

CONVERGENCE OF THE YAMABE FLOW FOR ARBITRARY INITIAL ENERGY

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Abstract

We consider the Yamabe flow $\frac{\partial g}{\partial t} = -(R_g - r_g)g$, where g is a Riemannian metric on a compact manifold M , R_g denotes its scalar curvature, and r_g denotes the mean value of the scalar curvature. We prove convergence of the Yamabe flow if the dimension n satisfies $3 \leq n \leq 5$ or the initial metric is locally conformally flat.

1. Introduction

In this paper, we present a general convergence result for the Yamabe flow in conformal geometry. Let M be a compact manifold of dimension $n \geq 3$ without boundary and let g be a Riemannian metric on M . Along the Yamabe flow, the Riemannian metric is deformed according to

$$(1) \quad \frac{\partial}{\partial t} g = -(R_g - r_g)g,$$

where R_g is the scalar curvature of g and r_g is the mean value of R_g , i.e.,

$$(2) \quad r_g = \frac{\int_M R_g dvol_g}{\int_M dvol_g}.$$

The Yamabe energy of a Riemannian metric g on M is defined as

$$(3) \quad \frac{\int_M R_g dvol_g}{\left(\int_M dvol_g\right)^{\frac{n-2}{n}}}.$$

Moreover, the Yamabe constant of a Riemannian metric g_0 is defined as the infimum of the Yamabe energy among metrics conformally equivalent to g_0 , i.e.,

$$(4) \quad Y(M, g_0) = \inf_{\substack{g = u^{\frac{4}{n-2}} g_0 \\ u \in C^\infty(M), u > 0}} \frac{\int_M R_g dvol_g}{\left(\int_M dvol_g\right)^{\frac{n-2}{n}}}.$$

By definition, $Y(M, g_0)$ depends only on the conformal class of g_0 .

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Since the Yamabe flow preserves the conformal structure, we may write $g = u^{\frac{4}{n-2}} g_0$, where g_0 is a fixed background metric on M and u is a positive function. The scalar curvature of g is related to the scalar curvature of g_0 by

$$(5) \quad R_g = -u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u \right).$$

Hence, the Yamabe flow reduces to the following evolution equation for the conformal factor:

$$(6) \quad \frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \frac{n+2}{4} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u + r_g u^{\frac{n+2}{n-2}} \right).$$

Moreover, the Yamabe constant of g_0 can be written as

$$(7) \quad Y(M, g_0) = \inf_{u \in C^\infty(M), u > 0} \frac{\int_M \left(\frac{4(n-1)}{n-2} |du|_{g_0}^2 + R_{g_0} u^2 \right) d\text{vol}_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}}}.$$

In case $Y(M, g_0) \leq 0$, it is not difficult to show that the conformal factor is uniformly bounded above and below. Moreover, the flow converges to a metric of constant scalar curvature as $t \rightarrow \infty$.

The case $Y(M, g_0) > 0$ is more interesting. Chow [6] proved the convergence of the flow for locally conformally flat metrics with positive Ricci curvature. Ye [20] later extended the result to all locally conformally flat metrics.

Recently, Struwe and Schwetlick [17] proved convergence of the Yamabe flow in lower dimensions ($3 \leq n \leq 5$) under the assumption that the Yamabe energy of the initial metric is less than $(Y(M, g_0)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}}$, where $Y(S^n)$ denotes the Yamabe energy of the standard sphere S^n . Under this assumption, it is shown that any singularity consists of at most one bubble, and the positive mass theorem precludes the formation of a singularity of this type.

In the present paper, we prove convergence of the flow for arbitrary initial energy.

Theorem 1.1. *Suppose that either $3 \leq n \leq 5$ or M is locally conformally flat. Moreover, assume that M is not conformally equivalent to the standard sphere S^n . Then, for every choice of the initial metric, the Yamabe flow exists for all time and converges to a metric with constant scalar curvature.*

We expect that the methods of this paper can be used to prove convergence of the Yamabe flow in dimension $n \geq 6$ under a technical condition on the Weyl tensor (compare [1], [11], Theorem B, and [16], Theorem 4.1 on p. 219).

2. Longtime existence

In light of the foregoing discussion, it suffices to consider the case $Y(M, g_0) > 0$. In this case, we can find a background metric g_0 , which is conformally equivalent to the initial metric and has positive scalar curvature R_{g_0} (see [15], Lemma 1.1). Let $g(t) = u(t)^{\frac{4}{n-2}} g_0$ be a solution of the Yamabe flow

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - r_{g(t)}) g(t).$$

Since the volume of M does not change under the evolution, we may assume that

$$\int_M dvol_{g(t)} = 1$$

for all $t \geq 0$. With this normalization, the mean value of the scalar curvature can be written as

$$r_{g(t)} = \int_M R_{g(t)} dvol_{g(t)}.$$

Using the identity

$$(8) \quad \frac{\partial}{\partial t} R_{g(t)} = (n - 1) \Delta_{g(t)} R_{g(t)} + R_{g(t)} (R_{g(t)} - r_{g(t)}),$$

we obtain

$$(9) \quad \frac{d}{dt} r_{g(t)} = -\frac{n-2}{2} \int_M (R_{g(t)} - r_{g(t)})^2 dvol_{g(t)}.$$

In particular, the function $t \mapsto r_{g(t)}$ is decreasing.

Proposition 2.1. *The scalar curvature of the metric $g(t)$ satisfies*

$$(10) \quad \inf_M R_{g(t)} \geq \min \left\{ \inf_M R_{g(0)}, 0 \right\}$$

for all $t \geq 0$.

Proof. The scalar curvature satisfies the evolution equation

$$\frac{\partial}{\partial t} R_{g(t)} = (n - 1) \Delta_{g(t)} R_{g(t)} + R_{g(t)} (R_{g(t)} - r_{g(t)}).$$

Observe that $r_{g(t)} > 0$ since $Y(M, g_0) > 0$. Hence, the assertion follows from the maximum principle.

For abbreviation, let

$$\sigma = \max \left\{ \sup_M (1 - R_{g(0)}), 1 \right\},$$

so that $R_{g(t)} + \sigma \geq 1$ for all $t \geq 0$.

The following two results are similar to Lemma 3.3 in [17]. Our arguments mostly follow those of Struwe and Schwetlick, but we do not require that the scalar curvature is positive everywhere.

Lemma 2.2. *For every $p > 2$, we have*

$$\begin{aligned}
(11) \quad & \frac{d}{dt} \int_M (R_{g(t)} + \sigma)^{p-1} d\text{vol}_{g(t)} \\
&= -\frac{4(n-1)(p-2)}{p-1} \int_M \left| d(R_{g(t)} + \sigma)^{\frac{p-1}{2}} \right|_{g(t)}^2 d\text{vol}_{g(t)} \\
&\quad - \left(\frac{n+2}{2} - p \right) \int_M ((R_{g(t)} + \sigma)^{p-1} - (r_{g(t)} + \sigma)^{p-1}) (R_{g(t)} - r_{g(t)}) d\text{vol}_{g(t)} \\
&\quad - (p-1) \int_M \sigma ((R_{g(t)} + \sigma)^{p-2} - (r_{g(t)} + \sigma)^{p-2}) (R_{g(t)} - r_{g(t)}) d\text{vol}_{g(t)}.
\end{aligned}$$

Proof. This follows immediately from the evolution equation for the scalar curvature.

Lemma 2.3. *For every $p > \max\{\frac{n}{2}, 2\}$, we have*

$$\begin{aligned}
(12) \quad & \frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p d\text{vol}_{g(t)} \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^p d\text{vol}_{g(t)} \right)^{\frac{2p-n+2}{2p-n}} \\
&\quad + C \int_M |R_{g(t)} - r_{g(t)}|^p d\text{vol}_{g(t)}
\end{aligned}$$

for some uniform constant C independent of t .

Proof. Using the evolution equation for the scalar curvature, we obtain

$$\begin{aligned}
& \frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p d\text{vol}_{g(t)} \\
&= -\frac{(n-2)(p-1)}{p} \\
&\quad \cdot \int_M \left(\frac{4(n-1)}{n-2} \left| d|R_{g(t)} - r_{g(t)}|^{\frac{p}{2}} \right|_{g(t)}^2 + R_{g(t)} |R_{g(t)} - r_{g(t)}|^p \right) d\text{vol}_{g(t)} \\
&\quad + \left(\frac{(n-2)(p-1)}{p} + p - \frac{n}{2} \right) \int_M |R_{g(t)} - r_{g(t)}|^p (R_{g(t)} - r_{g(t)}) d\text{vol}_{g(t)} \\
&\quad + \left(\frac{(n-2)(p-1)}{p} + p \right) \int_M r_{g(t)} |R_{g(t)} - r_{g(t)}|^p d\text{vol}_{g(t)}
\end{aligned}$$

$$\begin{aligned}
 & + \frac{(n-2)p}{2} \int_M (R_{g(t)} - r_{g(t)})^2 dvol_{g(t)} \\
 & \cdot \int_M |R_{g(t)} - r_{g(t)}|^{p-2} (R_{g(t)} - r_{g(t)}) dvol_{g(t)},
 \end{aligned}$$

hence

$$\begin{aligned}
 & \frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} \\
 & \leq -\frac{(n-2)(p-1)}{p} Y(M, g_0) \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{pn}{n-2}} dvol_{g(t)} \right)^{\frac{n-2}{n}} \\
 & + \left(\frac{(n-2)(p-1)}{p} + p - \frac{n}{2} \right) \int_M |R_{g(t)} - r_{g(t)}|^{p+1} dvol_{g(t)} \\
 & + \left(\frac{(n-2)(p-1)}{p} + p \right) \int_M r_{g(0)} |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} \\
 & + \frac{(n-2)p}{2} \left(\int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} \right)^{\frac{p+1}{p}}.
 \end{aligned}$$

Since $p > \frac{n}{2}$, this implies

$$\begin{aligned}
 \frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} & \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} \right)^{\frac{2p-n+2}{2p-n}} \\
 & + C \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)}
 \end{aligned}$$

by Hölder's inequality. From this, the assertion follows.

Proposition 2.4. *Given any $T > 0$, we can find positive constants $C(T)$ and $c(T)$ such that*

$$\sup_M u(t) \leq C(T)$$

and

$$\inf_M u(t) \geq c(T)$$

for all $0 \leq t \leq T$.

Proof. The function $u(t)$ satisfies

$$\frac{\partial}{\partial t} u(t) = -\frac{n-2}{4} (R_{g(t)} - r_{g(t)}) \leq \frac{n-2}{4} (r_{g(0)} + \sigma).$$

Thus, we conclude that

$$\sup_M u(t) \leq C(T)$$

for all $0 \leq t \leq T$. Hence, if we define

$$P = R_{g_0} + \sigma \left(\sup_{0 \leq t \leq T} \sup_M u(t) \right)^{\frac{4}{n-2}},$$

then we obtain

$$\begin{aligned} (13) \quad & -\frac{4(n-1)}{n-2} \Delta_{g_0} u(t) + P u(t) \\ & \geq -\frac{4(n-1)}{n-2} \Delta_{g_0} u(t) + R_{g_0} u(t) + \sigma u(t)^{\frac{n+2}{n-2}} \\ & = (R_{g(t)} + \sigma) u(t)^{\frac{n+2}{n-2}} \geq 0 \end{aligned}$$

for all $0 \leq t \leq T$. By Corollary A.3, we can find a positive constant $c(T)$ such that

$$\inf_M u(t) \left(\sup_M u(t) \right)^{\frac{n+2}{n-2}} \geq c(T)$$

for all $0 \leq t \leq T$. Since $\sup_M u(t) \leq C(T)$, the assertion follows.

