

CONVERGENCE OF THE MULTIGRID V -CYCLE ALGORITHM FOR SECOND-ORDER BOUNDARY VALUE PROBLEMS WITHOUT FULL ELLIPTIC REGULARITY

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ABSTRACT. The multigrid V -cycle algorithm using the Richardson relaxation scheme as the smoother is studied in this paper. For second-order elliptic boundary value problems, the contraction number of the V -cycle algorithm is shown to improve uniformly with the increase of the number of smoothing steps, without assuming full elliptic regularity. As a consequence, the V -cycle convergence result of Braess and Hackbusch is generalized to problems without full elliptic regularity.

1. INTRODUCTION

In this paper we consider the effect of the regularity of a second-order elliptic boundary value problem on the asymptotic behavior of the contraction number of a V -cycle multigrid algorithm with respect to the number of smoothing steps.

Let Ω be a polygonal domain in \mathbb{R}^2 with reentrant corners. Consider the variational problem of finding $u \in H_0^1(\Omega)$ such that

$$(1.1) \quad a(u, v) = F(v) \quad \forall v \in H_0^1(\Omega),$$

where $F \in H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$, and

$$(1.2) \quad a(u, v) = \int_{\Omega} [p(x)\nabla u \cdot \nabla v + r(x)uv] dx \quad \forall u, v \in H^1(\Omega).$$

We assume that $p(x)$ and $r(x)$ are C^1 on $\bar{\Omega}$, $p(x) > 0$ on $\bar{\Omega}$, and $r(x) \geq 0$ on $\bar{\Omega}$.

It is clear that

$$(1.3) \quad |a(v_1, v_2)| \leq C \|v_1\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} \quad \forall v_1, v_2 \in H^1(\Omega),$$

and the Poincaré inequality implies that

$$(1.4) \quad a(v, v) \geq c \|v\|_{H^1(\Omega)}^2 \quad \forall v \in H_0^1(\Omega).$$

Here C and c are positive constants which depend on Ω and the coefficients $p(x)$ and $r(x)$.

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The estimates (1.3)–(1.4) and the Riesz representation theorem imply (cf. [16, 14]) that (1.1) is uniquely solvable and

$$(1.5) \quad \|u\|_{H^1(\Omega)} \lesssim \|F\|_{H^{-1}(\Omega)}.$$

In order to avoid the proliferation of constants, we henceforth use the notation $A \lesssim B$ (or $B \gtrsim A$) to represent the inequality $A \leq \text{constant} \times B$, where the constant is positive and independent of all the variables in the inequality, and it is always assumed to be mesh-independent (i.e., it is independent of mesh sizes and mesh levels). The notation $A \approx B$ is equivalent to the statement that $A \lesssim B$ and $B \lesssim A$.

Because of the presence of reentrant corners, the solution of (1.1) does not have $H^2(\Omega)$ regularity for $F \in L_2(\Omega)$. Instead (cf. Corollary 5.12 and Section 14 of [18], and also the related work in [21, 27]), there exists a number α satisfying

$$\frac{1}{2} < \alpha < 1$$

such that $u \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$ for $F \in H^{-1+\alpha}(\Omega)$, and the following regularity estimate holds:

$$(1.6) \quad \|u\|_{H^{1+\alpha}(\Omega)} \lesssim \|F\|_{H^{-1+\alpha}(\Omega)}.$$

Let \mathcal{T}_1 be a triangulation of Ω , and the triangulations \mathcal{T}_k , $k = 2, 3, \dots$, be obtained from \mathcal{T}_{k-1} by connecting midpoints. The P_1 finite element spaces associated with the triangulations \mathcal{T}_k will be denoted by V_k . The k^{th} level discrete problem for (1.1) is to find $u_k \in V_k$ such that

$$(1.7) \quad a(u_k, v) = F(v) \quad \forall v \in V_k.$$

On each level we introduce a discrete inner product

$$(1.8) \quad (v_1, v_2)_k = h_k^2 \sum_{p \in \mathcal{V}_k} v_1(p)v_2(p) \quad \forall v_1, v_2 \in V_k,$$

where \mathcal{V}_k is the set of all the internal vertices of \mathcal{T}_k . We can then represent the variational form $a(\cdot, \cdot)$ on $V_k \times V_k$ by the operator $A_k : V_k \rightarrow V_k$ defined by

$$(1.9) \quad (A_k v_1, v_2)_k = a(v_1, v_2) \quad \forall v_1, v_2 \in V_k.$$

Note that A_k is symmetric positive definite with respect to $(\cdot, \cdot)_k$.

The discrete problem (1.7) can be rewritten as

$$(1.10) \quad A_k u_k = f_k,$$

where $f_k \in V_k$ is defined by

$$(1.11) \quad (f_k, v)_k = F(v) \quad \forall v \in V_k.$$

The V -cycle multigrid algorithm (cf. [23, 25, 5, 10]) is an iterative solver for equations of the form (1.10). Given $g \in V_k$ and initial guess $z_0 \in V_k$, it produces $MG_V(k, g, z_0, m_1, m_2)$ as an approximate solution for the equation

$$(1.12) \quad A_k z = g,$$

where m_1 (resp. m_2) is the number of pre-smoothing (resp. post-smoothing) steps. The following are the known results concerning the convergence of the V -cycle multigrid algorithm.

In the case of full elliptic regularity where $\alpha = 1$, Braess and Hackbusch [22, 4, 23] (cf. also [1, 25, 7]) proved that there exists a positive mesh-independent constant C such that

$$(1.13) \quad \begin{aligned} \|z - MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)\|_a & \\ & \leq \left[\frac{C}{C + [\max(m_1, 1) \max(m_2, 1)]^{1/2}} \right] \|z - z_0\|_a \end{aligned}$$

for $m_1 + m_2 \geq 1$, where the energy norm $\|\cdot\|_a$ is defined by

$$(1.14) \quad \|v\|_a = \sqrt{a(v, v)} \quad \forall v \in H_0^1(\Omega).$$

For the case where $\alpha < 1$, Zhang [33] and Bramble and Pasciak [8, 9] (cf. also [31, 32, 20, 11, 28]) showed that there exists a positive constant $\delta < 1$, independent of the meshes and the number of smoothing steps, such that

$$(1.15) \quad \|z - MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)\|_a \leq \delta \|z - z_0\|_a.$$

Note that, in contrast to (1.13), the estimate (1.15) does not indicate that the contraction number of the V -cycle algorithm decreases with the increase of the number of smoothing steps.

In this paper we develop a new additive approach to the convergence of the V -cycle algorithm and obtain the following estimate (cf. Lemma 6.8):

$$(1.16) \quad \|z - MG_{\mathcal{V}}(k, g, z_0, 0, m)\|_a \lesssim m^{-\alpha/2} \|z - z_0\|_a$$

for m sufficiently large. Combining (1.15) and (1.16), we have (cf. Theorem 6.9)

$$(1.17) \quad \begin{aligned} \|z - MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)\|_a & \\ & \leq \left[\frac{C}{C + [\max(m_1, 1) \max(m_2, 1)]^{\alpha/2}} \right] \|z - z_0\|_a \end{aligned}$$

for $m_1 + m_2 \geq 1$, where the positive constant C is mesh-independent. In other words, the estimate (1.13) is generalized to boundary value problems without full elliptic regularity.

Remark 1.1. The new result (1.17) is obtained only for the V -cycle algorithm using the Richardson relaxation scheme as the smoother (cf. Section 2), while the estimates (1.13) and (1.15) are valid for many other smoothers. The generalization of (1.17) to other smoothers will be investigated in future work.

The rest of the paper is organized as follows. The multigrid V -cycle algorithm with the Richardson relaxation scheme as the smoother is described in Section 2. Tools for the analysis of the multigrid V -cycle algorithm are developed in Sections 3–5. The estimates (1.16) and (1.17) are then proved in Section 6.

2. A MULTIGRID V-CYCLE ALGORITHM

In this section we describe the V -cycle multigrid algorithm with Richardson relaxation as the smoother. First we note that the P_1 finite element spaces V_k are nested, i.e.,

$$(2.1) \quad V_1 \subset V_2 \subset \dots \subset H_0^1(\Omega),$$

and the mesh sizes $h_k = \max_{T \in \mathcal{T}_k} \text{diam } T$ are related by

$$(2.2) \quad h_k = 2h_{k+1} \quad \text{for } k = 1, 2, 3, \dots$$

It is also easy to see that the mesh-dependent inner product defined by (1.8) satisfies

$$(2.3) \quad (v, v)_k \approx \|v\|_{L_2(\Omega)}^2 \quad \forall v \in V_k,$$

and then (1.3) and a standard inverse estimate (cf. [16, 14]) imply that the spectral radius $\rho(A_k)$ satisfies

$$(2.4) \quad \rho(A_k) \lesssim h_k^{-2}.$$

The computation of $MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)$ is defined recursively as follows.

