leq j \leq d$, consider the subfamily of $\mathcal{F}$ consisting in polynomials of degree j. Let w_j be the number of distinct coefficients of n^j in this subfamily of polynomials. The vector $(d, w_d, w_{d-1}, \dots, w_1)$ is called the *type* of the polynomial family.

We say that the P.F. is *standard* if the degree of the marked polynomial is equal to the degree of the family. We abbreviate "standard polynomial family with r parameters" as S.P.F$_r$ or S.P.F.

By definition, for a family of type $(d, w_d, w_{d-1}, \dots, w_1)$ and of length ℓ, we have $w_d > 0$ and $w_d + w_{d-1} + \cdots + w_1 \leq \ell$.

Let the set of all possible types be ordered lexicographically. This means that if $(d, w_d, w_{d-1}, \dots, w_1)$ and $(d', w'_{d'}, w'_{d'-1}, \dots, w'_1)$ are types, we have $(d, w_d, w_{d-1}, \dots, w_1) > (d', w'_{d'}, w'_{d'-1}, \dots, w'_1)$ if $d > d'$, or if $d = d'$ and $w_d > w'_d$, or if $d = d'$, $w_d = w'_d$ and $w_{d-1} > w'_{d-1}$, ...

The following Lemma is immediate:

Lemma 3. *Any decreasing sequence of types is eventually constant.*

This implies that the ordering of types is a *well ordering*: every non-empty set of types has a smallest element.

4.2. **Two properties of polynomial families.** In the next section we show by induction that some polynomial families satisfy two properties. Before stating the theorem we need some more notation:

For $r \geq 1$ we define inductively the notion of a *small subset* of $\mathbb{Z}^r$. A subset of $\mathbb{Z}$ is small if and only if it is finite. A subset E of $\mathbb{Z}^{r+1}$ is small if and only there exists a small subset F of $\mathbb{Z}^r$ so that the subset

$$\big\{n \in \mathbb{Z} : (n_1, n_2, \dots, n_r, n) \in E\big\}$$

of $\mathbb{Z}$ is finite for every $(n_1, n_2, \dots, n_r) \in \mathbb{Z}^r \setminus F$.

Note that if $p(n_1, n_2, \dots, n_r)$ is a non-identically zero integer polynomial in r variables, then its zero set

$$\big\{(n_1, n_2, \dots, n_r) \in \mathbb{Z}^d : p(n_1, n_2, \dots, n_r) = 0\big\}$$

is small.

We say that a property holds *for almost every* $(n_1, n_2, \dots, n_r) \in \mathbb{Z}^r$ if it holds outside a small subset of $\mathbb{Z}^r$. To avoid the need to consider some special cases separately, we use also this sentence for $r = 0$. In this case, the sentence "the property holds for almost every $(n_1, n_2, \dots, n_r) \in \mathbb{Z}^r$" simply means that the property holds.

The properties that we study for a P.F. are the following.

Definition 4. Let $k \geq 1$ be an integer and let $\mathcal{F}$ be a P.F.$_r$ as in (11), with mark m.

(i) We say that $\mathcal{F}$ satisfies property $\mathcal{I}$ (with constant k) if, for all functions $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$, if $|||f_m|||_k = 0$ then

$$\sup_M \Big| \frac{1}{N} \sum_{n=M}^{M+N-1} \int T^{p_1(h_1,\dots,h_r;n)} f_1 \cdot \ldots \cdot T^{p_\ell(h_1,\dots,h_r;n)} f_\ell \, d\mu \Big| \to 0 \tag{12}$$

as $N \to +\infty$, for almost every $(h_1, \dots, h_r) \in \mathbb{Z}^r$.

(ii) We say that $\mathcal{F}$ satisfies property $\mathcal{N}$ (with constant k) if, for all functions $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$, if $|||f_m|||_k = 0$ then

$$\sup_M \Big\| \frac{1}{N} \sum_{n=M}^{M+N-1} T^{p_1(h_1,\dots,h_r;n)} f_1 \cdot \ldots \cdot T^{p_\ell(h_1,\dots,h_r;n)} f_\ell \Big\|_2 \to 0 \tag{13}$$

as $N \to +\infty$, for almost every $(h_1, \dots, h_r) \in \mathbb{Z}^r$.

A P.F. satisfying property $\mathcal{N}$ also satisfies property $\mathcal{I}$ with the same constant. By Proposition 2 every P.F. of degree 1 satisfies property $(\mathcal{N})$.

5. The main induction

We procede by induction: starting with a P.F we modify it by applying alternately two transformations. We show that this algorithm stops after a finite number of steps.

