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Convergence of scaled renewal processes and a packet arrival model

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We study the superposition process of a class of independent renewal processes with long-range dependence. It is known that under two different scalings in time and space either fractional Brownian motion or a stable Lévy process may arise in the rescaling asymptotic limit. It is shown here that in a third, intermediate scaling regime a new limit process appears, which is neither Gaussian nor stable. The new limit process is characterized by its cumulant generating function and some of its properties are discussed.

Keywords: fractional Brownian motion; heavy tails; long-range dependence; renewal processes; weak convergence

1. Introduction

This study is concerned with the asymptotic scaling behaviour of sums of independent random processes with long-range dependence. Specifically, the long-range dependent process will be a renewal counting process with heavy-tailed inter-renewal times. It is known that the superposition process of independent copies of such processes, suitably scaled, may exhibit rescaling limits. In fact, different limit processes may arise depending on the details of the rescaling scheme.

To explain the framework, consider

$$\frac{1}{b(m,\, T)}\sum_{i=1}^{m}(X^i_{Tt} - cTt),$$

the summation process of m independent and identically distributed copies of a centred random process $\{X_t\}$, where $b(m, T)$ is a normalization constant as m and T tend to infinity. Taqqu and Levy (1986), and recently Levy and Taqqu (2000), take X_t to be a renewal–reward process and study the limit under assumptions of heavy tails of both renewal times and rewards, hence implementing Mandelbrot's (1969) idea of using renewal–reward processes with a heavy-tailed distribution of the inter-renewal times to study long-range dependence phenomena in physics and economics. The scaling limits investigated in Willinger *et al*. (1997) and Taqqu *et al*. (1997) refer to the case $X_t = \int_0^t Z_s\, \mathrm{d}s$, where $\{Z_t\}$ is an on/off process with heavy-tailed distributions for on periods, off periods, or both. This application is related to the nature of network traffic, with $\{Z_t\}$ being a traffic rate process, and the results discussed in Willinger *et al*. (1997) relate a global self-similarity property, claimed to be intrinsic to Internet traffic, to the local property of heavy-tailed distributions of on or off periods. The main result for both of these models is that if m tends to infinity first, followed by T, then the rescaled process converges to *fractional Brownian motion*, whereas if the order of the limit operations is reversed then the limit is a *stable Lévy motion*.

This motivates the search for a simultaneous limit regime in which one of the parameters m or T is a function of the other, with the hope of finding a new limiting process which would provide the 'missing link' between fractional Brownian motion and stable Lévy motion. This is the idea behind this paper, as well as those of Mikosch *et al*. (2002) for on/off processes (and the infinite-sources Poisson model) and Pipiras *et al*. (2002) for a class of renewal-rate processes. It is shown by Mikosch *et al*. (2002) that the limit process depends on the 'connection rate', i.e. on the rate of growth of the number of users $m = M(T)$ compared to the time scale $T^{\alpha-1}$, where α is the regular variation exponent of the tail of the distribution of the on periods. However, the limiting processes are still the same as in the double-limit case. That is, if the connection rate is fast, meaning that $M(T)$ grows faster than $T^{\alpha-1}$, the limit is fractional Brownian motion. If the connection rate is slow the rescaled process converges to a stable Lévy motion. Analogous results are reported in Pipiras *et al*. (2002) for the case of renewal-rate processes.

We investigate the third limiting regime corresponding to an 'intermediate' connection rate, using standard renewal processes with heavy-tailed inter-renewal time distribution, and derive a new non-Gaussian and non-stable limit process. Section 2 introduces the model, and Section 3 contains the main convergence result. Section 4 studies various properties of the limit process. In Section 5 we bring together the results for the underlying heavy-tailed renewal process required for the proof, some of which may also be of independent interest. Finally, in Section 6 we give a proof of the theorem based on cumulant generating functions.

2. A renewal-based model for the arrival process

Consider the renewal process $\{S_n\}$ generated by a sequence of independent non-negative random variables $\{U_k\}_{k\geqslant 1}$, i.e. $S_n = \sum_{k=1}^{n} U_k$. The inter-renewal times $\{U_k\}_{k\geqslant 2}$ are supposed to be identically distributed with distribution function $F(t)$, while the distribution of the first inter-renewal time U_1 can be different. Our basic assumption is that the inter-renewal distribution $F(t)$ has a regularly varying tail with exponent $1+\beta$, $0<\beta<1$, i.e. there exists a slowly varying function $L(t)$ such that

$$1 - F(t) \sim t^{-(1+\beta)} L(t), \qquad \text{as } t \to \infty. \tag{1}$$

Thus the variables U_k, $k \geqslant 2$, possess finite expectation μ but the variance is infinite. As for the distribution of the first renewal U_1, in the context of our applications it is natural to choose it to be equal to the equilibrium distribution

$$F_1(t) = \frac{1}{\mu}\int_0^t (1 - F(s))\mathrm{d}s,$$

so that the resulting renewal point process becomes stationary. In particular, this implies that the corresponding renewal counting process,

$$N_t = \max\{n : S_n \leqslant t\},$$

has stationary increments. The usual *pure* renewal processes, with the same distribution $F(t)$ for all renewal intervals, also emerge as we proceed. Henceforth, we denote by N_t the stationary renewal counting process, while $\tilde{N}_t$ corresponds to a pure renewal process.

Let $\{N_t^{(i)}\}$, $i = 1, \ldots, m$, be m independent copies of the stationary renewal counting process $\{N_t\}$. We are interested in the asymptotic properties of the superposition process

$$W(m, t) = \sum_{i=1}^{m} N_t^{(i)},$$

which counts the total number of renewal events occurring in corresponding renewal sequences $\{S_n^{(i)}\}$, $i = 1, \ldots, m$, up to time $t \geqslant 0$.

The summation processes discussed in Section 1 have been suggested as models for the total workload at a network node arising from m independent sources sharing a common medium, such as a local area network. In the light of such applications the process studied here can be thought of as a rudimentary packet arrival model with heavy tails, which in a sense acts as a skeleton for more detailed models designed to capture the workload behaviour in real systems. In this simple model source i generates one packet at each time epoch $\{S_n^{(i)}\}_{n\geqslant 1}$. The amount of work from source i up to time $t \geqslant 0$ is $N_t^{(i)}$, and $W(m, t)$ represents the cumulative arrival process of the system counting the accumulated work generated by all users. We will investigate the behaviour of the (rescaled) process $W(m, t)$ as the number of sources m grows to infinity and time is suitably rescaled, under the assumption that the distribution of packet inter-arrival times is heavy-tailed.

3. Convergence result

Our object of interest is the centred and rescaled process

$$Y^{(m)}(t) = \frac{1}{b_m}\left(W(m, a_m t) - \frac{m a_m t}{\mu}\right), \tag{2}$$

where the sequence a_m governs the rescaling of time and b_m is the corresponding normalization of 'space'. The centring sequence $m a_m t/\mu$ corresponds to the expected value $\mathrm{E}W(m, t) = mt/\mu$.

We assume that the scaling sequence a_m is any sequence such that $a_m \to \infty$. The rate of growth of a_m relative to m determines the asymptotic behaviour of $Y^{(m)}$ as $m \to \infty$, and it is known that the choice of such growth rate conditions results in a fundamental dichotomy in the asymptotic limit under the rescaling scheme (2). Essentially, if a_m grows slowly compared to m then the limit is fractional Brownian motion, and if a_m grows fast relative to m the limit is a stable Lévy process. The contribution in this work is the derivation of a new limit process under a third, intermediate, limit regime. Thus we will discuss the following three options:

- fast connection rate (number of users grows faster than rescaling of time),

$$\frac{mL(a_m)}{a_m^\beta} \to \infty; \tag{FCR}$$

- slow connection rate (time is rescaled faster than the number of users grows),

$$\frac{mL(a_m)}{a_m^\beta} \to 0; \tag{SCR}$$

- intermediate connection rate (time is rescaled proportionally to the growth of the number of users),

$$\frac{mL(a_m)}{a_m^\beta} \to \mu. \tag{ICR}$$

The conditions (FCR) and (SCR) are equivalent to the fast- and slow-growth conditions used in Mikosch *et al*. (2002), as can be verified similarly to their Lemma 1.

Some further remarks are in order concerning the choice of the rescaling scheme (2). In this paper the time scale a_m is taken to be a function of the number of users m, and the rescaled process $W(m, a_m t)$ is studied as m tends to infinity. In contrast, Mikosch *et al*. (2002) consider the number of users $M(T)$ in the on/off model and the connection intensity $\lambda(T)$ in the $M/G/\infty$ model to be functions of a time parameter T, and let T tend to infinity. The latter limiting scheme is also used by Taqqu and Levy (1986) and Pipiras *et al*. (2002) in the renewal–reward process setting. This is in effect an inverse scaling as compared to (2), which in our case would correspond to the process $W(m_T, Tt)$, where $T \to \infty$. In the limiting scheme (2) the sequence a_m can be regarded as an inverse function of $M(T)$.

The term 'connection rate' is used above in a descriptive sense, only indicating the relationship between the number of sources and the time interval over which the sources are active, and should not be thought of as a rate in the sense of number of connected users per unit of time.

We begin by stating in Theorem 1 the limit results for $Y^{(m)}$ under (FCR) and (SCR) scaling. The results are in complete analogy with those found by Mikosch *et al*. (2002) and Pipiras *et al*. (2002) for the more complex but related models studied in these papers, in the case where the heavy tails are parametrized by a single parameter. In our case of simple renewal processes the proof of convergence under (FCR) can be carried out by modifying selected parts of the proof of Theorem 2 below (see Section 6.5). The convergence result under (SCR) will be discussed in a more general framework elsewhere.

Theorem 1. *Under assumption* (1) *and either* (FCR) *or* (SCR) *the following limiting relations hold:*

(a) *Under condition* (FCR)*, as* $m \to \infty$*,*

$$\frac{W(m,\, a_m t) - m a_m t/\mu}{m^{1/2} a_m^{1-\beta/2} L(a_m)^{1/2}} \Rightarrow \mu^{-3/2}\sigma_\beta B_H(t), \qquad \sigma_\beta^2 = \frac{2}{\beta(1-\beta)(2-\beta)}, \tag{3}$$

where $\Rightarrow$ *denotes weak convergence of processes in the space of cadlag functions and* $B_H(t)$ *is standard fractional Brownian motion with index* $H = 1 - \beta/2$.

(b) *Under condition* (SCR), *let* $L^*(u)$ *be a slowly varying function at infinity such that* $L(u^{1/(1+\beta)} L^*(u))/L^*(u)^{1+\beta} \to 1$ *as* $u \to \infty$. *Then as* $m \to \infty$,

$$\frac{W(m,\, a_m t) - m a_m t/\mu}{(m a_m)^{1/(1+\beta)} L^*(m a_m)} \overset{fdd}{\to} -\mu^{-(2+\beta)/(1+\beta)} \Lambda_\alpha(t), \tag{4}$$

where $\overset{fdd}{\to}$ *denotes convergence of the finite-dimensional distributions and* $\Lambda_\alpha(t)$ *is* α-*stable Lévy motion with index* $\alpha = 1 + \beta$, *such that* $\Lambda_\alpha(t) \sim S_\alpha(c_\alpha^{-1/\alpha} t^{1/\alpha},\, 1,\, 0)$, $c_\alpha = (1-\alpha)/(\Gamma(2-\alpha)\cos(\pi\alpha/2))$.