Algorithm 2.1. For $k = 1$, we define

$$(2.5) \quad MG_{\mathcal{V}}(1, g, z_0, m_1, m_2) = A_1^{-1}g.$$

For $k \geq 2$, we obtain $MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \dots, m_1$, compute z_j by

$$(2.6) \quad z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

2. (Coarse Grid Correction) Compute z_{m_1+1} by

$$(2.7) \quad z_{m_1+1} = z_{m_1} + MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_{m_1}), 0, m_1, m_2).$$

3. (Post-Smoothing) For $j = m_1 + 2, \dots, m_1 + m_2 + 1$, compute z_j by

$$(2.8) \quad z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)$ to be $z_{m_1+m_2+1}$.

In (2.6) and (2.8), the number Λ_k satisfies

$$(2.9) \quad \rho(A_k) \leq \Lambda_k,$$

and the intergrid transfer operator $I_k^{k-1} : V_k \rightarrow V_{k-1}$ in (2.7) is defined by

$$(2.10) \quad (I_k^{k-1}v, w)_{k-1} = (v, w)_k \quad \forall v \in V_k, w \in V_{k-1}.$$

In view of (2.4), we can always take $\Lambda_k = Ch_k^{-2}$, where the positive constant C is mesh-independent.

Next we recall briefly some well-known formulas that describe the errors of V -cycle algorithms. More details can be found in [26, 23, 5, 10].

Let $E_{k, m_1, m_2} : V_k \rightarrow V_k$ be the operator connecting the initial error and the final error of the multigrid V -cycle algorithm applied to (1.12), i.e.,

$$(2.11) \quad E_{k, m_1, m_2}(z - z_0) = z - MG_{\mathcal{V}}(k, g, z_0, m_1, m_2).$$

The operator E_{k, m_1, m_2} can be described in terms of the Ritz projection operators $P_k : H_0^1(\Omega) \rightarrow V_k$ defined by

$$(2.12) \quad a(P_k \zeta, v) = a(\zeta, v) \quad \forall \zeta \in H_0^1(\Omega), v \in V_k,$$

and the operators $R_k : V_k \rightarrow V_k$ defined by

$$(2.13) \quad R_k = Id_k - \frac{1}{\Lambda_k}A_k,$$

where Id_k is the identity operator on V_k .

Remark 2.2. It follows immediately from (1.14) and (2.12) that

$$(2.14) \quad \|P_k \zeta\|_a \leq \|\zeta\|_a \quad \forall \zeta \in H_0^1(\Omega),$$

$$(2.15) \quad \|\zeta - P_k \zeta\|_a = \inf_{v \in V_k} \|\zeta - v\|_a \quad \forall \zeta \in H_0^1(\Omega).$$

Note that (1.9), (2.10) and (2.12) imply

$$(2.16) \quad I_k^{k-1} A_k v = A_{k-1} P_k v \quad \forall v \in V_k.$$

Using (2.11), (2.13) and (2.16), we can express the effects of (2.6)–(2.8) as

$$(2.17) \quad \begin{aligned} z - z_{m_1} &= R_k^{m_1}(z - z_0), \\ z - z_{m_1+1} &= (Id_k - P_{k-1} + E_{k-1,m_1,m_2} P_{k-1})(z - z_{m_1}), \\ z - z_{m_1+m_2+1} &= R_k^{m_2}(z - z_{m_1+1}). \end{aligned}$$

Comparing (2.11) and (2.17), we obtain the recursive relation

$$(2.18) \quad E_{k,m_1,m_2} v = R_k^{m_2} (Id_k - P_{k-1} + E_{k-1,m_1,m_2} P_{k-1}) R_k^{m_1} v \quad \forall v \in V_k, k \geq 2,$$

and of course we also have (cf. (2.5))

$$(2.19) \quad E_{1,m_1,m_2} v = 0 \quad \forall v \in V_1.$$

In particular the operators $E_{k,m} = E_{k,0,m}$ and $E_{k,m}^* = E_{k,m,0}$ satisfy

$$(2.20) \quad E_{k,m} v = R_k^m (Id_k - P_{k-1} + E_{k-1,m} P_{k-1}) v \quad \forall v \in V_k, k \geq 2,$$

$$(2.21) \quad E_{1,m} v = 0 \quad \forall v \in V_1,$$

$$(2.22) \quad E_{k,m}^* v = (Id_k - P_{k-1} + E_{k-1,m}^* P_{k-1}) R_k^m v \quad \forall v \in V_k, k \geq 2,$$

$$(2.23) \quad E_{1,m}^* v = 0 \quad \forall v \in V_1.$$

The following well-known relations (cf. [26, 23]) can be derived by mathematical induction using (2.18)–(2.23):

$$(2.24) \quad a(E_{k,m} v_1, v_2) = a(v_1, E_{k,m}^* v_2) \quad \forall v_1, v_2 \in V_k,$$

$$(2.25) \quad E_{k,m_1,m_2} = E_{k,m_2} E_{k,m_1}^* \quad \text{for } k = 1, 2, \dots$$

The estimate (1.16) can be rewritten as

$$(2.26) \quad \|E_{k,m} v\|_a \lesssim m^{-(\alpha/2)} \|v\|_a \quad \forall v \in V_k.$$

We will establish (2.26) in Section 6 using the following additive expression for $E_{k,m}$, which follows immediately from (2.20) and (2.21).

$$(2.27) \quad E_{k,m} v = \sum_{j=2}^k R_k^m R_{k-1}^m \cdots R_j^m (P_j - P_{j-1}) v \quad \forall v \in V_k.$$

Remark 2.3. The operators $E_{k,m}$ ($k \geq 2$) can also be written as (cf. [5, 10])

$$(2.28) \quad E_{k,m} v = R_k^m (Id_k - P_{k-1} + R_{k-1}^m P_{k-1}) \cdots (Id_k - P_2 + R_2^m P_2) (Id_k - P_1) v$$

for $v \in V_k$. This multiplicative representation of $E_{k,m}$ plays a key role in the proofs of many of the known V -cycle convergence results. Using (2.27) instead of (2.28) is the point of departure for the new approach to V -cycle convergence developed in this paper.

3. CONSEQUENCES OF THE ELLIPTIC REGULARITY

First we note that (1.5) and (1.6) imply a scale of elliptic regularity estimates.

Following the notation in [30, 21], we define, for $s \geq 0$, the space $\tilde{H}^s(\Omega) = \{v \in H^s(\Omega) : \tilde{v} \in H^s(\mathbb{R}^2)\}$, where \tilde{v} is the trivial extension of v to \mathbb{R}^2 , and $\|v\|_{\tilde{H}^s(\Omega)} = \|\tilde{v}\|_{H^s(\mathbb{R}^2)}$. The spaces $\tilde{H}^s(\Omega)$ for $s \geq 0$ form a scale of interpolation spaces. Moreover, for $s - \frac{1}{2} \notin \mathbb{Z}$, the space $\tilde{H}^s(\Omega)$ coincides with $H_0^s(\Omega)$, and the norms $\|\cdot\|_{\tilde{H}^s(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$ are equivalent on $\tilde{H}^s(\Omega) = H_0^s(\Omega)$. Consequently, for $s \geq 0$ and $s - \frac{1}{2} \notin \mathbb{Z}$, we have $H^{-s}(\Omega) = [H_0^s(\Omega)]' = [\tilde{H}^s(\Omega)]'$ with equivalent norms.

