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# TECHNICAL NOTE

# Convergence of the Steepest Descent Method for Minimizing Quasiconvex Functions[1,2]

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**Abstract.** To minimize a continuously differentiable quasiconvex function $f: \mathbb{R}^n \to \mathbb{R}$, Armijo's steepest descent method generates a sequence $x^{k+1} = x^k - t_k \nabla f(x^k)$, where $t_k > 0$. We establish strong convergence properties of this classic method: either $x^k \to \bar{x}$, s.t. $\nabla f(\bar{x}) = 0$; or $\arg\min f = \emptyset$, $\|x^k\| \to \infty$, and $f(x^k) \downarrow \inf f$. We also discuss extensions to other line searches.

**Key Words.** Steepest descent methods, convex programming, Armijo's line search.

## 1. Introduction

To minimize a continuously differentiable quasiconvex function $f: \mathbb{R}^n \to \mathbb{R}$, Cauchy's steepest descent method (Ref. 1) with Armijo's stepsizes (Ref. 2) generates a sequence $\{x^k\}$ via

$$x^{k+1} = x^k - t_k g^k, \quad g^k = \nabla f(x^k), \qquad k = 0, 1, \ldots, \tag{1}$$

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where

$$t_k = \arg\max\{t: f(x^k - tg^k) \le f(x^k) - \alpha t\|g^k\|^2, t = 2^{-i}, i = 0, 1, \dots\}, \tag{2}$$

with $\alpha \in (0, 1)$. We prove in Section 2 the following strong convergence result.

**Theorem 1.1.** Global Convergence. Either $x^k \to \bar{x} \in X := \{x: \nabla f(x) = 0\}$, or $\check{X} := \arg\min f = \emptyset$, $\|x^k\| \to \infty$, and $f(x^k) \downarrow \inf f$.

A closely related result appeared in Ref. 3 after an earlier version of this note was accepted. The present version provides a considerably simpler convergence proof that permits generalization to the quasiconvex case. Other related results for nondifferentiable optimization methods are given in Ref. 4 and Ref. 5, Remark 3.2. These relations and extensions are discussed in Section 3.

## 2. Global Convergence of Steepest Descent

We make the following standing assumption that generalizes Armijo's condition (2).

**Assumption 2.1.** Let $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that:

(A1) $\exists \alpha \in (0, 1)$, $\tau_\alpha > 0$, $\forall t \in (0, \tau_\alpha]$: $\phi(t) \le \alpha t$,

(A2) $\exists \beta > 0$, $\tau_\beta \in (0, \infty]$, $\forall t \in (0, \tau_\beta] \cap \mathbb{R}$: $\phi(t) \ge \beta t^2$,

(A3) $\forall k, f(x^{k+1}) \le f(x^k) - \phi(t_k)\|g^k\|^2$ and $0 < t_k \le \tau_\beta$ in (1),

(A4) $\exists \gamma > 1$, $\tau_\gamma > 0$, $\forall k$: $t_k \ge \tau_\gamma$ or
$[\exists \tilde{t}_k \in [t_k, \gamma t_k]: f(x^k - \tilde{t}_k g^k) \ge f(x^k) - \phi(\tilde{t}_k)\|g^k\|^2]$.

Note that (2) corresponds to

$$\phi(t) = \alpha t, \qquad \beta = \alpha, \qquad \gamma = 2, \qquad \tau_\alpha = \tau_\beta = \tau_\gamma = 1.$$

As in Ref. 4, we start by considering the condition

$$f(x^k) \ge f(\tilde{x}), \qquad \text{for some fixed } \tilde{x} \text{ and all } k, \tag{3}$$

which holds if $\check{X} \ne \emptyset$ or $\tilde{x}$ is a cluster point of $\{x^k\}$.

**Lemma 2.1.** If (3) holds, then

$$\sum_{k=0}^{\infty} t_k^2 \|g^k\|^2 \le [f(x^0) - f(\tilde{x})]/\beta. \tag{4}$$

Moreover, $x^k \to \bar{x}$ for some $\bar{x}$.

**Proof.** By (A2)–(A3),

$$\beta t_k^2 \|g^k\|^2 \le \phi(t_k)\|g^k\|^2 \le f(x^k) - f(x^{k+1});$$

adding these inequalities yields (4). Next, since $\langle g^k, \tilde{x} - x^k \rangle \le 0$ by (3) and quasiconvexity of $f$ [Ref. 6, Theorem 9.1.4], and since $x^k - x^{k+1} = t_k g^k$, we deduce that

$$\begin{aligned} \|\tilde{x} - x^{k+1}\|^2 &= \|\tilde{x} - x^k\|^2 + 2\langle \tilde{x} - x^k, x^k - x^{k+1}\rangle + \|x^{k+1} - x^k\|^2 \\ &\le \|\tilde{x} - x^k\|^2 + t_k^2 \|g^k\|^2, \end{aligned}$$

so that

$$\|\tilde{x} - x^l\|^2 \le \|\tilde{x} - x^k\|^2 + \sum_{j=k}^{\infty} t_j^2 \|g^j\|^2 < \infty,$$

if $l > k$. Hence, $\{x^k\}$ is bounded and has a cluster point $\bar{x}$, so we may set $\tilde{x} = \bar{x}$ above to deduce from (4) for any $\epsilon > 0$ the existence of $k$ such that $\|\bar{x} - x^k\|^2 \le \epsilon/2$ and

$$\sum_{j=k}^{\infty} t_j^2 \|g^j\|^2 \le \epsilon/2;$$

thus, $\|\bar{x} - x^l\|^2 \le \epsilon$ for all $l > k$, i.e., $x^k \to \bar{x}$. □