Our main result establishes a new limit process in the asymptotic regime between parts (a) and (b) of Theorem 1, corresponding to the condition (ICR), where the appropriate norming sequence in (2) is seen to be simply $b_m = a_m$.

Theorem 2. *Under assumption* (1) *and the intermediate growth condition* (ICR), *as* $m \to \infty$ *the weak convergence of processes*

$$Y^{(m)}(t) = \frac{1}{a_m} \sum_{i=1}^{m} \left(N_{a_m t}^{(i)} - \frac{a_m t}{\mu} \right) = \frac{W(m,\, a_m t) - m a_m t/\mu}{a_m} \Rightarrow -\mu^{-1} Y_\beta(t) \tag{5}$$

holds, where $Y_\beta(t)$ *is a zero-mean, non-Gaussian and non-stable process with stationary increments. The limit process is not self-similar, it is continuous and has finite moments of all orders. The finite-dimensional distributions of the increments of* $Y_\beta(t)$ *are characterized by the following cumulant generating function:*

$$\log \mathrm{E} \exp\left\{ \sum_{i=1}^{n} \theta_i (Y_\beta(t_i) - Y_\beta(t_{i-1})) \right\} = \frac{1}{\beta} \sum_{i=1}^{n} \theta_i^2 \int_0^{t_i - t_{i-1}} \int_0^{v} \exp(\theta_i u) u^{-\beta}\, \mathrm{d}u\, \mathrm{d}v$$

$$+ \frac{1}{\beta} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \theta_i \theta_j \exp\left(\sum_{k=i+1}^{j-1} \theta_k (t_k - t_{k-1}) \right)$$

$$\times \int_0^{t_i - t_{i-1}} \mathrm{d}v \int_0^{t_j - t_{j-1}} \exp(\theta_j u) \exp(\theta_i v)(t_{j-1} - t_i + u + v)^{-\beta}\, \mathrm{d}u, \tag{6}$$

where $0 = t_0 < t_1 < \ldots < t_n$.

Remark. We have chosen the constant in front of the process $Y_\beta(t)$ in (5) to be negative since it turns out that the limit process of $Y^{(m)}(t)$ has negatively skewed marginal distributions. Hence $Y_\beta(t)$ is 'positively skewed'.

4. Properties of the limit process

4.1. Elementary properties

Property 1 (Marginal distributions). *The cumulant generating function of the marginal distributions of $Y_\beta(t)$ is*

$$\log \mathrm{E}\mathrm{e}^{\theta Y_\beta(t)} = \frac{\theta^2}{\beta}\int_0^t\int_0^v \mathrm{e}^{\theta u} u^{-\beta}\,\mathrm{d}u\,\mathrm{d}v. \tag{7}$$

This also can be written as

$$\log \mathrm{E}\mathrm{e}^{\theta Y_\beta(t)} = \frac{\theta^2 t^{2-\beta}}{\beta(1-\beta)(2-\beta)} M(1-\beta, 3-\beta; \theta t), \tag{8}$$

where

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)z^k}{\Gamma(b+k)k!}$$

is Kummer's special function *from the family of confluent hypergeometric functions.*

Proof. Expression (7) follows from (6) with $n = 1$. To prove (8), observe that for parameter values with $\operatorname{Re} b > \operatorname{Re} a > 0$ Kummer's function $M(a, b; z)$ possesses the following integral representation (Abramowitz and Stegun 1992, Formula 13.2.1):

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 \mathrm{e}^{zu} u^{a-1}(1-u)^{b-a-1}\,\mathrm{d}u.$$

On the other hand, changing the order of integration in the double integral (7), together with the change of variables $u' = u/t$, gives

$$\int_0^t\int_0^v \mathrm{e}^{\theta u} u^{-\beta}\,\mathrm{d}u\,\mathrm{d}v = t^{2-\beta}\int_0^1 \mathrm{e}^{\theta t u} u^{-\beta}(1-u)\mathrm{d}u,$$

and (8) *follows.* □

Property 2 (Moments). *The process $Y_\beta(t)$ has finite moments of all orders. In particular,*

$$
\begin{aligned}
&\mathrm{E}Y_\beta(t) = 0,\\
&\mathrm{E}Y_\beta^2(t) = \frac{2}{\beta(1-\beta)(2-\beta)}\,t^{2-\beta}, \qquad (9)\\
&\text{Skewness} \;= \frac{\mathrm{E}[Y_\beta^3(t)]}{\mathrm{E}[Y_\beta^2(t)]^{3/2}} = \frac{3\sqrt{\beta(2-\beta)}(1-\beta)^{3/2}}{\sqrt{2}(3-\beta)}\,t^{\beta/2},\\
&\text{Kurtosis} \;= \frac{\mathrm{E}[Y_\beta^4(t)]}{\mathrm{E}[Y_\beta^2(t)]^2} = 3\left(1+\frac{\beta(1-\beta)^2(2-\beta)^2}{(3-\beta)(4-\beta)}\,t^\beta\right).
\end{aligned}
$$

In general, for $k \geqslant 1$,

$$\mathrm{E}Y_\beta^k(t) \sim \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)}\,t^{k-\beta}, \qquad \text{as } t \to \infty. \tag{10}$$

Proof. Since the moment generating function of $Y_\beta(t)$ exists for each real θ, all moments are finite. Relation (8) yields

$$\log \mathrm{E}e^{\theta Y_\beta(t)} = \frac{1}{\beta}\sum_{k=2}^{\infty}\frac{(k-1)k\,t^{k-\beta}\theta^k}{(k-1-\beta)(k-\beta)k!}.$$

The cumulants of the marginal distribution are now obtained by differentiation with respect to θ:

$$C_k(t) = \frac{\mathrm{d}^k}{\mathrm{d}\theta^k}\log \mathrm{E}e^{\theta Y_\beta(t)}\bigg|_{\theta=0} = \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)}\,t^{k-\beta}. \tag{11}$$

In our case, since $\mathrm{E}Y_\beta(t) = C_1(t) = 0$, the first three moments of $Y_\beta(t)$ are equal to the cumulants. Computations for any higher moment can also be carried out but do not lead to simple expressions. However, the moments have the same asymptotic behaviour when $t \to \infty$ as the cumulants, and thus (10) follows from (11). □

Property 3 (Covariance). *The covariance function of* $Y_\beta(t)$ *equals*

$$\mathrm{E}Y_\beta(t)Y_\beta(s) = \sigma_\beta^2(t^{2-\beta} + s^{2-\beta} - |t-s|^{2-\beta}), \tag{12}$$

where the constant σ_β^2 *is defined in* (3).

Proof. Since the process $Y_\beta(t)$ has stationary increments, (12) follows from (9):

$$\mathrm{E}Y_\beta(t)Y_\beta(s) = \frac{1}{2}(\mathrm{var}[Y_\beta(t)] + \mathrm{var}[Y_\beta(s)] - \mathrm{var}[Y_\beta(t-s)]).$$

□

Property 4 (Regularity). *The trajectories of the process $Y_\beta(t)$ are Hölder continuous of order γ, for any $0 < \gamma < 1$.*

Proof. This property follows from the general asymptotic form (10) of the moments by applying the Kolmogorov–Chentsov criterion (Karatzas and Shreve (1991), Theorem 2.8). □

4.2. Relation to fractional Brownian motion

We have seen above that the first two moments of $Y_\beta(t)$ coincide with the corresponding moments of the (multiple of) fractional Brownian motion $\sigma_\beta B_H(t)$ with index $H = 1 - \beta/2$, while higher-order cumulants and moments are different. For comparison with (10),

$$\mathrm{E}B_H^k(t) \sim \text{const}\, t^{k(1-\beta/2)}, \qquad \text{as } t \to \infty.$$

The Kolmogorov–Chentsov criterion applied to this case yields that fractional Brownian motion of index H is Hölder continuous of order γ only for $0 < \gamma < H$. Consequently, the process Y_β is more regular than fractional Brownian motion.

Note also that the process $Y_\beta(t)$ has the same covariance function as the (multiple of) fractional Brownian motion $\sigma_\beta B_H(t)$. Since this function is self-similar in the sense of self-similarity of deterministic functions, the process $Y_\beta(t)$ is *second-order self-similar*. However, it is not self-similar in general. The relationship of the limit process Y_β to fractional Brownian motion as well as its scaling properties are further clarified in the next result.

Corollary 1. *The process $Y_\beta(t)$ obeys the scaling limit relation*

$$c^H Y_\beta(t/c) \Rightarrow \sigma_\beta B_H(t), \qquad H = 1 - \frac{\beta}{2},\ c \to \infty, \tag{13}$$

in the sense of weak convergence of continuous processes.

Proof. This result is easily derived from (6), by observing that the cumulant generating function of the finite-dimensional distributions of increments of fractional Brownian motion with index $H = 1 - \beta/2$ can be written

$$\begin{aligned}
&\log \mathrm{E} \exp\left\{\sum_{i=1}^{n} \theta_i (B_H(t_i) - B_H(t_{i-1}))\right\} \\
&\qquad = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i \theta_j \operatorname{cov}(B_H(t_i) - B_H(t_{i-1}), B_H(t_j) - B_H(t_{j-1})) \\
&\qquad = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i \theta_j \Big(|t_{i-1} - t_j|^{2-\beta} - |t_i - t_j|^{2-\beta} + |t_i - t_{j-1}|^{2-\beta} - |t_{i-1} - t_{j-1}|^{2-\beta} \Big) \\
&\qquad = \sigma_1 \sum_{i=1}^{n} \theta_i^2 \int_0^{t_i - t_{i-1}} \int_0^{v} u^{-\beta} \, \mathrm{d}u \, \mathrm{d}v \\
&\qquad\quad + \sigma_1 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \theta_i \theta_j \int_0^{t_i - t_{i-1}} \int_0^{t_j - t_{j-1}} (t_{j-1} - t_i + u + v)^{-\beta} \, \mathrm{d}u \, \mathrm{d}v,
\end{aligned}$$

where $\sigma_1 = (1-\beta)(2-\beta)$. By comparing this representation with (6), it is seen that under the scaling (13) all the exponential factors appearing in (6) are wiped out, leaving only the Gaussian distribution of the fractional Brownian motion. □

4.3. Relation to stable Lévy motion

In order to indicate the relationship of $Y_\beta(t)$ to a stable Lévy motion, consider the stable Lévy motion $\Lambda_\alpha(t)$ with marginal distributions totally skewed to the right, which appears in (4). The cumulant generating function of $(\Lambda_\alpha(t_1), \ldots, \Lambda_\alpha(t_n))$ exists for $(\theta_1, \ldots, \theta_n)$ such that $\theta_i \leqslant 0$, $i = 1, \ldots, n$ (Samorodnitsky and Taqqu 1994, Proposition 1.2.12), even if it does not characterize the distribution. Since the increments of the process $\Lambda_\alpha(t)$ are independent and $\Lambda_\alpha(t_i) - \Lambda_\alpha(t_{i-1}) \sim S_\alpha(c_\alpha^{-1/\alpha}(t_i - t_{i-1})^{1/\alpha}, 1, 0)$, where c_α is as introduced in (4), the same proposition yields

$$\begin{aligned}
\log \mathrm{E} \exp\left\{\sum_{i=1}^{n} \theta_i (\Lambda_\alpha(t_i) - \Lambda_\alpha(t_{i-1}))\right\} &= \sum_{i=1}^{n} \log \mathrm{E} \exp\{\theta_i (\Lambda_\alpha(t_i) - \Lambda_\alpha(t_{i-1}))\} \\
&= -\left(c_\alpha \cos \frac{\pi\alpha}{2}\right)^{-1} \sum_{i=1}^{n} (-\theta_i)^\alpha (t_i - t_{i-1}) \\
&= -\frac{1}{\beta} \sum_{i=1}^{n} \theta_i^2 \int_0^{t_i - t_{i-1}} \int_0^{\infty} \mathrm{e}^{\theta_i u} u^{-\beta} \, \mathrm{d}u \, \mathrm{d}v.
\end{aligned}$$

This expression may now be compared with (6).