Let $0 \leq s \leq \alpha$ and $u \in H_0^1(\Omega)$ satisfy (1.1) for $F \in [\tilde{H}^{1-s}(\Omega)]'$. The interpolation theory of Sobolev spaces (cf. [30, 3]) and the estimates (1.5) and (1.6) imply that $u \in \tilde{H}^{1+s}(\Omega)$ and

$$(3.1) \quad \|u\|_{\tilde{H}^{1+s}(\Omega)} \lesssim \|F\|_{[\tilde{H}^{1-s}(\Omega)]'}$$

It follows from (3.1) that $T : H^{1+s}(\Omega) \cap H_0^1(\Omega) \rightarrow [\tilde{H}^{1-s}(\Omega)]'$ defined by

$$T\phi = -\nabla \cdot (p(x)\nabla\phi) + r(x)\phi$$

is an isomorphism for $0 \leq s \leq \alpha$; i.e., T is one-to-one, onto and the following estimate holds:

$$(3.2) \quad \|T\phi\|_{[\tilde{H}^{1-s}(\Omega)]'} \approx \|\phi\|_{H^{1+s}(\Omega)} \quad \forall \phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$$

Moreover, by a density argument, the variational form $a(\cdot, \cdot)$ has a unique extension from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $[H^{1+s}(\Omega) \cap H_0^1(\Omega)] \times \tilde{H}^{1-s}(\Omega)$ such that

$$(3.3) \quad a(\phi, \psi) = (T\phi)(\psi) \quad \forall \phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega), \psi \in \tilde{H}^{1-s}(\Omega)$$

The Sobolev spaces $H^{1+s}(\Omega) \cap H_0^1(\Omega)$ and $\tilde{H}^{1-s}(\Omega)$ satisfy a duality relation with respect to the extended variational form $a(\cdot, \cdot)$, as stated in the next lemma.

Lemma 3.1. *The following estimates hold for $0 \leq s \leq \alpha$:*

$$(3.4) \quad \|\phi\|_{H^{1+s}(\Omega)} \approx \sup_{\substack{\psi \in \tilde{H}^{1-s}(\Omega) \\ \psi \neq 0}} \frac{a(\phi, \psi)}{\|\psi\|_{\tilde{H}^{1-s}(\Omega)}} \quad \forall \phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$$

$$(3.5) \quad \|\psi\|_{\tilde{H}^{1-s}(\Omega)} \approx \sup_{\substack{\phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega) \\ \phi \neq 0}} \frac{a(\phi, \psi)}{\|\phi\|_{H^{1+s}(\Omega)}} \quad \forall \psi \in \tilde{H}^{1-s}(\Omega)$$

Proof. The estimates (3.4) and (3.5) follow immediately from (3.2), (3.3) and the standard duality formulas

$$\begin{aligned} \|F\|_{[\tilde{H}^{1-s}(\Omega)]'} &= \sup_{\substack{\psi \in \tilde{H}^{1-s}(\Omega) \\ \psi \neq 0}} \frac{F(\psi)}{\|\psi\|_{\tilde{H}^{1-s}(\Omega)}} & \quad \forall F \in [\tilde{H}^{1-s}(\Omega)]', \\ \|\psi\|_{\tilde{H}^{1-s}(\Omega)} &= \sup_{\substack{F \in [\tilde{H}^{1-s}(\Omega)]' \\ F \neq 0}} \frac{F(\psi)}{\|F\|_{[\tilde{H}^{1-s}(\Omega)]'}} & \quad \forall \psi \in \tilde{H}^{1-s}(\Omega). \end{aligned}$$

□

Lemma 3.1 can be used to measure the approximation properties of the finite element spaces V_k in lower order Sobolev norms.

Lemma 3.2. *The following estimate holds:*

$$(3.6) \quad \|\zeta - P_k \zeta\|_{\tilde{H}^{1-s}(\Omega)} \lesssim h_k^s \|\zeta\|_a \quad \forall 0 \leq s \leq \alpha, \zeta \in H_0^1(\Omega).$$

Proof. There exists an operator (cf. [17, 29]) $\mathcal{I}_k : H_0^1(\Omega) \rightarrow V_k$ such that

$$(3.7) \quad \|\phi - \mathcal{I}_k \phi\|_{H^1(\Omega)} \lesssim h_k^s \|\phi\|_{H^{1+s}(\Omega)} \quad \forall \phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega).$$

By (1.3), (2.15) and (3.7) we have

$$(3.8) \quad \|\phi - P_k \phi\|_a \lesssim h_k^s \|\phi\|_{H^{1+s}(\Omega)} \quad \forall \phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega).$$

Using (1.14), (2.12), (3.5), and (3.8), we can establish (3.6) as follows:

$$\begin{aligned} \|\zeta - P_k \zeta\|_{\tilde{H}^{1-s}(\Omega)} &\approx \sup_{\substack{\phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega) \\ \phi \neq 0}} \frac{a(\phi, \zeta - P_k \zeta)}{\|\phi\|_{H^{1+s}(\Omega)}} \\ &= \sup_{\substack{\phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega) \\ \phi \neq 0}} \frac{a(\phi - P_k \phi, \zeta)}{\|\phi\|_{H^{1+s}(\Omega)}} \\ &\lesssim \sup_{\substack{\phi \in H^{1+s}(\Omega) \cap H_0^1(\Omega) \\ \phi \neq 0}} \frac{\|\phi - P_k \phi\|_a \|\zeta\|_a}{\|\phi\|_{H^{1+s}(\Omega)}} \lesssim h_k^s \|\zeta\|_a. \end{aligned}$$

□

4. MESH DEPENDENT NORMS

Following [2] we define the mesh dependent norms $\|\cdot\|_{s,k}$ ($s \in \mathbb{R}$) by

$$(4.1) \quad \|v\|_{s,k} = \sqrt{(A_k^s v, v)_k} \quad \forall v \in V_k.$$

It is clear from (1.3), (1.4), (1.9), (1.14), (2.3), (2.4), and (4.1) that

$$(4.2) \quad \|v\|_{1,k} = \|v\|_a \approx \|v\|_{H^1(\Omega)} \quad \forall v \in V_k,$$

$$(4.3) \quad \|v\|_{0,k} \approx \|v\|_{L_2(\Omega)} \quad \forall v \in V_k,$$

$$(4.4) \quad \|v\|_{s,k} \lesssim h_k^{t-s} \|v\|_{t,k} \quad \forall v \in V_k, 0 \leq t \leq s \leq 2.$$

It also follows immediately from (4.1) that

$$(4.5) \quad \|v\|_{1+t,k} = \sup_{\substack{w \in V_k \\ w \neq 0}} \frac{a(v, w)}{\|w\|_{1-t,k}} \quad \forall v \in V_k, t \in \mathbb{R}.$$

In particular, we have the following generalized Cauchy-Schwarz inequality:

$$(4.6) \quad |a(v_1, v_2)| \leq \|v_1\|_{1+t,k} \|v_2\|_{1-t,k} \quad \forall v_1, v_2 \in V_k, t \in \mathbb{R}.$$

These mesh dependent norms are related to the Sobolev norms, in fact (cf. [2, 12]),

$$(4.7) \quad \|v\|_{s,k} \approx \|v\|_{\tilde{H}^s(\Omega)} \quad \forall v \in V_k, 0 \leq s \leq 1.$$

The following smoothing properties of R_k are well known (cf. [2, 23]):

$$(4.8) \quad \|R_k^m v\|_{s,k} \lesssim (h_k \sqrt{m})^{t-s} \|v\|_{t,k} \quad \forall v \in V_k, 0 \leq t \leq s \leq 2,$$

$$(4.9) \quad \|R_k v\|_{s,k} \leq \|v\|_{s,k} \quad \forall v \in V_k, s \in \mathbb{R}.$$

The combined effect of the smoothing and approximation properties is given in the next lemma.

