CONVERGENCE OF THE PML METHOD FOR ELASTIC WAVE SCATTERING PROBLEMS

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Abstract. In this paper we study the convergence of the perfectly matched layer (PML) method for solving the time harmonic elastic wave scattering problems. We introduce a simple condition on the PML complex coordinate stretching function to guarantee the ellipticity of the PML operator. We also introduce a new boundary condition at the outer boundary of the PML layer which allows us to extend the reflection argument of Bramble and Pasciak to prove the stability of the PML problem in the truncated domain. The exponential convergence of the PML method in terms of the thickness of the PML layer and the strength of PML medium property is proved. Numerical results are included.

1. Introduction

We study the convergence of the perfectly matched layer (PML) method for solving elastic wave scattering problems with the traction boundary condition:

$$\nabla\cdot\tau(\boldsymbol{u})+\gamma^2\boldsymbol{u}=-\boldsymbol{q}\quad\text{in }\mathbb{R}^3\backslash\bar{D},\tag{1.1}$$

$$\tau(\boldsymbol{u})\boldsymbol{n}_D=-\boldsymbol{g}\quad\text{on }\Gamma_D.\tag{1.2}$$

Here $D\subset\mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ_D, $\boldsymbol{q}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{D})'$ has support inside $B_l:=\{\boldsymbol{x}=(x_1,x_2,x_3)^T\in\mathbb{R}^3:|x_i|<l_i,i=1,2,3\}$ for some constants $l_i>0,i=1,2,3$, $\boldsymbol{g}\in\boldsymbol{H}^{-1/2}(\Gamma_D)$ is determined by the traction on the boundary, $\boldsymbol{n}_D$ is the unit outer normal to Γ_D, and $\gamma=\sqrt{\rho_0}\omega>0$ with the angular frequency $\omega>0$ and the constant density $\rho_0>0$. In this paper, for any Banach space X, we denote the boldfaced letter $\boldsymbol{X}=X^3$. $\|\cdot\|_X$ stands for the norm of X or $\boldsymbol{X}$. X' is the dual space of X.

In the region outside D, the medium is assumed to be linear, homogeneous, and isotropic with constant Lamé constants λ and μ. The stress tensor $\tau(\boldsymbol{u})$ relates to the displacement vector $\boldsymbol{u}=(u_1,u_2,u_3)^T$ by the generalized Hooke law:

$$\tau(\boldsymbol{u})=2\mu\,\varepsilon(\boldsymbol{u})+\lambda\mathrm{tr}(\varepsilon(\boldsymbol{u}))\mathrm{I},\quad\varepsilon(\boldsymbol{u})=\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^T),\tag{1.3}$$

where $\mathrm{I}\in\mathbb{R}^{3\times3}$ is the identity matrix and $\boldsymbol{\nabla}\boldsymbol{u}$ is the displacement gradient tensor whose elements are $(\boldsymbol{\nabla}\boldsymbol{u})_{ij}=\partial u_i/\partial x_j$, $i,j=1,2,3$. We remark that the results in this paper can be extended to solve the scattering problems with other boundary conditions such as Dirichlet or mixed boundary conditions on Γ_D.

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We now introduce the Kupradze-Sommerfeld radiation condition in order to complete the definition of the problem. It is known that under the constitutive relation (1.3), (1.1) can be rewritten to the following equation:

$$\boldsymbol{u}+\frac{1}{k_p^2}\nabla(\operatorname{div}\boldsymbol{u})-\frac{1}{k_s^2}\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u})=0\quad \text{in } \mathbb{R}^3\backslash\bar{B}_l,$$

where $k_p=\frac{\gamma}{\sqrt{\lambda+2\mu}}$ and $k_s=\frac{\gamma}{\sqrt{\mu}}$ are respectively the wave numbers of compressional and shear waves. Let $\boldsymbol{u}_p=-\frac{1}{k_p^2}\nabla(\operatorname{div}\boldsymbol{u})$ be the compressional part and $\boldsymbol{u}_s=\frac{1}{k_s^2}\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u})$ be the shear part of the wave field. They satisfy the Helmholtz equations

$$\boldsymbol{\Delta}\boldsymbol{u}_p+k_p^2\boldsymbol{u}_p=0,\qquad \boldsymbol{\Delta}\boldsymbol{u}_s+k_s^2\boldsymbol{u}_s=0\quad \text{in } \mathbb{R}^3\backslash\bar{B}_l.$$

It is clear that $\boldsymbol{u}=\boldsymbol{u}_p+\boldsymbol{u}_s$ in $\mathbb{R}^3\backslash\bar{B}_l$. The Kupradze-Sommerfeld radiation condition is given by the requirement that $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ should satisfy the Sommerfeld radiation condition

$$\lim_{|\boldsymbol{x}|\to\infty}|\boldsymbol{x}|\left(\frac{\partial\boldsymbol{u}_p}{\partial|\boldsymbol{x}|}-\mathbf{i}k_p\boldsymbol{u}_p\right)=0,\qquad \lim_{|\boldsymbol{x}|\to\infty}|\boldsymbol{x}|\left(\frac{\partial\boldsymbol{u}_s}{\partial|\boldsymbol{x}|}-\mathbf{i}k_s\boldsymbol{u}_s\right)=0.$$

The existence and uniqueness of the time harmonic elastic wave equation under the Kupradze-Sommerfeld radiation condition are considered in Kupradze [23] for smooth scatterers. For scatterers with Lipschitz boundary, the existence and uniqueness of the scattering solutions are proved in Bramble and Pasciak [7] for the Dirichlet boundary condition on Γ_D. For the Neumann boundary condition (1.2) on Γ_D, the existence of solutions will be considered briefly by the method of limiting absorption principle below (Theorem 2.1).

Since the work of Bérenger [4] which proposed a PML technique for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. [5] for the review). The basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that absorb all the waves that propagate from inside the computational domain.

The convergence of the PML method is studied in [25, 21, 3, 6, 8] for time harmonic acoustic, electromagnetic, and elastic wave scattering problems with circular or spherical PML layers. The convergence of the PML method was also studied in the context of the adaptive PML technique for grating problems in [17] and for acoustic and Maxwell scattering problems in [15, 16, 13, 14]. The main idea of the adaptive PML technique is to use the a posteriori error estimate to determine the PML parameters and to use the adaptive finite element method to solve the PML equations. The adaptive PML technique provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

The purpose of this paper is to study the convergence of the Cartesian PML method for the time harmonic elastic waves which was first proposed in [12] and also studied in [28]. The complex coordinate stretching to derive the Cartesian PML method is [11]:

$$\tilde{x}_j=x_j+\zeta\int_0^{x_j}\sigma_j(t)dt+\mathbf{i}\int_0^{x_j}\sigma_j(t)dt,\quad j=1,2,3, \tag{1.4}$$

where $\zeta \geq 0$ is a constant to be specified and $\sigma_j(t)$ is the PML medium property. The choice of a positive parameter ζ is equivalent to the complex frequency shifted PML method proposed in [24] which has the advantage of additional damping for the evanescent waves. The mathematical analysis in [9, 10, 14] reveals that an appropriately chosen parameter ζ guarantees the ellipticity of the PML operator without any constraint on the smallness of the PML medium property $\sigma_j(t)$ for 3D acoustic and electromagnetic waves. The first contribution in this paper is to show that the PML method with $\zeta \geq \sqrt{(\lambda+2\mu)/\mu}$ will guarantee the ellipticity of the elastic PML operator (Lemma 3.3 below).

The convergence of the Cartesian PML method is studied in [22, 10, 9, 16, 18] for time harmonic acoustic and Maxwell scattering problems. The key gradient in the analysis in [10, 9] is a reflection argument to show the inf-sup condition for the sesquilinear form associated with the PML equation in the truncated domain. This reflection argument cannot be directly extended to the elastic PML equations if one imposes homogeneous Dirichlet boundary condition at the outer boundary of the PML layer. In this paper we consider the following PML problem (see section 2 for the notation)

$$\nabla\cdot(\tilde{\tau}(\hat{\boldsymbol{u}})A)+\gamma^2 J\hat{\boldsymbol{u}}=-\boldsymbol{q} \quad \text{in } \Omega_L, \tag{1.5}$$

$$\tilde{\tau}(\hat{\boldsymbol{u}})A\boldsymbol{n}_D=-\boldsymbol{g} \quad \text{on } \Gamma_D, \tag{1.6}$$

$$\hat{\boldsymbol{u}}\cdot\boldsymbol{n}=0, \quad \tilde{\tau}(\hat{\boldsymbol{u}})A\boldsymbol{n}\times\boldsymbol{n}=0 \quad \text{on } \Gamma_L. \tag{1.7}$$

The mixed boundary condition (1.7) at the outer boundary of the PML layer Γ_L allows us to extend the reflection argument in Bramble and Pasciak [10, 9] for acoustic and electromagnetic scattering problems to solve the elastic scattering problems.

The layout of the paper is as follows. In section 2 we introduce the PML formulation for (1.1)-(1.2) by following the method of complex coordinate stretching. In section 3 we prove the well-posedness of the PML equation in $\mathbb{R}^3$. In section 4 we prove the stability of the PML equation in the truncated domain. In section 5 we prove the stability of the Dirichlet PML problem in the layer. In section 6 we show the convergence of the PML method. In section 7 we show some numerical results to illustrate the performance of the proposed PML method. In Section 8 we prove the existence of the scattering solution of (1.1)-(1.2) by the method of limiting absorption principle.

2. The PML equation

Let $B_l := \{\boldsymbol{x}=(x_1,x_2,x_3)^T \in \mathbb{R}^3 : |x_i| < l_i,\ i=1,2,3\}$ contain the scatterer D and the support of $\boldsymbol{q}$. Let $\Gamma_l = \partial B_l$ and $\boldsymbol{n}_l$ the unit outer normal to Γ_l. We start by introducing the Dirichlet-to-Neumann operator $\mathbb{T} : \boldsymbol{H}^{1/2}(\Gamma_l) \to \boldsymbol{H}^{-1/2}(\Gamma_l)$. Given $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\Gamma_l)$, we define $\mathbb{T}\boldsymbol{f} = \tau(\boldsymbol{\xi})\boldsymbol{n}_l$ with $\boldsymbol{\xi}$ being the solution of the following exterior Dirichlet problem:

$$\nabla\cdot\tau(\boldsymbol{\xi})+\gamma^2\boldsymbol{\xi}=0 \quad \text{in } \mathbb{R}^3\backslash\bar{B}_l, \tag{2.1}$$

$$\boldsymbol{\xi}=\boldsymbol{f} \quad \text{on } \Gamma_l, \tag{2.2}$$

(2.3) $\boldsymbol{\xi}$ satisfies the Kupradze-Sommerfeld radiation conditions at infinity.

Since (2.1)-(2.3) has a unique solution $\boldsymbol{\xi} \in \boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$ (cf. e.g. [7]), $\mathbb{T} : \boldsymbol{H}^{1/2}(\Gamma_l) \to \boldsymbol{H}^{-1/2}(\Gamma_l)$ is well-defined and is a continuous linear operator.

Let $\mathfrak{a} : \boldsymbol{H}^1(\Omega_l) \times \boldsymbol{H}^1(\Omega_l) \to \mathbb{C}$, where $\Omega_l = B_l \backslash \bar{D}$, be the sesquilinear form

$$\mathfrak{a}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \int_{\Omega_l} \left(\tau(\boldsymbol{\phi}) : \boldsymbol{\nabla} \bar{\boldsymbol{\psi}} - \gamma^2 \boldsymbol{\phi} \cdot \bar{\boldsymbol{\psi}} \right) d\boldsymbol{x} - \langle \mathbb{T}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{\Gamma_l}. \tag{2.4}$$

Here and in the following, for any Lipschitz domain $\mathcal{D} \subset \mathbb{R}^3$ with boundary Γ, we denote $(\cdot,\cdot)_{\mathcal{D}}$ the inner product on $\boldsymbol{L}^2(\mathcal{D})$ or the duality pairing between $\boldsymbol{H}^1(\mathcal{D})'$ and $\boldsymbol{H}^1(\mathcal{D})$ and $\langle \cdot,\cdot \rangle_\Gamma$ the inner product on $\boldsymbol{L}^2(\Gamma)$ or the duality pairing between $\boldsymbol{H}^{-1/2}(\Gamma)$ and $\boldsymbol{H}^{1/2}(\Gamma)$. The weak formulation of the scattering problem (1.1)-(1.2) is: Given $\boldsymbol{q} \in \boldsymbol{H}^1(\mathbb{R}^3 \backslash \bar{D})'$ and $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\Gamma_D)$, find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_l)$ such that

$$\mathfrak{a}(\boldsymbol{u}, \boldsymbol{\psi}) = (\boldsymbol{q}, \boldsymbol{\psi})_{\Omega_l} + \langle \boldsymbol{g}, \boldsymbol{\psi} \rangle_{\Gamma_D}, \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega_l). \tag{2.5}$$

The existence of a unique solution of the scattering problem (2.5) is a direct consequence of the following theorem whose proof will be discussed briefly in the Appendix of this paper.

Theorem 2.1. *For any $\boldsymbol{q} \in \boldsymbol{H}^1(\mathbb{R}^3 \backslash \bar{D})'$ with compact support and $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\Gamma_D)$, the problem* (1.1)-(1.2) *with the Kupradze-Sommerfeld radiation condition has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3 \backslash \bar{D})$ such that for any bounded open set $\mathcal{O} \subset \mathbb{R}^3 \backslash \bar{D}$ that contains the support of $\boldsymbol{q}$,*

$$\|\boldsymbol{u}\|_{H^1(\mathcal{O} \backslash \bar{D})} \le C(\|\boldsymbol{q}\|_{H^1(\mathbb{R}^3 \backslash \bar{D})'} + \|\boldsymbol{g}\|_{H^{-1/2}(\Gamma_D)}). \tag{2.6}$$

For the sesquilinear form $\mathfrak{a}(\cdot,\cdot)$, we associate with a bounded linear operator $\hat{A} : \boldsymbol{H}^1(\Omega_l) \to \boldsymbol{H}^1(\Omega_l)'$ such that

$$(\hat{A}\boldsymbol{\phi}, \boldsymbol{\psi})_{\Omega_l} = \mathfrak{a}(\boldsymbol{\phi}, \boldsymbol{\psi}), \quad \forall \boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega_l).$$

By Theorem 2.1, $\hat{A}$ is surjective and one-to-one. Thus, by the open mapping theorem, we know that there exists a constant $C > 0$ such that the following inf-sup condition is satisfied

$$\sup_{0 \neq \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega_l)} \frac{|\mathfrak{a}(\boldsymbol{\phi}, \boldsymbol{\psi})|}{\|\, \boldsymbol{\psi} \,\|_{H^1(\Omega_l)}} \ge \; C \,\|\, \boldsymbol{\phi} \,\|_{H^1(\Omega_l)}, \quad \forall\, \boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega_l). \tag{2.7}$$

2.1. **PML complex coordinate stretching.** The PML method is based on the complex coordinate stretching outside B_l. Let $\alpha_j(x_j) = 1 + \zeta\sigma_j(x_j) + \mathbf{i}\sigma_j(x_j)$, $j = 1, 2, 3$, be the model medium property. We require the following assumption on the parameter ζ to guarantee the ellipticity of the PML equation (see Lemma 3.3 below):

(H1) $\zeta \ge \sqrt{(\lambda + 2\mu)/\mu}$.

For $t \in \mathbb{R}$, $\sigma_j(t) \in C^1(\mathbb{R})$, $j = 1, 2, 3$, is an even function such that

$$\sigma_j'(t) \ge 0 \;\text{ for } t \ge 0, \quad \sigma_j = 0 \;\text{ for } |t| \le l_j, \quad \text{and } \sigma_j = \sigma_0 \;\text{ for } |t| \ge \bar{l}_j, \tag{2.8}$$

where $\bar{l}_j > l_j$ is fixed and $\sigma_0 > 0$ is a constant. The requirement that the medium property $\sigma_j(t)$ is constant for $|t| \ge \bar{l}_j$ has been also used in [10, 9] which is essential for using a reflection argument to prove the int-sup condition for the PML problem in the truncated domain.

For $\boldsymbol{x} \in \mathbb{R}^3$, denote by $\tilde{\boldsymbol{x}}(\boldsymbol{x}) = (\tilde{x}_1(x_1), \tilde{x}_2(x_2), \tilde{x}_3(x_3))^T$ the complex coordinate, where

$$\tilde{x}_j(x_j) = \int_0^{x_j} \alpha_j(t)dt = x_j + (\zeta + \mathbf{i}) \int_0^{x_j} \sigma_j(t)dt, \quad j = 1, 2, 3.$$

Note that $\tilde{x}_j(x_j)$ depends only on x_j. For any $z \in \mathbb{C}_{++} := \{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0, \mathrm{Im}\,(z) \geq 0\}$, denote

$$\tilde{x}_j^z(x_j) = x_j + z\int_0^{x_j} \sigma_j(t)dt, \quad j = 1, 2, 3. \tag{2.9}$$

Write $\tilde{\boldsymbol{x}}_z = (\tilde{x}_1^z(x_1), \tilde{x}_2^z(x_2), \tilde{x}_3^z(x_3))^T$ and $\tilde{\boldsymbol{y}}_z = (\tilde{y}_1^z(y_1), \tilde{y}_2^z(y_2), \tilde{y}_3^z(y_3))^T$. We define the complex distance

$$d(\tilde{\boldsymbol{x}}_z, \tilde{\boldsymbol{y}}_z) = \left[(\tilde{x}_1^z(x_1) - \tilde{y}_1^z(y_1))^2 + (\tilde{x}_2^z(x_2) - \tilde{y}_2^z(y_2))^2 + (\tilde{x}_3^z(x_3) - \tilde{y}_3^z(y_3))^2\right]^{1/2}.$$

Here and in the following, for any $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}\,(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash(-\infty, 0]$. It is obvious that $\tilde{\boldsymbol{x}}_{z_0} = \tilde{\boldsymbol{x}}$, where $z_0 = \zeta + \mathbf{i}$. The following lemma is a variant of [10, Lemma 3.1].