5.1. **The transformation A.** Let $\mathcal{F}$ be a P.F$_r$ as in (11), with mark m and type $(d, w_d, \ldots, w_1)$.

Let J be the set of $i \in \{1, 2, \ldots, \ell\}$ so that $p_i(h_1, \ldots, h_r; n)$ is of degree 1 (note that this set may be empty). Let $\mathcal{F}'$ be the sequence of I.P$_{r+1}$ with parameters $h_1, \ldots, h_r, h_{r+1}$ obtained by concatenation of the sequences

$$\big\{p_i(h_1, \ldots, h_r; n) : i \notin J\big\} \ ; \ \big\{p_i(h_1, \ldots, h_r; n + h_{r+1}) : i \notin J\big\} \ ;$$
$$\big\{p_i(h_1, \ldots, h_r; n) : i \in J\big\} \ .$$

It follows immediately that this sequence satisfies the condition of Definition 1 and so is a P.F$_{r+1}$. Take $p_m(h_1, \ldots, h_r; n)$ to be the marked polynomial. We say that that the family $\mathcal{F}'$ is the result of the *transformation* A applied to $\mathcal{F}$.

The type of $\mathcal{F}'$ is equal to the type $(d, w_d, \ldots, w_1)$ of $\mathcal{F}$. If $\mathcal{F}$ is a S.P.F. then $\mathcal{F}'$ is also a S.P.F. If $\mathcal{F}$ is of degree > 1, then the length of $\mathcal{F}'$ is strictly greater than the length of $\mathcal{F}$.

Lemma 4. *Let $\mathcal{F}$ be a* P.F$_r$ *as in* (11) *with mark m and assume that the degree of $p_m(h_1, \ldots, h_r; n)$ is > 1. Let $\mathcal{F}'$ be the* P.F$_{r+1}$ *obtained by transformation A applied to $\mathcal{F}$.*

If $\mathcal{F}'$ satisfies property ($\mathcal{I}$) with constant k, then $\mathcal{F}$ satisfies property ($\mathcal{N}$) with the same constant.

Proof. We proceed as in the proof of Proposition 2. Let $f_1, \ldots, f_\ell \in L^\infty(\mu)$ and assume that $|\!|\!| f_m |\!|\!|_k = 0$. For $(h_1, \ldots, h_r) \in \mathbb{Z}^r$ and $n \in \mathbb{Z}$ we write $u(h_1, \ldots, h_r; n)$ for the function

$$\prod_{i \le \ell} T^{p_i(h_1, \ldots, h_r; n)} f_i \ .$$

For $i \in J$ and for $(h_1, \ldots, h_{r+1}) \in \mathbb{Z}^{r+1}$ we have

$$p_i(h_1, \ldots, h_r; n + h_{r+1}) - p_i(h_1, \ldots, h_r; n) = h_{r+1} q_i(h_1, \ldots, h_r)$$

for some polynomial q_i in the variables $h_1, \dots, h_r$. We have

$$
\begin{aligned}
(14)\quad & \int u(h_1,\dots,h_r;n)\cdot u(h_1,\dots,h_r;n+h_{r+1})\,d\mu \\
&= \int \prod_{i\notin J} T^{p_i(h_1,\dots,h_r;n)} f_i \cdot \prod_{i\notin J} T^{p_i(h_1,\dots,h_r;n+h_{r+1})} f_i \cdot \\
&\qquad \prod_{i\in J} T^{p_i(h_1,\dots,h_r;n)}\big(f_i\cdot T^{h_{r+1}q_i(h_1,\dots,h_r)} f_i\big)\,d\mu\ .
\end{aligned}
$$

Note that $m \notin J$. By hypothesis, for almost every $(h_1, \dots, h_r, h_{r+1}) \in \mathbb{Z}^{r+1}$, the averages for n in an interval of this integral converge to zero when the length of the interval tends to $+\infty$.

Therefore, for almost every $(h_1, \dots, h_r) \in \mathbb{Z}^r$, the same property holds for all but a finite number of values of h_{r+1}. By the Van der Corput Lemma,

$$
\sup_M \Big\| \frac{1}{N} \sum_{n=M}^{M+N-1} u(h_1,\dots,h_r;n) \Big\|_2 \to 0
$$

as $N \to +\infty$. This is the announced result. $\square$

5.2. **The transformation B.** Let $\mathcal{F}$ be a S.P.F.$_r$ as in (11), with mark m and length $\ell > 1$.