Lemma 4.1. *The following estimates hold for $0 \leq s \leq \alpha$:*

$$(4.10) \quad \|R_k^m (Id_k - P_{k-1})v\|_{1-s,k} \lesssim m^{(s-\alpha)/2} h_k^s \|v\|_a \quad \forall v \in V_k .$$

Proof. Combining (2.2), (3.6), (4.2), (4.7) and (4.8), we have

$$\begin{aligned} \|R_k^m (Id_k - P_{k-1})v\|_{1-s,k} &\lesssim (h_k \sqrt{m})^{s-\alpha} \|v - P_{k-1}v\|_{1-\alpha,k} \\ &\approx (h_k \sqrt{m})^{s-\alpha} \|v - P_{k-1}v\|_{\tilde{H}^{1-\alpha}(\Omega)} \\ &\lesssim (h_k \sqrt{m})^{s-\alpha} h_{k-1}^\alpha \|v\|_a \lesssim m^{(s-\alpha)/2} h_k^s \|v\|_a . \end{aligned}$$

□

Let β be a fixed number satisfying $0 < \beta < \frac{1}{2}$. Note that (cf. [13, 30]) $V_k \subset H_0^{1+\beta}(\Omega) = H^{1+\beta}(\Omega) \cap H_0^1(\Omega)$. Later on we will need the relation between $\|\cdot\|_{1+\beta,k}$ and $\|\cdot\|_{H^{1+\beta}(\Omega)}$ stated in the next lemma, whose proof uses the error estimate (cf. [16, 19, 14])

$$(4.11) \quad \|\zeta - \Pi_k \zeta\|_a \lesssim h_k^\beta \|\zeta\|_{H^{1+\beta}(\Omega)} \quad \forall \zeta \in H_0^{1+\beta}(\Omega)$$

for the nodal interpolation operator Π_k , and also (cf. [13]) the estimate

$$(4.12) \quad \|\Pi_k \zeta\|_{H^{1+\beta}(\Omega)} \lesssim \|\zeta\|_{H^{1+\beta}(\Omega)} \quad \forall \zeta \in H_0^{1+\beta}(\Omega) .$$

Lemma 4.2. *The following estimate holds:*

$$(4.13) \quad \|v\|_{1+\beta,k} \approx \|v\|_{H^{1+\beta}(\Omega)} \quad \forall v \in V_k .$$

Proof. It follows from (3.4), (4.5) and (4.7) that

$$\begin{aligned} \|v\|_{1+\beta,k} &= \sup_{\substack{w \in V_k \\ w \neq 0}} \frac{a(v, w)}{\|w\|_{1-\beta,k}} \approx \sup_{\substack{w \in V_k \\ w \neq 0}} \frac{a(v, w)}{\|w\|_{\tilde{H}^{1-\beta}(\Omega)}} \\ &\lesssim \sup_{\substack{\psi \in \tilde{H}^{1-\beta}(\Omega) \\ \psi \neq 0}} \frac{a(v, \psi)}{\|\psi\|_{\tilde{H}^{1-\beta}(\Omega)}} \lesssim \|v\|_{H^{1+\beta}(\Omega)} \quad \forall v \in V_k . \end{aligned}$$

To prove the converse, we first observe that, since $V_k \subset H_0^{1+\beta}(\Omega)$, the Ritz projection operator P_k can be extended to $\tilde{H}^{1-\beta}(\Omega)$, by (3.3), so that

$$(4.14) \quad a(v, P_k \psi) = a(v, \psi) \quad \forall v \in V_k, \psi \in \tilde{H}^{1-\beta}(\Omega) .$$

Let $\phi \in H_0^{1+s}(\Omega)$ be arbitrary. We find, by (3.4), (3.8), (4.2), (4.11), (4.12), (4.14) and standard inverse estimates,

$$\begin{aligned} a(\phi, P_k \psi) &= a(P_k \phi - \Pi_k \phi, \psi) + a(\Pi_k \phi, \psi) \\ &\lesssim \|\psi\|_{\tilde{H}^{1-\beta}(\Omega)} (\|P_k \phi - \Pi_k \phi\|_{H^{1+\beta}(\Omega)} + \|\Pi_k \phi\|_{H^{1+\beta}(\Omega)}) \\ &\lesssim \|\psi\|_{\tilde{H}^{1-\beta}(\Omega)} (h^{-\beta} \|P_k \phi - \Pi_k \phi\|_{H^1(\Omega)} + \|\phi\|_{H^{1+\beta}(\Omega)}) \\ &\lesssim \|\psi\|_{\tilde{H}^{1-\beta}(\Omega)} \|\phi\|_{H^{1+\beta}(\Omega)} , \end{aligned}$$

which, in view of (3.5), implies

$$(4.15) \quad \|P_k \psi\|_{\tilde{H}^{1-\beta}(\Omega)} \lesssim \|\psi\|_{\tilde{H}^{1-\beta}(\Omega)} .$$

Let $v \in V_k$ be arbitrary. We have, by (4.6), (4.7), (4.14) and (4.15),

$$\begin{aligned} a(v, \psi) = a(v, P_k \psi) &\leq \|v\|_{1+\beta,k} \|P_k \psi\|_{1-\beta,k} \\ &\lesssim \|v\|_{1+\beta,k} \|P_k \psi\|_{\tilde{H}^{1-\beta}(\Omega)} \\ &\lesssim \|v\|_{1+\beta,k} \|\psi\|_{\tilde{H}^{1-\beta}(\Omega)} \quad \forall \psi \in \tilde{H}^{1-\beta}(\Omega) , \end{aligned}$$

and therefore, in view of (3.4), $\|v\|_{H^{1+\beta}(\Omega)} \lesssim \|v\|_{1+\beta,k} \quad \forall v \in V_k$. □

5. RELATIONS BETWEEN MESH DEPENDENT NORMS ON CONSECUTIVE LEVELS

Our goal in this section is to show that, given any $\theta \in (0, 1]$, we have

$$(5.1) \quad \|v\|_{1-\beta, k+1}^2 \leq (1 + \theta^2) \|v\|_{1-\beta, k}^2 + C\theta^{-4} h_k^{2\beta} \|v\|_a^2 \quad \forall v \in V_k,$$

$$(5.2) \quad \|\Pi_{k-1} v\|_{1-\beta, k-1}^2 \leq (1 + \theta^2) \|v\|_{1-\beta, k}^2 + C\theta^{-4} h_k^{2\beta} \|v\|_a^2 \quad \forall v \in V_k,$$

where β is the number chosen in Section 4, and C (and any other constant in this section and the next) is a generic positive constant independent of the meshes and θ , which can take different values at different places. The estimates (5.1) and (5.2) play a key role in the convergence analysis in Section 6. Their proofs involve several lemmas.

Let us first introduce another mesh dependent inner product $((\cdot, \cdot))_k$ on V_k :

$$(5.3) \quad ((v_1, v_2))_k = h_k^2 \sum_{p \in \mathcal{V}_k} n(p) v_1(p) v_2(p) \quad \forall v_1, v_2 \in V_k,$$

where the function $n(\cdot)$ is defined by

$$(5.4) \quad n(p) = \frac{1}{6} \times (\text{the number of triangles in } \mathcal{T}_k \text{ that have } p \text{ as a vertex}).$$

It is easy to see from (5.4) and the construction of \mathcal{T}_k that $n(\cdot)$ is independent of mesh levels, and

$$(5.5) \quad n(p) = 1 \quad \text{for } p \in \mathcal{V}_k \setminus \mathcal{V}_1.$$

The operator $\mathcal{A}_k : V_k \rightarrow V_k$ is defined by

$$(5.6) \quad ((\mathcal{A}_k v_1, v_2))_k = a(v_1, v_2) \quad \forall v_1, v_2 \in V_k.$$

We can then define the corresponding mesh dependent norms by

$$(5.7) \quad \|v\|_{s, k} = \sqrt{((\mathcal{A}_k^s v, v))_k}.$$

Remark 5.1. It is clear that all the properties of $\|\cdot\|$ stated in Section 4 also hold for $\| \cdot \|$. A V -cycle algorithm can also be defined by replacing the inner product $(\cdot, \cdot)_k$ with $((\cdot, \cdot))_k$ throughout Section 2. The two algorithms are different only by a diagonal preconditioner in the smoothing steps. For the algorithm based on $((\cdot, \cdot))_k$, the analysis in Section 6 can be carried out using the simpler estimates (5.8) and (5.11) below.

Our plan is to first prove the (simpler) analogs of (5.1) and (5.2) for the new mesh dependent norms, and then obtain (5.1) and (5.2) through the relation between the norms $\|\cdot\|_{1-\beta, k}$ and $\| \cdot \|_{1-\beta, k}$.