4.4. Interpretation as packet arrival model

Recall that an application we have in mind of the summation scheme in (2) is that $W(m, t)$ counts the accumulated number of packets generated by m independent users sharing a common medium, when the arrival stream from each source is characterized by a heavy-tailed interarrival distribution. It follows from Theorem 2, applying (ICR), that for large m,

$$W(m, t) \approx \frac{mt}{\mu} - \frac{a_m}{\mu} Y_\beta\left(\frac{t}{a_m}\right) \sim \frac{mt}{\mu} - \frac{1}{\mu^{3/2}} \sqrt{mL(a_m)} a_m^{1-\beta/2} Y_\beta\left(\frac{t}{a_m}\right).$$

Invoking Corollary 1 as well gives the coarser approximative representation

$$W(m, t) \approx \frac{1}{\mu} mt - \frac{\sigma_\beta}{\mu^{3/2}} \sqrt{mL(a_m)} B_{1-\beta/2}(t).$$

This provides a verification of the model for Ethernet-type traffic proposed by Norros (1995). A more comprehensive discussion of arrival process modeling with long-range dependence and further references can be found in Kaj (2002).

5. Some properties of renewal processes

To prepare for the proof of Theorem 2 we need structure results for the functionals $\mathrm{E}[\exp(\sum_{i=1}^n \theta_i N_{t_i})]$ as well as the precise asymptotics of high-order moments $\mathrm{E}(N_t - t/\mu)^k$, and the analogous results for the pure renewal process $\tilde{N}_t$. The technical key to our proof of Theorem 2 is Proposition 1 below, which is somewhat related to an integral equation in Kaj and Sagitov (1998, Lemma 3), derived in a different context.

5.1. Moment generating function for the *n*-dimensional distributions

We give two results for the multivariate moment generating functions of general renewal processes, not necessarily subject to a tail condition such as (1). It is assumed only that $\{N_t\}$ is a stationary renewal process with inter-renewal times $\{U_n\}_{n\geqslant 2}$ having distribution function $F(t)$ and finite mean value $\mu = \mathrm{E}(U_n)$, and that the first renewal time has the equilibrium distribution $F_1(t)$. The notation $\{\tilde{N}_t\}$ is used for the corresponding pure renewal process with all inter-renewal times having the same distribution function $F(t)$.

Proposition 1. *Fix* $n \geqslant 2$ *and a sequence of time points* $0 \leqslant t_1 \leqslant \ldots \leqslant t_n$. *The moment generating function of the finite-dimensional distributions of the stationary renewal counting process* $\{N_t\}$ *satisfies the recurrence relation*

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^n \theta_i N_{t_i}\right)\right] = \mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i N_{t_i}\right)\right] + \frac{1-\exp(-\theta_1)}{1-\exp\left(-\sum_{i=1}^n \theta_i\right)} \int_0^{t_1} \mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i \tilde{N}_{t_i-u}\right)\right] \mathrm{dE}\left[\exp\left(N_u \sum_{i=1}^n \theta_i\right)\right], \tag{14}$$

where $\tilde{N}_t$ is the corresponding pure renewal counting process.

Proof. We have

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^n \theta_i N_{t_i}\right)\right] - \mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i N_{t_i}\right)\right] = \mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i N_{t_i}\right)(\exp[\theta_1 N_{t_1}] - 1)\right].$$

By summing over all jumps in $(0, t_1]$ the term on the right-hand side can be written

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i N_{t_i}\right)(\exp[\theta_1 N_{t_1}] - 1)\right] \\
&= \mathrm{E}\left[\sum_{j=1}^\infty 1_{\{S_j \leqslant t_1\}} \exp\left(\sum_{i=2}^n \theta_i N_{t_i}\right)(\exp[\theta_1 N_{S_j}] - \exp[\theta_1 N_{S_j^-}])\right] \\
&= \mathrm{E}\left[\sum_{j=1}^\infty 1_{\{S_j \leqslant t_1\}} \exp\left(\sum_{i=2}^n \theta_i (N_{t_i} - N_{S_j})\right) - \exp\left(\sum_{i=1}^n \theta_i N_{S_j^-}\right) \exp\left(\sum_{i=2}^n \theta_i\right)(\exp[\theta_1] - 1)\right].
\end{aligned}$$

For any j and $i \geqslant 2$, on the set $\{S_j \leqslant t_1\}$ the increment $N_{t_i} - N_{S_j}$ has the same distribution as $N_{t_i - S_j}$, by stationarity. Now conditional on $\{S_j = u\}$, N_{t-u}, $t \geqslant u$, is the pure renewal process associated with the sequence $\{S_n\}$, $n \geqslant 2$. It follows that

$$\begin{aligned}
\mathrm{E}\left[\exp\left(\sum_{i=1}^n \theta_i N_{t_i}\right)\right] - \mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i N_{t_i}\right)\right] &= \exp\left(\sum_{i=2}^n \theta_i\right)(\exp[\theta_1] - 1) \\
&\quad \times \mathrm{E}\left[\sum_{j=1}^\infty 1_{\{S_j \leqslant t_1\}} \mathrm{E}\left[\exp\left(\sum_{i=2}^n \theta_i \tilde{N}_{t_i - S_j}\right)\middle|\mathcal{F}_{\mathrm{S_j}}\right] \exp\left(\sum_{i=1}^n \theta_i N_{S_j^-}\right)\right].
\end{aligned}$$

Moreover,

$$\exp\left(\sum_{i=1}^{n}\theta_i N_{S_j^-}\right) = \frac{\exp\left(\sum_{i=1}^{n}\theta_i N_{S_j}\right) - \exp\left(\sum_{i=1}^{n}\theta_i N_{S_j^-}\right)}{\exp\left(\sum_{i=1}^{n}\theta_i\right) - 1}$$

$$=: \frac{\Delta_{S_j}\exp\left(N_u\sum_{i=1}^{n}\theta_i\right)}{\exp\left(\sum_{i=1}^{n}\theta_i\right) - 1}$$

and so

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{n}\theta_i N_{t_i}\right)\right] - \mathrm{E}\left[\exp\left(\sum_{i=2}^{n}\theta_i N_{t_i}\right)\right]$$

$$= \frac{\exp\left(\sum_{i=2}^{n}\theta_i\right)(\exp[\theta_1]-1)}{\exp\left(\sum_{i=1}^{n}\theta_i\right) - 1}\mathrm{E}\left[\sum_{j=1}^{\infty}1_{\{S_j\leqslant t_1\}}\mathrm{E}\left[\exp\left(\sum_{i=2}^{n}\theta_i\tilde{N}_{t_i-S_j}\right)\Big|\mathcal{F}_{S_j}\right]\Delta_{S_j}\left(\exp\left[N_u\sum_{i=1}^{n}\theta_i\right]\right)\right]$$

$$= \frac{1-\exp(-\theta_1)}{1-\exp\left(-\sum_{i=1}^{n}\theta_i\right)}\mathrm{E}\left[\int_0^{t_1}\mathrm{E}\left[\exp\left(\sum_{i=2}^{n}\theta_i\tilde{N}_{t_i-u}\right)\right]\mathrm{d}\left(\exp\left[N_u\sum_{i=1}^{n}\theta_i\right]\right)\right]$$

$$= \frac{1-\exp(-\theta_1)}{1-\exp\left(-\sum_{i=1}^{n}\theta_i\right)}\int_0^{t_1}\mathrm{E}\left[\exp\left(\sum_{i=2}^{n}\theta_i\tilde{N}_{t_i-u}\right)\right]\mathrm{d}\mathrm{E}\exp\left(N_u\sum_{i=1}^{n}\theta_i\right)\Bigg].$$

□

Lemma 1. *The moment generating function of the finite-dimensional distributions of the process* $\{N_t\}$ *is differentiable in the time variable and relates to the corresponding function for the pure renewal process* $\{\tilde{N}_t\}$ *as follows:*

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{n}\theta_i\tilde{N}_{t_i}\right)\right] = \frac{\mu}{\exp\left(\sum_{i=1}^{n}\theta_i\right)-1}\sum_{j=1}^{n}\frac{\partial}{\partial t_j}\mathrm{E}\left[\exp\left(\sum_{i=1}^{n}\theta_i N_{t_i}\right)\right]. \tag{15}$$

Proof. We will use a counterpart to the one-dimensional renewal theory in higher dimensions developed by Hunter (1974). The author considered the two-dimensional case, but the ideas are based on two-dimensional Laplace transforms and convolutions, and carry over to any higher dimension. We will prove our claim for $n = 2$ as well; the proof for any other n follows the same pattern. Consider the two-dimensional convolution

$$A ** B(s, t) = \int_0^s \int_0^t A(s-u, t-v) \mathrm{d}B(u, v).$$

It is commutative with respect to $A(s, t)$ and $B(s, t)$ and, if $A(s, t)$ is differentiable almost everywhere, then

$$\frac{\partial}{\partial s}(A ** B)(s, t) = \left(\frac{\partial}{\partial s} A\right) ** B(s, t) + A(0, \cdot) ** B(s, t). \tag{16}$$

Proceeding with the proof, we have

$$\mathrm{E}\mathrm{e}^{\theta_1 N_s + \theta_2 N_t} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathrm{e}^{\theta_1 n + \theta_2 m} P(N_s = n, N_t = m), \tag{17}$$

and the same holds of course for the process $\{\tilde{N}_t\}$. We will prove that the probability $P(N_s = n, N_t = m)$ is differentiable in both s and t, and

$$\begin{aligned} &\frac{\partial}{\partial s} P(N_s = n, N_t = m) + \frac{\partial}{\partial t} P(N_s = n, N_t = m) \\ &= \begin{cases} \mu^{-1}(P(\tilde{N}_s = n-1, \tilde{N}_t = m-1) - P(\tilde{N}_s = n, \tilde{N}_t = m)), & \text{if } n \geqslant 1, m \geqslant 1, \\ -\mu^{-1} P(\tilde{N}_s = n, \tilde{N}_t = m), & \text{if } n = 0 \text{ or } m = 0. \end{cases} \end{aligned} \tag{18}$$

In view of these relations, the claim of the lemma will follow by differentiating both sides of (17).