Lemma 2.5. *For every $T > 0$, there exists a constant $C(T)$ such that*

$$(14) \quad \int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} d\text{vol}_{g(t)} \leq C(T)$$

for all $0 \leq t \leq T$.

Proof. Using Lemma 2.2 with $p = \frac{n+2}{2} > 2$, we obtain

$$\sup_{0 \leq t \leq T} \int_M (R_{g(t)} + \sigma)^{\frac{n}{2}} d\text{vol}_{g_0} \leq C$$

and

$$\int_0^T \int_M |d(R_{g(t)} + \sigma)^{\frac{n}{4}}|_{g(t)}^2 d\text{vol}_{g(t)} dt \leq C.$$

Using Proposition 2.4 and Sobolev's inequality, we conclude that

$$\int_0^T \left(\int_M (R_{g(t)} + \sigma)^{\frac{n^2}{2(n-2)}} d\text{vol}_{g(t)} \right)^{\frac{n-2}{n}} dt \leq C(T).$$

From this, it follows that

$$\int_0^T \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} d\text{vol}_{g(t)} \right)^{\frac{n-2}{n}} dt \leq C(T).$$

We now apply Lemma 2.3 with $p = \frac{n^2}{2(n-2)} > \max\{\frac{n}{2}, 2\}$. This implies

$$\begin{aligned} & \frac{d}{dt} \log \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} dvol_{g(t)} \right) \\ & \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} dvol_{g(t)} \right)^{\frac{n-2}{n}} + C. \end{aligned}$$

From this, the assertion follows.

Proposition 2.6. *Let $0 < \alpha < \min\{\frac{4}{n}, 1\}$. Given any $T > 0$, there exists a constant $C(T)$ such that*

$$(15) \quad |u(x_1, t_1) - u(x_2, t_2)| \leq C(T) \left((t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^\alpha \right)$$

for all $x_1, x_2 \in M$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$.

Proof. Let $\alpha = 2 - \frac{n}{p}$, where $\frac{n}{2} < p < \min\{\frac{n^2}{2(n-2)}, n\}$. Using Lemma 2.5 and Proposition 2.4, we obtain

$$(16) \quad \int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} u(t) - R_{g_0} u(t) \right|^p dvol_{g_0} \leq C(T)$$

and

$$(17) \quad \int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dvol_{g(t)} \leq C(T)$$

for all $t \in [0, T]$. The first inequality implies that

$$|u(x_1, t) - u(x_2, t)| \leq C(T) d(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in M$ and $t \in [0, T]$. Using the second inequality, we obtain

$$\begin{aligned} & |u(x, t_1) - u(x, t_2)| \\ & \leq C (t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(x, t_1) - u(x, t_2)| dvol_{g_0} \\ & \leq C (t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1) - u(t_2)| dvol_{g_0} + C(T) (t_1 - t_2)^{\frac{\alpha}{2}} \\ & \leq C (t_1 - t_2)^{-\frac{n-2}{2}} \sup_{t_1 \geq t \geq t_2} \int_{B_{\sqrt{t_1-t_2}}(x)} \left| \frac{\partial}{\partial t} u(t) \right| dvol_{g_0} + C(T) (t_1 - t_2)^{\frac{\alpha}{2}} \\ & \leq C (t_1 - t_2)^{\frac{\alpha}{2}} \sup_{t_1 \geq t \geq t_2} \left(\int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dvol_{g_0} \right)^{\frac{1}{p}} + C(T) (t_1 - t_2)^{\frac{\alpha}{2}} \\ & \leq C(T) (t_1 - t_2)^{\frac{\alpha}{2}} \end{aligned}$$

for all $x \in M$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. This proves the assertion.

We can now use the standard regularity theory for parabolic equations (see [7], Theorem 5 on p. 64) to show that all higher order derivatives of u are uniformly bounded on every fixed time interval $[0, T]$. Therefore, the flow exists for all time.

3. Proof of the main result assuming Proposition 3.3

Proposition 3.1. *Fix $\max\{\frac{n}{2}, 2\} < p < \frac{n+2}{2}$. Then, we have*

$$(18) \quad \lim_{t \rightarrow \infty} \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} = 0.$$

Proof. It follows from Lemma 2.2 that

$$\begin{aligned} & \frac{d}{dt} \int_M (R_{g(t)} + \sigma)^{p-1} dvol_{g(t)} \\ & \leq -\left(\frac{n+2}{2} - p\right) \int_M ((R_{g(t)} + \sigma)^{p-1} - (r_{g(t)} + \sigma)^{p-1}) \\ & \quad \cdot (R_{g(t)} - r_{g(t)}) dvol_{g(t)}. \end{aligned}$$

Since $p > 2$, we have

$$((R_{g(t)} + \sigma)^{p-1} - (r_{g(t)} + \sigma)^{p-1}) (R_{g(t)} - r_{g(t)}) \geq c |R_{g(t)} - r_{g(t)}|^p$$

for a suitable constant $c > 0$. Since $p < \frac{n+2}{2}$, it follows that

$$\frac{d}{dt} \int_M (R_{g(t)} + \sigma)^{p-1} dvol_{g(t)} \leq -c \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)}.$$

Thus, we conclude that

$$\int_0^\infty \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} dt \leq C,$$

hence

$$\liminf_{t \rightarrow \infty} \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} dt = 0.$$

On the other hand, since $p > \max\{\frac{n}{2}, 2\}$, we have

$$\begin{aligned} \frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} & \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} \right)^{\frac{2p-n+2}{2p-n}} \\ & \quad + C \int_M |R_{g(t)} - r_{g(t)}|^p dvol_{g(t)} \end{aligned}$$

by Lemma 2.3. From this, the assertion follows.

Hence, if we define

$$(19) \quad r_\infty = \lim_{t \rightarrow \infty} r_{g(t)},$$

then we obtain the following result:

Corollary 3.2. *For every $1 < p < \frac{n+2}{2}$, we have*

$$(20) \quad \lim_{t \rightarrow \infty} \int_M |R_{g(t)} - r_\infty|^p \, d\text{vol}_{g(t)} = 0.$$

The proof of Theorem 1.1 will be based on the following proposition. The proof of Proposition 3.3 will occupy Sections 4–7.

Proposition 3.3. *Let $\{t_\nu : \nu \in \mathbb{N}\}$ be a sequence of times such that $t_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$. Then, we can find a real number $0 < \gamma < 1$ and a constant C such that, after passing to a subsequence, we have*

$$(21) \quad r_{g(t_\nu)} - r_\infty \leq C \left(\int_M u(t_\nu)^{\frac{2n}{n-2}} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)}$$

for all integers ν in that subsequence. Note that γ and C may depend on the sequence $\{t_\nu : \nu \in \mathbb{N}\}$.

The following result is an immediate consequence of Proposition 3.3.

Proposition 3.4. *There exist real numbers $0 < \gamma < 1$ and $t_0 > 0$ such that*

$$(22) \quad r_{g(t)} - r_\infty \leq \left(\int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - r_\infty|^{\frac{2n}{n+2}} \, d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)}$$

for all $t \geq t_0$.

Proof. Suppose this is not true. Then, there exists a sequence of times $\{t_\nu : \nu \in \mathbb{N}\}$ such that $t_\nu \geq \nu$ and

$$r_{g(t_\nu)} - r_\infty \geq \left(\int_M u(t_\nu)^{\frac{2n}{n-2}} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\frac{1}{\nu})}$$

for all $\nu \in \mathbb{N}$. We now apply Proposition 3.3 to this sequence $\{t_\nu : \nu \in \mathbb{N}\}$. Hence, there exists an infinite subset $I \subset \mathbb{N}$, a real number $0 < \gamma < 1$ and a real number C such that

$$r_{g(t_\nu)} - r_\infty \leq C \left(\int_M u(t_\nu)^{\frac{2n}{n-2}} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)}$$

for all $\nu \in I$. Thus, we conclude that

$$1 \leq C \left(\int_M u(t_\nu)^{\frac{2n}{n-2}} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(\gamma - \frac{1}{\nu})}$$

for all $\nu \in I$. On the other hand, we have

$$\lim_{\nu \rightarrow \infty} \left(\int_M u(t_\nu)^{\frac{2n}{n-2}} |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(\gamma - \frac{1}{\nu})} = 0$$

by Corollary 3.2. This is a contradiction.

Proposition 3.5. *We have*

$$(23) \quad \int_0^\infty \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g_0} \right)^{\frac{1}{2}} dt \leq C.$$

Proof. It follows from Proposition 3.4 that

$$\begin{aligned} r_{g(t)} - r_\infty &\leq C \left(\int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)} \\ &\quad + C (r_{g(t)} - r_\infty)^{1+\gamma}, \end{aligned}$$

hence,

$$(24) \quad r_{g(t)} - r_\infty \leq C \left(\int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)}$$

if t is sufficiently large. Therefore, we obtain

$$(25) \quad \begin{aligned} \frac{d}{dt} (r_{g(t)} - r_\infty) &= -\frac{n-2}{2} \int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g_0} \\ &\leq -\frac{n-2}{2} \left(\int_M u(t)^{\frac{2n}{n-2}} |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} d\text{vol}_{g_0} \right)^{\frac{n+2}{n}} \\ &\leq -c (r_{g(t)} - r_\infty)^{\frac{2}{1+\gamma}}, \end{aligned}$$

where c is a positive constant independent of t . This implies

$$(26) \quad \frac{d}{dt} (r_{g(t)} - r_\infty)^{-\frac{1-\gamma}{1+\gamma}} \geq c.$$

From this, it follows that

$$(r_{g(t)} - r_\infty)^{-\frac{1-\gamma}{1+\gamma}} \geq ct,$$

hence

$$(27) \quad r_{g(t)} - r_\infty \leq Ct^{-\frac{1+\gamma}{1-\gamma}}$$

if t is sufficiently large. Using Hölder's inequality, we obtain

$$(28) \quad \begin{aligned} \int_T^{2T} \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g_0} \right)^{\frac{1}{2}} dt \\ &\leq \left(T \int_T^{2T} \int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 d\text{vol}_{g_0} dt \right)^{\frac{1}{2}} \\ &\leq \left(\frac{2}{n-2} T (r_{g(T)} - r_{g(2T)}) \right)^{\frac{1}{2}} \\ &\leq CT^{-\frac{\gamma}{1-\gamma}} \end{aligned}$$

if T is sufficiently large. Since $0 < \gamma < 1$, we conclude that

$$\begin{aligned}
 (29) \quad & \int_0^\infty \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 dvol_{g_0} \right)^{\frac{1}{2}} dt \\
 &= \int_0^1 \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 dvol_{g_0} \right)^{\frac{1}{2}} dt \\
 &+ \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 dvol_{g_0} \right)^{\frac{1}{2}} dt \\
 &\leq C \sum_{k=0}^\infty 2^{-\frac{\gamma}{1-\gamma} k} \leq C.
 \end{aligned}$$

This proves the assertion.

Proposition 3.6. *Given any $\eta_0 > 0$, we can find a real number $r > 0$ such that*

$$\int_{B_r(x)} u(t)^{\frac{2n}{n-2}} dvol_{g_0} \leq \eta_0$$

for all $x \in M$ and $t \geq 0$.

Proof. We can find a real number $T > 0$ such that

$$(30) \quad \int_T^\infty \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 dvol_{g_0} \right)^{\frac{1}{2}} dt \leq \frac{\eta_0}{n}.$$

We now choose a real number $r > 0$ such that

$$(31) \quad \int_{B_r(x)} u(t)^{\frac{2n}{n-2}} dvol_{g_0} \leq \frac{\eta_0}{2}$$

for all $x \in M$ and $0 \leq t \leq T$. Then, we have

$$\begin{aligned}
 (32) \quad & \int_{B_r(x)} u(t)^{\frac{2n}{n-2}} dvol_{g_0} \\
 &\leq \int_{B_r(x)} u(T)^{\frac{2n}{n-2}} dvol_{g_0} \\
 &+ \frac{n}{2} \int_T^\infty \left(\int_M u(t)^{\frac{2n}{n-2}} (R_{g(t)} - r_{g(t)})^2 dvol_{g_0} \right)^{\frac{1}{2}} dt \leq \eta_0
 \end{aligned}$$

for all $x \in M$ and $t \geq T$. This proves the assertion.