Lemma 5.2. *The following estimate holds:*

$$(5.8) \quad \|v\|_{1-\beta, k+1} \leq \|v\|_{1-\beta, k} \quad \forall v \in V_k.$$

Proof. It is clear from (5.6) and (5.7) that

$$(5.9) \quad \|v\|_{1, k+1} = \|v\|_a = \|v\|_{1, k} \quad \forall v \in V_k.$$

Let the parents of $q \in \mathcal{V}_{k+1} \setminus \mathcal{V}_k$ be denoted by q' and q'' , i.e., $q', q'' \in \mathcal{V}_k$ and q is the midpoint between them. Note that each $p \in \mathcal{V}_k$ is the parent of exactly

$6n(p)$ many vertices in $\mathcal{V}_{k+1} \setminus \mathcal{V}_k$. Let $v \in V_k$ be arbitrary. Using (2.2), (5.3) and (5.5) we can estimate $\|v\|_{0,k+1}^2$ by

$$\begin{aligned} \|v\|_{0,k+1}^2 &= h_{k+1}^2 \left(\sum_{p \in \mathcal{V}_k} n(p)[v(p)]^2 + \sum_{q \in \mathcal{V}_{k+1} \setminus \mathcal{V}_k} [v(q)]^2 \right) \\ &= h_{k+1}^2 \left(\sum_{p \in \mathcal{V}_k} n(p)[v(p)]^2 + \sum_{q \in \mathcal{V}_{k+1} \setminus \mathcal{V}_k} \left[\frac{v(q') + v(q'')}{2} \right]^2 \right) \\ &\leq h_{k+1}^2 \left(\sum_{p \in \mathcal{V}_k} n(p)[v(p)]^2 + \sum_{q \in \mathcal{V}_{k+1} \setminus \mathcal{V}_k} \frac{[v(q')]^2 + [v(q'')]^2}{2} \right) \\ &= h_{k+1}^2 \left(\sum_{p \in \mathcal{V}_k} n(p)[v(p)]^2 + \sum_{p \in \mathcal{V}_k} 6n(p) \frac{[v(p)]^2}{2} \right) \\ &= h_k^2 \sum_{p \in \mathcal{V}_k} n(p)[v(p)]^2 = \|v\|_{0,k}^2. \end{aligned}$$

Hence we have

$$(5.10) \quad \|v\|_{0,k+1} \leq \|v\|_{0,k} \quad \forall v \in V_k.$$

The estimate (5.8) follows from (5.9), (5.10) and interpolation between the (real) Hilbert scales $(V_k, \|\cdot\|_{s,k})$ and $(V_{k+1}, \|\cdot\|_{s,k+1})$ (cf. Theorem B.4 in [5]). \square

Lemma 5.3. *There exists a positive constant C such that*

$$(5.11) \quad \|\Pi_{k-1}v\|_{1-\beta,k-1}^2 \leq (1 + \theta^2)\|v\|_{1-\beta,k}^2 + C\theta^{-2}h_k^{2\beta}\|v\|_a^2$$

$\forall \theta \in (0, 1], v \in V_k$.

Proof. We have, by (1.14), (2.2), (4.11), (5.9) and the analog of (4.13) for $\|\cdot\|_{1+\beta,k}$,

$$\begin{aligned} \|\Pi_{k-1}v\|_{1,k-1}^2 &\leq (\|v\|_a + \|v - \Pi_{k-1}v\|_a)^2 \\ (5.12) \quad &\leq (1 + \theta^2)\|v\|_a^2 + (1 + \theta^{-2})\|v - \Pi_{k-1}v\|_a^2 \\ &\leq (1 + \theta^2)\|v\|_{1,k}^2 + c_1\theta^{-2}h_k^{2\beta}\|v\|_{1+\beta,k}^2 \quad \forall v \in V_k. \end{aligned}$$

Let the parents of $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$ be denoted by p' and p'' . Given any $v \in V_k$, we have the following elementary estimate:

$$\begin{aligned} &\frac{[v(p')]^2}{2} + \frac{[v(p'')]^2}{2} \\ (5.13) \quad &= \frac{(v(p) + [v(p') - v(p)])^2}{2} + \frac{(v(p) + [v(p'') - v(p)])^2}{2} \\ &\leq (1 + \theta^2)[v(p)]^2 + (1 + \theta^{-2}) \left(\frac{[v(p') - v(p)]^2 + [v(p'') - v(p)]^2}{2} \right). \end{aligned}$$

Summing up (5.13) over all $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$, we find

$$(5.14) \quad 3 \sum_{q \in \mathcal{V}_{k-1}} n(q)[v(q)]^2 \leq (1 + \theta^2) \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} [v(p)]^2 + c_2\theta^{-2}|v|_{H^1(\Omega)}^2.$$

It follows from (2.2), (5.3), (5.5), (5.14) and the analogs of (4.2) and (4.4) for $\|\cdot\|_{s,k}$ that

$$\begin{aligned}
 \|\Pi_{k-1} v\|_{0,k-1}^2 &= h_{k-1}^2 \sum_{q \in \mathcal{V}_{k-1}} n(q) [v(q)]^2 \\
 (5.15) \quad &\leq h_k^2 \left[\sum_{q \in \mathcal{V}_{k-1}} n(q) [v(q)]^2 + (1 + \theta^2) \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} [v(p)]^2 + c_2 \theta^{-2} |v|_{H^1(\Omega)}^2 \right] \\
 &\leq h_k^2 \left[(1 + \theta^2) \sum_{p \in \mathcal{V}_k} n(p) [v(p)]^2 + c_3 \theta^{-2} \|v\|_{1,k}^2 \right] \\
 &\leq (1 + \theta^2) \|v\|_{0,k}^2 + c_4 \theta^{-2} h_k^{2\beta} \|v\|_{\beta,k}^2 \quad \forall v \in V_k.
 \end{aligned}$$

Let $C = \max(c_1, c_4)$. Observe that

$$(1 + \theta^2) \|v\|_{s,k}^2 + C \theta^{-2} h_k^{2\beta} \|v\|_{s+\beta,k}^2 = \langle \mathcal{A}_k^s v, v \rangle_{\theta,k} \quad \forall v \in V_k,$$

where the inner product $\langle \cdot, \cdot \rangle_{\theta,k}$ is defined by

$$\langle v_1, v_2 \rangle_{\theta,k} = (1 + \theta^2) (v_1, v_2)_k + C \theta^{-2} h_k^{2\beta} (\mathcal{A}_k^\beta v_1, v_2)_k \quad \forall v_1, v_2 \in V_k.$$

Therefore, for each k , the spaces $(V_k, \sqrt{(1 + \theta^2) \|\cdot\|_{s,k}^2 + C \theta^{-2} h_k^{2\beta} \|\cdot\|_{s+\beta,k}^2})$ form a Hilbert scale.

By interpolating (5.12) and (5.15) between the Hilbert scales $(V_{k-1}, \|\cdot\|_{s,k-1})$ and $(V_k, \sqrt{(1 + \theta^2) \|\cdot\|_{s,k}^2 + C \theta^{-2} h_k^{2\beta} \|\cdot\|_{s+\beta,k}^2})$, we have

$$(5.16) \quad \|\Pi_{k-1} v\|_{1-\beta,k-1}^2 \leq (1 + \theta^2) \|v\|_{1-\beta,k}^2 + C \theta^{-2} h_k^{2\beta} \|v\|_{1,k}^2 \quad \forall v \in V_k.$$

The estimate (5.11) follows from (5.9) and (5.16). □

Remark 5.4. It is important that the interpolation between (real) Hilbert scales is exact (cf. Theorem B.4 in [5]) so that no additional constant appears in front of the term $(1 + \theta^2) \|v\|_{1-\beta,k}^2$ on the right-hand side of (5.11).

The following lemmas relate the two mesh dependent norms.

Lemma 5.5. *There exists a positive constant C such that*

$$(5.17) \quad \|v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k}^2 + C \theta^{-4} h_k^2 \|v\|_a^2 \quad \forall \theta \in (0, 1], v \in V_k,$$

$$(5.18) \quad \|v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k}^2 + C \theta^{-4} h_k^2 \|v\|_a^2 \quad \forall \theta \in (0, 1], v \in V_k.$$

Proof. We will only establish (5.18) since the proof of (5.17) is completely analogous. Also it suffices to show that

$$(5.19) \quad \|v\|_{0,k}^2 \leq (1 + C' \ell^{-2}) \|v\|_{0,k}^2 + C'' \ell^4 h_k^2 \|v\|_{H^1(\Omega)}^2 \quad \forall v \in V_k, \ell = 1, 2, \dots$$

Let $D = \text{diam } \Omega$ and

$$d = \min_{\substack{p, q \in \mathcal{V}_1 \\ p \neq q}} |p - q|$$

be the minimum of the distances among the distinct vertices of \mathcal{T}_1 .

It is clear from the analog of (2.3) for $\|\cdot\|_{0,k}$ that, for $\ell h_k \geq \frac{d}{2}$,

$$(5.20) \quad \|v\|_{0,k}^2 \leq c_1 \ell^4 h_k^2 \|v\|_{H^1(\Omega)}^2 \quad \forall v \in V_k,$$

and (5.19) follows.