Lemma 2.2. *For any $z \in U := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > |\mathrm{Im}\,(z)|\}$, we have*

$$|\boldsymbol{x} - \boldsymbol{y}| \leq |d(\tilde{\boldsymbol{x}}_z, \tilde{\boldsymbol{y}}_z)| \leq (1 + |z|\sigma_0)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3.$$

Proof. For the sake of completeness we recall the proof here. From the definition we know that

$$\tilde{x}_j^z(x_j) - \tilde{y}_j^z(y_j) = \alpha_j^z(\xi_j)(x_j - y_j), \quad \alpha_j^z(\xi_j) = 1 + z\sigma_j(\xi_j), \qquad j = 1, 2, 3,$$

where ξ_j is some number between x_j and y_j. It is clear that $0 \leq \sigma_j(\xi_j) \leq \sigma_0$. Thus

$$|d(\tilde{\boldsymbol{x}}_z, \tilde{\boldsymbol{y}}_z)|^2 = |\boldsymbol{x} - \boldsymbol{y}|^2 \Big|\sum_{j=1}^3 t_j \alpha_j^z(\xi_j)^2\Big|, \quad t_j = |x_j - y_j|^2/|\boldsymbol{x} - \boldsymbol{y}|^2. \tag{2.10}$$

The right half of the desired estimate follows now directly since $t_1 + t_2 + t_3 = 1$. To proceed, we let $z = a + \mathbf{i}b \in U$, $a, b \in \mathbb{R}$. Then $a > |b| \geq 0$. The left half of the desired inequality follows easily from the following observation:

$$\mathrm{Re}\,\alpha_j^z(\xi_j)^2 = 1 + (a^2 - b^2)\sigma_j(\xi_j)^2 + 2a\sigma_j(\xi_j) \geq 1.$$

This completes the proof. □

2.2. **The PML equation.** In this subsection we derive the PML equation based on the method of complex coordinate stretching. By Betti formula [23], the solution $\boldsymbol{\xi}$ of the exterior Dirichlet problem (2.1)-(2.3) satisfies:

$$\boldsymbol{\xi} = -\boldsymbol{\Psi}_{\mathrm{SL}}(\mathbb{T}\boldsymbol{f}) + \boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{f}) \quad \text{in } \mathbb{R}^3\backslash\bar{B}_l, \tag{2.11}$$

where $\boldsymbol{\Psi}_{\mathrm{SL}}, \boldsymbol{\Psi}_{\mathrm{DL}}$ are respectively the single and double layer potentials. For $n = 1, 2, 3$, the n-th component of the potentials are, for $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\Gamma_l)$, $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\Gamma_l)$,

$$\boldsymbol{\Psi}_{\mathrm{SL}}(\boldsymbol{\lambda})(\boldsymbol{x}) \cdot \boldsymbol{e}_n = \langle \boldsymbol{\lambda}, \overline{\Gamma(\boldsymbol{x}, \cdot)\boldsymbol{e}_n}\rangle_{\Gamma_l}, \quad \boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{f})(\boldsymbol{x}) \cdot \boldsymbol{e}_n = \langle \mathbb{T}[\Gamma(\boldsymbol{x}, \cdot)\boldsymbol{e}_n], \bar{\boldsymbol{f}}\rangle_{\Gamma_l}.$$

Here $\boldsymbol{e}_n$ is the unit vector in the x_n direction and $\Gamma(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{e}_n$ is the n-th column of the fundamental solution matrix $\Gamma(\boldsymbol{x}, \boldsymbol{y})$ of the time harmonic elastic wave equation satisfying the Kupradze-Sommerfeld radiation condition. The (j, k)-element of $\Gamma(\boldsymbol{x}, \boldsymbol{y})$ is

$$\Gamma_{jk}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\gamma^2}\left[k_s^2 G_{k_s}(\boldsymbol{x}, \boldsymbol{y})\delta_{jk} - \frac{\partial^2}{\partial x_j \partial x_k}(G_{k_p}(\boldsymbol{x}, \boldsymbol{y}) - G_{k_s}(\boldsymbol{x}, \boldsymbol{y}))\right],$$

where $G_k(\boldsymbol{x}, \boldsymbol{y}) = f_k(|\boldsymbol{x} - \boldsymbol{y}|), f_k(r) = \frac{e^{\mathbf{i}kr}}{4\pi r}$ for $r > 0$, is the fundamental solution of the Helmholtz equation of wave number k. It is known that $\boldsymbol{\Psi}_{\mathrm{SL}}(\boldsymbol{\lambda}) \in \boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$

for $\boldsymbol{\lambda}\in\boldsymbol{H}^{-1/2}(\Gamma_l)$ and $\boldsymbol{\Psi}_{\rm DL}(\boldsymbol{f})\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3)$ for $\boldsymbol{f}\in\boldsymbol{H}^{1/2}(\Gamma_l)$ (see e.g. McLean [27, Theorem 6.11] and also the proof in Lemma 5.1 below).

Straightforward calculation shows that

$$\Gamma_{jk}(\boldsymbol{x},\boldsymbol{y})=\Gamma_1(|\boldsymbol{x}-\boldsymbol{y}|)\delta_{jk}+\Gamma_2(|\boldsymbol{x}-\boldsymbol{y}|)\frac{(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^2},\tag{2.12}$$

where, for $r>0$,

$$\Gamma_1(r)=\frac{1}{\gamma^2}\left[k_s^2f_{k_s}(r)-\frac{f'_{k_p}(r)-f'_{k_s}(r)}{r}\right],\tag{2.13}$$

$$\Gamma_2(r)=\frac{1}{\gamma^2}\left[3\ \frac{f'_{k_p}(r)-f'_{k_s}(r)}{r}+(k_p^2f_{k_p}(r)-k_s^2f_{k_s}(r))\right].\tag{2.14}$$

The functions Γ_1 and Γ_2 can be extended to be analytic functions defined in $\mathbb{C}\backslash\{0\}$.

Lemma 2.3. *For $j=1,2$, $\Gamma_j(z)$ is analytic in $\mathbb{C}\backslash\{0\}$. Moreover, $|\Gamma_j(z)|\le C|z|^{-1}$, $|\Gamma'_j(z)|\le C|z|^{-2}$, and $|\Gamma''_j(z)|\le C|z|^{-3}$ uniformly for $z\in\mathbb{C}\backslash\{0\}$, $|z|\le 1$.*

Proof. $\Gamma_j(z)$ is obviously analytic in $\mathbb{C}\backslash\{0\}$. For $z\in\mathbb{C}\backslash\{0\}$, we have

$$\begin{aligned} f'_{k_p}(z)-f'_{k_s}(z) &= \frac{1}{4\pi z^2}\left[(\mathbf{i}k_pz-1)e^{\mathbf{i}k_pz}-(\mathbf{i}k_sz-1)e^{\mathbf{i}k_sz}\right]\\ &= \frac{1}{4\pi}\sum_{n=2}^{\infty}\frac{(n-1)[(\mathbf{i}k_p)^n-(\mathbf{i}k_s)^n]}{n!}z^{n-2}.\end{aligned}$$

This yields $|\Gamma_j(z)|\le C|z|^{-1}$, $|\Gamma'_j(z)|\le C|z|^{-2}$, and $|\Gamma''_j(z)|\le C|z|^{-3}$ for $|z|\le 1$, $z\ne 0$, $j=1,2$. □

For any $z\in U=\{z\in\mathbb{C}:\mathrm{Re}\,(z)>|\mathrm{Im}\,(z)|\}$ defined in Lemma 2.2, we define the modified single and double layer potentials $\tilde{\boldsymbol{\Psi}}^z_{\rm SL}$ and $\tilde{\boldsymbol{\Psi}}^z_{\rm DL}$ as follows. For $\boldsymbol{\lambda}\in\boldsymbol{H}^{-1/2}(\Gamma_l),\boldsymbol{f}\in\boldsymbol{H}^{1/2}(\Gamma_l)$, the n-th component, $n=1,2,3$, of the modified potentials are

$$\tilde{\boldsymbol{\Psi}}^z_{\rm SL}(\boldsymbol{\lambda})(\boldsymbol{x})\cdot\boldsymbol{e}_n=\langle\boldsymbol{\lambda},\overline{\tilde{\Gamma}_z(\boldsymbol{x},\cdot)\boldsymbol{e}_n}\rangle_{\Gamma_l},\quad \tilde{\boldsymbol{\Psi}}^z_{\rm DL}(\boldsymbol{f})(\boldsymbol{x})\cdot\boldsymbol{e}_n=\langle\mathbb{T}\,[\tilde{\Gamma}_z(\boldsymbol{x},\cdot)\boldsymbol{e}_n],\bar{\boldsymbol{f}}\rangle_{\Gamma_l},$$

where the (j,k)-element of the matrix $\tilde{\Gamma}_z(\boldsymbol{x},\boldsymbol{y})$ is

$$\tilde{\Gamma}^{jk}_z(\boldsymbol{x},\boldsymbol{y})=\Gamma_1(d(\tilde{\boldsymbol{x}}_z,\tilde{\boldsymbol{y}}_z))\delta_{jk}+\Gamma_2(d(\tilde{\boldsymbol{x}}_z,\tilde{\boldsymbol{y}}_z))\frac{(\tilde{x}^z_j-\tilde{y}^z_j)(\tilde{x}^z_k-\tilde{y}^z_k)}{d(\tilde{\boldsymbol{x}}_z,\tilde{\boldsymbol{y}}_z)^2}.\tag{2.15}$$

In the following, for $z_0=\zeta+\mathbf{i}$, we denote $\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})=\tilde{\Gamma}_{z_0}(\boldsymbol{x},\boldsymbol{y})$, $\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})=\tilde{\Gamma}^{jk}_{z_0}(\boldsymbol{x},\boldsymbol{y})$, and, for any $\boldsymbol{\lambda}\in\boldsymbol{H}^{-1/2}(\Gamma_l),\boldsymbol{f}\in\boldsymbol{H}^{1/2}(\Gamma_l)$,

$$\tilde{\boldsymbol{\Psi}}_{\rm SL}(\boldsymbol{\lambda})=\tilde{\boldsymbol{\Psi}}^{z_0}_{\rm SL}(\boldsymbol{\lambda}),\quad \tilde{\boldsymbol{\Psi}}_{\rm DL}(\boldsymbol{f})=\tilde{\boldsymbol{\Psi}}^{z_0}_{\rm DL}(\boldsymbol{f}).$$

Lemma 2.4. *Let (H1) be satisfied. For $j,k=1,2,3$, we have for any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$ such that $\mathrm{Im}\,d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})>0$,*

$$\begin{aligned}
&|\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})|\le C(1+|z_0|\sigma_0)^2|\boldsymbol{x}-\boldsymbol{y}|^{-1}e^{-k_p\mathrm{Im}\,d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})},\\
&|\nabla_{\boldsymbol{x}}\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})|\le C(1+|z_0|\sigma_0)^4(|\boldsymbol{x}-\boldsymbol{y}|^{-1}+|\boldsymbol{x}-\boldsymbol{y}|^{-2})e^{-k_p\mathrm{Im}\,d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})},\\
&|\nabla_{\boldsymbol{y}}\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})|\le C(1+|z_0|\sigma_0)^4(|\boldsymbol{x}-\boldsymbol{y}|^{-1}+|\boldsymbol{x}-\boldsymbol{y}|^{-2})e^{-k_p\mathrm{Im}\,d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})},\\
&|\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}}\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})|\le C(1+|z_0|\sigma_0)^6(|\boldsymbol{x}-\boldsymbol{y}|^{-1}+|\boldsymbol{x}-\boldsymbol{y}|^{-3})e^{-k_p\mathrm{Im}\,d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}.
\end{aligned}$$

Proof. Since $z_0 \in U = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > |\mathrm{Im}\,(z)|\}$, by Lemma 2.2 we have $|\tilde{x}_j - \tilde{y}_j|/|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \le 1 + |z_0|\sigma_0$ and consequently

$$|\tilde{\Gamma}_{jk}(\boldsymbol{x}, \boldsymbol{y})| \le C(1 + |z_0|\sigma_0)^2 \sum_{j=1}^{2} |\Gamma_j(d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}))|.$$

By Lemma 2.3 and Lemma 2.2, if $|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \le 1$, then $e^{-k_p \mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})} \ge e^{-k_p}$, and thus

$$|\Gamma_j(d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}))| \le C|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})|^{-1} \le C|\boldsymbol{x} - \boldsymbol{y}|^{-1} \le C|\boldsymbol{x} - \boldsymbol{y}|^{-1} e^{-k_p \mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})}.$$

On the other hand, if $|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \ge 1$, by Lemma 2.2 and simple calculations we have

$$|\Gamma_j(d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}))| \le C|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})|^{-1} e^{-k_p \mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})} \le C|\boldsymbol{x} - \boldsymbol{y}|^{-1} e^{-k_p \mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})}.$$

This shows the estimate for $|\tilde{\Gamma}_{jk}(\boldsymbol{x}, \boldsymbol{y})|$. The other estimates can be proved similarly. □

The following lemma which extends [16, Lemma 3.2] is proved in [14].

Lemma 2.5. *For any $z_i = a_i + \mathbf{i} b_i$ with $a_i, b_i \in \mathbb{R}, i = 1, 2, 3$, such that $a_1 b_1 + a_2 b_2 + a_3 b_3 \ge 0$ and $a_1^2 + a_2^2 + a_3^2 > 0$, we have*

$$\mathrm{Im}\,(z_1^2 + z_2^2 + z_3^2)^{1/2} \ge \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

Let $z_j = \tilde{x}_j - \tilde{y}_j = (x_j - y_j) + (\zeta + \mathbf{i}) \int_{y_j}^{x_j} \sigma_j(t) dt$, $j = 1, 2, 3$. By Lemma 2.5, $d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = (z_1^2 + z_2^2 + z_3^2)^{1/2}$ satisfies

$$\mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \ge \frac{\displaystyle\sum_{j=1}^{3} \left(|x_j - y_j| \Big| \int_{y_j}^{x_j} \sigma_j(t) dt \Big| + \zeta \Big| \int_{y_j}^{x_j} \sigma_j(t) dt \Big|^2 \right)}{(1 + \zeta\sigma_0)|\boldsymbol{x} - \boldsymbol{y}|}. \tag{2.16}$$

The following lemma which extends [10, Lemma 3.2] shows that $\mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is bounded below by $|\boldsymbol{x} - \boldsymbol{y}|$ if $\boldsymbol{x}, \boldsymbol{y}$ are far away.

Lemma 2.6. *Let $\beta > 1$ be a fixed number. If $|\boldsymbol{x} - \boldsymbol{y}| \ge 2\sqrt{3}\beta \bar{l}_{\max}$, where $\bar{l}_{\max} = \max_{j=1,2,3} \bar{l}_j$, where $\bar{l}_j$, $j = 1, 2, 3$, are defined in (2.8), we have $\mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \ge \frac{1}{3}(1 - \beta^{-1})^2 \sigma_0 \, |\boldsymbol{x} - \boldsymbol{y}|$.*

Proof. Let j be the index such that $|x_j - y_j| = \max_{i=1,2,3} |x_i - y_i|$. Then $|x_j - y_j|^2 \ge |\boldsymbol{x} - \boldsymbol{y}|^2/3$. It follows from the assumption $|\boldsymbol{x} - \boldsymbol{y}| \ge 2\sqrt{3}\beta \bar{l}_{\max}$ that $|x_j - y_j| \ge 2\beta \bar{l}_j$. Thus, since $\sigma_j(t) = \sigma_0$ for $|t| \ge \bar{l}_j$,

$$\left| \int_{y_j}^{x_j} \sigma(t) dt \right| \ge (|x_j - y_j| - 2\bar{l}_j)\sigma_0 \ge (1 - \beta^{-1})\sigma_0 |x_j - y_j|.$$

This implies by (2.16) that

$$\mathrm{Im}\, d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \ge \frac{1}{3}(1 - \beta^{-1})^2 \sigma_0 |\boldsymbol{x} - \boldsymbol{y}|.$$

This completes the proof. □

For any $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\Gamma_l)$, let $\mathbb{E}(\boldsymbol{f})(\boldsymbol{x})$ be the PML extension:

$$\mathbb{E}(\boldsymbol{f})(\boldsymbol{x}) = -\tilde{\boldsymbol{\Psi}}_{\rm SL}(\mathbb{T}\boldsymbol{f}) + \tilde{\boldsymbol{\Psi}}_{\rm DL}(\boldsymbol{f}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^3 \backslash \bar{B}_l. \tag{2.17}$$

By (2.11) we know that $\mathbb{E}(\boldsymbol{f}) = \boldsymbol{f}$ on Γ_l for any $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\Gamma_l)$. By Lemma 2.2, $|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \geq |\boldsymbol{x} - \boldsymbol{y}|$ for $\boldsymbol{x} \in \mathbb{R}^3 \backslash \bar{B}_l$, $\boldsymbol{y} \in \Gamma_l$. Thus since $\sigma_j \in C^1(\mathbb{R})$, $j = 1, 2, 3$, we have $\mathbb{E}(\boldsymbol{f}) \in C^2(\mathbb{R}^3 \backslash \bar{B}_l)$. Moreover, by Lemma 2.4 and Lemma 2.6 we know that $\mathbb{E}(\boldsymbol{f})$ decays exponentially as $|\boldsymbol{x}| \to \infty$.

For the solution $\boldsymbol{u}$ of the scattering problem (2.5), let $\tilde{\boldsymbol{u}} = \mathbb{E}(\boldsymbol{u}|_{\Gamma_l})$ be the PML extension of $\boldsymbol{u}|_{\Gamma_l}$. It satisfies $\tilde{\boldsymbol{u}} = \boldsymbol{u}|_{\Gamma_l}$ on Γ_l and the equation

$$\tilde{\nabla} \cdot \tilde{\tau}(\tilde{\boldsymbol{u}}) + \gamma^2 \tilde{\boldsymbol{u}} = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{B}_l, \tag{2.18}$$

where

$$\tilde{\tau}(\tilde{\boldsymbol{u}}) = 2\mu \tilde{\varepsilon}(\tilde{\boldsymbol{u}}) + \lambda \mathrm{tr}(\tilde{\varepsilon}(\tilde{\boldsymbol{u}}))\mathrm{I}, \quad \tilde{\varepsilon}(\tilde{\boldsymbol{u}}) = \frac{1}{2}(\tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{u}})^T).$$

Here $\tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{u}} \in \mathbb{C}^{3\times 3}$ whose elements are $(\partial \tilde{u}_i / \partial \tilde{x}_j)$, $i, j = 1, 2, 3$. For $\boldsymbol{x} \in \mathbb{R}^3$, let $\boldsymbol{F}(\boldsymbol{x}) = (F_1(x_1), F_2(x_2), F_3(x_3))^T$ with $F_j(x_j) = \tilde{x}_j(x_j)$, $j = 1, 2, 3$. Then $\tilde{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})$. Denote by $\boldsymbol{\nabla F}$ the Jacobi matrix of $\boldsymbol{F}$, then

$$\tilde{\nabla} \cdot = J^{-1} \nabla \cdot J \, (\boldsymbol{\nabla F})^{-1}, \quad J = \det(\boldsymbol{\nabla F}). \tag{2.19}$$

By (2.19) we easily obtain from (2.18) the desired PML equation

$$\nabla \cdot (\tilde{\tau}(\tilde{\boldsymbol{u}})A) + \gamma^2 J \tilde{\boldsymbol{u}} = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{B}_l.$$

Here

$$\tilde{\tau}(\tilde{\boldsymbol{u}}) = 2\mu \tilde{\varepsilon}(\tilde{\boldsymbol{u}}) + \lambda \mathrm{tr}(\tilde{\varepsilon}(\tilde{\boldsymbol{u}}))\mathrm{I}, \quad \tilde{\varepsilon}(\tilde{\boldsymbol{u}}) = \frac{1}{2}(\boldsymbol{\nabla} \tilde{\boldsymbol{u}} B^T + B(\boldsymbol{\nabla} \tilde{\boldsymbol{u}})^T),$$

where $B = (\boldsymbol{\nabla F})^{-T} = \mathrm{diag}(\alpha_1(x_1)^{-1}, \alpha_2(x_2)^{-1}, \alpha_3(x_3)^{-1}) \in \mathbb{C}^{3\times 3}$ is a diagonal matrix and $A = J(\boldsymbol{\nabla F})^{-T} = JB$. We notice that $\tilde{\tau}(\boldsymbol{\phi}) = \tilde{\tau}(\boldsymbol{x}, \boldsymbol{\phi})$, $\tilde{\varepsilon}(\boldsymbol{\phi}) = \tilde{\varepsilon}(\boldsymbol{x}, \boldsymbol{\phi})$ which satisfies $\tilde{\tau}(\boldsymbol{x}, \boldsymbol{\phi}) = \tau(\boldsymbol{\phi})$, $\tilde{\varepsilon}(\boldsymbol{x}, \boldsymbol{\phi}) = \varepsilon(\boldsymbol{\phi})$ for $\boldsymbol{x} \in B_l$.