Claim 1. *There exists $j \in \{1, \dots, \ell\}$, different than m, so that the family $\mathcal{F}'$ given by*

$$
(15)\qquad \big\{p_i(h_1,\dots,h_r;n) - p_j(h_1,\dots,h_r;n) : 1 \le i \le \ell,\ i \neq j\big\}
$$

with mark $p_m(h_1,\dots,h_r;n) - p_j(h_1,\dots,h_r;n)$ is a S.P.F.$_r$ *of type strictly less than the type of $\mathcal{F}$.*

Proof of the claim. We note that $\mathcal{F}'$ is a P.F.$_r$. Let $(d, w_d, w_{d-1}, \dots, w_1)$ and $(d', w'_{d'}, w'_{d'-1}, \dots, w'_1)$ be the types of $\mathcal{F}$ and of $\mathcal{F}'$, respectively. Note that $d' \le d$. We distinguish three cases.

1) *Assume that* $(d, w_d, w_{d-1}, \dots, w_1) = (d, 1, 0, \dots, 0)$.
This means that all polynomials of the given family have the same degree d and the same leading coefficient. Choose $j \neq m$ so that $p_m - p_j$ has the maximal possible degree. Then d' is equal to the degree of $p_m - p_j$ and $\mathcal{F}'$ is a S.P.F. Moreover $d' < d$ and thus $(d, w_d, w_{d-1}, \dots, w_1) > (d', w'_{d'}, w'_{d'-1}, \dots, w'_1)$.

2) *Assume that* $(d, w_d, w_{d-1}, \dots, w_1) = (d, w_d, 0, \dots, 0)$ *with* $w_d > 1$.
Then all polynomials of the given family have the same degree d but not the same leading coefficient. Choose $j \in \{1, \dots, \ell\}$ so that the leading

coefficients of p_j and p_m are different. Then $d' = d$, $\mathcal{F}'$ is a S.P.F. and $w'_d = w_d - 1$. Thus $(d, w_d, w_{d-1}, \dots, w_1) > (d', w'_{d'}, w'_{d'-1}, \dots, w'_1)$.

3) *Assume that $w_r > 0$ for some $r < d$.*
Choose j so that p_j has the smallest possible degree r. Then $r < d$ and $j \neq m$; $\mathcal{F}'$ is a S.P.F. and $(d', w'_{d'}, w'_{d'-1}, \dots, w'_1) = (d, w_d, w_{d-1}, \dots, w_{r+1}, w_r - 1, w'_{r-1}, \dots, w'_1)$ which is strictly less than $(d, w_d, w_{d-1}, \dots, w_1)$.

The claim is proven. □

Let $j \in \{1, \dots, \ell\}$ be as in the claim. If there are several possible choices for j, we take the smallest one. We say that the S.P.F. $\mathcal{F}'$ defined by (15) is the result of the *transformation* B applied to $\mathcal{F}$. The length of $\mathcal{F}'$ is $\ell - 1$.

Lemma 5. *Let $\mathcal{F}$ be a* S.P.F$_r$ *of length $\ell > 1$ and let $\mathcal{F}'$ be the* S.P.F$_r$ *obtained by applying the transformation B to $\mathcal{F}$.*

If $\mathcal{F}'$ satisfies property $(\mathcal{N})$ with constant k, then $\mathcal{F}$ satisfies property $(\mathcal{I})$ with the same constant.

Proof. Let the P.F. $\mathcal{F}$ with mark m be written as in (11). We use the same notation as above. For all integers M, N with $N > 0$ we have

$$\Big|\frac{1}{N}\sum_{n=M}^{M+N-1}\int\prod_{1\le i\le \ell} T^{p_i(h_1,\dots,h_r;n)} f_i\, d\mu\Big|$$

$$= \Big|\int f_j \cdot \frac{1}{N}\sum_{n=M}^{M+N-1}\prod_{\substack{1\le i\le \ell\\ i\neq j}} T^{p_i(h_1,\dots,h_r;n)-p_j(h_1,\dots,h_r;n)} f_i\, d\mu\Big|$$

$$\le \|f_j\|_2 \cdot \Big\|\frac{1}{N}\sum_{n=M}^{M+N-1}\prod_{\substack{1\le i\le \ell\\ i\neq j}} T^{p_i(h_1,\dots,h_r;n)-p_j(h_1,\dots,h_r;n)} f_i\Big\|_2$$

and the result is proven. □

5.3. **The iteration.**

Proposition 3.