Proposition 3.7. *The function $u(t)$ satisfies*

$$(33) \quad \sup_M u(t) \leq C$$

and

$$(34) \quad \inf_M u(t) \geq c$$

for all $t \geq 0$. Here, C and c are positive constants independent of t .

Proof. Fix $\frac{n}{2} < q < p < \frac{n+2}{2}$. By Corollary 3.2, we have

$$(35) \quad \int_M |R_{g(t)}|^p d\text{vol}_{g(t)} \leq C$$

for some constant C independent of t . By Proposition 3.6, we can find a constant $r > 0$ independent of t such that

$$(36) \quad \int_{B_r(x)} d\text{vol}_{g(t)} \leq \eta_0$$

for all $x \in M$ and $t \geq 0$. Using Hölder's inequality, we obtain

$$\int_{B_r(x)} |R_{g(t)}|^q d\text{vol}_{g(t)} \leq \left(\int_{B_r(x)} d\text{vol}_{g(t)} \right)^{\frac{p-q}{p}} \left(\int_{B_r(x)} |R_{g(t)}|^p d\text{vol}_{g(t)} \right)^{\frac{q}{p}}.$$

Hence, if we choose η_0 sufficiently small, then we have

$$(37) \quad \int_{B_r(x)} |R_{g(t)}|^q d\text{vol}_{g(t)} \leq \eta_1$$

for all $x \in M$ and $t \geq 0$. Here, η_1 is the constant appearing in Proposition A.1. Using Proposition A.1, we conclude that $u(t)$ is uniformly bounded from above. Hence, if we define

$$P = R_{g_0} + \sigma \left(\sup_{t \geq 0} \sup_M u(t) \right)^{\frac{4}{n-2}},$$

then we obtain

$$(38) \quad \begin{aligned} & -\frac{4(n-1)}{n-2} \Delta_{g_0} u(t) + P u(t) \\ & \geq -\frac{4(n-1)}{n-2} \Delta_{g_0} u(t) + R_{g_0} u(t) + \sigma u(t)^{\frac{n+2}{n-2}} \\ & = (R_{g(t)} + \sigma) u(t)^{\frac{n+2}{n-2}} \geq 0. \end{aligned}$$

According to Corollary A.3, we can find a positive constant c such that

$$\inf_M u(t) \left(\sup_M u(t) \right)^{\frac{n+2}{n-2}} \geq c$$

for all $t \geq 0$. Since $u(t)$ is uniformly bounded from above, we conclude that $u(t)$ is uniformly bounded from below.

Proposition 3.8. *Let $0 < \alpha < \frac{4}{n+2}$. Then, the function u satisfies*

$$(39) \quad |u(x_1, t_1) - u(x_2, t_2)| \leq C \left((t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^\alpha \right)$$

for all $x_1, x_2 \in M$ and $0 < t_1 - t_2 < 1$. Here, C is a positive constant independent of t_1 and t_2 .

Proof. Let $\alpha = 2 - \frac{n}{p}$, where $\frac{n}{2} < p < \frac{n+2}{2}$. Using Corollary 3.2 and Proposition 3.7, we obtain

$$(40) \quad \int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} u(t) - R_{g_0} u(t) \right|^p dvol_{g_0} \leq C$$

and

$$(41) \quad \int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dvol_{g(t)} \leq C,$$

where C is a positive constant independent of t . The first inequality implies that

$$|u(x_1, t) - u(x_2, t)| \leq C d(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in M$ and $t \geq 0$. Using the second inequality, we obtain

$$\begin{aligned} & |u(x, t_1) - u(x, t_2)| \\ & \leq C (t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(x, t_1) - u(x, t_2)| dvol_{g_0} \\ & \leq C (t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1) - u(t_2)| dvol_{g_0} + C (t_1 - t_2)^{\frac{\alpha}{2}} \\ & \leq C (t_1 - t_2)^{-\frac{n-2}{2}} \sup_{t_1 \geq t \geq t_2} \int_{B_{\sqrt{t_1-t_2}}(x)} \left| \frac{\partial}{\partial t} u(t) \right| dvol_{g_0} + C (t_1 - t_2)^{\frac{\alpha}{2}} \\ & \leq C (t_1 - t_2)^{\frac{\alpha}{2}} \sup_{t_1 \geq t \geq t_2} \left(\int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dvol_{g_0} \right)^{\frac{1}{p}} + C (t_1 - t_2)^{\frac{\alpha}{2}} \\ & \leq C (t_1 - t_2)^{\frac{\alpha}{2}} \end{aligned}$$

for all $x \in M$ and $0 < t_1 - t_2 < 1$. This proves the assertion.

In view of Proposition 3.8, we may apply the standard regularity theory for parabolic equations (see [7], Theorem 5 on p. 64) to derive uniform estimates for all higher order derivatives of u . The uniqueness of the asymptotic limit follows from Proposition 3.5. This completes the proof of Theorem 1.1.

4. Blow-up analysis

The remaining part of this paper will be concerned with the proof of Proposition 3.3. Let $\{t_\nu : \nu \in \mathbb{N}\}$ be a sequence of times such that $t_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$. For abbreviation, let $u_\nu = u(t_\nu)$ and $g_\nu = g(t_\nu) = u(t_\nu)^{\frac{4}{n-2}} g_0 = u_\nu^{\frac{4}{n-2}} g_0$. The normalization condition implies that

$$\int_M d\text{vol}_{g_\nu} = 1,$$

hence

$$(42) \quad \int_M u_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} = 1$$

for all $\nu \in \mathbb{N}$. Moreover, it follows from Corollary 3.2 that

$$\int_M |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} d\text{vol}_{g_\nu} \rightarrow 0,$$

hence

$$(43) \quad \int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} u_\nu - R_{g_0} u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right|^{\frac{2n}{n+2}} d\text{vol}_{g_0} \rightarrow 0$$

as $\nu \rightarrow \infty$.

At this point, we may apply the following compactness result due to Struwe [19]. A similar result for the harmonic map heat flow can be found in [12].

Proposition 4.1. *Let $\{u_\nu : \nu \in \mathbb{N}\}$ be a sequence of positive functions satisfying (42) and (43). After passing to a subsequence if necessary, we can find a non-negative integer m , a non-negative smooth function u_∞ and a sequence of m -tuplets $(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)_{1 \leq k \leq m}$ with the following properties:*

(i) *The function u_∞ satisfies the equation*

$$(44) \quad \frac{4(n-1)}{n-2} \Delta_{g_0} u_\infty - R_{g_0} u_\infty + r_\infty u_\infty^{\frac{n+2}{n-2}} = 0.$$

(ii) *For all $i \neq j$, we have*

$$(45) \quad \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{d(x_{i,\nu}^*, x_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*} \rightarrow \infty$$

as $\nu \rightarrow \infty$.

(iii) *We have*

$$(46) \quad \left\| u_\nu - u_\infty - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{H^1(M)} \rightarrow 0$$

as $\nu \rightarrow \infty$. Here, the functions $\bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}$ are the standard test functions constructed in Appendix B.

Proof. The first and the third statement follow from results of Struwe [19]. The second statement is due to Bahri and Coron [3].

Proposition 4.2. *If u_∞ vanishes at one point in M , then u_∞ vanishes everywhere.*

Proof. By Proposition 4.1, the function u_∞ satisfies

$$-\frac{4(n-1)}{n-2} \Delta_{g_0} u_\infty + R_{g_0} u_\infty = r_\infty u_\infty^{\frac{n+2}{n-2}} \geq 0.$$

By assumption, the background metric g_0 has positive scalar curvature. Hence, if u_∞ attains a non-positive minimum, then u_∞ is constant by the strong maximum principle (see [8], Theorem 8.19 on p. 198).

The cases $u_\infty \equiv 0$ and $u_\infty > 0$ need to be discussed separately. The case $u_\infty \equiv 0$ will be studied in Section 5. In Section 6, we deal with the case $u_\infty > 0$. The proof of Proposition 3.3 will be completed in Section 7.

For convenience, we define two functionals $E(u)$ and $F(u)$ by

$$(47) \quad E(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |du|_{g_0}^2 + R_{g_0} u^2 \right) dvol_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}}}$$

and

$$(48) \quad F(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |du|_{g_0}^2 + R_{g_0} u^2 \right) dvol_{g_0}}{\int_M u^{\frac{2n}{n-2}} dvol_{g_0}}.$$

Then, we have

$$\begin{aligned} 1 &= \lim_{\nu \rightarrow \infty} \int_M u_\nu^{\frac{2n}{n-2}} dvol_{g_0} \\ &= \lim_{\nu \rightarrow \infty} \left(\int_M u_\infty^{\frac{2n}{n-2}} dvol_{g_0} + \sum_{k=1}^m \int_M \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}^{\frac{2n}{n-2}} dvol_{g_0} \right) \\ &= \left(\frac{E(u_\infty)}{r_\infty} \right)^{\frac{n}{2}} + m \left(\frac{Y(S^n)}{r_\infty} \right)^{\frac{n}{2}}, \end{aligned}$$

hence,

$$(49) \quad r_\infty = \left(E(u_\infty)^{\frac{n}{2}} + m Y(S^n)^{\frac{n}{2}} \right)^{\frac{2}{n}}$$

(compare [17], Lemma 3.4).

5. The case $u_\infty \equiv 0$

Throughout this section, we will assume that $u_\infty \equiv 0$. For every $\nu \in \mathbb{N}$, we denote by \mathcal{A}_ν the set of all m -tuplets $(x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m$ such that

$$(50) \quad d(x_k, x_{k,\nu}^*) \leq \varepsilon_{k,\nu}^*, \quad \frac{1}{2} \leq \frac{\varepsilon_k}{\varepsilon_{k,\nu}^*} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2$$

for all $1 \leq k \leq m$. Moreover, we can find an m -tuple $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \in \mathcal{A}_\nu$ such that

$$(51) \quad \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) \right|_{g_0}^2 + R_{g_0} \left(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right)^2 \right) d\text{vol}_{g_0} \\ \leq \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \varepsilon_k)} \right) \right|_{g_0}^2 + R_{g_0} \left(u_\nu - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \varepsilon_k)} \right)^2 \right) d\text{vol}_{g_0}$$

for all $(x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in \mathcal{A}_\nu$.

Proposition 5.1.

(i) For all $i \neq j$, we have

$$(52) \quad \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \rightarrow \infty$$

as $\nu \rightarrow \infty$.

(ii) We have

$$(53) \quad \left\| u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{H^1(M)} \rightarrow 0$$

as $\nu \rightarrow \infty$.

Proof. (i) Since $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \in \mathcal{A}_\nu$, we have

$$32 \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 32 \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + 8 \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \\ \geq 8 \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 8 \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + 2 \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*}$$

$$\begin{aligned} &\geq 4 \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4 \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{(d(x_{i,\nu}, x_{j,\nu}) + \varepsilon_{i,\nu}^* + \varepsilon_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*} \\ &\geq 4 \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4 \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{d(x_{i,\nu}^*, x_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*}, \end{aligned}$$

and the expression on the right-hand side tends to ∞ as $\nu \rightarrow \infty$.

(ii) By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\begin{aligned} &\int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) \right|_{g_0}^2 \right. \\ &\quad \left. + R_{g_0} \left(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right)^2 \right) d\text{vol}_{g_0} \\ &\leq \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right) \right|_{g_0}^2 \right. \\ &\quad \left. + R_{g_0} \left(u_\nu - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right)^2 \right) d\text{vol}_{g_0}. \end{aligned}$$

By Proposition 4.1, the expression on the right-hand side tends to 0 as $\nu \rightarrow \infty$. This proves the assertion.