Let $B_L = \{\boldsymbol{x} \in \mathbb{R}^3 : |x_i| < L_i, i = 1, 2, 3\}$ be the domain containing B_l. The PML solution $\hat{\boldsymbol{u}}$ in $\Omega_L = B_L \backslash \bar{D}$ is defined as the weak solution of the following problem:

$$\nabla \cdot (\tilde{\tau}(\hat{\boldsymbol{u}})A) + \gamma^2 J \hat{\boldsymbol{u}} = -\boldsymbol{q} \quad \text{in } \Omega_L, \tag{2.20}$$

$$\tilde{\tau}(\hat{\boldsymbol{u}})A\boldsymbol{n}_D = -\boldsymbol{g} \quad \text{on } \Gamma_D, \tag{2.21}$$

$$\hat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0, \quad \tilde{\tau}(\hat{\boldsymbol{u}})A\boldsymbol{n} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma_L := \partial B_L. \tag{2.22}$$

The well-posedness of the PML problem (2.20)-(2.21) and the convergence of its solution to the solution of the original scattering problem will be studied in the following sections. We remark that the boundary condition (2.22) is different from the usual homogeneous Dirichlet condition $\hat{\boldsymbol{u}} = 0$ on Γ_L.

To conclude this section, we introduce the following assumption on the thickness of the PML layer which is rather mild in practical applications:

(H2) $d_j := L_j - l_j \geq 2(\bar{l}_j - l_j)$, $j = 1, 2, 3$. Set $d := \min(d_1, d_2, d_3)$.

Here $\bar{l}_j$, $j = 1, 2, 3$, are defined in (2.8). In the remainder of this paper we denote C the generic constant which is independent of d but may depend on σ_0 which, however, has at most polynomial growth in σ_0.

3. The PML equation in $\mathbb{R}^3$

In this section we will show that the PML equation

$$\nabla\cdot(\tilde{\tau}(\boldsymbol{u}_1)A)+\gamma^2 J\boldsymbol{u}_1=-J\boldsymbol{\Phi}\quad \text{in } \mathbb{R}^3, \tag{3.1}$$

has a unique weak solution $\boldsymbol{u}_1\in \boldsymbol{H}^1(\mathbb{R}^3)$ for any $\boldsymbol{\Phi}\in \boldsymbol{H}^1(\mathbb{R}^3)'$. The argument depends on the study of the fundamental solution matrix and the Newton potential of the PML equation which extends the study in [25, 10] for acoustic scattering problems.

We denote $\mathcal{A}(\cdot,\cdot):\boldsymbol{H}^1(\mathbb{R}^3)\times \boldsymbol{H}^1(\mathbb{R}^3)\to\mathbb{C}$ the sesquilinear form

$$\mathcal{A}(\boldsymbol{\phi},\boldsymbol{\psi})=\int_{\mathbb{R}^3}\tilde{\tau}(\boldsymbol{\phi})A:\boldsymbol{\nabla}\bar{\boldsymbol{\psi}}\,d\boldsymbol{x},\quad \forall \boldsymbol{\phi},\boldsymbol{\psi}\in \boldsymbol{H}^1(\mathbb{R}^3).$$

Our first goal is to show that under the assumption (H1), the sesquilinear form $\mathcal{A}$ is coercive in $\boldsymbol{H}^1(\mathbb{R}^3)$. We first prove some elementary lemmas.

Lemma 3.1. *Let (H1) be satisfied. Let $\mu'=\mu/(\lambda+\mu)$. Then we have*

$$(1+\mu')\mathrm{Re}\,\frac{\alpha_1\alpha_2\alpha_3}{\alpha_j^2}\ge\frac{\eta_1\eta_2\eta_3}{\eta_j^2}+\frac{\mu'}{|\alpha_j|^2},\quad j=1,2,3,$$

where $\eta_j(x_j)=1+\zeta\sigma_j(x_j)$ and thus $\alpha_j(x_j)=\eta_j(x_j)+\mathbf{i}\sigma_j(x_j)$, $j=1,2,3$.

Proof. We only prove the case when $j=1$. The other cases are similar. By direct calculation we have

$$\begin{aligned}&(1+\mu')\mathrm{Re}\,\frac{\alpha_2\alpha_3}{\alpha_1}-\frac{\eta_2\eta_3}{\eta_1}\\=&\ \frac{\mu'\eta_1^2(\eta_2\eta_3-\sigma_2\sigma_3)+(1+\mu')\sigma_1\eta_1(\sigma_2\eta_3+\sigma_3\eta_2)-\eta_2\eta_3\sigma_1^2-\eta_1^2\sigma_2\sigma_3}{\eta_1|\alpha_1|^2}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\sigma_1\eta_1(\sigma_2\eta_3+\sigma_3\eta_2)-\eta_2\eta_3\sigma_1^2-\eta_1^2\sigma_2\sigma_3 &= (\sigma_1-\sigma_3)(\sigma_2-\sigma_1)\\ &\ge -\sigma_2\sigma_3-\sigma_1^2.\end{aligned}$$

Thus

$$(1+\mu')\mathrm{Re}\,\frac{\alpha_2\alpha_3}{\alpha_1}-\frac{\eta_2\eta_3}{\eta_1}\ge\frac{\mu'\eta_1^2(\eta_2\eta_3-\sigma_2\sigma_3)-\sigma_2\sigma_3-\sigma_1^2}{\eta_1|\alpha_1|^2}.$$

The lemma follows since $\eta_1^2\ge\eta_1+\zeta^2\sigma_1^2$ and $\eta_2\eta_3-\sigma_2\sigma_3\ge 1+\mu'^{-1}\sigma_2\sigma_3$ by (H1). □

Lemma 3.2. *Let (H1) be satisfied. Let $\mu'=\mu/(\lambda+\mu)$. Then for any $\xi_1,\xi_2,\xi_3\in\mathbb{C}$,*

$$\begin{aligned}&(1+\mu')\sum_{i=1}^3\mathrm{Re}\,\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i^2}\ |\xi_i|^2+2\left[\eta_1\mathrm{Re}\,(\xi_2\bar{\xi}_3)+\eta_2\mathrm{Re}\,(\xi_1\bar{\xi}_3)+\eta_3\mathrm{Re}\,(\xi_1\bar{\xi}_2)\right]\\&\ge\sum_{i=1}^3\frac{\mu'}{|\alpha_i|^2}|\xi_i|^2.\end{aligned}$$

Proof. This is a direct consequence of Lemma 3.1 and the following identity

$$\begin{aligned}&\frac{\eta_2\eta_3}{\eta_1}|\xi|_1^2+\frac{\eta_1\eta_3}{\eta_2}|\xi|_2^2+\frac{\eta_1\eta_2}{\eta_3}|\xi|_3^2+2\left[\eta_1\mathrm{Re}\,(\xi_2\bar{\xi}_3)+\eta_2\mathrm{Re}\,(\xi_1\bar{\xi}_3)+\eta_3\mathrm{Re}\,(\xi_1\bar{\xi}_2)\right]\\&=\left|\sqrt{\frac{\eta_2\eta_3}{\eta_1}}\xi_1+\sqrt{\frac{\eta_1\eta_3}{\eta_2}}\xi_2+\sqrt{\frac{\eta_1\eta_2}{\eta_3}}\xi_3\right|^2.\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Let (H1) be satisfied. We have*

$$\mathrm{Re}\,\mathcal{A}(\boldsymbol{\phi},\boldsymbol{\phi}) \geq \min_{j=1,2,3}\min_{x_j\in\mathbb{R}} \frac{\mu}{|\alpha_j(x_j)|^2}||\boldsymbol{\nabla}\boldsymbol{\phi}||^2_{L^2(\mathbb{R}^3)}, \quad \forall \boldsymbol{\phi}\in \boldsymbol{H}^1(\mathbb{R}^3).$$

We remark that since $\alpha_j(x_j) = 1+\zeta\sigma_j(x_j)+\mathbf{i}\sigma_j(x_j)$, $j=1,2,3$, by (2.8) we know that $\min_{x_j\in\mathbb{R}}\frac{1}{|\alpha_j(x_j)|^2} \geq [(1+\zeta\sigma_0)^2+\sigma_0^2]^{-1}$.

Proof. We only need to prove the lemma for $\boldsymbol{\phi}\in \boldsymbol{C}_0^\infty(\mathbb{R}^3)$ by the density argument. First, since $B=\mathrm{diag}(\alpha_1^{-1},\alpha_2^{-1},\alpha_3^{-1})$ and $A=JB$, we have

$$\begin{aligned}\tilde{\tau}(\boldsymbol{\phi})A:\boldsymbol{\nabla}\bar{\boldsymbol{\phi}} &= J\tilde{\tau}(\boldsymbol{\phi}):\boldsymbol{\nabla}\bar{\boldsymbol{\phi}}B^T\\ &= \mu J(\boldsymbol{\nabla}\boldsymbol{\phi}B^T+B\boldsymbol{\nabla}\boldsymbol{\phi}^T):\boldsymbol{\nabla}\bar{\boldsymbol{\phi}}B^T+\lambda J\mathrm{tr}(\tilde{\varepsilon}(\boldsymbol{\phi}))\mathrm{tr}(\tilde{\varepsilon}(\bar{\boldsymbol{\phi}})).\end{aligned}$$

This yields

$$\begin{aligned}\int_{\mathbb{R}^3}\tilde{\tau}(\boldsymbol{\phi})A:\boldsymbol{\nabla}\bar{\boldsymbol{\phi}}\,d\boldsymbol{x} &= \int_{\mathbb{R}^3}\mu J\sum_{i,j=1}^3\left(\alpha_j^{-2}\Big|\frac{\partial\phi_i}{\partial x_j}\Big|^2+\alpha_i^{-1}\alpha_j^{-1}\frac{\partial\phi_j}{\partial x_i}\frac{\partial\bar{\phi}_i}{\partial x_j}\right)d\boldsymbol{x}\\ &+ \int_{\mathbb{R}^3}\lambda J\sum_{i,j=1}^3\alpha_i^{-1}\alpha_j^{-1}\frac{\partial\phi_i}{\partial x_i}\frac{\partial\bar{\phi}_j}{\partial x_j}d\boldsymbol{x}.\end{aligned}$$

Now since $\boldsymbol{\phi}\in\boldsymbol{C}_0^\infty(\mathbb{R}^3)$, we integrate by parts twice to obtain

$$\int_{\mathbb{R}^3}J\alpha_i^{-1}\alpha_j^{-1}\frac{\partial\phi_j}{\partial x_i}\frac{\partial\bar{\phi}_i}{\partial x_j}d\boldsymbol{x}=\int_{\mathbb{R}^3}J\alpha_i^{-1}\alpha_j^{-1}\frac{\partial\phi_i}{\partial x_i}\frac{\partial\bar{\phi}_j}{\partial x_j}d\boldsymbol{x},\quad \forall i\neq j,\ i,j=1,2,3,$$

where we have used the fact that for $i\neq j$, $J\alpha_i^{-1}\alpha_j^{-1}=\alpha_k$, where $k\neq i,j$, is independent of x_i,x_j. Thus

$$\begin{aligned}&\mathrm{Re}\int_{\mathbb{R}^3}\tilde{\tau}(\boldsymbol{\phi})A:\boldsymbol{\nabla}\bar{\boldsymbol{\phi}}\,d\boldsymbol{x}\\ =&\int_{\mathbb{R}^3}\mu\sum_{i=1}^3\sum_{j\neq i}\mathrm{Re}\,\frac{\alpha_1\alpha_2\alpha_3}{\alpha_j^2}\Big|\frac{\partial\phi_i}{\partial x_j}\Big|^2d\boldsymbol{x}+\int_{\mathbb{R}^3}(\lambda+2\mu)\sum_{i=1}^3\mathrm{Re}\,\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i^2}\Big|\frac{\partial\phi_i}{\partial x_i}\Big|^2d\boldsymbol{x}\\ +&\int_{\mathbb{R}^3}2(\lambda+\mu)\left[\eta_1\mathrm{Re}\left(\frac{\partial\phi_2}{\partial x_2}\frac{\partial\bar{\phi}_3}{\partial x_3}\right)+\eta_2\mathrm{Re}\left(\frac{\partial\phi_1}{\partial x_1}\frac{\partial\bar{\phi}_3}{\partial x_3}\right)+\eta_3\mathrm{Re}\left(\frac{\partial\phi_1}{\partial x_2}\frac{\partial\bar{\phi}_2}{\partial x_2}\right)\right].\end{aligned}$$

The lemma now follows from Lemma 3.2 and the fact that $\mathrm{Re}\,\frac{\alpha_1\alpha_2\alpha_3}{\alpha_j^2}\geq\frac{1}{|\alpha_j|^2}, j=1,2,3$, since $\zeta\geq 1$ which follows from (H1). ☐

Now we study the Newton potential for the PML equation (3.1). For $z\in U=\{z\in\mathbb{C}:\mathrm{Re}\,(z)>|\mathrm{Im}\,(z)|\}$ which is defined in Lemma 2.2, denote $\boldsymbol{F}_z(\boldsymbol{x})=\tilde{\boldsymbol{x}}_z(\boldsymbol{x})$ and $J_z=\det(\boldsymbol{\nabla}\boldsymbol{F}_z)$, where $\tilde{\boldsymbol{x}}_z(\boldsymbol{x})$ is defined in (2.9). For $\boldsymbol{\Phi}\in\boldsymbol{L}^2(\mathbb{R}^3)$ with compact support, we define

$$\boldsymbol{N}_z(\boldsymbol{\Phi})(\boldsymbol{x})=\int_{\mathbb{R}^3}J_z(\boldsymbol{y})\tilde{\Gamma}_z(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\Phi}(\boldsymbol{y})d\boldsymbol{y}\quad \text{in } \mathbb{R}^3. \tag{3.2}$$

To proceed, for any Banach space X with norm $\|\cdot\|_X$, we denote $A(U;X)$ the space of all X-valued analytic functions in U. A function $v(z)$ is called X-valued analytic function in U if for any $z\in U$, $\|(v(z+h)-v(z))/h\|_X\to 0$ as $|h|\to 0$, $h\in\mathbb{C}$.

Lemma 3.4. *Let $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$ with compact support. For any $z \in U = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > |\mathrm{Im}\,(z)|\}$ defined in Lemma 2.2, we have $\boldsymbol{N}_z(\boldsymbol{\Phi}) \in \boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies $\|\boldsymbol{N}_z(\boldsymbol{\Phi})\|_{H^1(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$. Moreover, $\boldsymbol{N}_z(\boldsymbol{\Phi}) \in A(U; \boldsymbol{H}^1(\mathcal{O}))$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$.*

Proof. For convenience we denote $\boldsymbol{u}_z = \boldsymbol{N}_z(\boldsymbol{\Phi})$. By Lemma 2.3 and Lemma 2.2, we know that $|\tilde{\Gamma}_z(\boldsymbol{x},\boldsymbol{y})| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-1}$ uniformly for $\boldsymbol{x},\boldsymbol{y}$ in bounded set of $\mathbb{R}^3$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Since $\boldsymbol{\Phi}$ has compact support, by well-known estimates for Riesz potentials (e.g. [20, Lemma 7.12]), we know that for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$, $\|\boldsymbol{u}_z\|_{L^2(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$. Similarly, by Lemma 2.3 and Lemma 2.2, we have $|\frac{\partial}{\partial x_j}\tilde{\Gamma}_z(\boldsymbol{x},\boldsymbol{y})| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-2}$ uniformly for $\boldsymbol{x},\boldsymbol{y}$ in bounded set of $\mathbb{R}^3$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Thus again by [20, Lemma 7.12], we have $\|\frac{\partial}{\partial x_j}\boldsymbol{u}_z\|_{L^2(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$. This shows $\boldsymbol{u}_z \in \boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\|\boldsymbol{u}_z\|_{H^1(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$.

Next, it is easy to see that $|\frac{\partial}{\partial z}d(\tilde{\boldsymbol{x}}_z,\tilde{\boldsymbol{y}}_z)| \le C|\boldsymbol{x}-\boldsymbol{y}|$ uniformly for $\boldsymbol{x},\boldsymbol{y}$ in bounded set. Thus, by Lemma 2.3 and Lemma 2.2, we can obtain that $|\frac{\partial}{\partial z}\tilde{\Gamma}^z_{jk}(\boldsymbol{x},\boldsymbol{y})| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-1}$, $|\frac{\partial}{\partial z}(\nabla_{\boldsymbol{x}}\Gamma_{jk}(\tilde{\boldsymbol{x}}_z,\tilde{\boldsymbol{y}}_z))| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-2}$ uniformly for $\boldsymbol{x},\boldsymbol{y}$ in bounded set and $\boldsymbol{x} \neq \boldsymbol{y}$. Consequently, $|\frac{\partial}{\partial z}(J_z(\boldsymbol{y})\tilde{\Gamma}^z_{jk}(\boldsymbol{x},\boldsymbol{y}))| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-1}$ and $|\frac{\partial}{\partial z}(J_z(\boldsymbol{y})\nabla_{\boldsymbol{x}}\tilde{\Gamma}^z_{jk}(\boldsymbol{x},\boldsymbol{y}))| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-2}$ uniformly for $\boldsymbol{x},\boldsymbol{y}$ in bounded set and $\boldsymbol{x} \neq \boldsymbol{y}$. This implies that for almost all $\boldsymbol{x}$ in the bounded open set $\mathcal{O}$, $\frac{\partial}{\partial z}(J_z(\cdot)\tilde{\Gamma}_z(\boldsymbol{x},\cdot))\boldsymbol{\Phi}(\cdot) \in \boldsymbol{H}^1(\mathcal{O})$. By Lebesgue dominated convergence theorem, we conclude that $\boldsymbol{u}_z \in A(U; \boldsymbol{H}^1(\mathcal{O}))$. □

The following lemma indicates that $J(\boldsymbol{y})\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})$ is the fundamental solution matrix of the PML equation.

Lemma 3.5. *For any $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$ with compact support, the Newton potential*

$$\boldsymbol{N}(\boldsymbol{\Phi})(\boldsymbol{x}) := \int_{R^3} J(\boldsymbol{y})\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\Phi}(\boldsymbol{y})d\boldsymbol{y} \tag{3.3}$$

satisfies $\boldsymbol{N}(\boldsymbol{\Phi}) \in \boldsymbol{H}^1(\mathbb{R}^3)$ and the PML equation in the weak sense

$$\mathcal{A}(\boldsymbol{N}(\boldsymbol{\Phi}),\boldsymbol{\psi}) - \gamma^2(J\boldsymbol{N}(\boldsymbol{\Phi}),\boldsymbol{\psi}) = (J\boldsymbol{\Phi},\boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1(\mathbb{R}^3), \tag{3.4}$$

where $(\cdot,\cdot)$ is the inner product on $\boldsymbol{L}^2(\mathbb{R}^3)$ or the duality pairing between $\boldsymbol{H}^1(\mathbb{R}^3)$ and $\boldsymbol{H}^1(\mathbb{R}^3)'$. Moreover, $\|\boldsymbol{N}(\boldsymbol{\Phi})\|_{H^1(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$.

Proof. Let $z_0 = \zeta + \mathbf{i}$, then $\boldsymbol{N}(\boldsymbol{\Phi}) = \boldsymbol{N}_{z_0}(\boldsymbol{\Phi})$. We again denote $\boldsymbol{u}_z = \boldsymbol{N}_z(\boldsymbol{\Phi})$ for $z \in U = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > |\mathrm{Im}\,(z)|\}$ which is defined in Lemma 2.2. By Lemma 3.4 we know $\boldsymbol{u}_{z_0} \in \boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\|\boldsymbol{u}_{z_0}\|_{H^1(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$. By the definition (3.3) and Lemma 2.4 we know that $\boldsymbol{u}_{z_0}$ decays exponentially and hence $\boldsymbol{u}_{z_0} \in \boldsymbol{H}^1(\mathbb{R}^3)$.