Introduce the bivariate renewal distribution

$$G(s, t) = P(U_n \leqslant s, U_n \leqslant t) = P(U_n \leqslant s \wedge t) = F(s \wedge t).$$

By Corollary 3.1.1 of Hunter (1974), for $n, m \geqslant 0$,

$$P(\tilde{N}_s = n, \tilde{N}_t = m) = H_{nm} ** G^{**\, n \wedge m}(s, t), \tag{19}$$

where

$$H_{nm}(s, t) = \begin{cases} 1 - \overline{G}(s, t) - \overline{\overline{G}}(s, t) + G(s, t), & \text{if } n = m, \\ \overline{G}_{n-m}(s, t) - \overline{G}_{n-m+1}(s, t) - \overline{G}_{n-m-1} ** G(s, t) + \overline{G}_{n-m} ** G(s, t), & \text{if } n > m, \\ \overline{\overline{G}}_{m-n}(s, t) - \overline{\overline{G}}_{m-n+1}(s, t) - \overline{\overline{G}}_{m-n-1} ** G(s, t) + \overline{\overline{G}}_{m-n} ** G(s, t), & \text{if } n < m, \end{cases}$$

with the notation $A_k(s, t) = A^{**\, k}(s, t)$, $\overline{A}(s, t) = A(s, \infty)$, and $\overline{\overline{A}}(s, t) = A(\infty, t)$ for any function $A(s, t)$. Observe that $\overline{G}_k(s, t) = G^{**\, k}(s, \infty)$.

In the stationary case the same argument, with slight modifications, gives for $n \geqslant 1$, $m \geqslant 1$,

$$P(N_s = n, N_t = m) = H_{nm} ** K ** G^{**\, n \wedge m - 1}(s, t), \tag{20}$$

where $K(s, t) = F_1(s \wedge t)$. Cases where $n = 0$ or $m = 0$ are special:

$$P(N_s = n, N_t = m)$$
$$= \begin{cases} 1 - \overline{K}(s, t) - \bar{\bar{K}}(s, t) + K(s, t), & \text{if } n = m = 0, \\ \overline{L}_{n-1}(s, t) - \overline{L}_n(s, t) - \overline{G}_{n-1} ** K(s, t) + \overline{G}_n ** K(s, t), & \text{if } n \geqslant 1, m = 0, \\ \bar{\bar{L}}_{m-1}(s, t) - \bar{\bar{L}}_m(s, t) - \bar{\bar{G}}_{m-1} ** K(s, t) + \bar{\bar{G}}_m ** K(s, t), & \text{if } n = 0, m \geqslant 1, \end{cases} \quad (21)$$

where $L_k(s, t) = K ** G_k(s, t)$.

To prove (18) for $n, m \geqslant 1$, consider the probability in (20). Since $F_1(t)$ is differentiable, so is the function $K(s, t)$ for $s \neq t$ and

$$\frac{\partial}{\partial s} K(s, t) = F_1'(s) 1_{\{s<t\}}.$$

In particular, for $s \neq t$,

$$\frac{\partial}{\partial s} K(s, t) + \frac{\partial}{\partial t} K(s, t) = F_1'(s \wedge t) = \frac{1 - G(s, t)}{\mu}.$$

Consequently, the convolution (20) is also differentiable, and an application of (16) with $A = K$ and $B = H_{nm} ** G^{**\, n\wedge m-1}$ yields

$$\frac{\partial}{\partial s} P(N_s = n, N_t = m) + \frac{\partial}{\partial t} P(N_s = n, N_t = m)$$
$$= H_{nm} ** \left(\frac{\partial}{\partial s} K\right) ** G^{**\, n\wedge m-1}(s, t) + H_{nm} ** \left(\frac{\partial}{\partial t} K\right) ** G^{**\, n\wedge m-1}(s, t)$$
$$= \mu^{-1} H_{nm} ** (1 - G) ** G^{**\, n\wedge m-1}(s, t). \quad (22)$$

The function $H_{nm}(s, t)$ depends only on the difference $|n - m|$, so that $H_{nm}(s, t) = H_{n-1\, m-1}(s, t)$. Hence, combining (22) and (19), we obtain the first part of (18).

In the case where $n = m = 0$, observe that $\overline{K}(s, t) = F_1(s)$, which implies

$$\frac{\partial}{\partial s} \overline{K}(s, t) + \frac{\partial}{\partial t} \overline{K}(s, t) = F_1'(s) = \frac{1 - \overline{G}(s, t)}{\mu}. \quad (23)$$

Thus, differentiating (21) in this case yields

$$\frac{\partial}{\partial s} P(N_s = 0, N_t = 0) + \frac{\partial}{\partial t} P(N_s = 0, N_t = 0) = -\mu^{-1}(1 - \overline{G}(s, t) - \bar{\bar{G}}(s, t) + G(s, t))$$
$$= -\mu^{-1} H_{00}(s, t),$$

which gives (18) for $n = m = 0$ in view of (19).

It remains to prove (18) in the case where only one of n and m is equal to zero. As earlier, since the function $K(s, t)$ is differentiable almost everywhere, so is the convolution $(\overline{G}_k ** K)(s, t)$, and due to (16),

$$\frac{\partial}{\partial s}(\overline{G}_k ** K)(s,\ t) + \frac{\partial}{\partial t}(\overline{G}_k ** K)(s,\ t) = \overline{G}_k ** \left(\frac{\partial}{\partial s}K\right)(s,\ t) + \overline{G}_k ** \left(\frac{\partial}{\partial t}K\right)(s,\ t)$$

$$= \mu^{-1}\overline{G}_k ** (1 - G)(s,\ t). \tag{24}$$

Further, since $L_k(s,\ t) = K ** G_k(s,\ t)$ is differentiable, we can interchange the limits:

$$\frac{\partial}{\partial s}\overline{L}_k(s,\ t) = \frac{\partial}{\partial s}\left(\lim_{t\to\infty}(K ** G_k)(s,\ t)\right) = \lim_{t\to\infty}\frac{\partial}{\partial s}(K ** G_k)(s,\ t).$$

Hence

$$\frac{\partial}{\partial s}\overline{L}_k(s,\ t) + \frac{\partial}{\partial t}\overline{L}_k(s,\ t) = \left(\frac{\partial}{\partial s}K\right) ** G_k(s,\ \infty) + \left(\frac{\partial}{\partial t}K\right) ** G_k(s,\ \infty)$$

$$= \mu^{-1}(1 - G) ** G_k(s,\ \infty)$$

$$= \mu^{-1}(\overline{G}_k(s,\ t) - \overline{G}_{k+1}(s,\ t)). \tag{25}$$

Thus differentiatiation of (21) in view of (24) and (25) yields

$$\frac{\partial}{\partial s}P(N_s = n,\ N_t = m) + \frac{\partial}{\partial t}P(N_s = n,\ N_t = m)$$

$$= -\mu^{-1}\begin{cases} \overline{G}_n(s,\ t) - \overline{G}_{n+1}(s,\ t) - \overline{G}_{\underline{n}-1} ** G(s,\ t) + \overline{G}_{\underline{n}} ** G(s,\ t), & \text{if } n \geqslant 1,\ m = 0, \\ \overline{\overline{G}}_m(s,\ t) - \overline{\overline{G}}_{m+1}(s,\ t) - \overline{\overline{G}}_{m-1} ** G(s,\ t) + \overline{\overline{G}}_m ** G(s,\ t), & \text{if } n = 0,\ m \geqslant 1, \end{cases}$$

The right-hand side is the function $-\mu^{-1}H_{nm}$, where either n or m is equal to zero. But in this case the right-hand side of (19) is equal to H_{nm}, and the second part of (18) follows. □

5.2. Moments of the renewal counting process

As we shall see, the limit of the one-dimensional distributions is determined by the asymptotic behaviour of the moments $\mathrm{E}N_t^k$ of the counting process. Introduce as usual the renewal function $U(t) = \sum_{n=1}^{\infty} F^{*n}(t)$. Observe that for a pure renewal process $\mathrm{E}\tilde{N}_t = U(t)$, while in the stationary case we have $\mathrm{E}N_t = F_1(t) + F_1 * U(t) = t/\mu$. A result due to Teugels (1968) states that if the renewal distribution $F(t)$ has a regularly varying tail with index α, $1 < \alpha < 2$, then for the pure renewal process, as $t \to \infty$,

$$U(t) - t/\mu \sim \frac{1}{(\alpha - 1)(2 - \alpha)}\frac{t^{2-\alpha}}{\mu^2}L(t) \tag{26}$$

and

$$\operatorname{var}\tilde{N}_t \sim \frac{2}{(2 - \alpha)(3 - \alpha)}\frac{t^{3-\alpha}}{\mu^3}L(t),$$

where the relation $f \sim cg$ for some functions f, g and a constant $c \in \mathbb{R}$ is defined as $\lim_{x\to\infty} f(x)/g(x) = c$.

It is a well-known fact that all moments of the renewal counting process exist and are finite (Asmussen 1987). The following proposition extends (26) to cover arbitrary moments $\mathrm{E}N_t^k$ and stationary renewal processes.

Proposition 2. *If the renewal distribution $F(t)$ has a regularly varying tail with index α, $1 < \alpha < 2$, then for any integer $k \geqslant 1$, as $t \to \infty$, we have:*

(a) *for the pure renewal process,*

$$\mathrm{E}\tilde{N}_t^k - \left(\frac{t}{\mu}\right)^k \sim \frac{k \cdot k!\Gamma(2-\alpha)}{(\alpha-1)\Gamma(k+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t), \tag{27}$$

$$\mathrm{E}(\tilde{N}_t - \mathrm{E}\tilde{N}_t)^k \sim \frac{(-1)^k k}{(k-\alpha)(k+1-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t); \tag{28}$$

(b) *for the stationary renewal process,*

$$\mathrm{E}N_t^k - \left(\frac{t}{\mu}\right)^k \sim \frac{(k-1) \cdot k!\Gamma(2-\alpha)}{(\alpha-1)\Gamma(k+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t), \tag{29}$$

$$\mathrm{E}\left(N_t - \frac{t}{\mu}\right)^k \sim \frac{(-1)^k (k-1)k}{(\alpha-1)(k-\alpha)(k+1-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t). \tag{30}$$

In the proof we need some properties of the class R_ρ of regularly varying functions with index ρ. The following are stated in Bingham *et al*. (1987, Proposition 1.5.7):

(i) If $f \in R_\rho$, then $f^\alpha \in R_{\alpha\rho}$.
(ii) If $f_i \in R_{\rho_i}$, $i = 1, 2$, then $f_1 + f_2 \in R_\rho$, where $\rho = \max\{\rho_1, \rho_2\}$.