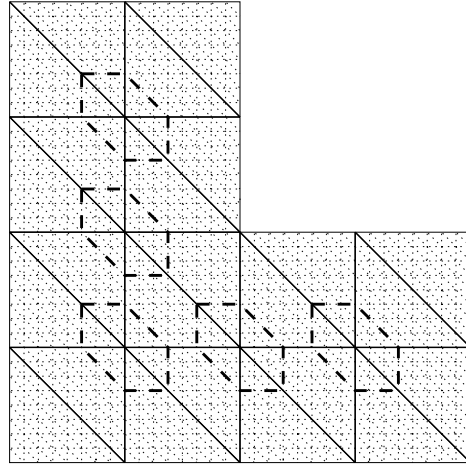


FIGURE 1.

In the case where $\ell h_k < \frac{d}{2}$, we define for each $q \in \mathcal{V}_1$ the (open) polygon Ω_q whose vertices are those points in \mathcal{V}_k that belong to the edges in \mathcal{T}_1 and which are ℓ steps away from q . The Ω_q 's for an L-shaped domain are depicted in Figure 1, where $k = 4$ and $\ell = 3$.

It is easy to see that the Ω_q 's are pairwise disjoint and

$$(5.21) \quad \text{diam } \Omega \approx \ell h_k .$$

Let $v \in V_k$ and $\mathcal{K} = \bigcup_{q \in \mathcal{V}_1} \bar{\Omega}_q$. By (5.5) we have

$$(5.22) \quad h_k^2 \sum_{p \in \mathcal{V}_k \setminus \mathcal{K}} n(p)[v(p)]^2 = h_k^2 \sum_{p \in \mathcal{V}_k \setminus \mathcal{K}} [v(p)]^2 .$$

Let $q \in \mathcal{V}_1$. We can estimate the contribution of the vertices of \mathcal{T}_k in $\bar{\Omega}_q$ to the left-hand side of (5.19) by

$$(5.23) \quad \begin{aligned} & h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p)[v(p)]^2 \\ &= h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p)(v_q + [v(p) - v_q])^2 \\ &\leq (1 + \ell^{-2})h_k^2 \left(\sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p) \right) v_q^2 + (1 + \ell^2)h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p)[v(p) - v_q]^2 , \end{aligned}$$

where the number v_q is defined by

$$(5.24) \quad v_q = \frac{1}{|\Omega_q|} \int_{\Omega_q} v \, dx .$$

A direct calculation using (5.21), (5.24) and the Bramble-Hilbert lemma (cf. [6, 19]) shows that

$$(5.25) \quad h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p)[v(p) - v_q]^2 \leq c_2 \|v - v_q\|_{L_2(\Omega_q)}^2 \leq c_3 \ell^2 h_k^2 |v|_{H^1(\Omega_q)}^2 .$$

Using (5.5) we have

$$(5.26) \quad \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p) = (n(q) - 1) + \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} 1 \leq (1 + c_4 \ell^{-2}) \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} 1.$$

We find from (5.23), (5.25) and (5.26) that

$$(5.27) \quad h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p) [v(p)]^2 \leq (1 + c_5 \ell^{-2}) h_k^2 \left(\sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} v_q^2 \right) + c_6 \ell^4 h_k^2 |v|_{H^1(\Omega_q)}^2.$$

On the other hand we also have, by (5.25),

$$(5.28) \quad \begin{aligned} h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} v_q^2 &\leq (1 + \ell^{-2}) h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} [v(p)]^2 + (1 + \ell^2) h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} [v_q - v(p)]^2 \\ &\leq (1 + \ell^{-2}) h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} [v(p)]^2 + c_7 \ell^4 h_k^2 |v|_{H^1(\Omega_q)}^2. \end{aligned}$$

Combining (5.27) and (5.28) we obtain

$$(5.29) \quad h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} n(p) [v(p)]^2 \leq (1 + c_8 \ell^{-2}) h_k^2 \sum_{p \in \mathcal{V}_k \cap \bar{\Omega}_q} [v(p)]^2 + c_9 \ell^4 h_k^2 |v|_{H^1(\Omega_q)}^2.$$

The estimate (5.19) is established by summing up (5.22) and (5.29) (over all $q \in \mathcal{V}_1$). \square

Lemma 5.6. *There exists a positive constant C such that*

$$(5.30) \quad \|v\|_{1-\beta,k}^2 \leq (1 + \theta^2) \|v\|_{1-\beta,k}^2 + C \theta^{-4} h_k^{2\beta} \|v\|_a^2 \quad \forall \theta \in (0, 1], v \in V_k,$$

$$(5.31) \quad \|v\|_{1-\beta,k}^2 \leq (1 + \theta^2) \|v\|_{1-\beta,k}^2 + C \theta^{-4} h_k^{2\beta} \|v\|_a^2 \quad \forall \theta \in (0, 1], v \in V_k.$$

Proof. The estimates (4.2), (4.4) and (5.18) imply that there exists a positive constant C such that

$$(5.32) \quad \|v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k}^2 + C \theta^{-4} h_k^{2\beta} \|v\|_{\beta,k}^2 \quad \forall v \in V_k.$$

On the other hand it follows from (4.2) and (5.9) that

$$(5.33) \quad \|v\|_{1,k}^2 = \|v\|_{1,k}^2 \leq (1 + \theta^2) \|v\|_{1,k}^2 + C \theta^{-4} h_k^{2\beta} \|v\|_{1+\beta,k}^2 \quad \forall v \in V_k.$$

The estimate (5.31) follows from interpolating (5.32) and (5.33) between the Hilbert scales $(V_k, \|\cdot\|_{s,k})$ and $(V_k, \sqrt{(1 + \theta^2) \|\cdot\|_{s,k}^2 + C \theta^{-4} h_k^{2\beta} \|\cdot\|_{s+\beta,k}^2})$.

The proof of (5.30) is similar. \square

The estimates (5.1) and (5.2) follow from Lemma 5.2, Lemma 5.3 and Lemma 5.6.

6. CONVERGENCE ANALYSIS OF THE V-CYCLE ALGORITHM

Let us first introduce some operators that will simplify many expressions in the convergence analysis. For $2 \leq j < k$, the operator $T_{k,j,m} : V_j \rightarrow V_k$ is defined by

$$(6.1) \quad T_{k,j,m} = R_k^m R_{k-1}^m \cdots R_{j+1}^m,$$

and the operator $T_{j,k,m} : V_k \rightarrow V_j$ defined by

$$(6.2) \quad T_{j,k,m} = P_j R_{j+1}^m \cdots P_{k-1} R_k^m$$

is the transpose of $T_{k,j,m}$ with respect to the variational form $a(\cdot, \cdot)$. The operator $T_{k,k,m}$ is defined to be the identity operator Id_k on V_k .

Remark 6.1. It is more precise to write

$$T_{k,j,m} = R_k^m I_{k-1}^k R_{k-1}^m I_{k-2}^{k-1} \cdots R_{j+1}^m I_j^{j+1},$$

where $I_\ell^{\ell+1} : V_\ell \rightarrow V_{\ell+1}$ is the natural injection. Such natural injections have been suppressed in Section 2 and (6.1).

Let $K \geq 2$ be an arbitrary but fixed integer and

$$(6.3) \quad v_k = (P_k - P_{k-1})v \quad \forall v \in V_K, 2 \leq k < K.$$

Note that the v_k 's are pairwise orthogonal with respect to $a(\cdot, \cdot)$, and

$$(6.4) \quad \sum_{k=2}^K \|v_k\|_a^2 = \|v - P_1 v\|_a^2 \leq \|v\|_a^2.$$

Moreover, we have

$$(6.5) \quad v_k = (Id_k - P_{k-1})v_k.$$

We can write, by (2.27) and (6.1)–(6.3),

$$(6.6) \quad \begin{aligned} & a(E_K v, E_K v) \\ &= \sum_{j,k=2}^K a(R_K^m \cdots R_{j+1}^m R_j^m (P_j - P_{j-1})v, R_K^m \cdots R_{k+1}^m R_k^m (P_k - P_{k-1})v) \\ &= \sum_{k=2}^K a(R_k^m v_k, T_{k,K,m} T_{K,k,m} R_k^m v_k) \\ & \quad + 2 \sum_{2 \leq j < k \leq K} a(R_j^m v_j, T_{j,k,m} T_{k,K,m} T_{K,k,m} R_k^m v_k). \end{aligned}$$

The properties of the operators $T_{j,k,m}$ are therefore crucial for the convergence analysis. From (2.14) and (4.9) we have, for $j, k \geq 2$, the trivial estimate

$$(6.7) \quad \|T_{j,k,m} v\|_{1,j} \leq \|v\|_{1,k} \quad \forall v \in V_k.$$

It follows from (4.2), (4.10), (6.5) and (6.7) that

$$(6.8) \quad a(R_k^m v_k, T_{k,K,m} T_{K,k,m} R_k^m v_k) \leq \|R_k^m v_k\|_{1,k}^2 \lesssim m^{-\alpha} \|v_k\|_a^2.$$

In order to estimate the remaining terms on the right-hand side of (6.6) we need some less trivial properties of the operators $T_{j,k,m}$. We begin the study of these properties with a lemma on the Ritz projection operators.