Proposition 5.2. *We have*

$$(54) \quad d(x_{k,\nu}, x_{k,\nu}^*) \leq o(1) \varepsilon_{k,\nu}^*, \quad \frac{\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^*} = 1 + o(1), \quad \alpha_{k,\nu} = 1 + o(1)$$

for all $1 \leq k \leq m$. In particular, $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$ is an interior point of \mathcal{A}_ν if ν is sufficiently large.

Proof. Observe that

$$\begin{aligned} &\left\| \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{H^1(M)} \\ &\leq \left\| u_\nu - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{H^1(M)} + \left\| u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{H^1(M)} \\ &= o(1) \end{aligned}$$

by Propositions 4.1 and 5.1. From this, the assertion follows.

In the sequel, we assume that

$$(55) \quad \varepsilon_{i,\nu} \leq \varepsilon_{j,\nu} \quad \text{for } i \leq j.$$

We now decompose the function u_ν as

$$(56) \quad u_\nu = v_\nu + w_\nu,$$

where

$$(57) \quad v_\nu = \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})},$$

and

$$(58) \quad w_\nu = u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}.$$

Note that the function w_ν satisfies

$$(59) \quad \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0} = o(1)$$

by Proposition 5.1.

Proposition 5.3.

(i) For every $1 \leq k \leq m$, we have

$$(60) \quad \left| \int_M \frac{\frac{n+2}{n-2}}{\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}} w_\nu dvol_{g_0} \right| \leq o(1) \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

(ii) For every $1 \leq k \leq m$, we have

$$(61) \quad \left| \int_M \frac{\frac{n+2}{n-2}}{\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}} \frac{\varepsilon_{k,\nu}^2 - d(x_{k,\nu}, x)^2}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2} w_\nu dvol_{g_0} \right| \leq o(1) \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

(iii) For all $1 \leq k \leq m$, we have

$$(62) \quad \left| \int_M \frac{\frac{n+2}{n-2}}{\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}} \frac{\varepsilon_{k,\nu} \exp^{-1}(x)}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2} w_\nu dvol_{g_0} \right| \leq o(1) \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

Proof. By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\int_M \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}, dw_\nu \rangle_{g_0} + R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} w_\nu \right) dvol_{g_0} = 0,$$

hence,

$$\int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) w_\nu dvol_{g_0} = 0$$

for all $1 \leq k \leq m$. Using the estimate

$$\left\| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} + r_\infty \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(M)} = o(1),$$

we conclude that

$$\left| \int_M \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} w_\nu \, dvol_{g_0} \right| \leq o(1) \|w_\nu\|_{L^{\frac{2n}{n-2}}(M)}$$

for all $1 \leq k \leq m$. This proves (i).

The remaining statements follow similarly.

In the next step, we prove uniform estimates for the second variation operator of the Yamabe functional at v_ν . A similar estimate was derived by Bahri (see [4], Proposition 3.1 on p. 64).

Proposition 5.4. *If ν is sufficiently large, then we have*

$$\begin{aligned} (63) \quad & \frac{n+2}{n-2} r_\infty \int_M \sum_{j=1}^m \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} w_\nu^2 \, dvol_{g_0} \\ & \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) \, dvol_{g_0} \end{aligned}$$

for some positive constant c independent of ν .

Proof. Suppose this is not true. Upon rescaling, we obtain a sequence of functions $\{\tilde{w}_\nu : \nu \in \mathbb{N}\}$ such that

$$(64) \quad \int_M \left(\frac{4(n-1)}{n-2} |d\tilde{w}_\nu|_{g_0}^2 + R_{g_0} \tilde{w}_\nu^2 \right) \, dvol_{g_0} = 1$$

and

$$(65) \quad \lim_{\nu \rightarrow \infty} \frac{n+2}{n-2} r_\infty \int_M \sum_{j=1}^m \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \tilde{w}_\nu^2 \, dvol_{g_0} \geq 1.$$

Note that

$$(66) \quad \int_M |\tilde{w}_\nu|^{\frac{2n}{n-2}} \, dvol_{g_0} \leq Y(M, g_0)^{-\frac{n}{n-2}}$$

by (64). In view of Proposition 5.1, we can find a sequence $\{N_\nu : \nu \in \mathbb{N}\}$ such that $N_\nu \rightarrow \infty$, $N_\nu \varepsilon_{j,\nu} \rightarrow 0$ for all $1 \leq j \leq m$, and

$$(67) \quad \frac{1}{N_\nu} \frac{\varepsilon_{j,\nu} + d(x_{i,\nu}, x_{j,\nu})}{\varepsilon_{i,\nu}} \rightarrow \infty$$

for all $i < j$. Let

$$(68) \quad \Omega_{j,\nu} = B_{N_\nu \varepsilon_{j,\nu}}(x_{j,\nu}) \setminus \bigcup_{i=1}^{j-1} B_{N_\nu \varepsilon_{i,\nu}}(x_{i,\nu})$$

for every $1 \leq j \leq m$. In view of (64) and (65), we can find an integer $1 \leq j \leq m$ such that

$$(69) \quad \lim_{\nu \rightarrow \infty} \int_M \bar{u}_{(x_j, \nu, \varepsilon_j, \nu)}^{\frac{4}{n-2}} \tilde{w}_\nu^2 d\text{vol}_{g_0} > 0$$

and

$$(70) \quad \begin{aligned} \lim_{\nu \rightarrow \infty} \int_{\Omega_{j, \nu}} \left(\frac{4(n-1)}{n-2} |d\tilde{w}_\nu|_{g_0}^2 + R_{g_0} \tilde{w}_\nu^2 \right) d\text{vol}_{g_0} \\ \leq \lim_{\nu \rightarrow \infty} \frac{n+2}{n-2} r_\infty \int_M \bar{u}_{(x_j, \nu, \varepsilon_j, \nu)}^{\frac{4}{n-2}} \tilde{w}_\nu^2 d\text{vol}_{g_0}. \end{aligned}$$

We now define a sequence of functions $\hat{w}_\nu : TM_{x_j, \nu} \rightarrow \mathbb{R}$ by

$$\hat{w}_\nu(\xi) = \varepsilon_{j, \nu}^{\frac{n-2}{2}} \tilde{w}_\nu(\exp_{x_j, \nu}(\varepsilon_{j, \nu} \xi))$$

for $\xi \in TM_{x_j, \nu}$. The sequence $\{\hat{w}_\nu : \nu \in \mathbb{N}\}$ satisfies

$$\lim_{\nu \rightarrow \infty} \int_{\{\xi \in TM_{x_j, \nu} : |\xi| \leq N_\nu\}} \frac{4(n-1)}{n-2} |d\hat{w}_\nu(\xi)|^2 d\xi \leq 1$$

and

$$\lim_{\nu \rightarrow \infty} \int_{\{\xi \in TM_{x_j, \nu} : |\xi| \leq N_\nu\}} |\hat{w}_\nu(\xi)|^{\frac{2n}{n-2}} d\xi \leq Y(M, g_0)^{-\frac{n}{n-2}}.$$

Hence, if we take the weak limit as $\nu \rightarrow \infty$, then we obtain a function $\hat{w} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 d\xi > 0$$

and

$$\int_{\mathbb{R}^n} |d\hat{w}(\xi)|^2 d\xi \leq n(n+2) \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 d\xi.$$

Moreover, it follows from Proposition 5.3 that

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \hat{w}(\xi) d\xi &= 0 \\ \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{1 - |\xi|^2}{1 + |\xi|^2} \hat{w}(\xi) d\xi &= 0 \\ \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{\xi}{1 + |\xi|^2} \hat{w}(\xi) d\xi &= 0. \end{aligned}$$

Using a result of Rey, we conclude that $\hat{w}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ (see [13], Appendix D, pp. 49–51). This is a contradiction.

Corollary 5.5. *If ν is sufficiently large, then we have*

$$(71) \quad \begin{aligned} & \frac{n+2}{n-2} r_\infty \int_M v_\nu^{\frac{4}{n-2}} w_\nu^2 dvol_{g_0} \\ & \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0} \end{aligned}$$

for some positive constant c independent of ν .

Proof. By definition of v_ν , we have

$$\int_M \left| v_\nu^{\frac{4}{n-2}} - \sum_{j=1}^m \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \right|^{\frac{n}{2}} dvol_{g_0} = o(1).$$

Therefore, the assertion follows from Proposition 5.4.

Proposition 5.6. *The Yamabe energy of v_ν satisfies the estimate*

$$(72) \quad E(v_\nu) \leq \left(\sum_{k=1}^m E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}}$$

if ν is sufficiently large.

Proof. Using the identity

$$(73) \quad \begin{aligned} & \int_M \left(\frac{4(n-1)}{n-2} |dv_\nu|_{g_0}^2 + R_{g_0} v_\nu^2 \right) dvol_{g_0} \\ & = \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 \left(\frac{4(n-1)}{n-2} |d\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}|_{g_0}^2 + R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^2 \right) dvol_{g_0} \\ & \quad + 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}, d\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \rangle_{g_0} \right. \\ & \quad \left. + R_{g_0} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right) dvol_{g_0}, \end{aligned}$$

we obtain

$$(74) \quad \begin{aligned} & E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}} \\ & = \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} dvol_{g_0} \\ & \quad - 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \end{aligned}$$

$$\cdot \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right) d\text{vol}_{g_0}.$$

Moreover, we have

$$\begin{aligned} (75) \quad & \left(\sum_{k=1}^m E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\ &= \left(\int_M \left(\sum_{k=1}^m F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right) d\text{vol}_{g_0} \right)^{\frac{2}{n}} \\ &\quad \cdot \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\ &\geq \int_M \left(\sum_{k=1}^m F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} v_\nu^2 d\text{vol}_{g_0} \\ &\geq \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} d\text{vol}_{g_0} \\ &\quad + 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \left(F(\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}^{\frac{2n}{n-2}} \right. \\ &\quad \left. + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} d\text{vol}_{g_0} \end{aligned}$$

by Hölder's inequality. Consider a pair $i < j$. We can find positive constants c and C independent of ν such that

$$\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}^{\frac{n+2}{n-2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}(x) \geq c \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \varepsilon_{i,\nu}^{-n}$$

and

$$\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}(x) \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \leq C \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n+2}{2}} \varepsilon_{i,\nu}^{-n}$$

if $d(x_{i,\nu}, x) \leq \varepsilon_{i,\nu}$ and ν is sufficiently large. From this, it follows that

$$\begin{aligned} & \left(F(\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \\ & \quad \cdot \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \\ & \geq F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \\ & \quad + c \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \varepsilon_{i,\nu}^{-n} 1_{\{d(x_{i,\nu}, x) \leq \varepsilon_{i,\nu}\}} \end{aligned}$$

for ν sufficiently large. Integration over M yields

$$\begin{aligned}
 (76) \quad & \int_M \left(F(\bar{u}(x_{i,\nu}, \varepsilon_{i,\nu}))^{\frac{n}{2}} \bar{u}(x_{i,\nu}, \varepsilon_{i,\nu})^{\frac{2n}{n-2}} + F(\bar{u}(x_{i,\nu}, \varepsilon_{i,\nu}))^{\frac{n}{2}} \bar{u}(x_{i,\nu}, \varepsilon_{i,\nu})^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \\
 & \cdot \bar{u}(x_{i,\nu}, \varepsilon_{i,\nu}) \bar{u}(x_{j,\nu}, \varepsilon_{j,\nu}) \, d\text{vol}_{g_0} \\
 & \geq \int_M F(\bar{u}(x_{j,\nu}, \varepsilon_{j,\nu})) \bar{u}(x_{i,\nu}, \varepsilon_{i,\nu}) \bar{u}(x_{j,\nu}, \varepsilon_{j,\nu})^{\frac{n+2}{n-2}} \, d\text{vol}_{g_0} \\
 & + c \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}
 \end{aligned}$$