It remains to show $\boldsymbol{u}_{z_0}$ satisfies (3.4). Here we use the argument in [25] for the Helmholtz equation. For that purpose, for any $\boldsymbol{\psi} \in \boldsymbol{C}^\infty_0(\mathbb{R}^3)$, consider

$$I(z) := \int_{\mathbb{R}^3} \left(\tilde{\tau}_z(\boldsymbol{u}_z)A_z : \boldsymbol{\nabla}\bar{\boldsymbol{\psi}} - \gamma^2 J_z\boldsymbol{u}_z \cdot \bar{\boldsymbol{\psi}} - J_z\boldsymbol{\Phi} \cdot \bar{\boldsymbol{\psi}}\right) d\boldsymbol{x}, \quad \forall z \in U,$$

where $\tilde{\tau}_z(\boldsymbol{u}_z) = 2\mu\tilde{\varepsilon}_z(\boldsymbol{u}_z) + \lambda\mathrm{tr}(\tilde{\varepsilon}_z(\boldsymbol{u}_z))\mathrm{I}$, $\tilde{\varepsilon}_z(\boldsymbol{u}_z) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}_z B_z^T + B_z^T(\boldsymbol{\nabla}\boldsymbol{u}_z)^T)$, $B_z = (\boldsymbol{\nabla}\boldsymbol{F}_z)^{-T}$, and $A_z = J_z B_z$. By Lemma 3.4, $I(z)$ is analytic in U. On the other hand, for $z \in \mathbb{R}_+\backslash\{0\} \subset U$, it is easy to see that $\boldsymbol{F}_z$ is C^2 smooth, injective, and maps $\mathbb{R}^3$ onto $\mathbb{R}^3$. Thus by using the formula of change of variable, we know that

$$\begin{aligned} I(z) &= \int_{\mathbb{R}^3} \left(\tilde{\tau}_z(\boldsymbol{u}_z) : \boldsymbol{\nabla}\bar{\boldsymbol{\psi}} B_z - \gamma^2 \boldsymbol{u}_z \cdot \bar{\boldsymbol{\psi}} - \boldsymbol{\Phi} \cdot \bar{\boldsymbol{\psi}} \right) J_z d\boldsymbol{x} \\ &= \int_{\mathbb{R}^3} \left(\tau(\boldsymbol{v}_z) : \boldsymbol{\nabla}\bar{\boldsymbol{\psi}}_z - \gamma^2 \boldsymbol{v}_z \cdot \bar{\boldsymbol{\psi}}_z - (\boldsymbol{\Phi} \circ \boldsymbol{F}_z^{-1}) \cdot \bar{\boldsymbol{\psi}}_z \right) d\boldsymbol{x}, \end{aligned}$$

where $\boldsymbol{\psi}_z := \boldsymbol{\psi} \circ \boldsymbol{F}_z^{-1}$, and

$$\boldsymbol{v}_z(\boldsymbol{x}) := (\boldsymbol{u}_z \circ \boldsymbol{F}_z^{-1})(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Gamma(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{\Phi} \circ \boldsymbol{F}_z^{-1})(\boldsymbol{y}) d\boldsymbol{y}.$$

It is clear that $\nabla \cdot \tau(\boldsymbol{v}_z) + \gamma^2 \boldsymbol{v}_z = -\boldsymbol{\Phi} \circ \boldsymbol{F}_z^{-1}$ in $\mathbb{R}^3$ for $z \in \mathbb{R}_+\backslash\{0\}$. Since $\boldsymbol{\psi}_z$ has compact support for $z \in \mathbb{R}_+\backslash\{0\}$, we obtain $I(z) = 0$ for $z \in \mathbb{R}_+\backslash\{0\}$ by integration by parts. Thus the analyticity of $I(z)$ yields that $I(z) = 0$ in U which implies that $\boldsymbol{u}_{z_0}$ satisfies the PML equation (3.4) in the weak sense. This completes the proof. □

We remark that in the lemma we have in fact proved that for any $z \in U = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > |\mathrm{Im}\,(z)|\}$ defined in Lemma 2.2, $\boldsymbol{u}_z = \boldsymbol{N}_z(\boldsymbol{\Phi})$, where $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$ has compact support, satisfies

$$\int_{\mathbb{R}^3} \left(\tilde{\tau}_z(\boldsymbol{u}_z) A_z : \boldsymbol{\nabla}\bar{\boldsymbol{\psi}} - \gamma^2 J_z \boldsymbol{u}_z \cdot \bar{\boldsymbol{\psi}} \right) d\boldsymbol{x} = \int_{\mathbb{R}^3} J_z \boldsymbol{\Phi} \cdot \bar{\boldsymbol{\psi}} d\boldsymbol{x}, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1(R^3). \tag{3.5}$$

Then by Lemma 3.3 and Lemma 3.4 we deduce that $\|\boldsymbol{u}_z\|_{H^1(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{H^1(R^3)'}$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$. Therefore, by the density argument we know that $\boldsymbol{N}_z(\boldsymbol{\Phi}) \in \boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ is well-defined for any $\boldsymbol{\Phi} \in \boldsymbol{H}^1(\mathbb{R}^3)'$ with compact support and satisfies $\|\boldsymbol{N}_z(\boldsymbol{\Phi})\|_{H^1(\mathcal{O})} \le C\|\boldsymbol{\Phi}\|_{H^1(\mathbb{R}^3)'}$ for any bounded open set $\mathcal{O} \subset \mathbb{R}^3$.

The following lemma shows that the Newton potential in (3.3) can also be defined for $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$.

Lemma 3.6. *For any $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$, we have $\boldsymbol{N}(\boldsymbol{\Phi}) \in \boldsymbol{H}^1(\mathbb{R}^3)$ which satisfies $\|\boldsymbol{N}(\boldsymbol{\Phi})\|_{H^1(\mathbb{R}^3)} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$ and*

$$\mathcal{A}(\boldsymbol{N}(\boldsymbol{\Phi}), \boldsymbol{\psi}) - \gamma^2 (J\boldsymbol{N}(\boldsymbol{\Phi}), \boldsymbol{\psi}) = (J\boldsymbol{\Phi}, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1(\mathbb{R}^3). \tag{3.6}$$

Proof. By Lemma 2.4 and Lemma 2.6 we know that for any $\boldsymbol{x} \in \mathbb{R}^3$, $j,k = 1,2,3$,

$$\int_{\mathbb{R}^3} |\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})| d\boldsymbol{y} \le C \int_{|\boldsymbol{x}-\boldsymbol{y}|<\beta_1} |\boldsymbol{x}-\boldsymbol{y}|^{-1} d\boldsymbol{y} + C \int_{|\boldsymbol{x}-\boldsymbol{y}|>\beta_1} e^{-k_p \mathrm{Im}\, d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})} d\boldsymbol{y} \le C,$$

where $\beta_1 = 2\sqrt{3}\beta\bar{l}_{\max}$. Now for any $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$ with compact support and $\boldsymbol{\psi} \in \boldsymbol{L}^2(\mathbb{R}^3)$, by Cauchy-Schwarz inequality,

$$\begin{aligned} & |(\boldsymbol{N}(\boldsymbol{\Phi}), \boldsymbol{\psi})| \\ =\ & \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} J(\boldsymbol{y}) \tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{\Phi}(\boldsymbol{y}) \overline{\boldsymbol{\psi}(\boldsymbol{x})} d\boldsymbol{x} d\boldsymbol{y} \right| \\ \le\ & C \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})| |\boldsymbol{\Phi}(\boldsymbol{y})|^2 d\boldsymbol{x} d\boldsymbol{y} \right)^{1/2} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})| |\boldsymbol{\psi}(\boldsymbol{x})|^2 d\boldsymbol{x} d\boldsymbol{y} \right)^{1/2} \\ \le\ & C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)} \|\boldsymbol{\psi}\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

This implies $\|\boldsymbol{N}(\boldsymbol{\Phi})\|_{L^2(\mathbb{R}^3)} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$. Similarly, one has $\|\boldsymbol{\nabla N}(\boldsymbol{\Phi})\|_{L^2(\mathbb{R}^3)} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$. Thus $\|\boldsymbol{N}(\boldsymbol{\Phi})\|_{H^1(\mathbb{R}^3)} \le C\|\boldsymbol{\Phi}\|_{L^2(\mathbb{R}^3)}$. This implies by the density that $\boldsymbol{N}(\boldsymbol{\Phi}) \in \boldsymbol{H}^1(\mathbb{R}^3)$ for any $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\mathbb{R}^3)$. The equality (3.6) follows now from (3.4) again by the density argument. This completes the proof. □

The following theorem is the main result of this section.

Theorem 3.7. *Let (H1) be satisfied. There exists a constant $C>0$ that*

$$\sup_{\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3)}\frac{|\mathcal{A}(\boldsymbol{\phi},\boldsymbol{\psi})-\gamma^2(J\boldsymbol{\phi},\boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_{H^1(\mathbb{R}^3)}}\ge C\|\boldsymbol{\phi}\|_{H^1(\mathbb{R}^3)},\qquad \forall\boldsymbol{\phi}\in\boldsymbol{H}^1(\mathbb{R}^3).$$

Proof. We follow the argument in [10, Theorem 5.2]. We only need to show that for any $\boldsymbol{F}_1\in\boldsymbol{H}^1(\mathbb{R}^3)'$, there exists a unique solution $\boldsymbol{w}\in\boldsymbol{H}^1(\mathbb{R}^3)$ that satisfies

$$\mathcal{A}(\boldsymbol{w},\boldsymbol{\psi})-\gamma^2(J\boldsymbol{w},\boldsymbol{\psi})=\boldsymbol{F}_1(\psi),\quad\forall\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3),\tag{3.7}$$

and the estimate $\|\boldsymbol{w}\|_{H^1(\mathbb{R}^3)}\le C\|\boldsymbol{F}_1\|_{\boldsymbol{H}^1(\mathbb{R}^3)'}$. We first show the existence and the estimate. By Lemma 3.3 and Lax-Milgram lemma we know that there is a unique $\boldsymbol{v}\in\boldsymbol{H}^1(\mathbb{R}^3)$ such that

$$\mathcal{A}(\boldsymbol{v},\boldsymbol{\psi})+(\boldsymbol{v},\boldsymbol{\psi})=\boldsymbol{F}_1(\boldsymbol{\psi}),\qquad\forall\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3).$$

Moreover, $\|\boldsymbol{v}\|_{H^1(\mathbb{R}^3)}\le C\|\boldsymbol{F}_1\|_{H^1(\mathbb{R}^3)'}$. For $\boldsymbol{F}_2:=\gamma^2\boldsymbol{v}+J^{-1}\boldsymbol{v}\in\boldsymbol{L}^2(\mathbb{R}^3)$, we introduce the Newton potential $\boldsymbol{v}_1=\boldsymbol{N}(\boldsymbol{F}_2)$ which satisfies $\|\boldsymbol{v}_1\|_{H^1(\mathbb{R}^3)}\le C\|\boldsymbol{F}_2\|_{L^2(\mathbb{R}^3)}\le C\|\boldsymbol{v}\|_{L^2(\mathbb{R}^3)}\le C\|\boldsymbol{F}_1\|_{H^1(\mathbb{R}^3)'}$, and by Lemma 3.6

$$\mathcal{A}(\boldsymbol{v}_1,\boldsymbol{\psi})-\gamma^2(J\boldsymbol{v}_1,\boldsymbol{\psi})=(\gamma^2J\boldsymbol{v}+\boldsymbol{v},\boldsymbol{\psi}),\quad\forall\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3),$$

which implies $\boldsymbol{w}=\boldsymbol{v}_1+\boldsymbol{v}\in\boldsymbol{H}^1(\mathbb{R}^3)$ satisfies (3.7) and the estimate $\|\boldsymbol{w}\|_{H^1(\mathbb{R}^3)}\le C\|\boldsymbol{F}_1\|_{H^1(\mathbb{R}^3)'}$.

It remains to show the uniqueness. Let $\boldsymbol{w}$ satisfy (3.7) with $\boldsymbol{F}_1=0$. For any $\boldsymbol{\Phi}\in\boldsymbol{L}^2(\mathbb{R}^3)$, let $\boldsymbol{N}(\boldsymbol{\Phi})\in\boldsymbol{H}^1(\mathbb{R}^3)$ be the Newton potential in Lemma 3.6. Then since $\mathcal{A}(\boldsymbol{N}(\boldsymbol{\Phi}),\bar{\boldsymbol{w}})=\mathcal{A}(\boldsymbol{w},\overline{\boldsymbol{N}(\boldsymbol{\Phi})})$, by (3.6) and (3.7) with $\boldsymbol{F}_1=0$, we have,

$$(J\boldsymbol{\Phi},\bar{\boldsymbol{w}})=\mathcal{A}(\boldsymbol{N}(\boldsymbol{\Phi}),\bar{\boldsymbol{w}})-\gamma^2(J\boldsymbol{\Phi},\bar{\boldsymbol{w}})=\mathcal{A}(\boldsymbol{w},\overline{\boldsymbol{N}(\boldsymbol{\Phi})})-\gamma^2(J\boldsymbol{w},\overline{\boldsymbol{N}(\boldsymbol{\Phi})})=0,$$

for any $\boldsymbol{\Phi}\in\boldsymbol{L}^2(\mathbb{R}^3)$. This shows $\boldsymbol{w}=0$ and completes the proof. □

We finally show that the Newton potential $\boldsymbol{N}(\boldsymbol{\Phi})$ can also be defined for $\boldsymbol{\Phi}\in\boldsymbol{H}^1(\mathbb{R}^3)'$.

Lemma 3.8. *Let (H1) be satisfied. The Newton potential $\boldsymbol{N}:\boldsymbol{L}^2(\mathbb{R}^3)\to\boldsymbol{H}^1(\mathbb{R}^3)$ defined in* (3.3) *extends as a continuous linear operator from $\boldsymbol{H}^1(\mathbb{R}^3)'$ to $\boldsymbol{H}^1(\mathbb{R}^3)$ and satisfies $\|\boldsymbol{N}(\boldsymbol{\Phi})\|_{H^1(\mathbb{R}^3)}\le C\|\boldsymbol{\Phi}\|_{H^1(\mathbb{R}^3)'}$. Moreover,*

$$\mathcal{A}(\boldsymbol{N}(\boldsymbol{\Phi}),\boldsymbol{\psi})-\gamma^2(J\boldsymbol{N}(\boldsymbol{\Phi}),\boldsymbol{\psi})=(J\boldsymbol{\Phi},\boldsymbol{\psi}),\qquad\forall\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3).\tag{3.8}$$

Proof. For $\boldsymbol{\Phi}\in\boldsymbol{L}^2(\mathbb{R}^3)$, Theorem 3.7 and Lemma 3.6 imply that $\|\boldsymbol{N}(\boldsymbol{\Phi})\|_{H^1(\mathbb{R}^3)}\le C\|\boldsymbol{\Phi}\|_{H^1(\mathbb{R}^3)'}$. The lemma follows then from the density of $\boldsymbol{L}^2(\mathbb{R}^3)$ in $\boldsymbol{H}^1(\mathbb{R}^3)'$. □

4. The PML equation in the truncated domain

We first introduce some notation. For any bounded domain $\mathcal{D}\subset\mathbb{R}^3$ with boundary Γ, we use the weighted H^1-norm $\|\varphi\|_{H^1(\mathcal{D})}=\left(d_{\mathcal{D}}^{-2}\|\varphi\|^2_{L^2(\mathcal{D})}+\|\nabla\varphi\|^2_{L^2(\mathcal{D})}\right)^{1/2}$ and the weighted $H^{1/2}$-norm $\|v\|_{H^{1/2}(\Gamma)}=\left(d_{\mathcal{D}}^{-1}\|v\|^2_{L^2(\Gamma)}+|v|^2_{\frac12,\Gamma}\right)^{1/2}$, where $d_{\mathcal{D}}$ is the diameter of $\mathcal{D}$, and

$$|v|^2_{\frac12,\Gamma}=\int_\Gamma\int_\Gamma\frac{|v(\boldsymbol{x})-v(\boldsymbol{x}')|^2}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,ds(\boldsymbol{x})\,ds(\boldsymbol{x}').$$

It is obvious that for any $v \in W^{1,\infty}(\Gamma)$,

$$\|v\|_{H^{1/2}(\Gamma)} \le (|\Gamma|\, d_{\mathcal{D}}^{-1})^{1/2}\|v\|_{L^\infty(\Gamma)} + (|\Gamma| d_{\mathcal{D}})^{1/2}\|\nabla v\|_{L^\infty(\Gamma)}. \tag{4.1}$$

By the scaling argument and the trace theorem we know that there exist constants C_1, C_2 independent of $d_{\mathcal{D}}$ such that

$$C_1\frac{|\mathcal{D}|}{d_{\mathcal{D}}^3}\|v\|_{H^{1/2}(\Gamma)} \le \inf_{\substack{\varphi|_\Gamma = v\\ \varphi\in H^1(\mathcal{D})}} \|\varphi\|_{H^1(\mathcal{D})} \le C_2\,\frac{d_{\mathcal{D}}^2}{|\Gamma|}\,\|v\|_{H^{1/2}(\Gamma)}. \tag{4.2}$$

We denote $\mathcal{A}_{\mathcal{D}} : \boldsymbol{H}^1(\mathcal{D}) \times \boldsymbol{H}^1(\mathcal{D}) \to \mathbb{C}$ the sesquilinear form:

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\phi},\boldsymbol{\psi}) = \int_{\mathcal{D}} \tilde{\tau}(\boldsymbol{\phi})A : \boldsymbol{\nabla}\bar{\boldsymbol{\psi}} d\boldsymbol{x}, \qquad \forall \boldsymbol{\phi},\boldsymbol{\psi} \in \boldsymbol{H}^1(\mathcal{D}).$$

Since $\tilde{\varepsilon}(\boldsymbol{\phi})$ is a symmetric matrix, we have, for any $\boldsymbol{\phi},\boldsymbol{\psi} \in \boldsymbol{H}^1(\mathcal{D})$,

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\phi},\boldsymbol{\psi}) = \int_{\mathcal{D}} \left(\mu J\tilde{\varepsilon}(\boldsymbol{\phi}) : \tilde{\varepsilon}(\bar{\boldsymbol{\psi}}) + \lambda J \widetilde{\mathrm{div}}\,\boldsymbol{\phi}\cdot \widetilde{\mathrm{div}}\,\bar{\boldsymbol{\psi}}\right) d\boldsymbol{x}, \tag{4.3}$$

where $\widetilde{\mathrm{div}}\,\boldsymbol{v} = \sum_{i=1}^3 \frac{1}{\alpha_i}\frac{\partial v_i}{\partial x_i}$ is the divergence operator with respect to the stretched coordinates.

Let $\boldsymbol{V}(B_L) = \{\boldsymbol{v} \in \boldsymbol{H}^1(B_L) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_L\}$. The purpose of this section is to show the following theorem which plays a key role in our subsequent analysis.

Theorem 4.1. *Let (H1)-(H2) be satisfied and $\sigma_0 d$ be sufficiently large. Then there exists a constant $C > 0$ such that*

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}(B_L)} \frac{|\mathcal{A}_{B_L}(\boldsymbol{\phi},\boldsymbol{v}) - \gamma^2(J\boldsymbol{\phi},\boldsymbol{v})_{B_L}|}{\|\boldsymbol{v}\|_{H^1(B_L)}} \ge C\|\boldsymbol{\phi}\|_{H^1(B_L)}, \quad \forall \boldsymbol{\phi} \in \boldsymbol{V}(B_L). \tag{4.4}$$

Proof. The argument extends the reflection argument in [10, 9] for the Helmholtz and Maxwell equations. For $\boldsymbol{\phi} \in \boldsymbol{V}(B_L)$, we define a functional $\boldsymbol{F}_1 \in \boldsymbol{V}(B_L)'$ by

$$\boldsymbol{F}_1(\boldsymbol{v}) := \mathcal{A}_{B_L}(\boldsymbol{\phi},\boldsymbol{v}) - \gamma^2(J\boldsymbol{\phi},\boldsymbol{v})_{B_L}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}(B_L).$$

Then the inf-sup condition (4.4) is equivalent to show $\|\boldsymbol{\phi}\|_{H^1(B_L)} \le C\|\boldsymbol{F}_1\|_{V(B_L)'}$.