Lemma 6.2. *There exists a positive constant C such that*

$$(6.9) \quad \|P_{k-1} v\|_{1-\beta, k-1}^2 \leq (1 + \theta^2) \|v\|_{1-\beta, k}^2 + C \theta^{-4} h_k^{2\beta} \|v\|_a^2$$

$\forall \theta \in (0, 1], v \in V_k$.

Proof. It follows from (2.14) and (5.2) that

$$(6.10) \quad \begin{aligned} \|P_{k-1} v\|_{1-\beta, k-1}^2 &\leq (1 + \theta^2) \|P_{k-1} v\|_{1-\beta, k}^2 + c_1 \theta^{-4} h_k^{2\beta} \|P_{k-1} v\|_a^2 \\ &\leq (1 + c_2 \theta^2) \|v\|_{1-\beta, k}^2 \\ & \quad + \left[c_3 \theta^{-2} \|P_{k-1} v - v\|_{1-\beta, k}^2 + c_1 \theta^{-4} h_k^{2\beta} \|v\|_a^2 \right]. \end{aligned}$$

The estimate (6.9) follows from (2.2), (3.6), (4.7) and (6.10). \square

The properties of the operators $T_{j,k,m}$ are given in the next three lemmas.

Lemma 6.3. *The following estimate holds:*

$$(6.11) \quad \|T_{k,K,m}v\|_{1-\beta,k} \lesssim \|v\|_{1-\beta,K} + h_k^\beta \|v\|_a \quad \forall v \in V_K, 2 \leq k \leq K.$$

Proof. The case where $k = K$ is trivial. For $2 \leq k < K$, it follows from (2.2), (4.9), (6.2), (6.7) and (6.9) that, for any $v \in V_K$,

$$(6.12) \quad \begin{aligned} & \|T_{k,K,m}v\|_{1-\beta,k}^2 \\ &= \|P_k R_{k+1}^m T_{k+1,K,m}v\|_{1-\beta,k}^2 \\ &\leq (1 + \theta_k^2) \|R_{k+1}^m T_{k+1,K,m}v\|_{1-\beta,k+1}^2 + c_* \theta_k^{-4} h_k^{2\beta} \|R_{k+1}^m T_{k+1,K,m}v\|_a^2 \\ &\leq (1 + \theta_k^2) \|T_{k+1,K,m}v\|_{1-\beta,k+1}^2 + c_* \theta_k^{-4} h_k^{2\beta} \|v\|_a^2, \end{aligned}$$

where $\theta_k \in (0, 1]$ is arbitrary, and c_* is independent of the meshes and θ_k .

Iterating (6.12) we find

$$(6.13) \quad \begin{aligned} \|T_{k,K,m}v\|_{1-\beta,k}^2 &\leq \left[\prod_{\ell=k}^{K-1} (1 + \theta_\ell^2) \right] \|v\|_{1-\beta,K}^2 \\ &\quad + c_* \left[\sum_{\ell=k}^{K-1} h_\ell^{2\beta} \theta_\ell^{-4} \prod_{r=k}^{\ell-1} (1 + \theta_r^2) \right] \|v\|_a^2, \end{aligned}$$

where the θ_ℓ 's ($k \leq \ell \leq K - 1$) are arbitrary numbers in $(0, 1]$.

By taking $\theta_\ell = 3^{-\beta(\ell-k)/4}$, we deduce from (2.2) and (6.13) that

$$(6.14) \quad \|T_{k,K,m}v\|_{1-\beta,k}^2 \leq \omega \|v\|_{1-\beta,K}^2 + c_* \omega \eta h_k^{2\beta} \|v\|_a^2 \quad \forall v \in V_K,$$

where

$$(6.15) \quad \omega = \prod_{n=0}^{\infty} \left(1 + 3^{-(\beta/2)n} \right) \quad \text{and} \quad \eta = \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^{\beta n}.$$

The estimate (6.11) follows from (6.14) and (6.15). □

Lemma 6.4. *The following estimate holds:*

$$(6.16) \quad \|T_{K,k,m}v\|_{1-\beta,K} \lesssim \|v\|_{1-\beta,k} \quad \forall v \in V_k, 2 \leq k \leq K.$$

Proof. The case where $k = K$ is trivial. For $2 \leq k < K$, using (4.9), (5.1), (6.1) and (6.7) we obtain

$$(6.17) \quad \begin{aligned} \|T_{K,k,m}v\|_{1-\beta,K}^2 &= \|R_K^m T_{K-1,k,m}v\|_{1-\beta,K}^2 \\ &\leq \|T_{K-1,k,m}v\|_{1-\beta,K}^2 \\ &\leq (1 + \theta_{K-1}^2) \|T_{K-1,k,m}v\|_{1-\beta,K-1}^2 + c_* \theta_{K-1}^{-4} h_{K-1}^{2\beta} \|v\|_a^2 \end{aligned}$$

$\forall v \in V_k$, where $\theta_{K-1} \in (0, 1]$ is arbitrary and the positive constant c_* is independent of the meshes and θ_K .

By iterating (6.17) we find

$$(6.18) \quad \begin{aligned} \|T_{K,k,m}v\|_{1-\beta,K}^2 &\leq c_* \left[\sum_{\ell=k}^{K-1} h_\ell^{2\beta} \theta_\ell^{-4} \prod_{\ell+1 \leq r \leq K-1} (1 + \theta_r^2) \right] \|v\|_a^2 \\ &\quad + \left[\prod_{\ell=k}^{K-1} (1 + \theta_\ell^2) \right] \|v\|_{1-\beta,k}^2 \quad \forall v \in V_k, \end{aligned}$$

where $\theta_k, \dots, \theta_{K-1} \in (0, 1]$ are arbitrary.

As in the proof of Lemma 6.3, by taking $\theta_\ell = 3^{-\beta(\ell-k)/4}$ for $k \leq \ell \leq K-1$, we obtain from (6.18) the estimate

$$\|T_{K,k,m}v\|_{1-\beta,K}^2 \lesssim \|v\|_{1-\beta,k}^2 + h_k^{2\beta} \|v\|_a^2 \quad \forall v \in V_k,$$

which together with (4.2) and (4.4) imply (6.16). \square

Lemma 6.5. *The following estimate holds:*

$$(6.19) \quad \|T_{k,K,m}T_{K,k,m}v\|_{1-\beta,k} \lesssim \|v\|_{1-\beta,k} \quad \forall v \in V_k, 2 \leq k \leq K.$$

Proof. From (4.2), (4.4), (6.7), (6.11) and (6.16) we have, for any $v \in V_k$,

$$\begin{aligned} \|T_{k,K,m}T_{K,k,m}v\|_{1-\beta,k} &\lesssim \|T_{K,k,m}v\|_{1-\beta,K} + h_k^\beta \|T_{K,k,m}v\|_a \\ &\lesssim \|v\|_{1-\beta,k} + h_k^\beta \|v\|_a \\ &\lesssim \|v\|_{1-\beta,k}. \end{aligned}$$

\square

Lemma 6.6. *Let $2 \leq j < k$ and $\theta \in (0, 1]$. There exists a mesh-independent positive integer m_θ such that*

$$(6.20) \quad \|T_{j,k,m}v\|_{1-\beta,j} \leq (1 + 2\theta^2)^{(k-j)/2} \|v\|_{1-\beta,k} \quad \forall v \in V_k, m \geq m_\theta.$$

Proof. From (4.2), (4.8), (4.9), (6.2) and (6.9), we have

$$\begin{aligned} \|T_{j,k,m}v\|_{1-\beta,j}^2 &= \|P_j R_{j+1}^m T_{j+1,k,m}v\|_{1-\beta,j}^2 \\ &\leq (1 + \theta^2) \|R_{j+1}^m T_{j+1,k,m}v\|_{1-\beta,j+1}^2 + C\theta^{-4} h_{j+1}^{2\beta} \|R_{j+1}^m T_{j+1,k,m}v\|_a^2 \\ &\leq (1 + \theta^2) \|T_{j+1,k,m}v\|_{1-\beta,j+1}^2 + Cm^{-\beta} \theta^{-4} \|T_{j+1,k,m}v\|_{1-\beta,j+1}^2, \end{aligned}$$

where C is independent of the meshes and θ .