**Lemma 2.2.** If $\bar{x}$ is a cluster point of $\{x^k\}$, then $\bar{x} \in X$, i.e., $\nabla f(\bar{x}) = 0$.

**Proof.** Suppose that $x^k \xrightarrow{K} \bar{x}$, but $\bar{g} := \nabla f(\bar{x}) \neq 0$. Then, $t_k \xrightarrow{K} 0$ from [cf. (A2)–(A3)]

$$0 \le \beta t_k^2 \|g^k\|^2 \le f(x^k) - f(x^{k+1}) \xrightarrow{K} 0,$$

with $g^k \xrightarrow{K} \bar{g} \neq 0$ and $f(x^k) \downarrow f(\bar{x})$ by continuity. Thus, for all large $k \in K$,

$$f(x^k - \tilde{t}_k g^k) - f(x^k) \ge -\phi(\tilde{t}_k)\|g^k\|^2 \ge -\alpha \tilde{t}_k \|g^k\|^2, \tag{5}$$

by (A4) and (A1), where the left side equals $-\tilde{t}_k \langle g^k, \nabla f(x^k - \breve{t}_k g^k)\rangle$ for some $\breve{t}_k \in [0, \tilde{t}_k]$ by the mean-value theorem, and by (A4), $0 \le \tilde{t}_k \le \gamma t_k \xrightarrow{K} 0$. Hence, dividing (5) by $\tilde{t}_k$ and letting $k \xrightarrow{K} \infty$ yields $-\|\bar{g}\|^2 \ge -\alpha \|\bar{g}\|^2$, a contradiction with $\alpha < 1$ [cf. (A1)]. □

We can now prove Theorem 1.1 under Assumption 2.1 that generalizes (2).

**Proof of Theorem 1.1.** If (3) holds, e.g., $\check{X} \neq \varnothing$ or $\{x^k\}$ has a cluster point, then the preceding results yield $x^k \to \bar{x} \in X$. If $\|x^k\| \not\to \infty$, then $\{x^k\}$ has a cluster point. If $\lim_{k\to\infty} f(x^k) > \inf f$, then (3) holds. □

## 3. Discussion of Other Line Searches

First, suppost that $\alpha \in (\frac{1}{2}, 1)$ and $f$ is convex. Then, the proof of Lemma 2.2 simplifies, since

$$\langle g^k, x^k - \tilde{x} \rangle \geq f(x^k) - f(\tilde{x}) \geq f(x^k) - f(x^{k+1}) \geq \alpha t_k \|g^k\|^2,$$

$$\begin{aligned} \|\tilde{x} - x^{k+1}\|^2 - \|\tilde{x} - x^k\|^2 &\leq -2\alpha t_k^2 \|g^k\|^2 + t_k^2 \|g^k\|^2 \\ &= -(2\alpha - 1)\|x^{k+1} - x^k\|^2 \leq 0. \end{aligned}$$

This observation is used in Ref. 7 to prove that $x^k \to \bar{x} \in \check{X}$ if $\check{X} \neq \varnothing$ and $\nabla f$ is Lipschitz continuous; thus, our result improves that of Ref. 7.

Second, it is easy to verify Theorem 1.1 for any line search for which Lemma 2.3 holds and for all $k$,

$$f(x^{k+1}) \leq f(x^k) - \alpha t_k \|g^k\|^2$$

and

$$t_k \in (0, t_{\max}], \qquad \text{for some fixed } t_{\max} > 0.$$

Such stepsizes may be found by many procedures [Refs. 8–12]. Note that exact line searches are not admissible, but one may use, as in Ref. 12, Section 10.7.2,

$$t_k \approx \arg\min\{f(x^k - tg^k) : f(x^k - tg^k) \leq f(x^k) - \alpha t \|g^k\|^2, 0 < t \leq t_{\max}\}.$$

Third, under (A1)–(A2) to satisfy (A3)–(A4), one may let (cf. the proof of Lemma 2.3)

$$\begin{aligned} t_k = \arg\max\{t : f(x^k - tg^k) \leq f(x^k) - \phi(t)\|g^k\|^2, \\ t = 2^{-i} \min[\tau_\alpha, \tau_\beta], i = 0, 1, \ldots\}. \end{aligned} \tag{6}$$

We note that (6) with $\phi(t) = \alpha t^2$ was used in Ref. 13. Again, the Armijo-type search (6) may be relaxed as in the preceding paragraph. In particular, one may use

$$t_k \approx \check{t}_k := \arg\min\{f(x^k - tg^k) + \alpha t^2 \|g^k\|^2 : t > 0\}.$$

If $f$ is pseudoconvex, then $X = \check{X}$ [Ref. 6, Theorem 9.3.3]; so if $\check{X} \neq \varnothing$ and $t_k = \check{t}_k$ for all $k$, then $x^k \to \bar{x} \in X$; thus, we recover the result of Ref. 14.

Fourth, one may verify Assumption 2.1 for the algorithms of Ref. 3; in their notation, let $\phi(t)=\beta t^2$ with $\beta=L\delta_2/2(1-\delta_2)$ for Algorithm A, $\phi=\psi$ for Algorithm B. Theorem 1.1 is stronger than Theorem 3 of Ref. 3, and our proof is simpler.

We note that quasiconvexity of $f$ is necessary for Lemma 2.2, and consequently Theorem 1.1. For example, let

$$n=2, \qquad f(x)=e^{x_1}-x_2^2, \qquad x^0=(0,0)^T.$$

Each of the above methods generates

$$x^k=(x_1^k,0)^T, \text{ with } x_1^k\downarrow -\infty \text{ and } f(x^k)\downarrow 0, \text{ while } \inf f=-\infty.$$

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