if ν is sufficiently large. From this, it follows that

$$\begin{aligned}
 (77) \quad & \left(\sum_{k=1}^m E(\bar{u}(x_{k,\nu}, \varepsilon_{k,\nu}))^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
 & \geq \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\bar{u}(x_{k,\nu}, \varepsilon_{k,\nu})) \bar{u}(x_{k,\nu}, \varepsilon_{k,\nu})^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \\
 & + 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} F(\bar{u}(x_{j,\nu}, \varepsilon_{j,\nu})) \bar{u}(x_{i,\nu}, \varepsilon_{i,\nu}) \bar{u}(x_{j,\nu}, \varepsilon_{j,\nu})^{\frac{n+2}{n-2}} \, d\text{vol}_{g_0} \\
 & + c \sum_{i < j} \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}.
 \end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}
 (78) \quad & E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
 & \leq \left(\sum_{k=1}^m E(\bar{u}(x_{k,\nu}, \varepsilon_{k,\nu}))^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
 & - 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \bar{u}(x_{i,\nu}, \varepsilon_{i,\nu}) \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}(x_{j,\nu}, \varepsilon_{j,\nu}) \right. \\
 & \quad \left. - R_{g_0} \bar{u}(x_{j,\nu}, \varepsilon_{j,\nu}) + F(\bar{u}(x_{j,\nu}, \varepsilon_{j,\nu})) \bar{u}(x_{j,\nu}, \varepsilon_{j,\nu})^{\frac{n+2}{n-2}} \right) \, d\text{vol}_{g_0} \\
 & - c \sum_{i < j} \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}.
 \end{aligned}$$

Since $F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) = r_\infty + o(1)$, it follows from Lemmas B.4 and B.5 that

(79)

$$\begin{aligned}
& \int_M \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right. \\
& \quad \left. - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \right| d\text{vol}_{g_0} \\
& \leq \int_M \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right. \\
& \quad \left. - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} + r_\infty \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \right| d\text{vol}_{g_0} \\
& \quad + |F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) - r_\infty| \int_M \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} d\text{vol}_{g_0} \\
& \leq C \left(\delta^4 + \delta^{n-2} + \frac{\varepsilon_{j,\nu}^2}{\delta^2} \right) \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \\
& \quad + o(1) \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}
\end{aligned}$$

for $i < j$. Hence, if we choose δ sufficiently small, the assertion follows.

Corollary 5.7. *If ν is sufficiently large, then the Yamabe energy of v_ν satisfies the estimate*

$$(80) \quad E(v_\nu) \leq (m Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}}.$$

Proof. Using Proposition B.3, we obtain

$$E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \leq Y(S^n)$$

for all $1 \leq k \leq m$. Hence, the assertion follows from Proposition 5.6.

6. The case $u_\infty > 0$

We next discuss the case $u_\infty > 0$.

Proposition 6.1. *There exists a sequence of smooth functions $\{\psi_a : a \in \mathbb{N}\}$ and a sequence of positive real numbers $\{\lambda_a : a \in \mathbb{N}\}$ with the following properties:*

(i) *For every $a \in \mathbb{N}$, the function ψ_a satisfies the equation*

$$(81) \quad \frac{4(n-1)}{n-2} \Delta_{g_0} \psi_a - R_{g_0} \psi_a + \lambda_a u_\infty^{\frac{4}{n-2}} \psi_a = 0.$$

(ii) For all $a, b \in \mathbb{N}$, we have

$$(82) \quad \int_M u_\infty^{\frac{4}{n-2}} \psi_a \psi_b d\text{vol}_{g_0} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b \end{cases}.$$

(iii) The span of $\{\psi_a : a \in \mathbb{N}\}$ is dense in $L^2(M)$.

(iv) $\lambda_a \rightarrow \infty$ as $a \rightarrow \infty$.

Proof. Consider the linear operator

$$\psi \mapsto u_\infty^{-\frac{4}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \psi - R_{g_0} \psi \right).$$

This operator is symmetric with respect to the inner product

$$(\psi_1, \psi_2) \mapsto \int_M u_\infty^{\frac{4}{n-2}} \psi_1 \psi_2 d\text{vol}_{g_0}$$

on $L^2(M)$. Hence, the assertion follows from the spectral theorem.

Let A be a finite subset of \mathbb{N} such that $\lambda_a > \frac{n+2}{n-2} r_\infty$ for all $a \notin A$. We denote by Π the projection operator

$$(83) \quad \begin{aligned} \Pi f &= \sum_{a \notin A} \left(\int_M \psi_a f d\text{vol}_{g_0} \right) u_\infty^{\frac{4}{n-2}} \psi_a \\ &= f - \sum_{a \in A} \left(\int_M \psi_a f d\text{vol}_{g_0} \right) u_\infty^{\frac{4}{n-2}} \psi_a. \end{aligned}$$

Lemma 6.2. For every $1 \leq p < \infty$, we can find a constant C such that

$$(84) \quad \begin{aligned} \|f\|_{L^p(M)} &\leq C \left\| \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right\|_{L^p(M)} \\ &\quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f d\text{vol}_{g_0} \right|. \end{aligned}$$

Proof. Assume that this is not true. By compactness, we can find a function $f \in L^p(M)$ satisfying $\|f\|_{L^p(M)} = 1$,

$$(85) \quad \int_M u_\infty^{\frac{4}{n-2}} \psi_a f d\text{vol}_{g_0} = 0$$

for all $a \in A$ and

$$(86) \quad \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f = 0$$

in the sense of distributions. Hence, if we use the function ψ_a as a test function, then we obtain

$$\left(\lambda_a - \frac{n+2}{n-2} r_\infty \right) \int_M u_\infty^{\frac{4}{n-2}} \psi_a f d\text{vol}_{g_0} = 0$$

for all $a \in \mathbb{N}$. In particular, we have

$$(87) \quad \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} = 0$$

for all $a \notin A$. Thus, we conclude that $f = 0$. This is a contradiction.

Lemma 6.3.

(i) *There exists a constant C such that*

$$(88) \quad \|f\|_{L^{\frac{n+2}{n-2}}(M)} \leq C \left\| \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right) \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right|.$$

(ii) *There exists a constant C such that*

$$(89) \quad \|f\|_{L^1(M)} \leq C \left\| \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right) \right\|_{L^1(M)} \\ + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right|.$$

Proof. (i) It follows from standard elliptic regularity theory that

$$(90) \quad \|f\|_{L^{\frac{n+2}{n-2}}(M)} \leq C \left\| \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ + C \|f\|_{L^{\frac{n(n+2)}{n^2+4}}(M)}.$$

Using Lemma 6.2, we obtain

$$(91) \quad \|f\|_{L^{\frac{n+2}{n-2}}(M)} \leq C \left\| \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right|.$$

By definition of Π , we have

$$\begin{aligned}
 (92) \quad & \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \\
 & = \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + r_\infty u_\infty^{\frac{4}{n-2}} f \right) \\
 & \quad - \sum_{a \in A} \left(\lambda_a - \frac{n+2}{n-2} r_\infty \right) \left(\int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right) u_\infty^{\frac{4}{n-2}} \psi_a.
 \end{aligned}$$

This implies

$$\begin{aligned}
 (93) \quad & \left\| \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right\|_{L^q(M)} \\
 & \leq \left\| \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + r_\infty u_\infty^{\frac{4}{n-2}} f \right) \right\|_{L^q(M)} \\
 & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right|.
 \end{aligned}$$

Putting these facts together, the assertion follows.

(ii) It follows from Lemma 6.2 that

$$\begin{aligned}
 (94) \quad \|f\|_{L^1(M)} & \leq C \left\| \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right\|_{L^1(M)} \\
 & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right|.
 \end{aligned}$$

By definition of Π , we have

$$\begin{aligned}
 (95) \quad & \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \\
 & = \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + r_\infty u_\infty^{\frac{4}{n-2}} f \right) \\
 & \quad - \sum_{a \in A} \left(\lambda_a - \frac{n+2}{n-2} r_\infty \right) \left(\int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right) u_\infty^{\frac{4}{n-2}} \psi_a.
 \end{aligned}$$

This implies

$$\begin{aligned}
 (96) \quad & \left\| \frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} f \right\|_{L^1(M)} \\
 & \leq \left\| \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} f - R_{g_0} f + r_\infty u_\infty^{\frac{4}{n-2}} f \right) \right\|_{L^1(M)} \\
 & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a f \, d\text{vol}_{g_0} \right|.
 \end{aligned}$$

From this, the assertion follows.

Lemma 6.4. *There exists a positive real number ζ with the following significance: for every vector $z \in \mathbb{R}^A$ with $|z| \leq \zeta$, there exists a smooth function \bar{u}_z such that*

$$(97) \quad \int_M u_\infty^{\frac{4}{n-2}} (\bar{u}_z - u_\infty) \psi_a \, d\text{vol}_{g_0} = z_a$$

for all $a \in A$ and

$$(98) \quad \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) = 0.$$

Furthermore, the map $z \mapsto \bar{u}_z$ is real analytic.

Proof. This is a consequence of the implicate function theorem.

Lemma 6.5. *There exists a real number $0 < \gamma < 1$ such that*

$$(99) \quad \begin{aligned} & E(\bar{u}_z) - E(u_\infty) \\ & \leq C \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) \, d\text{vol}_{g_0} \right|^{1+\gamma} \end{aligned}$$

if z is sufficiently small.

Proof. Note that the function $z \mapsto E(\bar{u}_z)$ is real analytic. According to results of Lojasiewicz (see [18], equation (2.4) on p. 538), there exists a real number $0 < \gamma < 1$ such that

$$(100) \quad |E(\bar{u}_z) - E(u_\infty)| \leq \sup_{a \in A} \left| \frac{\partial}{\partial z_a} E(\bar{u}_z) \right|^{1+\gamma}$$

if z is sufficiently small. The partial derivatives of the function $z \mapsto E(\bar{u}_z)$ are given by

$$(101) \quad \begin{aligned} \frac{\partial}{\partial z_a} E(\bar{u}_z) &= -2 \frac{\int_M \tilde{\psi}_{a,z} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) \, d\text{vol}_{g_0}}{\left(\int_M \bar{u}_z^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \right)^{\frac{n-2}{n}}} \\ &\quad - 2 (F(\bar{u}_z) - r_\infty) \frac{\int_M \bar{u}_z^{\frac{n+2}{n-2}} \tilde{\psi}_{a,z} \, d\text{vol}_{g_0}}{\left(\int_M \bar{u}_z^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \right)^{\frac{n-2}{n}}}, \end{aligned}$$

where $\tilde{\psi}_{a,z} = \frac{\partial}{\partial z_a} \bar{u}_z$ for $a \in A$. The function $\tilde{\psi}_{a,z}$ satisfies

$$(102) \quad \int_M u_\infty^{\frac{4}{n-2}} \tilde{\psi}_{a,z} \psi_b \, dvol_{g_0} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b \end{cases}$$

for all $b \in A$ and

$$(103) \quad \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \tilde{\psi}_{a,z} - R_{g_0} \tilde{\psi}_{a,z} + \frac{n+2}{n-2} r_\infty \bar{u}_z^{\frac{4}{n-2}} \tilde{\psi}_{a,z} \right) = 0.$$

Using the identity

$$\Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) = 0,$$

we obtain

$$(104) \quad \begin{aligned} \frac{\partial}{\partial z_a} E(\bar{u}_z) = & -2 \frac{\int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) \, dvol_{g_0}}{\left(\int_M \bar{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0} \right)^{\frac{n-2}{n}}} \\ & + 2 \sum_{b \in A} \frac{\int_M \bar{u}_z^{\frac{n+2}{n-2}} \tilde{\psi}_{a,z} \, dvol_{g_0} \int_M u_\infty^{\frac{4}{n-2}} \bar{u}_z \psi_b \, dvol_{g_0}}{\int_M \bar{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0}} \\ & \cdot \frac{\int_M \psi_b \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) \, dvol_{g_0}}{\left(\int_M \bar{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0} \right)^{\frac{n-2}{n}}} \end{aligned}$$

for all $a \in A$. Thus, we conclude that

$$(105) \quad \begin{aligned} & \sup_{a \in A} \left| \frac{\partial}{\partial z_a} E(\bar{u}_z) \right| \\ & \leq C \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_z - R_{g_0} \bar{u}_z + r_\infty \bar{u}_z^{\frac{n+2}{n-2}} \right) \, dvol_{g_0} \right|. \end{aligned}$$

From this, the assertion follows.