Lemma 2. *Further properties of regularly varying functions are as follows:*

(iii) *If $f_i \in R_{\rho_i}$, $i = 1, \ldots, n$, $\rho_i \neq \rho_j$ for $i \neq j$, and $c_i \in \mathbb{R}$, then $\sum_{i=1}^n c_i f_i(x) \sim c_k f_k(x)$, as $x \to \infty$, where k is the index of the largest ρ_i, $i = 1, \ldots, n$.*
(iv) *If $f_i(x) \sim a_i x^\rho L(x)$, as $x \to \infty$, with $a_i \in \mathbb{R}$, $L(x)$ slowly varying and $c_i \in \mathbb{R}$, then $\sum_{i=1}^n c_i f_i(x) \sim \sum_{i=1}^n c_i a_i x^\rho L(x)$, $x \to \infty$.*

Proof. (iii) Property (ii) yields that $\sum_{i=1}^n c_i f_i(x) \in R_{\rho_k}$, where $\rho_k = \max\{\rho_1, \ldots, \rho_n\}$ and hence $\sum_{i=1}^n c_i f_i(x) \sim c_k f_k(x)\tilde{L}(x)$, for some slowly varying $\tilde{L}(x)$. We claim that if $\rho_i \neq \rho_j$ for $i \neq j$, even more is true: $\tilde{L}(x) = 1$. This follows trivially from the representation $\sum_{i=1}^n c_i f_i(x) = \sum_{i=1}^n c_i x^{\rho_i} L_i(x)$ by dividing by $c_k f_k(x) = c_k x^{\rho_k} L_k(x)$ and taking $x \to \infty$.
(iv) Follows from the equality $\lim_{x\to\infty} (c_1 f_1(x) + c_2 f_2(x))/(x^\rho L(x)) = c_1 a_1 + c_2 a_2$. □

The idea of the proof of Proposition 2 is that arbitrary moments $\mathrm{E}N_t^k$ can be expressed as 'polynomials' with respect to convolutions of the renewal function, hence the asymptotics for them can be obtained from Teugels's result (26). Standard techniques when dealing with convolutions involve Laplace–Stieltjes (LS) transforms and Karamata's

Tauberian theorem (Bingham *et al*. 1987, Theorem 1.7.1), which states that the behaviour of a function at infinity is determined by the behaviour of its LS transform at zero.

Denote $V_k(t) = \mathrm{E}N_t^k$, where, to start with, no assumptions are made about the distribution of the first renewal. Note that the LS transforms of $V_k(t)$ and $U(t)$,

$$v_k(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}V_k(t), \qquad u(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}U(t),$$

exist and are finite. We proceed with some supplementary lemmas.

Lemma 3. *The LS transforms $v_k(s)$ of $V_k(t)$, $k \geqslant 1$, satisfy the recurrence relation*

$$v_k(s) = (1 + u(s)) \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} v_i(s) + (-1)^{k+1} v_1(s), \qquad k \geqslant 2. \tag{31}$$

Proof. For $k \geqslant 0$, consider the generalized renewal measure

$$Z_k(t) = \sum_{n=1}^\infty n^k F_1 * F^{*n-1}(t).$$

Observe that $Z_0(t) = V_1(t)$ and for $k \geqslant 1$,

$$\begin{aligned} V_k(t) &= \sum_{n=1}^\infty n^k P(N(t) = n) = \sum_{n=1}^\infty n^k (F_1 * F^{*n-1}(t) - F_1 * F^{*n}(t)) \\ &= Z_k(t) - Z_k * F(t). \end{aligned}$$

Thus $V_k(t)$ is the coefficient of the renewal equation

$$Z_k(t) = V_k(t) + Z_k * F(t), \tag{32}$$

which involves the function $Z_k(t)$. By the classical renewal theorem (Asmussen 1987), the unique solution of (32) is given by

$$Z_k(t) = V_k(t) + V_k * U(t). \tag{33}$$

On the other hand, the relation $n^k - (n-1)^k = -\sum_{i=0}^{k-1} \binom{k}{i} n^i (-1)^{k-i}$ inserted into $V_k(t)$ yields

$$\begin{aligned} V_k(t) &= \sum_{n=1}^\infty (n^k - (n-1)^k) F_1 * F^{*n-1}(t) \\ &= \sum_{n=1}^\infty \left(-\sum_{i=0}^{k-1} \binom{k}{i} n^i (-1)^{k-i} \right) F_1 * F^{*n-1}(t) \\ &= \sum_{i=0}^{k-1} \binom{k}{i} (-1)^{k-i+1} Z_i(t). \end{aligned} \tag{34}$$

Hence, by (33),

$$V_k(t) = \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} (V_i(t) + V_i * U(t)) + (-1)^{k+1} V_1(t), \tag{35}$$

and the recurrence property (31) is just the LS transform counterpart of (35). ☐

Lemma 4. *For any integer $k \geqslant 1$, the LS transform $\upsilon_k(s)$ can be expressed in terms of the LS transform $u(s)$ as follows:*

$$\upsilon_k(s) = \upsilon_1(s) \sum_{j=1}^{k} c_{kj} u(s)^{j-1}, \tag{36}$$

where $c_{kj} = j!\left\{ {k \atop j} \right\} = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} i^k$, $j = 1, \ldots, k$, and $\left\{ {k \atop j} \right\}$ is the Stirling number of the second kind.

Proof. The existence of constants c_{kj}, $j = 1, \ldots, k$, such that (36) holds follows from the recurrence property (31) for $\upsilon_k(s)$ by a standard induction argument. Indeed, assuming that (36) is true for $\upsilon_i(s)$, $i \leqslant k-1$, we have

$$\begin{aligned}
\upsilon_k(s) &= (1 + u(s)) \sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} \upsilon_1(s) \sum_{j=1}^{i} c_{ij} u(s)^{j-1} + (-1)^{k+1} \upsilon_1(s) \\
&= \upsilon_1(s) \left(\sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{i1} + (-1)^{k+1} \right) \\
&\quad + \upsilon_1(s) \sum_{j=2}^{k} \left(\sum_{i=j}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{ij} + \sum_{i=j-1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{ij-1} \right) u(s)^{j-1} \\
&\quad + \upsilon_1(s) k c_{k-1\,k-1} u(s)^{k-1},
\end{aligned}$$

so that (36) is true for $\upsilon_k(s)$ as well, with the constants

$$c_{kj} = \begin{cases} \displaystyle\sum_{i=1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{i1} + (-1)^{k+1}, & j = 1, \\ \displaystyle\sum_{i=j}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{ij} + \sum_{i=j-1}^{k-1} \binom{k}{i} (-1)^{k-i+1} c_{ij-1}, & 2 \leqslant j \leqslant k-1, \\ k c_{k-1 k-1}, & j = k. \end{cases}$$

Now $c_{11} = 1$ yields $c_{k1} = 1$ and $c_{kk} = k!$ for all $k \geqslant 1$. Also, for $k \geqslant 2$,

$$c_{kj} = j(c_{k-1j} + c_{k-1j-1}), \qquad j = 2, \ldots, k-1.$$

Thus if we define the triangular sequence $a_{kj} = c_{kj}/j!$, for $j = 1, \ldots, k$, $k \geqslant 1$, it will satisfy $a_{kk} = 1$, $k \geqslant 1$, and

$$a_{kj} = ja_{k-1j} + a_{k-1j-1}, \qquad j = 2, \ldots, k-1.$$

This recurrence relation defines the sequence $\left\{ {k \atop j} \right\}$ of Stirling numbers of the second kind (see Rosen *et al*. 2000), which gives us the claim of the lemma. □

Lemma 5. *If the renewal distribution $F(t)$ has a regularly varying tail with index α, $1 < \alpha < 2$, then, for any integer $j \geqslant 1$,*

$$u(s)^j - (\mu s)^{-j} \sim \frac{j\Gamma(2-\alpha)}{\alpha - 1} \frac{1}{\mu^{j+1} s^{j+1-\alpha}} L(s^{-1}), \qquad \text{as } s \downarrow 0. \tag{37}$$

Proof. By binomial expansion,

$$u(s)^j - (\mu s)^{-j} = \sum_{r=1}^{j} \binom{j}{r} \left(u(s) - \frac{1}{\mu s} \right)^r \left(\frac{1}{\mu s} \right)^{j-r}. \tag{38}$$

Karamata's Tauberian theorem applied to Teugels's estimate (26) gives

$$u(s) - (\mu s)^{-1} \sim \frac{\Gamma(2-\alpha)}{\alpha - 1} \frac{1}{\mu^2 s^{2-\alpha}} L(s^{-1}), \qquad \text{as } s \downarrow 0.$$

In particular, $u(1/s) - s/\mu \in R_{2-\alpha}$ and property (i) implies $(u(1/s) - s/\mu)^r (s/\mu)^{j-r} \in R_{r(2-\alpha)+j-r}$. Property (iii) of Lemma 2 now shows that the dominating term in the expansion (38) is

$$\binom{j}{1} \left(u(s) - \frac{1}{\mu s} \right) \left(\frac{1}{\mu s} \right)^{j-1},$$

corresponding to the regular variation index

$$j + 1 - \alpha = \max_{1 \leqslant r \leqslant j} \{ r(2-\alpha) + j - r \}.$$

Thus, as $s \downarrow 0$,

$$\begin{aligned} u(s)^j - (\mu s)^{-j} &\sim \binom{j}{1} \left(u(s) - \frac{1}{\mu s} \right) \left(\frac{1}{\mu s} \right)^{j-1} \\ &\sim j \frac{\Gamma(2-\alpha)}{\alpha - 1} \frac{1}{\mu^2 s^{2-\alpha}} L(s^{-1}) \left(\frac{1}{\mu s} \right)^{j-1} \end{aligned}$$

and the claim of the lemma follows. □

To prove parts (27) and (29) of Proposition 2, we will use Lemma 4 to express the function $v_k(s)$ in terms of the function $u(s)$ and then employ Lemma 5, which describes the behaviour of the terms of the expansion at zero. Indeed, for a pure renewal process we have $\tilde{v}_1(s) = u(s)$, and by Lemma 4,

$$\tilde{v}_k(s) = \sum_{j=1}^{k} c_{kj} u(s)^j,$$

hence

$$\tilde{v}_k(s) - c_{kk}(\mu s)^{-k} = c_{kk}(u(s)^k - (\mu s)^{-k}) + \sum_{j=1}^{k-1} c_{kj} u(s)^j. \tag{39}$$

Lemma 5 yields $u(1/s)^k - (s/\mu)^k \in R_{k+1-\alpha}$. Moreover, $u(1/s)^j \in R_j$. Hence, again by property (iii) of Lemma 2, the dominating term in (39) as $s \downarrow 0$ is $k!(u(s)^k - (\mu s)^{-k})$ with the regular variation index

$$k + 1 - \alpha = \max\{1, \ldots, k-1, k+1-\alpha\},$$

so that

$$\tilde{v}_k(s) - k!(\mu s)^{-k} \sim k!(u(s)^k - (\mu s)^{-k}) \sim k! \frac{k\,\Gamma(2-\alpha)}{\alpha - 1} \frac{1}{\mu^{k+1} s^{k+1-\alpha}} L(s^{-1}). \tag{40}$$

To finish the proof of (27), it remains to apply Karamata's Tauberian theorem to (40) (note that the LS transform of t^k is equal to $k!s^{-k}$).