(i) *Every S.P.F. satisfies property $(\mathcal{N})$.*
(ii) *Every P.F. satisfies property $(\mathcal{I})$.*
(iii) *Every P.F. so that the marked polynomial has degree > 1 satisfies property $(\mathcal{N})$.*

Proof. (i) Consider first a P.F$_r$ of degree 1. The family of polynomials in the variable n obtained by fixing the values of the parameters satisfies property $(\mathcal{H})$ for almost every choice of these values. By Proposition 2, property ($\mathcal{N}$) holds for this P.F. Thus we can restrict to polynomial families of degree > 1.

Starting with a given S.P.F.$_r$ $\mathcal{F}$ of degree > 1 we alternately apply transformations A and B, starting with transformation A.

Since $\mathcal{F}$ is of degree > 1, the S.P.F. obtained after the first transformation A has length > 1 and transformation B can be applied. Assume now that the P.F. obtained after some of the transformations B of the iteration has degree $d' > 1$, length $\ell' \geq 1$ and is a S.P.F. Then the S.P.F. obtained by transformation A has degree $d' > 1$ and of length $\geq \ell' + 1 > 1$. Again, applying transformation B is possible. The result of this transformation is a S.P.F. of length $\geq \ell' \geq 1$.

Therefore it is possible to continue the iteration as long as the S.P.F. is of degree > 1. A S.P.F. of degree 1 can occur only after a transformation B.

The type is preserved by transformation A and decreases strictly when the transformation B is applied. By Lemma 3, the iteration stops after a finite number of steps, resulting in a S.P.F. of degree 1.

Each time we apply transformation A, the S.P.F. has degree > 1. Thus the marked polynomial is of degree > 1 and we can use Lemma 4.

At the end of the iteration we obtain a S.P.F. of degree 1 and as already noted, property ($\mathcal{N}$) holds for this S.P.F. By alternating Lemmas 4 and 5, we have that the initial P.F. satisfies property ($\mathcal{N}$).

(ii) Let $\mathcal{F}$ be a P.F.$_r$ as in (11) of type $(d, w_d, \ldots, w_1)$. Define $\mathcal{F}'$ to be

$$\left\{n^{d+1} + p_i(h_1, \ldots, h_r; n) : 1 \leq i \leq \ell\right\} .$$

Then $\mathcal{F}'$ is a S.P.F. and satisfies property ($\mathcal{N}$) by part (i) of the Proposition and thus also property ($\mathcal{I}$). For all functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$ and integers M, N with $N > 0$, the integral in (12) remains unchanged when the P.F. $\mathcal{F}'$ is substituted for $\mathcal{F}$. Therefore $\mathcal{F}$ satisfies property ($\mathcal{I}$).

Part (iii) of the Proposition follows immediately from part (ii) and Lemma 4. $\square$

5.4. **The case of a totally ergodic system.**

Proposition 4. *Assume that (X, μ, T) is totally ergodic. Then property ($\mathcal{N}$) holds for every P.F.*

Proof. We assume that (X, μ, T) is totally ergodic.

The proof follows along the same lines as the proof of Proposition 3, by using a quantitative version of the properties ($\mathcal{I}$) and ($\mathcal{N}$) and corresponding modifications of Lemmas 4 and 5.

Definition 5. Let $k \geq 1$ be an integer, $\alpha \in (0, 1]$ a real and $\mathcal{F}$ a P.F$_r$ as in (11), with mark m.

(i) We say that $\mathcal{F}$ satisfies property $\mathcal{I}'$ (with constants k and α) if, for all functions $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$ with $|f_i| \leq 1$ for each i,

(16)
$$\limsup_{N\to+\infty} \sup_M \Big| \frac{1}{N} \sum_{n=M}^{M+N-1} \int T^{p_1(h_1,\dots,h_r;n)} f_1 \cdot \ldots \cdot T^{p_\ell(h_1,\dots,h_r;n)} f_\ell \, d\mu \Big| \leq |\!|\!| f_m |\!|\!|_k^\alpha$$

for almost every $(h_1, \dots, h_r) \in \mathbb{Z}^r$.

(ii) We say that $\mathcal{F}$ satisfies property $\mathcal{N}'$ (with constants k and α) if, for all functions $f_1, f_2, \dots, f_\ell \in L^\infty(\mu)$ with $|f_i| \leq 1$ for each i,

(17)
$$\limsup_{N\to+\infty} \sup_M \Big\| \frac{1}{N} \sum_{n=M}^{M+N-1} T^{p_1(h_1,\dots,h_r;n)} f_1 \cdot \ldots \cdot T^{p_\ell(h_1,\dots,h_r;n)} f_\ell \Big\|_2 \leq |\!|\!| f_m |\!|\!|_k^\alpha$$

for almost every $(h_1, \dots, h_r) \in \mathbb{Z}^r$.