We introduce an extension of $\boldsymbol{\phi}$ to the domain $B_L^{R_1} = (-2L_1 + \bar{l}_1, 2L_1 - \bar{l}_1) \times (-L_2, L_2) \times (-L_3, L_3)$ as follows, where $\bar{l}_j, j = 1, 2, 3$, are defined in (2.8). For any $\boldsymbol{x} \in B_L^{R_1}$, we denote

$$\boldsymbol{x}^{R_1} = \begin{cases} (2L_1 - x_1, x_2, x_3)^T & \text{if } |x_1 - L_1| \le L_1 - \bar{l}_1;\\ (-2L_1 - x_1, x_2, x_3)^T & \text{if } |x_1 + L_1| \le L_1 - \bar{l}_1.\end{cases}$$

$\boldsymbol{x}^{R_1}$ is the image point of $\boldsymbol{x}$ with respect to $x_1 = L_1$ or $x_1 = -L_1$. For $\boldsymbol{x} \in B_L^{R_1}\backslash\bar{B}_L$, let

$$\phi_1^{R_1}(\boldsymbol{x}) = -\phi_1(\boldsymbol{x}^{R_1}), \quad \phi_2^{R_1}(\boldsymbol{x}) = \phi_2(\boldsymbol{x}^{R_1}), \quad \phi_3^{R_1}(\boldsymbol{x}) = \phi_3(\boldsymbol{x}^{R_1}).$$

$\phi_1^{R_1}$ is the extension of ϕ_1 in B_L to $B_L^{R_1}$ by odd reflection with respect to $x_1 = \pm L_1$. For $j = 2, 3$, $\phi_j^{R_1}$ is the extension of ϕ_j in B_L to $B_L^{R_1}$ by even reflection with respect to $x_1 = \pm L_1$. Obviously $\boldsymbol{\phi}^{R_1} = (\phi_1^{R_1}, \phi_2^{R_1}, \phi_3^{R_1})^T \in \boldsymbol{H}^1(B_L^{R_1})$ since $\boldsymbol{\phi}\cdot\boldsymbol{n} = 0$ on Γ_L. For any $\boldsymbol{v} \in \boldsymbol{H}_0^1(B_L^{R_1})$, we define $\boldsymbol{F}^{R_1} \in \boldsymbol{H}^{-1}(B_L^{R_1})$ by

$$\boldsymbol{F}^{R_1}(\boldsymbol{v}) := \int_{B_L^{R_1}} \left(\mu J\tilde{\varepsilon}(\boldsymbol{\phi}^{R_1}) : \tilde{\varepsilon}(\bar{\boldsymbol{v}}) + \lambda J\widetilde{\mathrm{div}}\,\boldsymbol{\phi}^{R_1}\cdot\widetilde{\mathrm{div}}\,\bar{\boldsymbol{v}} - \gamma^2 J\boldsymbol{\phi}^{R_1}\cdot\bar{\boldsymbol{v}}\right) d\boldsymbol{x}.$$

Since $\sigma_1(x_1) = \sigma_0$ for $|x_1| \geq \bar{l}_1$, we have, for $\boldsymbol{x} \in B_L^{R_1}$,

$$\begin{aligned}
&\tilde{\varepsilon}_{jj}(\boldsymbol{\phi}^{R_1})(\boldsymbol{x}) = \tilde{\varepsilon}_{jj}(\boldsymbol{\phi})(\boldsymbol{x}^{R_1}), \quad j = 1, 2, 3,\\
&\tilde{\varepsilon}_{12}(\boldsymbol{\phi}^{R_1})(\boldsymbol{x}) = -\tilde{\varepsilon}_{12}(\boldsymbol{\phi})(\boldsymbol{x}^{R_1}), \quad \tilde{\varepsilon}_{13}(\boldsymbol{\phi}^{R_1})(\boldsymbol{x}) = -\tilde{\varepsilon}_{13}(\boldsymbol{\phi})(\boldsymbol{x}^{R_1}),\\
&\tilde{\varepsilon}_{23}(\boldsymbol{\phi}^{R_1})(\boldsymbol{x}) = \tilde{\varepsilon}_{23}(\boldsymbol{\phi})(\boldsymbol{x}^{R_1}),
\end{aligned}$$

which imply by the change of variables that

$$\boldsymbol{F}^{R_1}(\boldsymbol{v}) = \int_{B_L} \left(\mu J \tilde{\varepsilon}(\boldsymbol{\phi}) : \tilde{\varepsilon}(\bar{\tilde{\boldsymbol{v}}}) + \lambda J \widetilde{\operatorname{div}}\, \boldsymbol{\phi} \cdot \widetilde{\operatorname{div}}\, \bar{\tilde{\boldsymbol{v}}} - \gamma^2 J \boldsymbol{\phi} \cdot \bar{\tilde{\boldsymbol{v}}} \right) d\boldsymbol{x},$$

where $\tilde{\boldsymbol{v}} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T$ is defined in B_L as

$$\tilde{v}_1(\boldsymbol{x}) = \begin{cases} v_1(\boldsymbol{x}) - v_1(\boldsymbol{x}^{R_1}) & \text{if } x_1 \in (\bar{l}_1, L_1);\\ v_1(\boldsymbol{x}) & \text{if } x_1 \in (-\bar{l}_1, \bar{l}_1);\\ v_1(\boldsymbol{x}) - v_1(\boldsymbol{x}^{R_1}) & \text{if } x_1 \in (-L_1, -\bar{l}_1), \end{cases}$$

and for $j = 2, 3$,

$$\tilde{v}_j(\boldsymbol{x}) = \begin{cases} v_j(\boldsymbol{x}) + v_j(\boldsymbol{x}^{R_1}) & \text{if } x_1 \in (\bar{l}_1, L_1);\\ v_j(\boldsymbol{x}) & \text{if } x_1 \in (-\bar{l}_1, \bar{l}_1);\\ v_j(\boldsymbol{x}) + v_j(\boldsymbol{x}^{R_1}) & \text{if } x_1 \in (-L_1, -\bar{l}_1). \end{cases}$$

Since $\boldsymbol{v} \in \boldsymbol{H}_0^1(B_L^{R_1})$, we know that $\tilde{\boldsymbol{v}} \in \boldsymbol{V}(B_L)$ and $\|\tilde{\boldsymbol{v}}\|_{H^1(B_L)} \leq C\|\boldsymbol{v}\|_{H^1(B_L^{R_1})}$. Thus

$$(4.5)\quad \|\boldsymbol{F}^{R_1}\|_{H^{-1}(B_L^{R_1})} \leq C \sup_{\tilde{\boldsymbol{v}} \in \boldsymbol{V}(B_L)} \frac{|\mathcal{A}_{B_L}(\boldsymbol{\phi}, \tilde{\boldsymbol{v}}) - \gamma^2 (J\boldsymbol{\phi}, \tilde{\boldsymbol{v}})_{B_L}|}{\|\tilde{\boldsymbol{v}}\|_{H^1(B_L)}} = C\|\boldsymbol{F}_1\|_{\boldsymbol{V}(B_L)'}.$$

Now we extend $\phi_2^{R_1}$ by odd reflection, $\phi_1^{R_1}, \phi_3^{R_1}$ by even reflection with respect to $x_2 = \pm L_2$ to obtain a function $\boldsymbol{\phi}^{R_1R_2}$ defined in $B_L^{R_1R_2} = (-2L_1 + \bar{l}_1, 2L_1 - \bar{l}_1) \times (-2L_2 + \bar{l}_2, 2L_2 - \bar{l}_2) \times (-L_3, L_3)$. We further extend $\phi_3^{R_1R_2}$ by odd reflection, $\phi_1^{R_1R_2}, \phi_2^{R_1R_2}$ by even reflection with respect to $x_3 = \pm L_3$ to obtain a function $\boldsymbol{\phi}^R$ defined in $B_L^R = (-2L_1 + \bar{l}_1, 2L_1 - \bar{l}_1) \times (-2L_2 + \bar{l}_2, 2L_2 - \bar{l}_2) \times (-2L_3 + \bar{l}_3, 2L_3 - \bar{l}_3)$. For any $\boldsymbol{v} \in \boldsymbol{H}_0^1(B_L^R)$, we then define a functional $\boldsymbol{F}^R \in \boldsymbol{H}^{-1}(B_L^R)$ by

$$(4.6)\quad \boldsymbol{F}^R(\boldsymbol{v}) := \int_{B_L^R} \left(\mu J \tilde{\varepsilon}(\boldsymbol{\phi}^R) : \tilde{\varepsilon}(\bar{\boldsymbol{v}}) + \lambda J \widetilde{\operatorname{div}}\, \boldsymbol{\phi}^R \cdot \widetilde{\operatorname{div}}\, \bar{\boldsymbol{v}} - \gamma^2 J \boldsymbol{\phi}^R \cdot \bar{\boldsymbol{v}} \right) d\boldsymbol{x}.$$

By a similar argument leading to (4.5) one can prove $\|\boldsymbol{F}^R\|_{H^{-1}(B_L^R)} \leq C\|\boldsymbol{F}_1\|_{V(B_L)'}$.

Now we extend $\boldsymbol{F}^R \in \boldsymbol{H}^{-1}(B_L^R)$ to a bounded linear functional $\boldsymbol{F}_2 \in \boldsymbol{H}^1(\mathbb{R}^3)'$ by Hahn-Banach theorem such that $\|\boldsymbol{F}_2\|_{H^1(\mathbb{R}^3)'} = \|\boldsymbol{F}^R\|_{H^{-1}(B_L^R)}$. For $\boldsymbol{F}_2 \in H^1(\mathbb{R}^3)'$ we use Lemma 3.7 to conclude that there exists a $\boldsymbol{w} \in \boldsymbol{H}^1(\mathbb{R}^3)$ such that

$$\mathcal{A}(\boldsymbol{w}, \boldsymbol{v}) - \gamma^2 (J\boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{F}_2(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\mathbb{R}^3),$$

and $\|\boldsymbol{w}\|_{H^1(\mathbb{R}^3)} \leq C\|\boldsymbol{F}_2\|_{H^1(\mathbb{R}^3)'} \leq C\|\boldsymbol{F}_1\|_{V(B_L)'}$. This yields, by using (4.6), for $\boldsymbol{w}_1 = \boldsymbol{w} - \boldsymbol{\phi}^R \in \boldsymbol{H}^1(B_L^R)$,

$$(4.7)\qquad \mathcal{A}(\boldsymbol{w}_1, \boldsymbol{v}) - \gamma^2 (J\boldsymbol{w}_1, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(B_L^R).$$

Since $J(\boldsymbol{y})\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})$ is the fundamental solution matrix of the PML equation, by the integral representation formula we have for $\boldsymbol{x}\in B_L$,

$$
\begin{aligned}
J(\boldsymbol{x})\boldsymbol{w}_1(\boldsymbol{x})\cdot\boldsymbol{e}_n &= \int_{\partial B_L^R}\tilde{\tau}(\boldsymbol{w}_1(\boldsymbol{y}))A\boldsymbol{n}\cdot J\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_n d\boldsymbol{s}(\boldsymbol{y}) \\
&- \int_{\partial B_L^R}\tilde{\tau}(J\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_n)A\boldsymbol{n}\cdot\boldsymbol{w}_1(\boldsymbol{y})d\boldsymbol{s}(\boldsymbol{y}).
\end{aligned}
\tag{4.8}
$$

Denote $d_j^R=(L_j-\bar{l}_j)$, $j=1,2,3$. Then $d^R:=\min(d_1^R,d_2^R,d_3^R)$ is the distance between B_L and ∂B_L^R. Clearly $d_j^R\geq d_j/2$ by (H2), $j=1,2,3$. Denote by $B_{L+d^R/2}:=\{x\in\mathbb{R}^3:|x_j|<L_j+d_j^R/2, j=1,2,3\}$. Since $\sigma_j(t)=\sigma_0$ for $L_j\leq|t|\leq L_j+d_j^R$, we have $\left|\int_{x_j}^{y_j}\sigma_j(t)dt\right|\geq\sigma_0 d_j^R/2$ for $\boldsymbol{x}\in B_L$, $|y_j|\geq L_j+d_j^R/2$. By (2.16) we have then for any $\boldsymbol{x}\in B_L,\boldsymbol{y}\in B_L^R\backslash\bar{B}_{L+d^R/2}$,

$$
\operatorname{Im} d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\geq\frac{d^R/2}{\sqrt{\sum_{j=1}^3(2L_j+d_j^R)^2}}\,\sigma_0 d^R/2:=\gamma_1\sigma_0 d^R.
\tag{4.9}
$$

Let $\chi\in C_0^\infty(\mathbb{R}^3)$ be the cut-off function such that $\chi=0$ in $B_{L+d^R/2}$, $\chi=1$ near ∂B_L^R, and $|\nabla\chi|\leq C(d^R)^{-1}\leq Cd^{-1}$. Then by integrating by parts and using $\nabla\cdot(\tilde{\tau}(\boldsymbol{w}_1)A)-\gamma^2 J\boldsymbol{w}_1=0$ in B_L^R which is a consequence of (4.7), we obtain, for any $\boldsymbol{x}\in B_L$,

$$
\begin{aligned}
&\left|\int_{\partial B_L^R}\tilde{\tau}(\boldsymbol{w}_1(\boldsymbol{y}))A\boldsymbol{n}\cdot J\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_n d\boldsymbol{s}(\boldsymbol{y})\right| \\
=\;&\left|\int_{B_L^R}\Big(\gamma^2 J\boldsymbol{w}_1\cdot(J\chi\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_n)-\tilde{\tau}(\boldsymbol{w}_1)A:D(J\chi\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_n)\Big)\,d\boldsymbol{y}\right| \\
\leq\;& Cd^{3/2}\|\boldsymbol{w}_1\|_{H^1(B_L^R)}\max_{j,k=1,2,3}\max_{\boldsymbol{y}\in B_L^R\backslash\bar{B}_{L+d^R/2}}\Big(d|\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})|+|\nabla_{\boldsymbol{y}}\tilde{\Gamma}_{jk}(\boldsymbol{x},\boldsymbol{y})|\Big) \\
\leq\;& Cd^{3/2}e^{-k_p\gamma_1\sigma_0 d^R}\|\boldsymbol{w}_1\|_{H^1(B_L^R)},
\end{aligned}
$$

where we have used (4.9) and Lemma 2.4. A similar argument for the second term in (4.8) implies that $\|\boldsymbol{w}_1\|_{L^\infty(B_L)}\leq Cd^{3/2}e^{-k_p\gamma_1\sigma_0 d^R}\|\boldsymbol{w}_1\|_{H^1(B_L^R)}$. One can obtain a similar bound for $\boldsymbol{\nabla}\boldsymbol{w}_1$ to get $\|\boldsymbol{\nabla}\boldsymbol{w}_1\|_{L^\infty(B_L)}\leq Cd^{3/2}e^{-k_p\gamma_1\sigma_0 d^R}\|\boldsymbol{w}_1\|_{H^1(B_L^R)}$. Thus

$$
\begin{aligned}
\|\boldsymbol{w}_1\|_{H^1(B_L)} &\leq Cd^{3/2}(d^{-1}\|\boldsymbol{w}_1\|_{L^\infty(B_L)}+\|\boldsymbol{\nabla}\boldsymbol{w}_1\|_{L^\infty(B_L)}) \\
&\leq Cd^3e^{-k_p\gamma_1\sigma_0 d^R}\|\boldsymbol{w}_1\|_{H^1(B_L^R)} \\
&\leq Cd^3e^{-k_p\gamma_1\sigma_0 d^R}(\|\boldsymbol{w}\|_{H^1(B_L^R)}+\|\boldsymbol{\phi}^R\|_{H^1(B_L^R)}) \\
&\leq Cd^3e^{-k_p\gamma_1\sigma_0 d^R}(\|\boldsymbol{w}\|_{H^1(\mathbb{R}^3)}+\|\boldsymbol{\phi}\|_{H^1(B_L)}).
\end{aligned}
\tag{4.10}
$$

Therefore,

$$
\begin{aligned}
\|\boldsymbol{\phi}\|_{H^1(B_L)} &\leq \|\boldsymbol{w}\|_{H^1(B_L)}+\|\boldsymbol{w}_1\|_{H^1(B_L)} \\
&\leq \|\boldsymbol{w}\|_{H^1(B_L)}+Cd^3e^{-k_p\gamma_1\sigma_0 d^R}(\|\boldsymbol{w}\|_{H^1(\mathbb{R}^3)}+\|\boldsymbol{\phi}\|_{H^1(B_L)}).
\end{aligned}
$$

This shows $\|\boldsymbol{\phi}\|_{H^1(B_L)}\leq C\|\boldsymbol{w}\|_{H^1(\mathbb{R}^3)}\leq C\|\boldsymbol{F}_1\|_{V(B_L)'}$ if $\sigma_0 d$ and thus $\sigma_0 d^R\geq\sigma_0 d/2$ is sufficiently large. This completes the proof. □

5. The PML equation in the layer

In this section we consider the following problem of the PML equation in the layer $\Omega_{\rm PML} := B_L\backslash\bar{B}_l$

$$\nabla\cdot(\tilde{\tau}(\boldsymbol{w})A)+\gamma^2J\boldsymbol{w}=0 \quad \text{in } \Omega_{\rm PML}, \tag{5.1}$$

$$\boldsymbol{w}=0 \quad \text{on } \Gamma_l, \tag{5.2}$$

$$\boldsymbol{w}\cdot\boldsymbol{n}=\boldsymbol{f}_1\cdot\boldsymbol{n}, \quad \tilde{\tau}(\boldsymbol{w})A\boldsymbol{n}\times\boldsymbol{n}=\boldsymbol{g}_1\times\boldsymbol{n} \quad \text{on } \Gamma_L, \tag{5.3}$$

where $\boldsymbol{f}_1\in\boldsymbol{H}^{1/2}(\Gamma_L),\boldsymbol{g}_1\in\boldsymbol{H}^{-1/2}(\Gamma_L)$.