Therefore, by choosing m_θ large enough, we have

$$(6.21) \quad \|T_{j,k,m}v\|_{1-\beta,j}^2 \leq (1 + 2\theta^2) \|T_{j+1,k,m}v\|_{1-\beta,j+1}^2 \quad \forall v \in V_k, m \geq m_\theta.$$

The estimate (6.20) follows by iterating (6.21). \square

Given any $\theta \in (0, 1]$, we can now estimate the terms in the second sum on the right-hand side of (6.6) by using (2.2), (4.6), (4.8), Lemma 6.5 and Lemma 6.6 as follows.

For $2 \leq j < k \leq K$, we have

$$\begin{aligned} &a(R_j^m v_j, T_{j,k,m}T_{k,K,m}T_{K,k,m}R_k^m v_k) \\ &\leq \|R_j^m v_j\|_{1+\beta,j} \|T_{j,k,m}T_{k,K,m}T_{K,k,m}R_k^m v_k\|_{1-\beta,j} \\ &\leq C(1 + 2\theta^2)^{(k-j)/2} \|R_j^m v_j\|_{1+\beta,j} \|R_k^m v_k\|_{1-\beta,k} \\ (6.22) \quad &\leq Cm^{-\beta} (1 + 2\theta^2)^{(k-j)/2} \left(\frac{h_k}{h_j}\right)^\beta (h_j^{-\beta} \|R_j^{\lceil m/2 \rceil} v_j\|_{1-\beta,j}) \\ &\quad \times (h_k^{-\beta} \|R_k^m v_k\|_{1-\beta,k}) \\ &\leq Cm^{-\beta} \left[(1 + 2\theta^2)^{1/(2\beta)} 2^{-1}\right]^{\beta(k-j)} (h_j^{-\beta} \|R_j^{\lceil m/2 \rceil} v_j\|_{1-\beta,j}) \\ &\quad \times (h_k^{-\beta} \|R_k^m v_k\|_{1-\beta,k}), \end{aligned}$$

where $m \geq m_\theta$ for a sufficiently large m_θ .

The following strengthened Cauchy-Schwarz inequality provides the last key estimate for the convergence analysis.

Lemma 6.7. *Let $v \in V_K$ be arbitrary and v_k ($2 \leq k \leq K$) be defined by (6.3). There exists a positive integer M independent of the meshes and v such that*

$$(6.23) \quad a(R_j^m v_j, T_{j,k,m} T_{k,K,m} T_{K,k,m} R_k^m v_k) \lesssim m^{-\alpha} \left(\frac{2}{3}\right)^{\beta(k-j)} \|v_j\|_a \|v_k\|_a$$

for $2 \leq j < k \leq K$ and $m \geq M$.

Proof. Combining (4.10), (6.5) and (6.22), we find

$$\begin{aligned} & a(R_j^m v_j, T_{j,k,m} T_{k,K,m} T_{K,k,m} R_k^m v_k) \\ & \leq C m^{-\beta} \left[(1 + 2\theta^2)^{1/(2\beta)} 2^{-1} \right]^{\beta(k-j)} (m^{(\beta-\alpha)/2} \|v_j\|_a) (m^{(\beta-\alpha)/2} \|v_k\|_a) \\ & \leq C m^{-\alpha} \left[(1 + 2\theta^2)^{1/(2\beta)} 2^{-1} \right]^{\beta(k-j)} \|v_j\|_a \|v_k\|_a \quad \forall m \geq m_\theta. \end{aligned}$$

The estimate (6.23) follows by choosing a small θ so that $(1 + 2\theta^2)^{1/(2\beta)} 2^{-1} < \frac{2}{3}$. \square

We are now ready to prove the main results of this paper.

Lemma 6.8. *There exists a positive integer M such that*

$$(6.24) \quad \|E_{K,m} v\|_a \lesssim m^{-\alpha/2} \|v\|_a \quad \forall v \in V_K, m \geq M, K = 1, 2, 3, \dots$$

Proof. The case $K = 1$ is trivial since $E_{1,m} = 0$. For $K \geq 2$, we have, by (6.4), (6.6), (6.8), (6.23) and the discrete Young's inequality (cf. [24]),

$$\begin{aligned} \|E_{K,m} v\|_a^2 &= a(E_{K,m} v, E_{K,m} v) \lesssim m^{-\alpha} \sum_{j,k=2}^K \left(\frac{2}{3}\right)^{\beta|k-j|} \|v_j\|_a \|v_k\|_a \\ &\lesssim m^{-\alpha} \left[\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{\beta n} \right] \sum_{k=2}^K \|v_k\|_a^2 \\ &\lesssim m^{-\alpha} \sum_{k=2}^K \|v_k\|_a^2 \lesssim m^{-\alpha} \|v\|_a^2. \end{aligned}$$

\square

As a corollary to Lemma 6.8 and the result (1.15), we have an error estimate for the V-cycle algorithm with any number of smoothing steps.

Theorem 6.9. *Given any $g \in V_k$ and any initial guess $z_0 \in V_k$, the approximate solution $MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)$ of (1.12) obtained by Algorithm 2.1 satisfies the error estimate*

$$(6.25) \quad \begin{aligned} & \|z - MG_{\mathcal{V}}(k, g, z_0, m_1, m_2)\|_a \\ & \leq \left[\frac{C}{C + [\max(m_1, 1) \max(m_2, 1)]^{\alpha/2}} \right] \|z - z_0\|_a \end{aligned}$$

for $m_1 + m_2 \geq 1$, where the positive constant C is mesh independent.

Proof. It follows from (1.15) that

$$(6.26) \quad \|E_{k,m} v\|_a \leq \delta \|v\|_a \quad \forall v \in V_k, k = 1, 2, \dots,$$

where $\delta \in (0, 1)$ is independent of the meshes and the number of smoothing steps.

From Lemma 6.8 we have

$$(6.27) \quad \|E_{k,m}v\|_a \leq \tilde{C}m^{-\alpha/2}\|v\|_a \quad \forall v \in V_k, m \geq M, k = 1, 2, \dots$$

Let $M_* \geq M$ be chosen large enough so that

$$(6.28) \quad \tilde{C}m^{-\alpha/2} \leq \frac{2\tilde{C}}{2\tilde{C} + m^{\alpha/2}} \quad \forall m \geq M_* + 1,$$

and then let C be chosen large enough so that

$$(6.29) \quad C \geq 2\tilde{C} \quad \text{and} \quad \delta \leq \frac{C}{C + M_*^{\alpha/2}}.$$

It follows from (6.26)–(6.29) that

$$\|E_{k,m}v\|_a \leq \left[\frac{C}{C + m^{\alpha/2}} \right] \|v\|_a \quad \forall v \in V_k, k = 1, 2, \dots,$$

i.e., the estimate (6.25) holds for $m_1 = 0$ and $m_2 = m$.

The general case of (6.25) follows easily from this special case and the relations (2.24) and (2.25). \square

Remark 6.10. The results of this paper can be easily generalized to two- and three-dimensional elliptic boundary value problems discretized by the Q_1 element.

Remark 6.11. In the case where Ω also has cracks, the elliptic regularity estimate is valid for $0 < \alpha < \frac{1}{2}$ (α can be taken to be arbitrarily close to $\frac{1}{2}$, cf. [21, 18, 27]). All the statements in this paper remain valid provided the concept of Sobolev spaces on cracked domains are defined appropriately (cf. [15]). In particular, Theorem 6.9 also holds for domains with cracks.

Remark 6.12. The new approach to V -cycle convergence analysis developed in this paper can also be applied to nonconforming finite elements. This will be addressed in a forthcoming paper.

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