Property $\mathcal{N}'$ with constants k and α implies property $\mathcal{I}'$ with the same constants. It also implies property $\mathcal{N}'$ with constants k' and α' if $k' \geq k$ and $0 < \alpha' \leq \alpha$ (because $|\!|\!| f_m |\!|\!|_k \leq |\!|\!| f_m |\!|\!|_{k'} \leq 1$).

Lemma 6. *Let $\mathcal{F}$ be a* P.F$_r$ *as in* (11) *with mark m. Let $\mathcal{F}'$ be the* P.F$_{r+1}$ *obtained by transformation A applied to $\mathcal{F}$.*

If $\mathcal{F}'$ satisfies property ($\mathcal{I}'$) with constants k and α, then $\mathcal{F}$ satisfies property ($\mathcal{N}$) with constants $k+1$ and $\alpha/2$.

Proof of Lemma 6. Let J, the polynomials q_i and $u(h_1, \dots, h_r; n)$ be defined as in the proof of Lemma 4. We distinguish two cases.

Assume first that $m \notin J$. By hypothesis, for almost every $(h_1, \dots, h_r, h_{r+1})$, the lim sup of the absolute value of the averages in n of the integral (14) is bounded by $|\!|\!| f_m |\!|\!|_k^\alpha$.

By the Van der Corput Lemma, for almost every $(h_1, \dots, h_r)$, the lim sup of the L^2-norm of the averages in n of the functions $u(h_1, \dots, h_r; n)$ is bounded by $|\!|\!| f_m |\!|\!|_k^{\alpha/2} \leq |\!|\!| f_m |\!|\!|_{k+1}^{\alpha/2}$. This is the announced result.

Assume now that $m \in J$. By hypothesis, for almost every $(h_1, \dots, h_r, h_{r+1})$, the lim sup of the absolute value of the averages in n of the integral (14) is bounded by $|\!|\!| f_m \cdot T^{h_{r+1} q_m(h_1,\dots,h_r)} f_m |\!|\!|_k^\alpha$. By the Van der Corput Lemma, the lim sup of the L^2-norm of the averages in n of

$u(h_1,\dots,h_r;n)$ is bounded by

$$\limsup_{H\to+\infty}\Bigl(\frac{1}{H}\sum_{h_{r+1}=0}^{H-1}|\!|\!| f_m\cdot T^{h_{r+1}q_m(h_1,\dots,h_r)}f_m|\!|\!|_k^{\alpha}\Bigr)^{1/2}$$
$$\leq\limsup_{H\to+\infty}\Bigl(\frac{1}{H}\sum_{h_{r+1}=0}^{H-1}|\!|\!| f_m\cdot T^{h_{r+1}q_m(h_1,\dots,h_r)}f_m|\!|\!|_k^{2^k}\Bigr)^{\alpha/2^{k+1}}=|\!|\!| f_m|\!|\!|_{k+1}^{\alpha}$$

by Proposition 1. This means that $\mathcal{F}$ satisfies property $\mathcal{N}$ with constants $k+1$ and α and so also with constants $k+1$ and $\alpha/2$. $\square$

The proof of the following Lemma is similar to the proof of Lemma 5:

Lemma 7. *Let $\mathcal{F}$ be a* $\mathrm{S.P.F}_r$ *of length $\ell>1$ and let $\mathcal{F}'$ be the* $\mathrm{S.P.F}_r$ *obtained by applying transformation B to $\mathcal{F}$.*

If $\mathcal{F}'$ satisfies property ($\mathcal{N}'$) with constants k and α, then $\mathcal{F}$ satisfies property ($\mathcal{I}'$) with the same constants.

We continue exactly as in the proof of Proposition 3. Property ($\mathcal{N}'$) is satisfied by a P.F. of degree 1 by last part of Proposition 2. The same iteration as in the proof of Proposition 3 shows that every S.P.F. satisfies property ($\mathcal{N}'$). We deduce that every P.F. satisfies property ($\mathcal{I}'$) and by using Lemma 6 that every P.F. satisfies property ($\mathcal{N}'$). Property ($\mathcal{N}$) follows immediately. $\square$

5.5. **End of the proof of Theorem 3.** Any family of integer polynomials satisfying hypothesis ($\mathcal{H}$) is a $\mathrm{P.F}_0$. Therefore part (a) of Theorem 3 follows from part (ii) of Proposition 3. Part (b) of this Theorem for linear polynomials follows from Proposition 2. If all polynomials are of degree >1 this statement follows from part (iii) of Proposition 3 and when the system is totally ergodic from Proposition 4.

As noted, Theorem 1 follows from Theorem 3 and Corollary 2.

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