For every $\nu \in \mathbb{N}$, we denote by \mathcal{A}_ν the set of all pairs $(z, (x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m}) \in \mathbb{R}^A \times (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m$ such that

$$(106) \quad |z| \leq \zeta$$

and

$$(107) \quad d(x_k, x_{k,\nu}^*) \leq \varepsilon_{k,\nu}^*, \quad \frac{1}{2} \leq \frac{\varepsilon_k}{\varepsilon_{k,\nu}^*} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2$$

for all $1 \leq k \leq m$. Moreover, we can find a pair

$$(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}) \in \mathcal{A}_\nu$$

such that

$$(108) \quad \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) \right|_{g_0}^2 + R_{g_0} \left(u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right)^2 \right) d\text{vol}_{g_0} \\ \leq \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \bar{u}_z - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \varepsilon_k)} \right) \right|_{g_0}^2 + R_{g_0} \left(u_\nu - \bar{u}_z - \sum_{k=1}^m \alpha_k \bar{u}_{(x_k, \varepsilon_k)} \right)^2 \right) d\text{vol}_{g_0}$$

for all $(z, (x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m}) \in \mathcal{A}_\nu$.

Proposition 6.6.

(i) For all $i \neq j$, we have

$$(109) \quad \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \rightarrow \infty$$

as $\nu \rightarrow \infty$.

(ii) We have

$$(110) \quad \left\| u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{H^1(M)} \rightarrow 0$$

as $\nu \rightarrow \infty$.

Proof. (i) Since $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \in \mathcal{A}_\nu$, we have

$$32 \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 32 \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + 8 \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \\ \geq 8 \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 8 \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + 2 \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*} \\ \geq 4 \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4 \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{(d(x_{i,\nu}, x_{j,\nu}) + \varepsilon_{i,\nu}^* + \varepsilon_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*} \\ \geq 4 \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4 \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{d(x_{i,\nu}^*, x_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^* \varepsilon_{j,\nu}^*},$$

and the expression on the right-hand side tends to ∞ as $\nu \rightarrow \infty$.

(ii) By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\begin{aligned} & \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) \right|_{g_0}^2 \right. \\ & \quad \left. + R_{g_0} \left(u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right)^2 \right) dvol_{g_0} \\ & \leq \int_M \left(\frac{4(n-1)}{n-2} \left| d \left(u_\nu - u_\infty - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right) \right|_{g_0}^2 \right. \\ & \quad \left. + R_{g_0} \left(u_\nu - u_\infty - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right)^2 \right) dvol_{g_0}. \end{aligned}$$

By Proposition 4.1, the expression on the right-hand side tends to 0 as $\nu \rightarrow \infty$. This proves the assertion.

Proposition 6.7. *We have*

$$(111) \quad |z_\nu| = o(1)$$

and

$$(112) \quad d(x_{k,\nu}, x_{k,\nu}^*) \leq o(1) \varepsilon_{k,\nu}^*, \quad \frac{\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^*} = 1 + o(1), \quad \alpha_{k,\nu} = 1 + o(1)$$

for all $1 \leq k \leq m$. In particular, $(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m})$ is an interior point of \mathcal{A}_ν if ν is sufficiently large.

Proof. Observe that

$$\begin{aligned} & \left\| \bar{u}_{z_\nu} + \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - u_\infty - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{H^1(M)} \\ & \leq \left\| u_\nu - u_\infty - \sum_{k=1}^m \bar{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \right\|_{H^1(M)} \\ & \quad + \left\| u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{H^1(M)} \\ & = o(1) \end{aligned}$$

by Propositions 4.1 and 6.6. From this, the assertion follows.

As above, we assume that

$$(113) \quad \varepsilon_{i,\nu} \leq \varepsilon_{j,\nu} \quad \text{for } i \leq j.$$

We now decompose the function u_ν as

$$(114) \quad u_\nu = v_\nu + w_\nu,$$

where

$$(115) \quad v_\nu = \bar{u}_{z_\nu} + \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})},$$

and

$$(116) \quad w_\nu = u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}.$$

Note that the function w_ν satisfies

$$(117) \quad \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0} = o(1)$$

by Proposition 6.6.

Proposition 6.8.

(i) For every $a \in A$, we have

$$(118) \quad \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a w_\nu dvol_{g_0} \right| \leq o(1) \int_M |w_\nu| dvol_{g_0}.$$

(ii) For every $1 \leq k \leq m$, we have

$$(119) \quad \left| \int_M \frac{\frac{n+2}{n-2}}{\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}} w_\nu dvol_{g_0} \right| \leq o(1) \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

(iii) For every $1 \leq k \leq m$, we have

$$(120) \quad \left| \int_M \frac{\frac{n+2}{n-2}}{\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}} \frac{\varepsilon_{k,\nu}^2 - d(x_{k,\nu}, x)^2}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2} w_\nu dvol_{g_0} \right| \leq o(1) \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

(iv) For all $1 \leq k \leq m$, we have

$$(121) \quad \left| \int_M \frac{\frac{n+2}{n-2}}{\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}} \frac{\varepsilon_{k,\nu} \exp_{x_{k,\nu}}^{-1}(x)}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2} w_\nu dvol_{g_0} \right| \leq o(1) \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

Proof. (i) As above, let $\tilde{\psi}_{a,z} = \frac{\partial}{\partial z_a} \bar{u}_z$. By definition of

$$(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}),$$

we have

$$\int_M \left(\frac{4(n-1)}{n-2} \langle d\tilde{\psi}_{a,z_\nu}, dw_\nu \rangle_{g_0} + R_{g_0} \tilde{\psi}_{a,z_\nu} w_\nu \right) dvol_{g_0} = 0.$$

This implies

$$\begin{aligned} & \lambda_a \int_M u_\infty^{\frac{4}{n-2}} \psi_a w_\nu dvol_{g_0} \\ &= - \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \psi_a - R_{g_0} \psi_a \right) w_\nu dvol_{g_0} \\ &= \int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} (\tilde{\psi}_{a,z_\nu} - \psi_a) - R_{g_0} (\tilde{\psi}_{a,z_\nu} - \psi_a) \right) w_\nu dvol_{g_0}. \end{aligned}$$

Since $\lambda_a > 0$, we conclude that

$$\left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a w_\nu dvol_{g_0} \right| \leq o(1) \|w_\nu\|_{L^1(M)}$$

for all $a \in A$.

(ii) By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\int_M \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}, dw_\nu \rangle + R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} w_\nu \right) dvol_{g_0} = 0,$$

hence

$$\int_M \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) w_\nu dvol_{g_0} = 0$$

for all $1 \leq k \leq m$. Using the estimate

$$\left\| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} + r_\infty \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(M)} = o(1),$$

we conclude that

$$\left| \int_M \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} w_\nu dvol_{g_0} \right| \leq o(1) \|w_\nu\|_{L^{\frac{2n}{n-2}}(M)}$$

for all $1 \leq k \leq m$.

The remaining statements follow similarly.

As above, we need an estimate for the second variation operator of the Yamabe functional at v_ν .

Proposition 6.9. *If ν is sufficiently large, then we have*

$$\begin{aligned} (122) \quad & \frac{n+2}{n-2} r_\infty \int_M \left(u_\infty^{\frac{4}{n-2}} + \sum_{j=1}^m \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \right) w_\nu^2 dvol_{g_0} \\ & \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0} \end{aligned}$$

for some positive constant c independent of ν .

Proof. Suppose this is not true. Upon rescaling, we obtain a sequence of functions $\{\tilde{w}_\nu : \nu \in \mathbb{N}\}$ such that

$$(123) \quad \int_M \left(\frac{4(n-1)}{n-2} |d\tilde{w}_\nu|_{g_0}^2 + R_{g_0} \tilde{w}_\nu^2 \right) dvol_{g_0} = 1$$

and

$$(124) \quad \lim_{\nu \rightarrow \infty} \frac{n+2}{n-2} r_\infty \int_M \left(u_\infty^{\frac{4}{n-2}} + \sum_{j=1}^m \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \right) \tilde{w}_\nu^2 dvol_{g_0} \geq 1.$$

Observe that

$$(125) \quad \int_M |\tilde{w}_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \leq Y(M, g_0)^{-\frac{n}{n-2}}$$

by (123). In view of Proposition 6.6, we can find a sequence $\{N_\nu : \nu \in \mathbb{N}\}$ such that $N_\nu \rightarrow \infty$, $N_\nu \varepsilon_{j,\nu} \rightarrow 0$ for all $1 \leq j \leq m$, and

$$(126) \quad \frac{1}{N_\nu} \frac{\varepsilon_{j,\nu} + d(x_{i,\nu}, x_{j,\nu})}{\varepsilon_{i,\nu}} \rightarrow \infty$$

for all $i < j$. Let

$$(127) \quad \Omega_{j,\nu} = B_{N_\nu \varepsilon_{j,\nu}}(x_{j,\nu}) \setminus \bigcup_{i=1}^{j-1} B_{N_\nu \varepsilon_{i,\nu}}(x_{i,\nu})$$

for every $1 \leq j \leq m$. In view of (123) and (124), there are only two possibilities:

Case 1. Suppose that

$$(128) \quad \lim_{\nu \rightarrow \infty} \int_M u_\infty^{\frac{4}{n-2}} \tilde{w}_\nu^2 dvol_{g_0} > 0$$

and

$$(129) \quad \begin{aligned} & \lim_{\nu \rightarrow \infty} \int_{M \setminus \bigcup_{j=1}^m \Omega_{j,\nu}} \left(\frac{4(n-1)}{n-2} |d\tilde{w}_\nu|_{g_0}^2 + R_{g_0} \tilde{w}_\nu^2 \right) dvol_{g_0} \\ & \leq \lim_{\nu \rightarrow \infty} \frac{n+2}{n-2} r_\infty \int_M u_\infty^{\frac{4}{n-2}} \tilde{w}_\nu^2 dvol_{g_0}. \end{aligned}$$

Let \tilde{w} be the weak limit of the sequence $\{\tilde{w}_\nu : \nu \in \mathbb{N}\}$. Then, the function \tilde{w} satisfies

$$\int_M u_\infty^{\frac{4}{n-2}} \tilde{w}^2 dvol_{g_0} > 0$$

and

$$\begin{aligned} & \int_M \left(\frac{4(n-1)}{n-2} |d\tilde{w}|_{g_0}^2 + R_{g_0} \tilde{w}^2 \right) dvol_{g_0} \\ & \leq \frac{n+2}{n-2} r_\infty \int_M u_\infty^{\frac{4}{n-2}} \tilde{w}^2 dvol_{g_0}. \end{aligned}$$

This implies

$$\begin{aligned} & \sum_{a \in \mathbb{N}} \lambda_a \left(\int_M u_\infty^{\frac{4}{n-2}} \psi_a \tilde{w} dvol_{g_0} \right)^2 \\ & \leq \sum_{a \in \mathbb{N}} \frac{n+2}{n-2} r_\infty \left(\int_M u_\infty^{\frac{4}{n-2}} \psi_a \tilde{w} dvol_{g_0} \right)^2. \end{aligned}$$

Using Proposition 6.8, we obtain

$$\int_M u_\infty^{\frac{4}{n-2}} \psi_a \tilde{w} dvol_{g_0} = 0$$

for all $a \in A$. Thus, we conclude that $\tilde{w}(x) = 0$ for all $x \in M$. This is a contradiction.