The proof of (29) is completely analogous; we have only to take into account that $v_1(s) = 1/(\mu s)$. Again, Lemma 4 implies

$$v_k(s) = \frac{1}{\mu s} \sum_{j=1}^{k} c_{kj} u(s)^{j-1},$$

and consequently

$$v_k(s) - c_{kk}(\mu s)^{-k} = c_{kk} \frac{1}{\mu s}(u(s)^{k-1} - (\mu s)^{-(k-1)}) + \frac{1}{\mu s} \sum_{j=1}^{k-1} c_{kj} u(s)^{j-1}.$$

By an argument analogous to the pure renewal case,

$$\begin{aligned} v_k(s) - k!(\mu s)^{-k} &\sim k! \frac{1}{\mu s}(u(s)^{k-1} - (\mu s)^{-(k-1)}) \\ &\sim k! \frac{1}{\mu s} \frac{(k-1)\Gamma(2-\alpha)}{\alpha - 1} \frac{1}{\mu^k s^{k-\alpha}} L(s^{-1}), \end{aligned} \tag{41}$$

which is equivalent to (29) by Karamata's Tauberian theorem.

To prove (28), we first consider the shifted moment $\mathrm{E}(\tilde{N}_t - t/\mu)^k$. Exploiting the relation $\sum_{j=0}^{k} \binom{k}{j} (-1)^j = 0$, we obtain

$$\mathrm{E}\left(\tilde{N}_t - \frac{t}{\mu}\right)^k = \sum_{j=0}^{k} \binom{k}{j} \mathrm{E}\tilde{N}_t^j \left(\frac{-t}{\mu}\right)^{k-j}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left(\mathrm{E}\tilde{N}_t^j - \left(\frac{t}{\mu}\right)^j\right) \left(\frac{-t}{\mu}\right)^{k-j}. \qquad (42)$$

By (27), just proved, for any $j = 0, \ldots, k$, we have

$$\left(\mathrm{E}\tilde{N}_t^j - \left(\frac{t}{\mu}\right)^j\right)\left(\frac{-t}{\mu}\right)^{k-j} \sim \frac{(-1)^{k-j} j \cdot j! \Gamma(2-\alpha)}{(\alpha-1)\Gamma(j+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t).$$

Thus property (iv) of Lemma 2 applied to (42) yields

$$\mathrm{E}\left(\tilde{N}_t - \frac{t}{\mu}\right)^k \sim \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j} j \cdot j! \Gamma(2-\alpha)}{(\alpha-1)\Gamma(j+2-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t)$$

$$= \frac{k(-1)^k}{(k-\alpha)(k+1-\alpha)} \frac{t^{k+1-\alpha}}{\mu^{k+1}} L(t). \qquad (43)$$

But due to property (iii) of Lemma 2, the expansion

$$\mathrm{E}(\tilde{N}_t - \mathrm{E}\tilde{N}_t)^k = \sum_{j=0}^{k} \binom{k}{j} \mathrm{E}(\tilde{N}_t - t/\mu)^{k-j} (-(\mathrm{E}\tilde{N}_t - t/\mu))^j$$

now gives

$$\mathrm{E}(\tilde{N}_t - \mathrm{E}\tilde{N}_t)^k \sim \mathrm{E}(\tilde{N}_t - t/\mu)^k,$$

hence (28). The proof of (30) is analogous to that of (43). □

6. Proof of Theorem 2

Consider

$$Z^{(m)}(t) = -\mu Y^{(m)}(t) = -\frac{\mu}{a_m}\left(W(m, a_m t) - \frac{m a_m t}{\mu}\right), \qquad t \geqslant 0. \qquad (44)$$

It will be proved that the process $\{Z^{(m)}(t)\}$ converges weakly in the (ICR) scaling regime to a process $\{Y_\beta(t)\}$ with finite-dimensional distributions characterized by (6). For convenience, we write $a_m = a$ in this section.

6.1. Convergence of one-dimensional distributions

The proof is based on the method of moments. We will prove that all cumulants of the marginal distributions of $Z^{(m)}(t)$ defined in (44) converge to those of the limit process $Y_\beta(t)$, from which convergence of the one-dimensional distributions follows.

Indeed, due to independence, for the cumulant generating function of $Z^{(m)}(t)$ we have

$$\log \mathrm{E} \exp\{\theta Z^{(m)}(t)\} = m \log \mathrm{E} \exp\{-\theta\mu(N_{at} - at/\mu)/a\}.$$

The k-th order cumulants of $Z^{(m)}(t)$,

$$C_k^{(m)}(t) = m \frac{\mathrm{d}^k}{\mathrm{d}\theta^k} (\log \mathrm{E} \mathrm{e}^{-\theta\mu(N_{at}-at/\mu)/a})|_{\theta=0}, \qquad k \geqslant 1,$$

are hence determined by the cumulants of the process $-\mu(N_{at} - at/\mu)/a$, which is a rescaled and centred renewal counting process up to a constant. It is well known that such cumulants can be expressed as polynomials with respect to the moments, i.e. there exist constants α_{kj}, $j = 0, \ldots, k$, such that

$$C_k^{(m)}(t) = m \sum_{j=0}^{k} \alpha_{kj} (-1)^j \mu^j \mathrm{E}(N_{at} - at/\mu)^j / a^j. \tag{45}$$

Moreover, it can be proved that $\alpha_{kk} = 1$ for all k. Since $a \to \infty$ as $m \to \infty$, by Proposition 2 we have

$$\mathrm{E}\left(N_{at} - \frac{at}{\mu}\right)^k \sim \frac{(-1)^k (k-1)k}{\beta(k-1-\beta)(k-\beta)} \frac{(at)^{k-\beta}}{\mu^{k+1}} L(a), \qquad \text{as } m \to \infty.$$

Hence the terms of expansion (45) are regularly varying and property (iii) of Lemma 2 yields

$$\begin{aligned} C_k^{(m)}(t) &\sim \alpha_{kk} m (-1)^k \mu^k \mathrm{E}(N_{at} - at/\mu)^k / a^k \\ &\sim m a^{-\beta} L(a) \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)} \frac{t^{k-\beta}}{\mu}. \end{aligned}$$

But $ma^{-\beta} L(a) \to \mu$ by assumption (ICR), thus

$$C_k^{(m)}(t) \to \frac{(k-1)k}{\beta(k-1-\beta)(k-\beta)} t^{k-\beta},$$

where the limit expressions are the cumulants of $Y_\beta(t)$ by formula (11). Hence $Z^{(m)}(t)$ converges in distribution to a random variable $Y_\beta(t)$, where $Y_\beta(t)$ satisfies (7).

6.2. Convergence of *n*-dimensional distributions

Here we use the asymptotic relation

$$\log \mathrm{E}\exp\left\{\sum_{i=1}^{n}\theta_i Z^{(m)}(t_i)\right\} = m\log \mathrm{E}\exp\left\{-\sum_{i=1}^{n}\theta_i\mu(N_{at_i}-at_i/\mu)/a\right\}$$
$$\sim m\mathrm{E}\left[\exp\left(-\sum_{i=1}^{n}\theta_i\mu(N_{at_i}-at_i/\mu)/a\right)-1\right]+\mathcal{O}\left(\frac{m}{a}\right), \qquad \text{as } m\to\infty. \quad (46)$$

For $n \geqslant 1$ and $1 \leqslant k \leqslant n$, put

$$\bar{\theta}_{k,n} = (\theta_k, \ldots, \theta_n), \qquad \bar{t}_{k,n} = (t_k, \ldots, t_n),$$

where $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n$, and let

$$\Phi_n(\bar{\theta}_{1,n};\, \bar{t}_{1,n}) = \mathrm{E}\left[\exp\left(\sum_{i=1}^{n}\theta_i(N_{t_i}-t_i/\mu\right)\right]$$

be the multivariate moment generating function for the centred renewal process $\{N_t - t/\mu\}_{t\geqslant 0}$. Similarly, let $\tilde{\Phi}_n(\bar{\theta}_{1,n};\, \bar{t}_{1,n})$ denote the corresponding function for the pure renewal process $\{\tilde{N}_t - t/\mu\}_{t\geqslant 0}$. By Proposition 1,

$$\Phi_n(\bar{\theta}_{1,n};\, \bar{t}_{1,n}) = \mathrm{E}\Phi_{n-1}(\bar{\theta}_{2,n};\, \bar{t}_{2,n})\exp(-\theta_1 t_1/\mu) + \frac{1-\exp(-\theta_1)}{1-\exp\left(-\sum_{i=1}^{n}\theta_i\right)} \quad (47)$$
$$\times\int_0^{t_1}\exp[-\theta_1(t_1-u)/\mu]\tilde{\Phi}_{n-1}(\bar{\theta}_{2,n};\, \bar{t}_{2,n}-u)\exp\left(-u\sum_{i=1}^{n}\theta_i/\mu\right)\mathrm{dE}\left[\exp\left(N_u\sum_{i=1}^{n}\theta_i\right)\right].$$

Here and in what follows the sequel the subtraction $\bar{t}_{k,n} - u = (t_k - u, \ldots, t_n - u)$ is interpreted componentwise.

For $m \geqslant 1$ and the given scaling sequence $a = a_m$, consider the scaled functions

$$\Lambda_n^{(m)}(\bar{\theta}_{1,n};\, \bar{t}_{1,n}) = m(\Phi_n(-\mu\bar{\theta}_{1,n}/a;\, a\bar{t}_{1,n}) - 1)$$
$$= m\mathrm{E}\left[\exp\left(-\sum_{i=1}^{n}\theta_i\mu(N_{at_i}-at_i/\mu)/a\right)-1\right],$$

and, analogously,

$$\tilde{\Lambda}_n^{(m)}(\bar{\theta}_{1,n};\, \bar{t}_{1,n}) = m(\tilde{\Phi}_n(-\mu\bar{\theta}_{1,n}/a;\, a\bar{t}_{1,n}) - 1).$$

According to (46) it is the limit functions of $\Lambda_n^{(m)}(\bar{\theta}_{1,n};\, \bar{t}_{1,n})$ as $m\to\infty$ that determine the cumulant generating function of Y_β.

Lemma 6. *The limit functions*

$$\Lambda_n(\bar{\theta}_{1,n};\, \bar{t}_{1,n}) = \lim_{m\to\infty}\Lambda_n^{(m)}(\bar{\theta}_{1,n};\, \bar{t}_{1,n}), \qquad n \geqslant 1,$$

exist and satisfy the system of recursive integral equations

$$\Lambda_n(\bar\theta_{1,n};\ \bar t_{1,n}) = \Lambda_{n-1}(\bar\theta_{2,n};\ \bar t_{2,n} - t_1) + \Lambda_1\left(\sum_{i=1}^n \theta_i;\ t_1\right)$$
$$- \left(1 - \frac{\theta_1}{\sum_{i=1}^n \theta_i}\right)\int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_1\left(\sum_{i=1}^n \theta_i;\ u\right)$$
$$- \left(1 + \frac{\theta_1}{\sum_{i=2}^n \theta_i}\right)\int_0^{t_1} e^{\theta_1(t_1-u)}\, d\Lambda_{n-1}(\bar\theta_{2,n};\ \bar t_{2,n} - u),$$

where, in the case $n = 1$, *we put* $\Lambda_{n-1} = 0$.