Lemma 5.1. *Let (H1) be satisfied. Given $\boldsymbol{f}\in\boldsymbol{H}^{1/2}(\Gamma_l)$, let $\tilde{\boldsymbol{\xi}}=\mathbb{E}(\boldsymbol{f})$ be the PML extension of $\boldsymbol{f}$ defined in* (2.17)*. Then $\tilde{\boldsymbol{\xi}}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{B}_l)$ and*

$$\langle\mathbb{T}\boldsymbol{f},\boldsymbol{\psi}\rangle_{\Gamma_l}=-\int_{\mathbb{R}^3\backslash\bar{B}_l}\left(\tilde{\tau}(\tilde{\boldsymbol{\xi}})A:\boldsymbol{\nabla}\bar{\boldsymbol{\psi}}-\gamma^2J\tilde{\boldsymbol{\xi}}\cdot\bar{\boldsymbol{\psi}}\right)d\boldsymbol{x}, \quad \forall\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{B}_l). \tag{5.4}$$

Proof. For $z\in U=\{z\in\mathbb{C}:{\rm Re}\,(z)>|{\rm Im}\,(z)|\}$ defined in Lemma 2.2, we first prove the modified single and double layer potentials $\tilde{\boldsymbol{\Psi}}^z_{\rm SL}(\boldsymbol{\lambda})\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$, $\tilde{\boldsymbol{\Psi}}^z_{\rm DL}(\boldsymbol{f})\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$ for any $\boldsymbol{\lambda}\in\boldsymbol{H}^{-1/2}(\Gamma_l),\boldsymbol{f}\in\boldsymbol{H}^{1/2}(\Gamma_l)$ by using an argument in [27, Theorem 6.11]. Since the trace operator $\gamma_0:\boldsymbol{H}^1(B_l)\to\boldsymbol{H}^{1/2}(\Gamma_l)$ is surjective and continuous, its conjugate operator $\gamma_0':\boldsymbol{H}^{-1/2}(\Gamma_l)\to\boldsymbol{H}^1(B_l)'$ is a continuous linear operator. Thus the modified single layer potential operator can be decomposed as $\tilde{\boldsymbol{\Psi}}^z_{\rm SL}=\boldsymbol{N}_z\circ\gamma_0'$ which implies by the remark after Lemma 3.5 that $\tilde{\boldsymbol{\Psi}}^z_{\rm SL}(\boldsymbol{\lambda})\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$ and satisfies $\|\tilde{\boldsymbol{\Psi}}^z_{\rm SL}(\boldsymbol{\lambda})\|_{H^1(\mathcal{O}\backslash\bar{B}_l)}\le C\|\boldsymbol{\lambda}\|_{H^{-1/2}(\Gamma_l)}$ for any bounded open set $\mathcal{O}$ in $\mathbb{R}^3$. For $\boldsymbol{f}\in\boldsymbol{H}^{1/2}(\Gamma_l)$, we denote $\boldsymbol{v}\in\boldsymbol{H}^1(B_l)$ the weak solution of the Dirichlet problem $\nabla\cdot\tau(\boldsymbol{v})=0$ in B_l, $\boldsymbol{v}=\boldsymbol{f}$ on Γ_l. Thus $\tau(\boldsymbol{v})\boldsymbol{n}_l\in\boldsymbol{H}^{-1/2}(\Gamma_l)$. It is easy to see by integration by parts that $\tilde{\boldsymbol{\Psi}}^z_{\rm DL}(\boldsymbol{f})=-\gamma^2\boldsymbol{N}(\boldsymbol{v})+\tilde{\boldsymbol{\Psi}}^z_{\rm SL}(\tau(\boldsymbol{v})\boldsymbol{n}_l)$. This shows by Lemma 3.4 that $\tilde{\boldsymbol{\Psi}}^z_{\rm DL}(\boldsymbol{f})\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$ and satisfies $\|\tilde{\boldsymbol{\Psi}}^z_{\rm DL}(\boldsymbol{f})\|_{H^1(\mathcal{O}\backslash\bar{B}_l)}\le C\|\boldsymbol{v}\|_{L^2(B_l)}+C\|\tau(\boldsymbol{v})\boldsymbol{n}_l\|_{H^{-1/2}(\Gamma_l)}\le C\|\boldsymbol{f}\|_{H^{1/2}(\Gamma_l)}$ for any bounded open set $\mathcal{O}$ in $\mathbb{R}^3$. Therefore, $\tilde{\boldsymbol{\xi}}_z:=-\tilde{\boldsymbol{\Psi}}^z_{\rm SL}(\mathbb{T}\boldsymbol{f})+\tilde{\boldsymbol{\Psi}}^z_{\rm DL}(\boldsymbol{f})\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$. Since $\tilde{\boldsymbol{\xi}}=\tilde{\boldsymbol{\xi}}_{z_0}$, $z_0=\zeta+\mathbf{i}$, we know that $\tilde{\boldsymbol{\xi}}\in\boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{B}_l)$. This implies $\tilde{\boldsymbol{\xi}}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{B}_l)$ since $\tilde{\boldsymbol{\xi}}$ decays exponentially as $|\boldsymbol{x}|\to\infty$.

Now we prove (5.4). It follows from (3.5) that $\tilde{\boldsymbol{\xi}}_z$ satisfies

$$\int_{\mathbb{R}^3\backslash\bar{B}_l}\left(\tilde{\tau}_z(\tilde{\boldsymbol{\xi}}_z)A_z:\boldsymbol{\nabla}\bar{\boldsymbol{\psi}}-\gamma^2J_z\tilde{\boldsymbol{\xi}}_z\cdot\bar{\boldsymbol{\psi}}\right)d\boldsymbol{x}=0, \quad \forall\boldsymbol{\psi}\in\boldsymbol{C}_0^\infty(\mathbb{R}^3\backslash\bar{B}_l).$$

Thus by the definition of weak derivative, $\nabla\cdot(\tilde{\tau}_z(\tilde{\boldsymbol{\xi}}_z)A_z)=-\gamma^2J_z\tilde{\boldsymbol{\xi}}_z\in\boldsymbol{L}^2(\mathbb{R}^3\backslash\bar{B}_l)$, which implies $\tilde{\tau}_z(\tilde{\boldsymbol{\xi}}_z)A_z\boldsymbol{n}_l\in\boldsymbol{H}^{-1/2}(\Gamma_l)$ and for any $\boldsymbol{\psi}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{B}_l)$,

$$\langle\tilde{\tau}_z(\tilde{\boldsymbol{\xi}}_z)A_z\boldsymbol{n}_l,\boldsymbol{\psi}\rangle_{\Gamma_l}=-\int_{\mathbb{R}^3\backslash\bar{B}_l}\left(\tilde{\tau}_z(\tilde{\boldsymbol{\xi}}_z)A_z:\boldsymbol{\nabla}\bar{\boldsymbol{\psi}}-\gamma^2J_z\tilde{\boldsymbol{\xi}}_z\cdot\bar{\boldsymbol{\psi}}\right)d\boldsymbol{x}. \tag{5.5}$$

Here we remark that $\boldsymbol{n}_l$ is the unit outer normal to Γ_l which is opposite to the unit outer normal to $\partial(\mathbb{R}^3\backslash\bar{B}_l)$.

The following argument is the same as that in Lemma 3.5. For any $z\in\mathbb{R}_+\backslash\{0\}\subset U$, $\boldsymbol{F}_z$ is C^2 smooth, injective, and maps $\mathbb{R}^3\backslash\bar{B}_l$ onto $\mathbb{R}^3\backslash\bar{B}_l$. Thus by using the formula of change of variable and integration by parts, we know that for any $\boldsymbol{\psi}\in$

$\boldsymbol{C}_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} I_1(z): &= \int_{\mathbb{R}^3\setminus\bar{B}_l}\left(\tilde{\tau}_z(\tilde{\boldsymbol{\xi}}_z)A_z:\boldsymbol{\nabla}\bar{\boldsymbol{\psi}}-\gamma^2 J_z\boldsymbol{u}_z\cdot\bar{\boldsymbol{\psi}}\right)d\boldsymbol{x}\\ &= \int_{\mathbb{R}^3\setminus\bar{B}_l}\left(\tau(\boldsymbol{v}_z):\boldsymbol{\nabla}\bar{\boldsymbol{\psi}}_z-\gamma^2\boldsymbol{v}_z\cdot\bar{\boldsymbol{\psi}}_z\right)d\boldsymbol{x}, \end{aligned}$$

where $\boldsymbol{\psi}_z=\boldsymbol{\psi}\circ\boldsymbol{F}_z^{-1}$ has compact support and

$$\begin{aligned} \boldsymbol{v}_z(\boldsymbol{x}):=(\tilde{\boldsymbol{\xi}}_z\circ\boldsymbol{F}_z^{-1})(\boldsymbol{x}) &= -\langle\mathbb{T}\boldsymbol{f}\circ\boldsymbol{F}_z^{-1},\overline{\Gamma(\boldsymbol{x},\cdot)\boldsymbol{e}_n}\rangle_{\Gamma_l}+\langle\mathbb{T}[\Gamma(\boldsymbol{x},\cdot)\boldsymbol{e}_n],\overline{\boldsymbol{f}\circ\boldsymbol{F}_z^{-1}}\rangle_{\Gamma_l}\\ &= -\langle\mathbb{T}\boldsymbol{f},\overline{\Gamma(\boldsymbol{x},\cdot)\boldsymbol{e}_n}\rangle_{\Gamma_l}+\langle\mathbb{T}[\Gamma(\boldsymbol{x},\cdot)\boldsymbol{e}_n],\bar{\boldsymbol{f}}\rangle_{\Gamma_l}\\ &= \boldsymbol{\xi}(\boldsymbol{x}),\quad\forall\boldsymbol{x}\in\mathbb{R}^3\setminus\bar{B}_l, \end{aligned}$$

where we have used $\boldsymbol{F}_z(\boldsymbol{x})=\boldsymbol{x}$ on Γ_l, the Betti formula (2.11) with $\boldsymbol{\xi}$ being the solution of (2.1)-(2.3). Thus by integration by parts we obtain $I_1(z)=-\langle\mathbb{T}f,\boldsymbol{\psi}_z\rangle_{\Gamma_l}=-\langle\mathbb{T}f,\boldsymbol{\psi}\rangle_{\Gamma_l}$ for $z\in\mathbb{R}_+\setminus\{0\}$. By Lemma 3.4, $I_1(z)$ is analytic in U which yields that $I_1(z)=-\langle\mathbb{T}f,\boldsymbol{\psi}\rangle_{\Gamma_l}$ for any $z\in U$. This completes the proof by (5.5) and noticing that $\tilde{\boldsymbol{\xi}}=\tilde{\boldsymbol{\xi}}_{z_0}$ and $\tilde{\tau}(\tilde{\boldsymbol{\xi}})A\boldsymbol{n}_l=\tilde{\tau}_{z_0}(\tilde{\boldsymbol{\xi}}_{z_0})A_{z_0}\boldsymbol{n}_l$ on Γ_l. □

Let $\boldsymbol{X}(\Omega_{\rm PML})=\{v\in\boldsymbol{H}^1(\Omega_{\rm PML}):\boldsymbol{v}=0\text{ on }\Gamma_l,\boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\Gamma_L\}$. The following theorem is the main result of this section.

Theorem 5.2. *Let (H1)-(H2) be satisfied and $\sigma_0 d$ be sufficiently large. Then there exists a constant $C>0$ such that*

$$\sup_{\boldsymbol{v}\in\boldsymbol{X}(\Omega_{\rm PML})}\frac{|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},\boldsymbol{v})-\gamma^2(J\boldsymbol{\phi},\boldsymbol{v})_{\Omega_{\rm PML}}|}{\|\boldsymbol{v}\|_{H^1(\Omega_{\rm PML})}}\geq C\|\boldsymbol{\phi}\|_{H^1(\Omega_{\rm PML})},\quad\forall\boldsymbol{\phi}\in\boldsymbol{X}(\Omega_{\rm PML}).$$

Moreover, the PML problem in the layer (5.1)-(5.3) *has a unique weak solution $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega_{\rm PML})$ which satisfies $\|\boldsymbol{w}\|_{H^1(\Omega_{\rm PML})}\leq C(\|\boldsymbol{f}_1\|_{H^{1/2}(\Gamma_L)}+\|\boldsymbol{g}_1\|_{H^{-1/2}(\Gamma_L)})$.*

Proof. We extend any $\boldsymbol{\phi}\in\boldsymbol{X}(\Omega_{\rm PML})$ to be zero in B_l and thus obtain a function (still denoted as $\boldsymbol{\phi}$) in $\boldsymbol{V}(B_L)$. By using Theorem 4.1

$$\|\boldsymbol{\phi}\|_{H^1(\Omega_{\rm PML})}=\|\boldsymbol{\phi}\|_{H^1(B_L)}\leq C\sup_{\boldsymbol{v}\in\boldsymbol{V}(B_L)}\frac{|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},\boldsymbol{v})-\gamma^2(J\boldsymbol{\phi},\boldsymbol{v})_{\Omega_{\rm PML}}|}{\|\boldsymbol{v}\|_{H^1(B_L)}}.$$

Here we notice that since $\boldsymbol{\phi}$ vanishes in B_l, the integration in the sesquilinear form $\mathcal{A}_{B_L}(\boldsymbol{\phi},\boldsymbol{v})-\gamma^2(J\boldsymbol{\phi},\boldsymbol{v})_{B_L}$ is restricted to $\Omega_{\rm PML}$. Now for any $\boldsymbol{v}\in\boldsymbol{V}(B_L)$, we define $\boldsymbol{w}=\overline{\mathbb{E}(\bar{\boldsymbol{v}}|_{\Gamma_l})}\in\boldsymbol{H}^1(\mathbb{R}^3\setminus\bar{B}_l)$. It is easy to see that it satisfies

$$\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},\boldsymbol{w})-\gamma^2(J\boldsymbol{\phi},\boldsymbol{w})_{\Omega_{\rm PML}}=\langle\tilde{\tau}(\bar{\boldsymbol{w}})A\boldsymbol{n},\bar{\boldsymbol{\phi}}\rangle_{\Gamma_L}.$$

Let $\chi\in C^\infty(\mathbb{R}^3)$ be the cut-off function such that $\chi=1$ in $B_{l+d/2}=\{\boldsymbol{x}\in\mathbb{R}^3:|x_i|<l_i+d_i/2,i=1,2,3\}$, $\chi=0$ on Γ_L, and $|\nabla\chi|\leq Cd^{-1}$ in $\Omega_{\rm PML}$. Then $\boldsymbol{v}_1=\boldsymbol{v}-\chi\boldsymbol{w}\in\boldsymbol{X}(\Omega_{\rm PML})$ and

$$\begin{aligned} &|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},\boldsymbol{v})-\gamma^2(J\boldsymbol{\phi},\boldsymbol{v})_{\Omega_{\rm PML}}|\\ \leq\ &|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},\boldsymbol{v}_1)-\gamma^2(J\boldsymbol{\phi},\boldsymbol{v}_1)_{\Omega_{\rm PML}}|\\ +\ &|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},(1-\chi)\boldsymbol{w})-\gamma^2(J\boldsymbol{\phi},(1-\chi)\boldsymbol{w})_{\Omega_{\rm PML}}|+|\langle\tilde{\tau}(\bar{\boldsymbol{w}})A\boldsymbol{n},\bar{\boldsymbol{\phi}}\rangle_{\Gamma_L}|\\ \leq\ &|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi},\boldsymbol{v}_1)-\gamma^2(J\boldsymbol{\phi},\boldsymbol{v}_1)_{\Omega_{\rm PML}}|+Cd^2\|\boldsymbol{\phi}\|_{H^1(\Omega_{\rm PML})}\|\boldsymbol{w}\|_{H^1(B_L\setminus\bar{B}_{l+d/2})}. \end{aligned}$$

Since $\sigma(x_i) = \sigma_0$ for $|x_i| \geq l_i + d_i/2 \geq \bar{l}_i$, where we have used (H2), we know by (2.16) that for any $\boldsymbol{x} \in B_L \backslash \bar{B}_{l+d/2}, \boldsymbol{y} \in \Gamma_l$,

$$\operatorname{Im} d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \geq \frac{d/2}{\sqrt{\sum_{i=1}^{3}(2l_i + d_i)^2}} \sigma_0 d/2 := \gamma_2 \sigma_0 d.$$

By the the definition of the PML extension in (2.17) and Lemma 2.4, we have

$$\begin{aligned} & \|\boldsymbol{w}\|_{H^1(B_L \backslash \bar{B}_{l+d/2})} \\ \leq\ & C d^{3/2} (d^{-1} \|\boldsymbol{w}\|_{L^\infty(B_L \backslash \bar{B}_{l+d/2})} + \|\boldsymbol{\nabla w}\|_{L^\infty(B_L \backslash \bar{B}_{l+d/2})}) \\ \leq\ & C d^{3/2} \max_{\substack{\boldsymbol{x} \in B_L \backslash \bar{B}_{l+d/2} \\ j,k=1,2,3}} (\|\tilde{\Gamma}_{jk}(\boldsymbol{x}, \cdot)\|_{W^{1,\infty}(\Gamma_l)} + \|\nabla_{\boldsymbol{x}} \tilde{\Gamma}_{jk}(\boldsymbol{x}, \cdot)\|_{W^{1,\infty}(\Gamma_l)}) \|\boldsymbol{v}\|_{H^{1/2}(\Gamma_l)} \\ \leq\ & C d^{1/2} e^{-k_p \gamma_2 \sigma_0 d} \|\boldsymbol{v}\|_{H^{1/2}(\Gamma_l)}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\boldsymbol{\phi}\|_{H^1(\Omega_{\rm PML})} \ &\leq\ C \sup_{\boldsymbol{v}_1 \in \boldsymbol{X}(\Omega_{\rm PML})} \frac{|\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\phi}, \boldsymbol{v}_1) - \gamma^2 (J\boldsymbol{\phi}, \boldsymbol{v}_1)_{\Omega_{\rm PML}}|}{\|\boldsymbol{v}_1\|_{H^1(\Omega_{\rm PML})}} \\ &\quad +\ C d^{5/2} e^{-k_p \gamma_2 \sigma_0 d} \|\boldsymbol{\phi}\|_{H^1(\Omega_{\rm PML})}. \end{aligned}$$

This shows the desired inf-sup condition if $\sigma_0 d$ is sufficiently large. □

6. Convergence of the PML problem

We start by introducing the approximate Dirichlet-to-Neumann operator $\hat{\mathbb{T}} : \boldsymbol{H}^{1/2}(\Gamma_l) \to \boldsymbol{H}^{-1/2}(\Gamma_l)$ associated with the PML problem. Given $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\Gamma_l)$, let $\boldsymbol{\zeta} \in \boldsymbol{H}^1(\Omega_{\rm PML})$ such that $\boldsymbol{\zeta} = \boldsymbol{f}$ on Γ_l, $\boldsymbol{\zeta} \cdot \boldsymbol{n} = 0$ on Γ_L, and

$$\mathcal{A}_{\Omega_{\rm PML}}(\boldsymbol{\zeta}, \boldsymbol{\psi}) - \gamma^2 (J\boldsymbol{\zeta}, \boldsymbol{\psi})_{\Omega_{\rm PML}} = 0, \quad \forall \boldsymbol{\psi} \in \boldsymbol{X}(\Omega_{\rm PML}). \tag{6.1}$$

By Theorem 5.2, $\hat{\mathbb{T}}$ is well-defined for sufficiently large $\sigma_0 d$. We define $\hat{\mathbb{T}}\boldsymbol{f} \in \boldsymbol{H}^{-1/2}(\Gamma_l)$ through the relation

$$\langle \hat{\mathbb{T}}\boldsymbol{f}, \boldsymbol{\psi} \rangle_{\Gamma_l} = -\int_{\Omega_{\rm PML}} \left(\tilde{\tau}(\boldsymbol{\zeta}) A : \boldsymbol{\nabla} \bar{\boldsymbol{\psi}} - \gamma^2 J \boldsymbol{\zeta} \cdot \bar{\boldsymbol{\psi}} \right) d\boldsymbol{x}, \tag{6.2}$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega_{\rm PML})$ such that $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ_L. By (6.1) we know that the right-hand side of (6.2) depends only on $\boldsymbol{\psi}|_{\Gamma_l}$. Moreover, $\hat{\mathbb{T}}\boldsymbol{f} = \tilde{\tau}(\boldsymbol{\zeta}) A \boldsymbol{n}_l$ in $\boldsymbol{H}^{-1/2}(\Gamma_l)$.