Case 2. Suppose that there exists an integer $1 \leq j \leq m$ such that

$$(130) \quad \lim_{\nu \rightarrow \infty} \int_M \frac{4}{\bar{u}_{(x_j, \nu, \varepsilon_j, \nu)}^{\frac{4}{n-2}}} \tilde{w}_\nu^2 dvol_{g_0} > 0$$

and

$$(131) \quad \begin{aligned} & \lim_{\nu \rightarrow \infty} \int_{\Omega_{j, \nu}} \left(\frac{4(n-1)}{n-2} |d\tilde{w}_\nu|_{g_0}^2 + R_{g_0} \tilde{w}_\nu^2 \right) dvol_{g_0} \\ & \leq \lim_{\nu \rightarrow \infty} \frac{n+2}{n-2} r_\infty \int_M \frac{4}{\bar{u}_{(x_j, \nu, \varepsilon_j, \nu)}^{\frac{4}{n-2}}} \tilde{w}_\nu^2 dvol_{g_0}. \end{aligned}$$

We now define a sequence of functions $\hat{w}_\nu : TM_{x_j, \nu} \rightarrow \mathbb{R}$ by

$$\hat{w}_\nu(\xi) = \varepsilon_{j, \nu}^{\frac{n-2}{2}} \tilde{w}_\nu(\exp_{x_j, \nu}(\varepsilon_{j, \nu} \xi))$$

for $\xi \in TM_{x_j, \nu}$. The sequence $\{\hat{w}_\nu : \nu \in \mathbb{N}\}$ satisfies

$$\lim_{\nu \rightarrow \infty} \int_{\{\xi \in TM_{x_j, \nu} : |\xi| \leq N_\nu\}} \frac{4(n-1)}{n-2} |d\hat{w}_\nu(\xi)|^2 d\xi \leq 1$$

and

$$\lim_{\nu \rightarrow \infty} \int_{\{\xi \in TM_{x_j, \nu} : |\xi| \leq N_\nu\}} |\hat{w}_\nu(\xi)|^{\frac{2n}{n-2}} d\xi \leq Y(M, g_0)^{-\frac{n}{n-2}}.$$

Hence, if we take the weak limit as $\nu \rightarrow \infty$, then we obtain a function $\hat{w} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 d\xi > 0$$

and

$$\int_{\mathbb{R}^n} |d\hat{w}(\xi)|^2 d\xi \leq n(n + 2) \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 d\xi.$$

Moreover, it follows from Proposition 5.3 that

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \hat{w}(\xi) d\xi &= 0 \\ \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{1 - |\xi|^2}{1 + |\xi|^2} \hat{w}(\xi) d\xi &= 0 \\ \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{\xi}{1 + |\xi|^2} \hat{w}(\xi) d\xi &= 0. \end{aligned}$$

Using a result of Rey, we conclude that $\hat{w}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ (see [13], Appendix D, pp. 49–51). This is a contradiction.

Corollary 6.10. *If ν is sufficiently large, then we have*

$$\begin{aligned} (132) \quad & \frac{n + 2}{n - 2} r_\infty \int_M v_\nu^{\frac{4}{n-2}} w_\nu^2 dvol_{g_0} \\ & \leq (1 - c) \int_M \left(\frac{4(n - 1)}{n - 2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0} \end{aligned}$$

for some positive constant c independent of ν .

Proof. By definition of v_ν , we have

$$\int_M \left| v_\nu^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}} - \sum_{j=1}^m \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \right|^{\frac{n}{2}} dvol_{g_0} = o(1).$$

Hence, the assertion follows from Proposition 6.9.

Lemma 6.11. *The difference $u_\nu - \bar{u}_{z_\nu}$ satisfies the estimate*

$$(133) \quad \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}$$

if ν is sufficiently large.

Proof. Using the identities

$$\frac{4(n - 1)}{n - 2} \Delta_{g_0} u_\nu - R_{g_0} u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} = -u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)$$

and

$$\Pi\left(\frac{4(n-1)}{n-2}\Delta_{g_0}\bar{u}_{z_\nu} - R_{g_0}\bar{u}_{z_\nu} + r_\infty\bar{u}_{z_\nu}^{\frac{n+2}{n-2}}\right) = 0,$$

we obtain

$$(134) \quad \begin{aligned} & \Pi\left(\frac{4(n-1)}{n-2}\Delta_{g_0}(u_\nu - \bar{u}_{z_\nu}) - R_{g_0}(u_\nu - \bar{u}_{z_\nu})\right. \\ & \quad \left. + \frac{n+2}{n-2}r_\infty u_\infty^{\frac{4}{n-2}}(u_\nu - \bar{u}_{z_\nu})\right) \\ & = \Pi\left(-u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - r_\infty) - \frac{n+2}{n-2}r_\infty(\bar{u}_{z_\nu}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}})(u_\nu - \bar{u}_{z_\nu})\right. \\ & \quad \left. + r_\infty\left(\bar{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\bar{u}_{z_\nu}^{\frac{4}{n-2}}(u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\right)\right). \end{aligned}$$

Using the inequality

$$(135) \quad \begin{aligned} & \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \\ & \leq C \left\| \Pi\left(\frac{4(n-1)}{n-2}\Delta_{g_0}(u_\nu - \bar{u}_{z_\nu}) - R_{g_0}(u_\nu - \bar{u}_{z_\nu})\right. \right. \\ & \quad \left. \left. + \frac{n+2}{n-2}r_\infty u_\infty^{\frac{4}{n-2}}(u_\nu - \bar{u}_{z_\nu})\right) \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a(u_\nu - \bar{u}_{z_\nu}) d\text{vol}_{g_0} \right|, \end{aligned}$$

we conclude that

$$(136) \quad \begin{aligned} & \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \\ & \leq C \|u_\nu^{\frac{n+2}{n-2}}(R_{g_\nu} - r_\infty)\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ & \quad + C \left\| (\bar{u}_{z_\nu}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}})(u_\nu - \bar{u}_{z_\nu}) \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ & \quad + C \left\| \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\bar{u}_{z_\nu}^{\frac{4}{n-2}}(u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a(u_\nu - \bar{u}_{z_\nu}) d\text{vol}_{g_0} \right|. \end{aligned}$$

Using the pointwise estimate

$$(137) \quad \begin{aligned} & \left| \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\bar{u}_{z_\nu}^{\frac{4}{n-2}}(u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right| \\ & \leq C \bar{u}_{z_\nu}^{\max\{0, \frac{4}{n-2}-1\}} |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + C |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}}, \end{aligned}$$

we obtain

$$(138) \quad \left\| \frac{n+2}{\bar{u}_{z_\nu}^{\frac{n+2}{n-2}}} + \frac{n+2}{n-2} \bar{u}_{z_\nu}^{\frac{4}{n-2}} (u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ \leq C \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)}.$$

Note that

$$(139) \quad \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ \leq \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(\bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ + \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ \leq C \sum_{k=1}^m (N\varepsilon_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(M)} \\ + C \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{4}{n-2}, 1\}} + |u_\nu - \bar{u}_{z_\nu}|^{\frac{4}{n-2}} \right\|_{L^{\frac{n}{2}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ \cdot \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}$$

by Hölder's inequality. Since

$$(140) \quad \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{2n}{n-2}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ = \left\| \sum_{k=1}^m \alpha_{k,\nu} \bar{u}(x_{k,\nu}, \varepsilon_{k,\nu}) + w_\nu \right\|_{L^{\frac{2n}{n-2}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ \leq \sum_{k=1}^m \alpha_{k,\nu} \|\bar{u}(x_{k,\nu}, \varepsilon_{k,\nu})\|_{L^{\frac{2n}{n-2}}(M \setminus B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} + \|w_\nu\|_{L^{\frac{2n}{n-2}}(M)} \\ \leq C N^{-\frac{n-2}{2}} + o(1),$$

it follows that

$$(141) \quad \left\| \frac{n+2}{\bar{u}_{z_\nu}^{\frac{n+2}{n-2}}} + \frac{n+2}{n-2} \bar{u}_{z_\nu}^{\frac{4}{n-2}} (u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ \leq C \sum_{k=1}^m (N\varepsilon_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} + (C N^{-2} + o(1)) \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}.$$

Moreover, we have

$$\begin{aligned}
 (142) \quad & \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a (u_\nu - \bar{u}_{z_\nu}) d\text{vol}_{g_0} \right| \\
 &= \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a \left(\sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} + w_\nu \right) d\text{vol}_{g_0} \right| \\
 &\leq C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + o(1) \|w_\nu\|_{L^1(M)} \\
 &\leq C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + o(1) \left\| u_\nu - \bar{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right\|_{L^1(M)} \\
 &\leq C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + o(1) \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)}.
 \end{aligned}$$

Putting these facts together, we conclude that

$$\begin{aligned}
 (143) \quad & \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\
 &\quad + C \sum_{k=1}^m (N \varepsilon_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \\
 &\quad + (C N^{-2} + o(1)) \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}.
 \end{aligned}$$

Hence, if we choose N sufficiently large, then we obtain

$$(144) \quad \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{(n-2)^2}{2(n+2)}}.$$

From this, the assertion follows.

Lemma 6.12. *The difference $u_\nu - \bar{u}_{z_\nu}$ satisfies the estimate*

$$(145) \quad \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}$$

if ν is sufficiently large.

Proof. Using the inequality

$$(146) \quad \begin{aligned} & \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} \\ & \leq C \left\| \Pi \left(\frac{4(n-1)}{n-2} \Delta_{g_0} (u_\nu - \bar{u}_{z_\nu}) - R_{g_0} (u_\nu - \bar{u}_{z_\nu}) \right. \right. \\ & \quad \left. \left. + \frac{n+2}{n-2} r_\infty u_\infty^{\frac{4}{n-2}} (u_\nu - \bar{u}_{z_\nu}) \right) \right\|_{L^1(M)} \\ & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a (u_\nu - \bar{u}_{z_\nu}) \, d\text{vol}_{g_0} \right| \end{aligned}$$

and (134) we conclude that

$$(147) \quad \begin{aligned} & \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} \\ & \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^1(M)} \\ & \quad + C \|(\bar{u}_{z_\nu}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}}) (u_\nu - \bar{u}_{z_\nu})\|_{L^1(M)} \\ & \quad + C \left\| \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2} \bar{u}_{z_\nu}^{\frac{4}{n-2}} (u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right\|_{L^1(M)} \\ & \quad + C \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a (u_\nu - \bar{u}_{z_\nu}) \, d\text{vol}_{g_0} \right|. \end{aligned}$$

Using the pointwise estimate

$$(148) \quad \begin{aligned} & \left| \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2} \bar{u}_{z_\nu}^{\frac{4}{n-2}} (u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right| \\ & \leq C \bar{u}_{z_\nu}^{\max\{0, \frac{4}{n-2}-1\}} |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + C |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}}, \end{aligned}$$

we obtain

$$(149) \quad \begin{aligned} & \left\| \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2} \bar{u}_{z_\nu}^{\frac{4}{n-2}} (u_\nu - \bar{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \right\|_{L^1(M)} \\ & \leq C \left\| |u_\nu - \bar{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2}, 2\}} + C |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^1(M)} \\ & \leq C \left\| |u_\nu - \bar{u}_{z_\nu}| \right\|_{L^1(M)}^{\max\{0, 1 - \frac{n-2}{4}\}} \left\| |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^1(M)}^{\min\{1, \frac{n-2}{4}\}} \\ & \quad + C \left\| |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^1(M)} \\ & \leq C \left\| |u_\nu - \bar{u}_{z_\nu}| \right\|_{L^1(M)}^{\max\{0, 1 - \frac{n-2}{4}\}} \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\min\{1, \frac{n-2}{4}\}} \\ & \quad + C \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \end{aligned}$$

by Hölder's inequality. Moreover, we have

$$(150) \quad \sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \psi_a (u_\nu - \bar{u}_{z_\nu}) dvol_{g_0} \right| \\ \leq C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + o(1) \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)}.$$

Putting these facts together, we conclude that

$$(151) \quad \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ + C \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)}^{\max\{0, 1 - \frac{n-2}{4}\}} \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\min\{1, \frac{n-2}{4}\}} \\ + C \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \\ + o(1) \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)}.$$

Since $\max\{0, 1 - \frac{n-2}{4}\} < 1$, this implies

$$(152) \quad \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ + C \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}.$$

The assertion follows now from Lemma 6.11.