Proof. The proof is by induction on n. The relation (7), established in the previous subsection, provides the existence of a limit function $\Lambda_1(\theta;\ t)$ for the case $n = 1$. The integral equation for $n = 1$ is trivial. Fix $n \geqslant 2$ and assume that $\Lambda_{n'}(\bar\theta_{1,n'};\ \bar t_{1,n'})$ exist for $n' \leqslant n-1$. It follows, in particular, that the limit functions $\Lambda_{n-k+1}(\bar\theta_{k,n};\ \bar t_{k,n})$ exist for $2 \leqslant k \leqslant n$.

To study the asymptotic behaviour of $\Lambda_n^{(m)}(\bar\theta_{1,n};\ \bar t_{1,n})$ as $m \to \infty$, we apply the defining scaling relation to (47). This gives

$$\Lambda_n^{(m)}(\bar\theta_{1,n};\ \bar t_{1,n}) = \Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\ \bar t_{2,n})e^{\theta_1 t_1} + I_1^{(m)} + I_2^{(m)} + I_3^{(m)}, \tag{48}$$

where

$$I_1^{(m)} = -\frac{a}{\mu}(e^{\mu\theta_1/a} - 1)\int_0^{t_1} e^{\theta_1(t_1-u)}\tilde\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\ \bar t_{2,n} - u)du,$$
$$I_2^{(m)} = -\frac{a}{\mu}(e^{\mu\theta_1/a} - 1)\int_0^{t_1} e^{\theta_1(t_1-u)}\left(1 + \frac{1}{m}\tilde\Lambda_{n-1}^{(m)}(\bar\theta_{2,n};\ \bar t_{2,n} - u)\right)H^{(m)}(du),$$
$$I_3^{(m)} = m\left(1 - \frac{a(e^{\mu\theta_1/a} - 1)}{\mu\theta_1}\right)(e^{\theta_1 t_1} - 1),$$

and the integration in $I_2^{(m)}$ is with respect to the signed measure

$$H^{(m)}(du) = m\left(\frac{\mu\exp\left(u\sum_{i=1}^n \theta_i\right)}{a\left(1 - \exp\left[\mu\sum_{i=1}^n \theta_i/a\right]\right)}\, dE\left[\exp\left(-\mu N_{au}\sum_{i=1}^n \theta_i/a\right)\right] - du\right).$$

Since

$$I_3^{(m)} = O\left(\frac{m}{a}\right), \qquad \text{as } m \to \infty,$$

we need only to investigate the terms $I_1^{(m)}$ and $I_2^{(m)}$.

To evaluate $I_1^{(m)}$ we apply Lemma 1, which implies

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{n}\theta_i\tilde{N}_{t_i-u}\right)\right] = -\frac{\mu}{\exp\left(\sum_{i=1}^{n}\theta_i - 1\right)}\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{E}\left[\exp\left(\sum_{i=1}^{n}\theta_i N_{t_i-u}\right)\right].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{E}\left[\exp\left\{\sum_{i=1}^{n}\theta_i\left(N_{t_i-u}-\frac{1}{\mu}(t_i-u)\right)\right\}\right]$$

$$= -\frac{1}{\mu}\left(\exp\left[\sum_{i=1}^{n}\theta_i - 1\right]\right)\mathrm{E}\left[\exp\left\{\sum_{i=1}^{n}\theta_i\left(\tilde{N}_{t_i-u}-\frac{1}{\mu}(t_i-u)\right)\right\}\right] + \frac{1}{\mu}\sum_{i=1}^{n}\theta_i\,\mathrm{E}\left[\exp\left\{\sum_{i=1}^{n}\theta_i\left(N_{t_i-u}-\frac{1}{\mu}(t_i-u)\right)\right\}\right],$$

which can be written

$$\tilde{\Phi}_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n}-u) = \frac{\sum_{i=1}^{n}\theta_i}{\exp\left(\sum_{i=1}^{n}\theta_i\right)-1}\Phi_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n}-u) - \frac{\mu}{\exp\left(\sum_{i=1}^{n}\theta_i\right)-1}\frac{\mathrm{d}}{\mathrm{d}u}\Phi_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n}-u).$$

Under the given rescaling scheme the same relation takes the form

$$\tilde{\Lambda}_n^{(m)}(\bar{\theta}_{1,n};\ \bar{t}_{1,n}-u) = \frac{\mu \sum_{i=1}^n \theta_i}{a\left(1-\exp\left[-\mu \sum_{i=1}^n \theta_i/a\right]\right)} \Lambda_n^{(m)}(\bar{\theta}_{1,n};\ \bar{t}_{1,n}-u)$$

$$- \frac{\mu}{a\left(\exp\left[-\mu \sum_{i=1}^n \theta_i/a\right]-1\right)} \frac{\mathrm{d}}{\mathrm{d}u} \Lambda_n^{(m)}(\bar{\theta}_{1,n};\ \bar{t}_{1,n}-u)$$

$$+ m\left(\frac{\mu \sum_{i=1}^n \theta_i}{a\left(1-\exp\left[-\mu \sum_{i=1}^n \theta_i/a\right]\right)} - 1\right).$$

If we apply the preceeding identity with the choice of index $n-1$, then the induction hypothesis allows us to replace the coefficients with their asymptotic limits, adding a remainder term, thus

$$\tilde{\Lambda}_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u) = \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u) + \frac{1}{\sum_{i=2}^n \theta_i} \frac{\mathrm{d}}{\mathrm{d}u} \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u) + O\left(\frac{m}{a}\right).$$

It follows that

$$I_1^{(m)} = -\theta_1 \int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)} \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u)\mathrm{d}u - \frac{\theta_1}{\sum_{i=2}^n \theta_i} \int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\, \mathrm{d}\Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u) + O\left(\frac{m}{a}\right).$$

Moreover, the integration by parts

$$\theta_1 \int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)} \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u)\mathrm{d}u = \mathrm{e}^{\theta_1 t_1} \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}) - \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-t_1) + \int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\, \mathrm{d}\Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-u)$$

gives

$$I_1^{(m)} = \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - t_1) - \mathrm{e}^{\theta_1 t_1}\Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n})$$
$$- \left(1 + \frac{\theta_1}{\sum_{i=2}^{n}\theta_i}\right)\int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - u) + O\left(\frac{m}{a}\right).$$

Turning to $I_2^{(m)}$, this integral is evaluated with respect to the measure

$$H^{(m)}(\mathrm{d}u) = m\left(\frac{\mu\exp\left(u\sum_{i=1}^{n}\theta_i\right)}{a\left(1-\exp\left[\mu\sum_{i=1}^{n}\theta_i/a\right]\right)}\,\mathrm{dE}\left[\exp\left(-\mu N_{au}\sum_{i=1}^{n}\theta_i/a\right)\right] - \mathrm{d}u\right)$$
$$= -\frac{1}{\sum_{i=1}^{n}\theta_i}\,\mathrm{d}\left(m\mathrm{E}\left[\exp\left\{-\mu\frac{N_{au}-au/\mu}{a}\sum_{i=1}^{n}\theta_i\right\} - 1\right]\right)$$
$$+ m\mathrm{E}\left[\exp\left(-\mu\frac{N_{au}-au/\mu}{a}\sum_{i=1}^{n}\theta_i\right) - 1\right]\mathrm{d}u + O\left(\frac{m}{a}\right)$$
$$= -\frac{1}{\sum_{i=1}^{n}\theta_i}\,\mathrm{d}\Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ u\right) + \Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ u\right)\mathrm{d}u + O\left(\frac{m}{a}\right).$$

Since

$$\theta_1\int_0^{t_1}\mathrm{e}^{\theta_1(t_1-u)}\Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ u\right)\mathrm{d}u = -\Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ t_1\right) + \int_0^{t_1}\mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ u\right),$$

by a partial integration, it follows that

$$I_2^{(m)} = \Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ t_1\right) - \left(1 - \frac{\theta_1}{\sum_{i=1}^{n}\theta_i}\right)\int_0^{t_1}\mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ u\right) + O\left(\frac{m}{a}\right).$$

Summarizing,

$$\Lambda_n^{(m)}(\bar{\theta}_{1,n};\ \bar{t}_{1,n}) = \Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - t_1) + \Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ t_1\right)$$

$$- \left(1 + \frac{\theta_1}{\sum_{i=2}^{n}\theta_i}\right)\int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_{n-1}^{(m)}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - u)$$

$$- \left(1 - \frac{\theta_1}{\sum_{i=1}^{n}\theta_i}\right)\int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_1^{(m)}\left(\sum_{i=1}^{n}\theta_i;\ u\right) + O\left(\frac{m}{a}\right).$$

Now take $m \to \infty$ and apply the induction hypothesis to conclude that the limit function $\Lambda_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n})$ exists and satisfies the equation

$$\Lambda_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n}) = \Lambda_{n-1}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - t_1) + \Lambda_1\left(\sum_{i=1}^{n}\theta_i;\ t_1\right)$$

$$- \left(1 + \frac{\theta_1}{\sum_{i=2}^{n}\theta_i}\right)\int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_{n-1}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - u)$$

$$- \left(1 - \frac{\theta_1}{\sum_{i=1}^{n}\theta_i}\right)\int_0^{t_1} \mathrm{e}^{\theta_1(t_1-u)}\,\mathrm{d}\Lambda_1\left(\sum_{i=1}^{n}\theta_i;\ u\right).$$

This completes the proof of the lemma and the proof of convergence of the finite-dimensional distributions. □

6.3. Cumulant generating function for the increment process

The logarithmic moment generating function for the increments of the limit process Y_β is given by

$$\Gamma_n(\bar{\theta}_{1,n},\ \bar{t}_{1,n}) = \log \mathrm{E}\exp\left\{\sum_{i=1}^{n}\theta_i(Y_\beta(t_i) - Y_\beta(t_{i-1}))\right\}.$$

In particular,

$$\Gamma_n(\bar{\theta}_{1,n},\ \bar{t}_{1,n}) = \Lambda_n((\theta_1 - \theta_2,\ \ldots,\ \theta_{n-1} - \theta_n,\ \theta_n);\ \bar{t}_{1,n}).$$

Lemma 6 shows that these functions satisfy the recursive system

$$\begin{aligned}\Gamma_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n}) = \Gamma_{n-1}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - t_1) + \Lambda_1(\theta_1;\ t_1)\\ + \frac{\theta_2}{\theta_1}\int_0^{t_1} \mathrm{e}^{(\theta_1-\theta_2)u}\,\mathrm{d}\Lambda_1(\theta_1;\ t_1 - u)\\ + \frac{\theta_1}{\theta_2}\int_0^{t_1} \mathrm{e}^{(\theta_1-\theta_2)u}\,\mathrm{d}\Gamma_{n-1}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - t_1 + u)\\ =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{49}$$

Note that in the integral terms of Lemma 6 we also made the change of variable $u \to t_1 - u$.