To proceed we notice by (2.16) that for $\boldsymbol{x} \in \Gamma_L, \boldsymbol{y} \in \Gamma_l$,

$$\operatorname{Im} d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \geq \frac{\min_{i=1,2,3}(L_i - \bar{l}_i)}{\sqrt{\sum_{i=1}^{3}(2l_i + d_i)^2}} \bar{\sigma} := \gamma_0 \bar{\sigma}, \quad \bar{\sigma} = \min_{i=1,2,3} \int_0^{L_i} \sigma_i(t) dt. \tag{6.3}$$

Lemma 6.1. *Let (H1)-(H2) be satisfied. For any $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\Gamma_l)$, let $\mathbb{E}(\boldsymbol{f})$ be the PML extension defined in (2.17). Then we have*

$$\|\,\mathbb{E}(\boldsymbol{f})\,\|_{H^{1/2}(\Gamma_L)} + \|\tilde{\tau}(\mathbb{E}(\boldsymbol{f})) A \boldsymbol{n}\|_{H^{-1/2}(\Gamma_L)} \leq C\, d^{1/2} e^{-k_p \gamma_0 \bar{\sigma}}\, \|\,\boldsymbol{f}\,\|_{H^{1/2}(\Gamma_l)}\,.$$

Proof. Since

$$\mathbb{E}(\boldsymbol{f})(\boldsymbol{x}) \cdot \boldsymbol{e}_n = -\left\langle \mathbb{T}\boldsymbol{f}, \overline{\tilde{\Gamma}(\boldsymbol{x}, \cdot)\boldsymbol{e}_n} \right\rangle_{\Gamma_l} + \left\langle \mathbb{T}[\tilde{\Gamma}(\boldsymbol{x}, \cdot)\boldsymbol{e}_n], \bar{\boldsymbol{f}} \right\rangle_{\Gamma_l}.$$

By (4.1) we have

$$\|\mathbb{E}(\boldsymbol{f})\|_{H^{1/2}(\Gamma_L)} \leq C L^{1/2} \|\mathbb{E}(\boldsymbol{f})\|_{L^\infty(\Gamma_L)} + C L^{3/2} \|\boldsymbol{\nabla} \mathbb{E}(\boldsymbol{f})\|_{L^\infty(\Gamma_L)}, \tag{6.4}$$

where L is the diameter of B_L. Clearly $L \le Cd$. For $\boldsymbol{x} \in \Gamma_L$, we have again by (4.1)

$$\begin{aligned}|\mathbb{E}(\boldsymbol{f})(\boldsymbol{x})| &\le \max_{n=1,2,3} \|\mathbb{T}\boldsymbol{f}\|_{H^{-1/2}(\Gamma_l)} \|\tilde{\Gamma}(\boldsymbol{x},\cdot)\boldsymbol{e}_n\|_{H^{1/2}(\Gamma_l)} \\ &+ \max_{n=1,2,3} \|\mathbb{T}[\tilde{\Gamma}(\boldsymbol{x},\cdot)\boldsymbol{e}_n]\|_{H^{-1/2}(\Gamma_l)} \|\boldsymbol{f}\|_{H^{1/2}(\Gamma_l)} \\ &\le C \max_{j,k=1,2,3} \|\tilde{\Gamma}_{jk}(\boldsymbol{x},\cdot)\|_{W^{1,\infty}(\Gamma_l)} \|\boldsymbol{f}\|_{H^{1/2}(\Gamma_l)}.\end{aligned}$$

Now by Lemma 2.4 and (6.3) we obtain

$$\|\mathbb{E}(\boldsymbol{f})\|_{L^\infty(\Gamma_L)} \le Cd^{-1}e^{-k_p\gamma_0\bar{\sigma}}\|\boldsymbol{f}\|_{H^{1/2}(\Gamma_l)}.$$

Similarly, one can prove $\|\boldsymbol{\nabla}\mathbb{E}(\boldsymbol{f})\|_{L^\infty(\Gamma_L)} \le Cd^{-1}e^{-k_p\gamma_0\bar{\sigma}}\|\boldsymbol{f}\|_{H^{1/2}(\Gamma_l)}$. This shows the estimate for $\|\mathbb{E}(\boldsymbol{f})\|_{H^{1/2}(\Gamma_L)}$ by (6.4).

For the estimate of $\|\tilde{\tau}(\mathbb{E}(\boldsymbol{f}))A\boldsymbol{n}\|_{H^{-1/2}(\Gamma_L)}$, we notice that by the definition of $\boldsymbol{H}^{-1/2}(\Gamma_L)$ norm that

$$\begin{aligned}&\|\tilde{\tau}(\mathbb{E}(\boldsymbol{f}))A\boldsymbol{n}\|_{H^{-1/2}(\Gamma_L)} \\ \le\ & CL^{3/2}\|\tilde{\tau}(\mathbb{E}(\boldsymbol{f}))A\boldsymbol{n}\|_{L^\infty(\Gamma_L)} \\ \le\ & CL^{3/2}\max_{1\le n\le 3}\left(\|\nabla_{\boldsymbol{x}}\langle \mathbb{T}\boldsymbol{f}, \overline{\tilde{\Gamma}(\boldsymbol{x},\cdot)\boldsymbol{e}_n}\rangle_{\Gamma_l}\|_{L^\infty(\Gamma_L)} + \|\nabla_{\boldsymbol{x}}\langle \mathbb{T}[\tilde{\Gamma}(\boldsymbol{x},\cdot)\boldsymbol{e}_n], \bar{\boldsymbol{f}}\rangle_{\Gamma_l}\|_{L^\infty(\Gamma_L)}\right).\end{aligned}$$

The proof can now be completed using a similar argument for the estimate of $\|\mathbb{E}(\boldsymbol{f})\|_{H^{1/2}(\Gamma_L)}$ as above. □

Lemma 6.2. *Let (H1)-(H2) be satisfied and $\sigma_0 d$ is sufficiently large. Then we have*

$$\|\mathbb{T}\boldsymbol{f} - \hat{\mathbb{T}}\boldsymbol{f}\|_{\boldsymbol{H}^{-1/2}(\Gamma_l)} \le Cd^{5/2}e^{-k_p\gamma_0\bar{\sigma}}\parallel \boldsymbol{f}\parallel_{H^{1/2}(\Gamma_l)}, \quad \forall \boldsymbol{f}\in \boldsymbol{H}^{1/2}(\Gamma_l).$$

Proof. For any $\boldsymbol{\psi} \in \boldsymbol{H}^{1/2}(\Gamma_l)$, we extend it to be a function $\tilde{\boldsymbol{\psi}} \in \boldsymbol{H}^1(\Omega_{\rm PML})$ which satisfies $\tilde{\boldsymbol{\psi}}\cdot\boldsymbol{n} = 0$ on Γ_L and $\|\tilde{\boldsymbol{\psi}}\|_{H^1(\Omega_{\rm PML})} \le C\|\boldsymbol{\psi}\|_{H^{1/2}(\Gamma_l)}$. By (6.2) and Lemma 5.1 we know that for $\tilde{\boldsymbol{\xi}} = \mathbb{E}(\boldsymbol{f})$,

$$\begin{aligned}&|\langle \mathbb{T}\boldsymbol{f} - \hat{\mathbb{T}}\boldsymbol{f}, \boldsymbol{\psi}\rangle_{\Gamma_l}| \\ =\ & \left|\int_{\Omega_{\rm PML}} \left(\tilde{\tau}(\tilde{\boldsymbol{\xi}}-\boldsymbol{\zeta})A : \boldsymbol{\nabla}\bar{\tilde{\boldsymbol{\psi}}} - \gamma^2 J(\tilde{\boldsymbol{\xi}}-\boldsymbol{\zeta})\cdot\bar{\tilde{\boldsymbol{\psi}}}\right)d\boldsymbol{x}\right| + |\langle\tilde{\tau}(\tilde{\boldsymbol{\xi}})A\boldsymbol{n}, \tilde{\boldsymbol{\psi}}\rangle_{\Gamma_L}| \\ \le\ & Cd^2\|\tilde{\boldsymbol{\xi}}-\boldsymbol{\zeta}\|_{H^1(\Omega_{\rm PML})}\ \|\tilde{\boldsymbol{\psi}}\|_{H^{1/2}(\Gamma_l)} + C\|\tilde{\tau}(\tilde{\boldsymbol{\xi}})A\boldsymbol{n}\|_{H^{-1/2}(\Gamma_L)}\|\tilde{\boldsymbol{\psi}}\|_{H^{1/2}(\Gamma_L)}.\end{aligned}$$

Since $\tilde{\boldsymbol{\xi}}-\boldsymbol{\zeta}$ satisfies the PML problem (5.1)-(5.2) with $\boldsymbol{f}_1 = \mathbb{E}(\boldsymbol{f})$, $\boldsymbol{g}_1 = \tilde{\tau}(\mathbb{E}(\boldsymbol{f}))A\boldsymbol{n}$, by Theorem 5.2 and Lemma 6.1, we have

$$\begin{aligned}\|\tilde{\boldsymbol{\xi}}-\boldsymbol{\zeta}\|_{H^1(\Omega_{\rm PML})} &\le C(\|\mathbb{E}(\boldsymbol{f})\|_{H^{1/2}(\Gamma_l)} + \|\tilde{\tau}(\mathbb{E}(\boldsymbol{f}))A\boldsymbol{n}\|_{H^{-1/2}(\Gamma_L)}) \\ &\le Cd^{1/2}e^{-k_p\gamma_0\bar{\sigma}}\|\boldsymbol{f}\|_{H^{1/2}(\Gamma_l)}.\end{aligned}$$

This completes the proof. □

Let $\mathfrak{b} : \boldsymbol{H}^1(\Omega_L)\times\boldsymbol{H}^1(\Omega_L) \to \mathbb{C}$ be the sesquilinear form given by

$$\mathfrak{b}(\boldsymbol{\phi},\boldsymbol{\psi}) = \int_{\Omega_L}\left(\tilde{\tau}(\boldsymbol{\phi})A : \boldsymbol{\nabla}\bar{\boldsymbol{\psi}} - \gamma^2 J\boldsymbol{\phi}\cdot\bar{\boldsymbol{\psi}}\right)d\boldsymbol{x}. \tag{6.5}$$

Denote by $\boldsymbol{V}(\Omega_L) = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega_L) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_L\}$. Then the weak formulation of (2.20)-(2.21) is: Given $\boldsymbol{q}\in\boldsymbol{H}^1(\Omega_l)'$, $\boldsymbol{g}\in\boldsymbol{H}^{-1/2}(\Gamma_D)$, find $\hat{\boldsymbol{u}}\in\boldsymbol{V}(\Omega_L)$ such that

$$\mathfrak{b}(\hat{\boldsymbol{u}},\boldsymbol{\psi}) = (\boldsymbol{q},\boldsymbol{\psi})_{\Omega_l} + \langle\boldsymbol{g},\boldsymbol{\psi}\rangle_{\Gamma_D}, \quad \forall\,\boldsymbol{\psi}\in\boldsymbol{V}(\Omega_L). \tag{6.6}$$

Theorem 6.3. *Let (H1)-(H2) be satisfied and $\sigma_0 d$ is sufficiently large. Then the PML problem* (6.6) *has a unique solution $\hat{\boldsymbol{u}} \in \boldsymbol{V}(\Omega_L)$. Moreover, we have the following error estimate*

$$\| \boldsymbol{u} - \hat{\boldsymbol{u}} \|_{H^1(\Omega_L)} \le C d^{5/2} e^{-k_p \gamma_0 \bar{\sigma}} \| \hat{\boldsymbol{u}} \|_{H^{1/2}(\Gamma_l)}, \tag{6.7}$$

where $\boldsymbol{u}$ is the solution of (2.5).

Proof. We first show that any solution $\hat{\boldsymbol{u}}$ of the PML problem (6.6) satisfies the estimate (6.7). By (6.2) we have

$$\mathfrak{a}(\hat{\boldsymbol{u}}, \boldsymbol{\psi}) + \langle \mathbb{T}\hat{\boldsymbol{u}} - \hat{\mathbb{T}}\hat{\boldsymbol{u}}, \boldsymbol{\psi} \rangle_{\Gamma_l} = (\boldsymbol{q}, \boldsymbol{\psi})_{\Omega_l} + \langle \boldsymbol{g}, \boldsymbol{\psi} \rangle_{\Gamma_D}, \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega_l).$$

Subtracting with (2.5) we get

$$\mathfrak{a}(\boldsymbol{u} - \hat{\boldsymbol{u}}, \boldsymbol{\psi}) = \langle \mathbb{T}\hat{\boldsymbol{u}} - \hat{\mathbb{T}}\hat{\boldsymbol{u}}, \boldsymbol{\psi} \rangle_{\Gamma_l}, \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega_l).$$

Now (6.7) follows from the inf-sup condition (2.7) and Lemma 6.2.

By the Fredholm alternative theorem we know that the uniqueness of the solution of the PML problem (6.6) implies the existence of the solution. To show the uniqueness, we let $\boldsymbol{q} = 0, \boldsymbol{g} = 0$ in (6.6). By the uniqueness of the scattering problem we know that the corresponding scattering solution $\boldsymbol{u} = 0$ in Ω_l. Thus (6.7) implies

$$\| \hat{\boldsymbol{u}} \|_{H^1(\Omega_l)} \le C d^{5/2} e^{-k_p \gamma_0 \bar{\sigma}} \| \hat{\boldsymbol{u}} \|_{H^{1/2}(\Gamma_l)} \le C d^{5/2} e^{-k_p \gamma_0 \bar{\sigma}} \| \hat{\boldsymbol{u}} \|_{H^1(\Omega_l)}.$$

Thus for sufficiently large $\sigma_0 d$ we conclude that $\hat{\boldsymbol{u}} = 0$ on Ω_l. That $\hat{\boldsymbol{u}}$ also vanishes in $\Omega_{\rm PML}$ is a direct consequence of Theorem 5.2. Thus $\hat{\boldsymbol{u}} = 0$ in Ω_L. This completes the proof. ☐

7. Numerical results

In this section we present a 2D example to illustrate the performance of the proposed PML method with respect to the change of the PML parameters. The computations are all carried out in MATLAB on ThinkStation D30 with Intel(R) Xeon(R) CPU 2.4GHz and 128GB memory.

We first introduce the finite element approximation of the PML problem (2.20)-(2.21). We assume $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega_l)$, $\boldsymbol{g} \in \boldsymbol{L}^2(\Gamma_D)$. Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_L. We assume the elements $K \in \mathcal{M}_h$ may have one curved side align with Γ_D so that $\Omega_L = \cup_{K \in \mathcal{M}_h} K$. Let $\boldsymbol{V}_h \subset \boldsymbol{H}^1(\Omega_L)$ be the conforming quadratic finite element space over Ω_L, and $\mathring{\boldsymbol{V}}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : \boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_L\}$. The finite element approximation to the PML problem (2.20)-(2.21) reads as follows: Find $\boldsymbol{u}_h \in \mathring{\boldsymbol{V}}_h$ such that

$$\mathfrak{b}(\boldsymbol{u}_h, \boldsymbol{\psi}_h) = (\boldsymbol{q}, \boldsymbol{\psi}_h)_{\Omega_l} + \langle \boldsymbol{g}, \boldsymbol{\psi}_h \rangle_{\Gamma_D}, \quad \forall\, \boldsymbol{\psi}_h \in \mathring{\boldsymbol{V}}_h. \tag{7.1}$$

In our example, we set $D = (-0.5, 0.5)^2$, $l_1 = l_2 = 2$, $\bar{l}_1 = \bar{l}_2 = 2.5$, and $d := d_1 = d_2$. Let $\lambda = 1$, $\mu = 1$, $\rho_0 = 3$, and $\omega = 5$, then $k_p = 5$. Let $\zeta = 1.8$. For the medium property $\sigma_j(t)$, $j = 1, 2$, we define

$$\beta_j(t) = \begin{cases} 4t, & 0 \le t \le 0.25, \\ 2 - 4t, & 0.25 \le t \le 0.5, \end{cases}$$

and for $l_j \le t \le \bar{l}_j$,

$$\sigma_j(t) = \sigma_0 \left(\int_{l_j}^{\bar{l}_j} \beta_j(s - l_j) ds \right)^{-1} \int_{l_j}^{t} \beta_j(s - l_j) ds.$$

We consider the scattering problem whose exact solution is known:

$$\boldsymbol{u} = \nabla G_{k_p}(|\boldsymbol{x}|), \quad G_{k_p}(|\boldsymbol{x}|) = \frac{\mathbf{i}}{4} H_0^1(k_p|\boldsymbol{x}|).$$

We follow a similar idea in [10] to construct the finite element mesh. Figure 7.1 shows a sample of the mesh used which maintains the same number of elements in the PML layer for different choices of the PML thickness d. In our numerical experiments, we take $1 \le d \le 4$ and thus the elements in the PML layer keep the shape regularity.

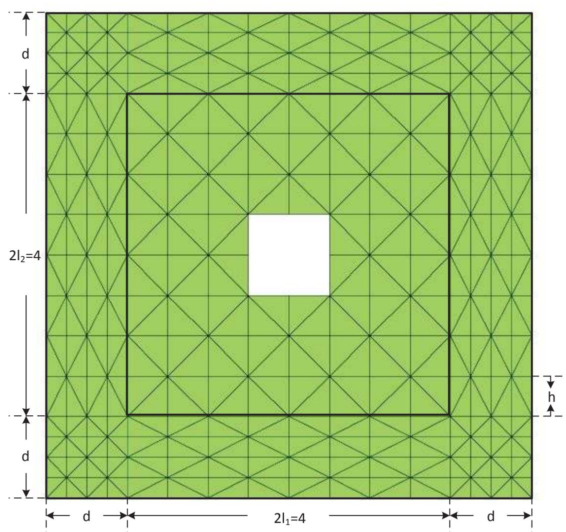

FIGURE 7.1. *The mesh when $h = 1/2$ and $d = 1$.*

We remark that error $\| \boldsymbol{u} - \boldsymbol{u}_h \|_{H^1(\Omega_l)}$ comes form two parts: the PML truncation error and finite element approximation error. It is clear that one can not expect the decrease of error when either one of the two parts of the error dominates. Figure 7.2 shows clearly the exponential decay of the error $\|\boldsymbol{u} - \boldsymbol{u_h}\|_{H^1(\Omega_l)}$ with respect to $k_p\gamma_0\bar{\sigma}$ when the finite element discretization error is negligible compared to the PML error. This is in conform with Theorem 6.3. Figure 7.3 shows the decay of the finite element error $\|\boldsymbol{u} - \boldsymbol{u_h}\|_{H^1(\Omega_l)}$ when the mesh is refined and we keep the product of the PML thickness d and PML strength σ_0 constant: $\sigma_0 d = 4$. We observe the expected second order convergence for the quadratic finite element. In Figure 7.4, we plot the real part of $\boldsymbol{u}_h$ and $\boldsymbol{u}_I$, the interpolation of the exact solution, when $\sigma_0 = 4$, $d = 1$ and $h = 1/32$. Note that the solution $\boldsymbol{u}_h$ goes rapidly to zero in the PML layer.

To conclude this section we remark that similar numerical results are also observed if we take the boundary condition $\boldsymbol{u} = 0$ at the outer boundary of the

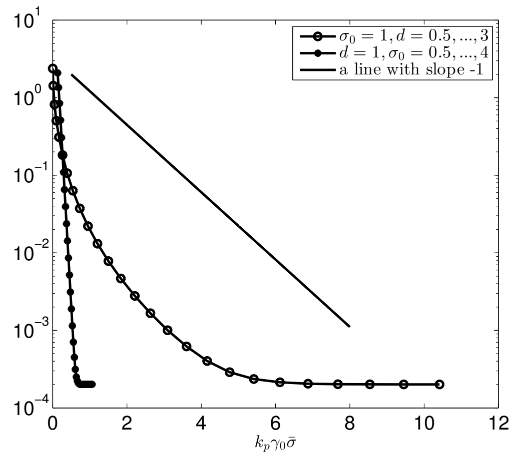

FIGURE 7.2. The $\log \|\boldsymbol{u} - \boldsymbol{u_h}\|_{H^1(\Omega_l)}$ - $k_p\gamma_0\bar{\sigma}$ plot of the finite element solution $\boldsymbol{u_h}$ when $h = 1/128$ and the degrees of freedom DOF=8266752.

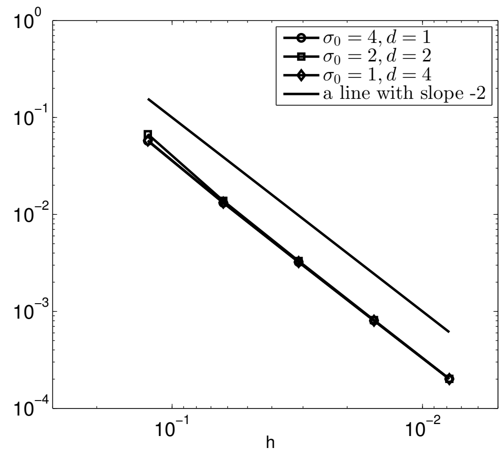

FIGURE 7.3. The $\log \|\boldsymbol{u} - \boldsymbol{u_h}\|_{H^1(\Omega_l)}$ - $\log h$ plot of the finite element solution $\boldsymbol{u_h}$ when $\sigma_0 d = 4$. The mesh size $h = 1/8, 1/16, 1/32, 1/64, 1/128$ and the corresponding degrees of freedom DOF $= 32832, 130176, 518400, 2068992, 8266752$.