To complete the proof of the characterization of the limit process Y_β it remains to verify that the functions given in (6) are the solutions of the integral equation stated above. To this end, assume that $\Gamma_{n-1}(\bar{\theta}_{2,n};\ \bar{t}_{2,n} - t_1)$ is given by the representation (6). Then, by (49),

$$\begin{aligned}\Gamma_n(\bar{\theta}_{1,n};\ \bar{t}_{1,n}) = \frac{1}{\beta}\sum_{i=1}^{n}\theta_i^2\int_0^{t_i-t_{i-1}}\int_0^{v}\exp(\theta_i u)u^{-\beta}\,\mathrm{d}u\,\mathrm{d}v\\ + \frac{1}{\beta}\sum_{i=2}^{n-1}\sum_{j=i+1}^{n}\theta_i\theta_j\exp\left[\sum_{k=i+1}^{j-1}\theta_k(t_k - t_{k-1})\right]\\ \times\int_0^{t_i-t_{i-1}}\int_0^{t_j-t_{j-1}}\exp(\theta_j u)\exp(\theta_i v)(t_{j-1} - t_i + u + v)^{-\beta}\,\mathrm{d}u\,\mathrm{d}v\\ + I_3 + I_4,\end{aligned}$$

and it remains to establish that

$$\begin{aligned}I_3 + I_4 = \frac{1}{\beta}\sum_{j=2}^{n}\theta_1\theta_j\exp\left[\sum_{k=2}^{j-1}\theta_k(t_k - t_{k-1})\right]\\ \times\int_0^{t_1}\int_0^{t_j-t_{j-1}}\exp(\theta_j u)\exp(\theta_1 v)(t_{j-1} - t_1 + u + v)^{-\beta}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned} \tag{50}$$

But by the induction hypothesis, changing the order of integration, and with the change of variable $u' = t_1 + v - u$,

$$
\begin{aligned}
I_3 &= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1} \mathrm{e}^{-(\theta_1-\theta_2)(u-t_1)}\int_0^u \mathrm{e}^{\theta_1 v} v^{-\beta}\,\mathrm{d}v\,\mathrm{d}u \\
&= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1}\int_v^{t_1} \mathrm{e}^{-(\theta_1-\theta_2)(u-v-t_1)}\,\mathrm{d}u\,\mathrm{e}^{\theta_2 v} v^{-\beta}\,\mathrm{d}v \\
&= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1}\int_v^{t_1} \mathrm{e}^{(\theta_1-\theta_2)u}\,\mathrm{d}u\,\mathrm{e}^{\theta_2 v} v^{-\beta}\,\mathrm{d}v \\
&= -\frac{\theta_1\theta_2}{\beta}\int_0^{t_1} \mathrm{e}^{(\theta_1-\theta_2)u}\int_0^u \mathrm{e}^{\theta_2 v} v^{-\beta}\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned}
$$

Further, since

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}w}\Gamma_{n-1}(\bar{\theta}_{2,n};\ \bar{t}_{2,n}-t_1+w) &= \frac{\theta_2^2}{\beta}\int_0^{t_2-t_1+w}\exp(\theta_2 v)v^{-\beta}\,\mathrm{d}v + \frac{1}{\beta}\sum_{j=3}^{n}\theta_2\theta_j\exp\left[\sum_{k=3}^{j-1}\theta_k(t_k-t_{k-1})\right] \\
&\quad\times\int_0^{t_j-t_{j-1}}\exp(\theta_j v)\exp[\theta_2(t_2-t_1+w)](t_{j-1}-t_1+w+v)^{-\beta}\,\mathrm{d}v,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
I_4 = &\ \frac{\theta_1\theta_2}{\beta}\int_0^{t_1}\exp[(\theta_1-\theta_2)u]\int_0^{t_2-t_1+u}\exp(\theta_2 v)v^{-\beta}\,\mathrm{d}v\,\mathrm{d}u \\
&+\frac{1}{\beta}\sum_{j=3}^{n}\theta_1\theta_j\exp\left[\sum_{k=3}^{j-1}\theta_k(t_k-t_{k-1})\right] \\
&\times\int_0^{t_1}\exp[(\theta_1-\theta_2)u]\exp[\theta_2(t_2-t_1+u)]\int_0^{t_j-t_{j-1}}\exp(\theta_j v)(t_{j-1}-t_1+u+v)^{-\beta}\,\mathrm{d}v\,\mathrm{d}u
\end{aligned}
$$

and (50) follows. □

6.4. Tightness

To finish the proof of the convergence result in Theorem 2, it remains to establish tightness as the sequence of laws of $Y^{(m)}$ converges to the law of $-\mu^{-1}Y_\beta$. As usual the trajectories of $Y^{(m)}$ are considered to be elements in the Skorokhod space $D(0, T)$ of cadlag functions on a real interval $[0, T]$, equipped with the Skorokhod topology. To prove tightness in $D(0, T)$ for any fixed T, fix $0<\beta<1$ and consider time points $0<t_1<t<t_2<T$. By stationarity of $Y^{(m)}$,

$$
\begin{aligned}
&\mathrm{E}(|Y^{(m)}(t)-Y^{(m)}(t_1)|\,|Y^{(m)}(t_2)-Y^{(m)}(t)|) \\
&\quad\leqslant \mathrm{var}(Y^{(m)}(t-t_1))^{1/2}\,\mathrm{var}(Y^{(m)}(t_2-t))^{1/2}
\end{aligned}
$$

By (45),

$$\mathrm{var}(Y^{(m)}(t)) = \frac{C_2^{(m)}(t)}{\mu^2} = \frac{m}{a^2}\mathrm{E}\left(N_{at} - \frac{at}{\mu}\right)^2.$$

As proved in Proposition 2, the function $\mathrm{E}(N_t - t/\mu)^2$ is regularly varying and

$$\mathrm{E}\left(N_t - \frac{t}{\mu}\right)^2 \sim \frac{2}{\beta(1-\beta)(2-\beta)\mu^3} t^{2-\beta} L(t), \qquad \text{as } t \to \infty. \tag{51}$$

The Potter bounds for a regularly varying function (Bingham *et al*. 1987, Theorem 1.5.6) yield that, for any $\epsilon > 0$, there exists m_0 such that

$$\frac{\mathrm{E}(N_{at} - at/\mu)^2}{\mathrm{E}(N_a - a/\mu)^2} < (1+\epsilon)\max\{t^{2-\beta+\epsilon}, t^{2-\beta-\epsilon}\}, \qquad \text{as } m \geqslant m_0.$$

Hence, for $m \geqslant m_0$,

$$\begin{aligned}
&\mathrm{E}(|Y^{(m)}(t) - Y^{(m)}(t_1)|\,|Y^{(m)}(t_2) - Y^{(m)}(t)|) \\
&\qquad \leqslant \frac{m}{a^2}(\mathrm{E}[N_{a(t-t_1)} - a(t-t_1)/\mu]^2)^{1/2}(\mathrm{E}[N_{a(t_2-t)} - a(t_2-t)/\mu]^2)^{1/2} \\
&\qquad \leqslant \frac{m}{a^2}(1+\epsilon)\mathrm{E}(N_a - a/\mu)^2 C(t_1, t, t_2),
\end{aligned}$$

where

$$\begin{aligned}
C(t_1, t, t_2) &= \max\{[(t-t_1)(t_2-t)]^{1-(\beta-\epsilon)/2}, [(t-t_1)(t_2-t)]^{1-(\beta+\epsilon)/2}\} \\
&\leqslant \max\{(t_2-t_1)^{2-\beta+\epsilon}, (t_2-t_1)^{2-\beta-\epsilon}\}.
\end{aligned}$$

Since (51) and the condition (ICR) imply that $(m/a^2)\mathrm{E}(N_a - a/\mu)^2 \to \sigma_\beta^2\mu^{-2}$, we have that, for any $\delta > 0$, there exists m_1 such that

$$\begin{aligned}
&\mathrm{E}(|Y^{(m)}(t) - Y^{(m)}(t_1)|\,|Y^{(m)}(t_2) - Y^{(m)}(t)|) \\
&\qquad\qquad \leqslant (1+\epsilon)(\sigma_\beta^2\mu^{-2} + \delta)\max\{(t_2-t_1)^{2-\beta+\epsilon}, (t_2-t_1)^{2-\beta-\epsilon}\}.
\end{aligned}$$

for $m \geqslant m_1$. Take $\epsilon < 1 - \beta$. Then the desired tightness property follows from Billingsley 1968, Theorem 15.6).

6.5. Proof of the convergence of $Y^{(m)}$ under condition (FCR)

Selected parts of the proof of Theorem 2 can be modified to provide the limit result (3) for the process $Y^{(m)}(t)$ under condition (FCR) with the normalizing sequence

$$b_m = (ma_m^{2-\beta}L(a_m))^{1/2}. \tag{52}$$

Since in this case the limit process is Gaussian, it is enough to show that the marginal distributions of $Y^{(m)}(t)$ converge to Gaussian distributions and that the covariance function converges to that of a multiple of fractional Brownian motion.

The convergence of the marginal distributions of $Y^{(m)}(t)$ under the new scaling $b_m = b$ can be obtained by the method of moments along the same lines as for the marginals of $Z^{(m)}(t)$ in Section 6.1. Now the cumulants of $Y^{(m)}(t)$ read as follows:

$$D_k^{(m)}(t) = m\frac{\mathrm{d}^k}{\mathrm{d}\theta^k}(\log \mathrm{E}\mathrm{e}^{\theta(N_{at}-at/\mu)/b})|_{\theta=0} = m\sum_{j=0}^{k}\alpha_{kj}\mathrm{E}(N_{at}-at/\mu)^j/b^j.$$

Continuing as earlier, due to Proposition 2 we have

$$\begin{aligned} D_k^{(m)}(t) &\sim \alpha_{kk}m\mathrm{E}(N_{at}-at/\mu)^k/b^k \\ &\sim \frac{ma^{k-\beta}L(a)}{b^k}\frac{(-1)^k(k-1)k}{\beta(k-1-\beta)(k-\beta)}\frac{t^{k-\beta}}{\mu^{k+1}}. \end{aligned}$$

Observe that (52) and condition (FCR) yield

$$\frac{ma^{k-\beta}L(a)}{b^k} = \begin{cases} 1, & \text{if } k=2, \\ (ma^{-\beta}L(a))^{1-k/2} \to 0, & \text{if } k>2. \end{cases}$$

Hence,

$$D_k^{(m)}(t) \to \begin{cases} \mu^{-3}\sigma_\beta^2 t^{2-\beta}, & \text{if } k=2, \\ 0 & \text{if } k>2, \end{cases} \tag{53}$$

and since $D_1^{(m)}(t) = 0$, it follows that the cumulants of the random variable $Y^{(m)}(t)$ converge to those of a Gaussian random variable with the same distribution as $\mu^{-3/2}\sigma_\beta B_H(t)$.

It remains to prove that the covariance function of the process $Y^{(m)}(t)$ converges to that of $\mu^{-3/2}\sigma_\beta B_H(t)$. But the process $Y^{(m)}(t)$ has stationary increments, whence

$$\mathrm{E}[Y^{(m)}(t)Y^{(m)}(s)] = \frac{1}{2}(\mathrm{var}[Y^{(m)}(t)] + \mathrm{var}[Y^{(m)}(s)] - \mathrm{var}[Y^{(m)}(t-s)]),$$

and the convergence follows from (53).

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