PML layer instead of the mixed boundary condition (1.7) introduced in this paper. The convergence of the PML method for the time harmonic elastic waves with the boundary condition $\boldsymbol{u} = 0$ at the outer boundary of the PML layer remains an interesting open problem.

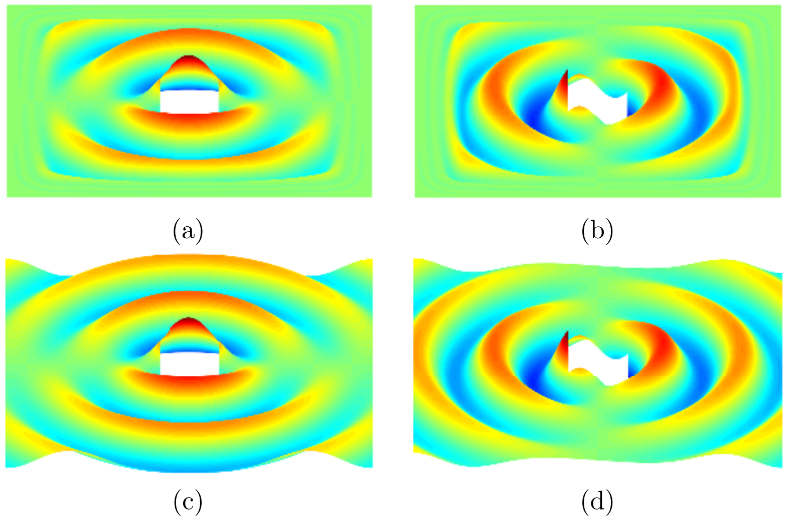

FIGURE 7.4. Numerical results when $\sigma_0 = 4$, $d = 1$, and $h = 1/32$. (a) The real part of the numerical solution $\boldsymbol{u}_{h,1}$; (b) The real part of the numerical solution $\boldsymbol{u}_{h,2}$; (c) The real part of the interpolation solution $\boldsymbol{u}_{I,1}$; (d) The real part of the interpolation solution $\boldsymbol{u}_{I,2}$.

8. Appendix

In this section we prove Theorem 2.1. We start with the following uniqueness result that is proved in [23, 26].

Lemma 8.1. *The scattering problem* (1.1)-(1.2) *with Kupradze-Sommerfeld radiation condition has at most one solution* $\boldsymbol{u} \in \boldsymbol{H}^1_{\rm loc}(\mathbb{R}^3\backslash\bar{D})$.

The existence of the solution can be proved by the method of limiting absorption principle by extending the argument for Helmholtz scattering problems (cf. e.g. [26]). Here we briefly recall the argument. For any $z = 1 + \mathbf{i}\varepsilon$, $\varepsilon > 0$, $\boldsymbol{q}_1 \in \boldsymbol{H}^1(\mathbb{R}^3)'$ with compact support in B_l, we consider the problem

$$\nabla \cdot \tau(\boldsymbol{u}_z) + z\gamma^2\boldsymbol{u}_z = -\boldsymbol{q}_1 \quad \text{in } \mathbb{R}^3. \tag{8.1}$$

It is easy to see by Lax-Milgram lemma that (8.1) has a unique solution $\boldsymbol{u}_z \in \boldsymbol{H}^1(\mathbb{R}^3)$. For any domain $\mathcal{D} \subset \mathbb{R}^3$, we define the weighted space $L^{2,s}(\mathcal{D})$, $s \in \mathbb{R}$, by

$$L^{2,s}(\mathcal{D}) = \{v \in L^2_{\rm loc}(\mathcal{D}) : (1 + |\boldsymbol{x}|^2)^{s/2}v \in L^2(\mathcal{D})\}$$

with the norm $\|v\|_{L^{2,s}(\mathcal{D})} = \left(\int_{\mathcal{D}}(1 + |\boldsymbol{x}|^2)^s|v|^2d\boldsymbol{x}\right)^{1/2}$. The weighted Sobolev space $H^{1,s}(\mathcal{D})$, $s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(\mathcal{D})$ whose first derivative is also in $L^{2,s}(\mathcal{D})$. The norm $\|v\|_{H^{1,s}(\mathcal{D})} = \left(\|v\|^2_{L^{2,s}(\mathcal{D})} + \|\nabla v\|^2_{L^{2,s}(\mathcal{D})}\right)^{1/2}$.

Lemma 8.2. *Let* $\boldsymbol{q}_1 \in \boldsymbol{L}^2(\mathbb{R}^3)$ *with support in* B_l. *For any* $z = 1+\mathbf{i}\varepsilon, 0 < \varepsilon < 1$, *we have, for any* $s > 1/2$, $\|\boldsymbol{u}_z\|_{H^{1,-s}(\mathbb{R}^3)} \le C\|\boldsymbol{q}_1\|_{L^2(\mathbb{R}^3)}$ *for some constant independent of* $\varepsilon, \boldsymbol{u}_z$, *and* $\boldsymbol{q}_1$.

Proof. We first observe that by testing (8.1) by $(1 + |\boldsymbol{x}|^2)^{-s}\bar{\boldsymbol{u}}_z$, $s > 1/2$, one can obtain $\|\boldsymbol{u}_z\|_{H^{1,-s}(\mathbb{R}^3)} \le C\|\boldsymbol{u}_z\|_{L^{2,-s}(\mathbb{R}^3)} + C\|\boldsymbol{q}_1\|_{L^2(\mathbb{R}^3)}$ by standard argument. Now we show $\|\boldsymbol{u}_z\|_{L^{2,-s}(\mathbb{R}^3)} \le C\|\boldsymbol{q}_1\|_{L^2(\mathbb{R}^3)}$. It is obvious that we only need to prove the estimate for $\boldsymbol{q}_1 \in \mathbb{C}_0^\infty(\mathbb{R}^3)^3$ for which we have the integral representation formula

$$\boldsymbol{u}_z(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Gamma^z(\boldsymbol{x},\boldsymbol{y})\boldsymbol{q}_1(\boldsymbol{y})d\boldsymbol{y}, \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

Here $\Gamma^z(\boldsymbol{x},\boldsymbol{y})$ is the fundamental solution matrix of (8.1) which has the complex wave number $\gamma z^{1/2}$, where $\operatorname{Im} z^{1/2} > 0$ for $\varepsilon > 0$. Similar to (2.12), we have

$$\Gamma^z_{jk}(\boldsymbol{x},\boldsymbol{y}) = \Gamma^z_1(|\boldsymbol{x}-\boldsymbol{y}|)\delta_{jk} + \Gamma^z_2(|\boldsymbol{x}-\boldsymbol{y}|)\frac{(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^2}, \tag{8.2}$$

where, for $r > 0$,

$$\Gamma^z_1(r) = \frac{1}{\gamma^2 z}\left[(k^z_s)^2 f_{k^z_s}(r) - \frac{f'_{k^z_p}(r) - f'_{k^z_s}(r)}{r}\right],$$
$$\Gamma^z_2(r) = \frac{1}{\gamma^2 z}\left[3\,\frac{f'_{k^z_p}(r) - f'_{k^z_s}(r)}{r} + ((k^z_p)^2 f_{k_p}(r) - (k^z_s)^2 f_{k^z_s}(r))\right].$$

Here $k^z_p = \gamma z^{1/2}/\sqrt{\lambda+2\mu}, k^z_s = \gamma z^{1/2}/\sqrt{\mu}$. It is easy to show that

$$|\Gamma^z_{jk}(|\boldsymbol{x}-\boldsymbol{y}|)| \le C|\boldsymbol{x}-\boldsymbol{y}|^{-1}, \qquad \text{for } \boldsymbol{x} \ne \boldsymbol{y}, \tag{8.3}$$

for some constant C independent of $\varepsilon \in (0,1)$.

For any $\boldsymbol{\phi} \in \boldsymbol{L}^{2,s}(\mathbb{R}^3)$, denote $\boldsymbol{\psi}(\boldsymbol{y}) = \int_{\mathbb{R}^3} \Gamma^z(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{x})d\boldsymbol{x}$. Since $\boldsymbol{q}_1$ is supported in B_l, we have $|(\boldsymbol{u}_z,\boldsymbol{\phi})_{\mathbb{R}^3}| \le \|\boldsymbol{\psi}\|_{L^2(B_l)}\|\boldsymbol{q}_1\|_{L^2(\mathbb{R}^3)}$. Now we estimate $\|\boldsymbol{\psi}\|_{L^2(B_l)}$. Write

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2 := \int_{B_{l+1}} \Gamma^z(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{x})d\boldsymbol{x} + \int_{\mathbb{R}^3\setminus\overline{B_{l+1}}} \Gamma^z(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{x})d\boldsymbol{x},$$

where $B_{l+1} := \{\boldsymbol{x} \in \mathbb{R}^3 : |x_i| < l_i + 1, i = 1,2,3\}$. By (8.3) and Cauchy-Schwarz inequality we have

$$\begin{aligned}\|\boldsymbol{\psi}_1\|^2_{L^2(B_l)} &\le C\int_{B_l}\left(\int_{B_{l+1}} |\boldsymbol{\phi}(\boldsymbol{x})|^2 d\boldsymbol{x} \cdot \int_{B_{l+1}} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2} d\boldsymbol{x}\right) d\boldsymbol{y}\\ &\le C\|\boldsymbol{\phi}\|^2_{L^2(B_{l+1})}.\end{aligned}$$

On the other hand, since by (8.3), $|\Gamma^z(\boldsymbol{x},\boldsymbol{y})| \le C$ for $\boldsymbol{x} \in \mathbb{R}^3\setminus\overline{B_{l+1}}, \boldsymbol{y} \in B_l$,

$$\begin{aligned}\|\boldsymbol{\psi}_2\|^2_{L^2(B_l)} &\le C\int_{B_l}\left|\int_{\mathbb{R}^3\setminus\overline{B_{l+1}}} |\boldsymbol{\phi}(\boldsymbol{x})| d\boldsymbol{x}\right|^2 d\boldsymbol{y}\\ &\le C\|\boldsymbol{\phi}\|^2_{L^{2,s}(\mathbb{R}^3)}.\end{aligned}$$

This yields $\|\boldsymbol{\psi}\|_{L^2(B_l)} \le C\|\boldsymbol{\phi}\|_{L^{2,s}(\mathbb{R}^3)}$. Therefore,

$$|(\boldsymbol{u}_z,\boldsymbol{\phi})_{\mathbb{R}^3}| \le C\|\boldsymbol{\phi}\|_{L^{2,s}(\mathbb{R}^3)}\|\boldsymbol{q}_1\|_{L^2(\mathbb{R}^3)}.$$

This shows $\|\boldsymbol{u}_z\|_{L^{2,-s}(\mathbb{R}^3)} \le C\|\boldsymbol{q}_1\|_{L^2(\mathbb{R}^3)}$ and completes the proof. □

Now we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. The argument is standard and we just give an outline below, see e.g. [26] for the consideration for Helmholtz equations. For any $0 < \varepsilon < 1$, we consider the problem

$$\nabla\cdot\tau(\boldsymbol{u}_\varepsilon)+(1+\mathbf{i}\varepsilon)\gamma^2\boldsymbol{u}_\varepsilon=-\boldsymbol{q}\quad\text{in }\mathbb{R}^3\backslash\bar{D},\tag{8.4}$$

$$\tau(\boldsymbol{u}_\varepsilon)\boldsymbol{n}_D=-\boldsymbol{g}\quad\text{on }\Gamma_D.\tag{8.5}$$

By Lax-Milgram lemma we know that the above problem has a unique solution $\boldsymbol{u}_\varepsilon\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{D})$. Let $\chi\in C_0^\infty(\mathbb{R}^3)$ be the cut-off function such that $0\le\chi\le1$, $\chi=0$ in B_l, and $\chi=1$ outside B_{l+1}. Let $\boldsymbol{v}_\varepsilon=\chi\boldsymbol{u}_\varepsilon$. Then $\boldsymbol{v}_\varepsilon$ satisfies (8.1) with $z=1+\mathbf{i}\varepsilon$ and $\boldsymbol{q}_1=\tau(\boldsymbol{u}_\varepsilon)\nabla\chi+(\lambda+\mu)(\boldsymbol{\nabla}^2\chi\boldsymbol{u}_\varepsilon+\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon\nabla\chi)+\mu\Delta\chi\boldsymbol{u}_\varepsilon+\mu\mathrm{div}\,\boldsymbol{u}_\varepsilon\nabla\chi$, where $\boldsymbol{\nabla}^2\chi$ is the Hessian matrix of χ. Clearly $\boldsymbol{q}_1$ has compact support. By Lemma 8.1 we can obtain

$$\|\boldsymbol{v}_\varepsilon\|_{H^{1,-s}(\mathbb{R}^3)}\le C\|\boldsymbol{u}_\varepsilon\|_{H^1(B_{l+1}\backslash\bar{D})}\tag{8.6}$$

for some constant C independent of $\varepsilon>0$. Now let $\chi_1\in C_0^\infty(\mathbb{R}^3)$ be the cut-off function such that $0\le\chi_1\le1$, $\chi_1=1$ in B_{l+1}, and $\chi_1=0$ outside B_{l+2}. Denote $\boldsymbol{w}_{\boldsymbol{g}}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{D})$ as the lifting of the function $\boldsymbol{g}\in\boldsymbol{H}^{-1/2}(\Gamma_D)$ such that $\tau(\boldsymbol{w}_{\boldsymbol{g}})\boldsymbol{n}_D=\boldsymbol{g}$ on Γ_D and $\|\boldsymbol{w}_{\boldsymbol{g}}\|_{H^1(\mathbb{R}^3\backslash\bar{D})}\le C\|\boldsymbol{g}\|_{H^{-1/2}(\Gamma_D)}$. By multiplying (8.4) with $\chi_1^2\overline{(\boldsymbol{u}_\varepsilon-\boldsymbol{w}_{\boldsymbol{g}})}$ and using the standard argument we have

$$\|\boldsymbol{u}_\varepsilon\|_{H^1(B_{l+1}\backslash\bar{D})}\le C(\|\boldsymbol{q}\|_{H^1(\mathbb{R}^3)'}+\|\boldsymbol{g}\|_{H^{-1/2}(\Gamma_D)}+\|\boldsymbol{u}_\varepsilon\|_{L^2(B_{l+2}\backslash\bar{D})}).$$

A combination of (8.6) and the above estimate yields

$$\|\boldsymbol{v}_\varepsilon\|_{H^{1,-s}(\mathbb{R}^3)}\le C(\|\boldsymbol{q}\|_{H^1(\mathbb{R}^3)'}+\|\boldsymbol{g}\|_{H^{-1/2}(\Gamma_D)}+\|\boldsymbol{u}_\varepsilon\|_{L^2(B_{l+2}\backslash\bar{D})}).\tag{8.7}$$

Now we claim

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(B_{l+2}\backslash\bar{D})}\le C(\|\boldsymbol{q}\|_{H^1(\mathbb{R}^3)'}+\|\boldsymbol{g}\|_{H^{-1/2}(\Gamma_D)}),\tag{8.8}$$

for any $\boldsymbol{q}\in\boldsymbol{H}^1(\mathbb{R}^3)'$ with the support inside B_l, $\boldsymbol{g}\in\boldsymbol{H}^{-1/2}(\Gamma_D)$, and $\varepsilon\in(0,1)$. If (8.8) were false, there would exist sequences $\{\boldsymbol{q}_m\}\subset\boldsymbol{H}^1(\mathbb{R}^3)'$ with support in B_l, $\{\boldsymbol{g}_m\}\subset\boldsymbol{H}^{-1/2}(\Gamma_D)$, $\{\varepsilon_m\}\subset(0,1)$, and $\{\boldsymbol{u}_{\varepsilon_m}\}$ the corresponding solution of (8.4)-(8.5) such that

$$\|\boldsymbol{u}_{\varepsilon_m}\|_{L^2(B_{l+2}\backslash\bar{D})}=1\quad\text{and}\quad\|\boldsymbol{q}_m\|_{H^1(\mathbb{R}^3)'}+\|\boldsymbol{g}_m\|_{H^{-1/2}(\Gamma_D)}\le1/m.\tag{8.9}$$

Then by (8.7), $\|\boldsymbol{u}_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}^3\backslash\bar{D})}\le C$ and thus there is a subsequence of $\{\varepsilon_m\}$, which is still denoted by $\{\varepsilon_m\}$, such that $\varepsilon_m\to\varepsilon'\in[0,1]$, and a subsequence of $\{\boldsymbol{u}_{\varepsilon_m}\}$, which is still denoted by $\{\boldsymbol{u}_{\varepsilon_m}\}$, such that $\{\boldsymbol{u}_{\varepsilon_m}\}$ converges weakly to some $\boldsymbol{u}_{\varepsilon'}\in\boldsymbol{H}^{1,-s}(\mathbb{R}^3\backslash\bar{D})$ which satisfies (8.4)-(8.5) with $\boldsymbol{q}=0$, $\boldsymbol{g}=0$, and $\varepsilon=\varepsilon'$.

By the integral representation satisfied by $\boldsymbol{u}_{\varepsilon_m}$ we know that for $n=1,2,3$,

$$\boldsymbol{u}_{\varepsilon'}(\boldsymbol{x})\cdot\boldsymbol{e}_n=\langle\mathbb{T}[\Gamma^{1+\mathbf{i}\varepsilon'}(\boldsymbol{x},\cdot)\boldsymbol{e}_n],\overline{\boldsymbol{u}_{\varepsilon'}|_{\Gamma_D}}\rangle_{\Gamma_D},\quad\forall\boldsymbol{x}\in\mathbb{R}^3\backslash\bar{D}.\tag{8.10}$$

If $\varepsilon'>0$, we deduce from (8.10) that $\boldsymbol{u}_{\varepsilon'}$ decays exponentially and thus in $\boldsymbol{u}_{\varepsilon'}\in\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{D})$. Now the uniqueness of the solution in $\boldsymbol{H}^1(\mathbb{R}^3\backslash\bar{D})$ indicates that $\boldsymbol{u}_{\varepsilon'}=0$. If $\varepsilon'=0$, (8.10) implies that $\boldsymbol{u}_{\varepsilon'}$ satisfies the Kupradze-Sommerfeld radiation condition and we conclude by Lemma 8.1 that $\boldsymbol{u}_{\varepsilon'}=0$. In any case $\boldsymbol{u}_{\varepsilon'}=0$, however, this contradicts to (8.9). Therefore, we have (8.8) and consequently by (8.7)

$$\|\boldsymbol{u}_\varepsilon\|_{H^{1,-s}(\mathbb{R}^3\backslash\bar{D})}\le C(\|\boldsymbol{q}\|_{H^1(\mathbb{R}^3)'}+\|\boldsymbol{g}\|_{H^{-1/2}(\Gamma_D)}).\tag{8.11}$$

Now it is easy to see that $\boldsymbol{u}_\varepsilon$ has a convergent subsequence which converges weakly to some $\boldsymbol{u}$ in $\boldsymbol{H}^{1,-s}(\mathbb{R}^3\backslash\bar{D})$ that satisfies (1.1)-(1.2) and the Kupradze-Sommerfeld radiation condition. The desired estimate follows from (8.11). This completes the proof. □

We remark that the above arguments extends easily to show that the existence of radiating solutions to the time harmonic elastic wave problem with other types of boundary conditions such as Dirichlet or mixed boundary conditions on Γ_D.

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