Lemma 6.13. *We have*

$$(153) \quad \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} + r_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) dvol_{g_0} \right| \\ \leq C \left(\int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{2n}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}$$

if ν is sufficiently large.

Proof. Integration by parts yields

$$(154) \quad \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} + r_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) dvol_{g_0} \\ = \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u_\nu - R_{g_0} u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} \right) dvol_{g_0} \\ + \lambda_a \int_M u_\infty^{\frac{4}{n-2}} \psi_a (u_\nu - \bar{u}_{z_\nu}) dvol_{g_0} - r_\infty \int_M \psi_a (u_\nu^{\frac{n+2}{n-2}} - \bar{u}_{z_\nu}^{\frac{n+2}{n-2}}) dvol_{g_0}.$$

Using the identity

$$\frac{4(n-1)}{n-2} \Delta_{g_0} u_\nu - R_{g_0} u_\nu + r_\infty u_\nu^{\frac{n+2}{n-2}} = -u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty),$$

we obtain

$$(155) \quad \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} + r_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) d\text{vol}_{g_0} \\ = - \int_M \psi_a u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty) d\text{vol}_{g_0} \\ + \lambda_a \int_M u_\infty^{\frac{4}{n-2}} \psi_a (u_\nu - \bar{u}_{z_\nu}) d\text{vol}_{g_0} - r_\infty \int_M \psi_a (u_\nu^{\frac{n+2}{n-2}} - \bar{u}_{z_\nu}^{\frac{n+2}{n-2}}) d\text{vol}_{g_0}.$$

Using the pointwise estimate

$$(156) \quad \left| u_\nu^{\frac{n+2}{n-2}} - \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right| \leq C \bar{u}_{z_\nu}^{\frac{4}{n-2}} |u_\nu - \bar{u}_{z_\nu}| + C |u_\nu - \bar{u}_{z_\nu}|^{\frac{n+2}{n-2}},$$

we conclude that

$$(157) \quad \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} + r_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) d\text{vol}_{g_0} \right| \\ \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ + C \|u_\nu - \bar{u}_{z_\nu}\|_{L^1(M)} + C \|u_\nu - \bar{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}}.$$

Hence, it follows from Lemma 6.11 and Lemma 6.12 that

$$(158) \quad \sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} + r_\infty \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) d\text{vol}_{g_0} \right| \\ \leq C \|u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}.$$

This proves the assertion.

Proposition 6.14. *The Yamabe energy of \bar{u}_{z_ν} satisfies the estimate*

$$(159) \quad E(\bar{u}_{z_\nu}) - E(u_\infty) \\ \leq C \left(\int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} d\text{vol}_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}(1+\gamma)}$$

if ν is sufficiently large.

Proof. This follows immediately from Lemmas 6.5 and 6.13.

Proposition 6.15. *The Yamabe energy of v_ν satisfies the estimate*

$$(160) \quad E(v_\nu) \leq \left(E(\bar{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}} - c \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}$$

if ν is sufficiently large.

Proof. Using the identity

$$(161) \quad \begin{aligned} & \int_M \left(\frac{4(n-1)}{n-2} |dv_\nu|_{g_0}^2 + R_{g_0} v_\nu^2 \right) dvol_{g_0} \\ &= \int_M \left(\frac{4(n-1)}{n-2} |d\bar{u}_{z_\nu}|_{g_0}^2 + R_{g_0} \bar{u}_{z_\nu}^2 \right) dvol_{g_0} \\ & \quad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 \left(\frac{4(n-1)}{n-2} |d\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}|_{g_0}^2 + R_{g_0} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^2 \right) dvol_{g_0} \\ & \quad + 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{z_\nu}, d\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \rangle_{g_0} \right. \\ & \quad \quad \left. + R_{g_0} \bar{u}_{z_\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \right) dvol_{g_0} \\ & \quad + 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \left(\frac{4(n-1)}{n-2} \langle d\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}, d\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \rangle_{g_0} \right. \\ & \quad \quad \left. + R_{g_0} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right) dvol_{g_0}, \end{aligned}$$

we obtain

$$(162) \quad \begin{aligned} & E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}} \\ &= \int_M F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{2n}{n-2}} dvol_{g_0} \\ & \quad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} dvol_{g_0} \\ & \quad - 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} \right) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} dvol_{g_0} \\ & \quad - 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \\ & \quad \quad \cdot \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right) dvol_{g_0}. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
(163) \quad & \left(E(\bar{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
& = \left(\int_M \left(F(\bar{u}_{z_\nu})^{\frac{n}{2}} \bar{u}_{z_\nu}^{\frac{2n}{n-2}} + \sum_{k=1}^m F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right) d\text{vol}_{g_0} \right)^{\frac{2}{n}} \\
& \quad \cdot \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
& \geq \int_M \left(F(\bar{u}_{z_\nu})^{\frac{n}{2}} \bar{u}_{z_\nu}^{\frac{2n}{n-2}} + \sum_{k=1}^m F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} v_\nu^2 d\text{vol}_{g_0} \\
& \geq \int_M F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{2n}{n-2}} d\text{vol}_{g_0} \\
& \quad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} d\text{vol}_{g_0} \\
& \quad + 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \left(F(\bar{u}_{z_\nu})^{\frac{n}{2}} \bar{u}_{z_\nu}^{\frac{2n}{n-2}} + F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \\
& \quad \cdot \bar{u}_{z_\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} d\text{vol}_{g_0} \\
& \quad + 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \left(F(\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}^{\frac{2n}{n-2}} \right. \\
& \quad \left. + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} d\text{vol}_{g_0}
\end{aligned}$$

by Hölder's inequality. Let $1 \leq k \leq m$. Using the inequality

$$\begin{aligned}
& \left(F(\bar{u}_{z_\nu})^{\frac{n}{2}} \bar{u}_{z_\nu}^{\frac{2n}{n-2}} + F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \bar{u}_{z_\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \\
& \geq F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} + c \varepsilon_{k,\nu}^{-\frac{n+2}{2}} 1_{\{d(x_{k,\nu}, x) \leq \varepsilon_{k,\nu}\}},
\end{aligned}$$

we obtain

$$\begin{aligned}
(164) \quad & \int_M \left(F(\bar{u}_{z_\nu})^{\frac{n}{2}} \bar{u}_{z_\nu}^{\frac{2n}{n-2}} + F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \bar{u}_{z_\nu} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} d\text{vol}_{g_0} \\
& \geq \int_M F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} d\text{vol}_{g_0} + c \varepsilon_{k,\nu}^{\frac{n-2}{2}}
\end{aligned}$$

if ν is sufficiently large. We next consider a pair $i < j$. We can find positive constants c and C independent of ν such that

$$\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}(x)^{\frac{n+2}{n-2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}(x) \geq c \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \varepsilon_{i,\nu}^{-n}$$

and

$$\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}(x) \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}(x)^{\frac{n+2}{n-2}} \leq C \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n+2}{2}} \varepsilon_{i,\nu}^{-n}$$

if $d(x_{i,\nu}, x) \leq \varepsilon_{i,\nu}$ and ν is sufficiently large. From this, it follows that

$$\begin{aligned} & \left(F(\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \\ & \quad \cdot \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \\ & \geq F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \\ & \quad + c \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \varepsilon_{i,\nu}^{-n} \mathbf{1}_{\{d(x_{i,\nu}, x) \leq \varepsilon_{i,\nu}\}} \end{aligned}$$

for ν sufficiently large. Integration over M yields

$$\begin{aligned} (165) \quad & \int_M \left(F(\bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})})^{\frac{n}{2}} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \\ & \quad \cdot \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} d\text{vol}_{g_0} \\ & \geq \int_M F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} d\text{vol}_{g_0} \\ & \quad + c \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \end{aligned}$$

if ν is sufficiently large. From this, it follows that

$$\begin{aligned} (166) \quad & \left(E(\bar{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\ & \geq \int_M F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{2n}{n-2}} d\text{vol}_{g_0} \\ & \quad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 F(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{2n}{n-2}} d\text{vol}_{g_0} \\ & \quad + 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} d\text{vol}_{g_0} \end{aligned}$$

$$\begin{aligned}
& + 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} dvol_{g_0} \\
& + c \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + c \sum_{i < j} \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}.
\end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}
(167) \quad & E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}} \\
& \leq \left(E(\bar{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\int_M v_\nu^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}} \\
& - 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} \right. \\
& \quad \left. - R_{g_0} \bar{u}_{z_\nu} + F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} dvol_{g_0} \\
& - 2 \int_M \sum_{i < j} \alpha_{i,\nu} \alpha_{j,\nu} \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right. \\
& \quad \left. - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \right) dvol_{g_0} \\
& - c \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} - c \sum_{i < j} \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}.
\end{aligned}$$

Note that

$$\begin{aligned}
(168) \quad & \int_M \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{z_\nu} - R_{g_0} \bar{u}_{z_\nu} + F(\bar{u}_{z_\nu}) \bar{u}_{z_\nu}^{\frac{n+2}{n-2}} \right| \bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} dvol_{g_0} \\
& \leq o(1) \varepsilon_{k,\nu}^{\frac{n-2}{2}}.
\end{aligned}$$

Moreover, since $F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) = r_\infty + o(1)$, it follows from Lemmas B.4 and B.5 that

$$\begin{aligned}
(169) \quad & \int_M \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right. \\
& \quad \left. - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} + F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \right| dvol_{g_0} \\
& \leq \int_M \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} \right. \\
& \quad \left. - R_{g_0} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})} + r_\infty \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \right| dvol_{g_0}
\end{aligned}$$

$$\begin{aligned}
 & + |F(\bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}) - r_\infty| \int_M \bar{u}_{(x_{i,\nu}, \varepsilon_{i,\nu})} \bar{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} dvol_{g_0} \\
 & \leq C \left(\delta^4 + \delta^{n-2} + \frac{\varepsilon_{j,\nu}^2}{\delta^2} \right) \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}} \\
 & \quad + o(1) \left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \right)^{-\frac{n-2}{2}}
 \end{aligned}$$

for $i < j$. Hence, if we choose δ sufficiently small, the assertion follows.

Corollary 6.16. *If ν is sufficiently large, then the Yamabe energy of v_ν satisfies the estimate*

$$\begin{aligned}
 (170) \quad E(v_\nu) & \leq (E(u_\infty)^{\frac{n}{2}} + m Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}} \\
 & \quad + C \left(\int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{2n} (1+\gamma)}.
 \end{aligned}$$

Proof. Using Proposition 6.14 and Proposition B.3, we obtain

$$\begin{aligned}
 E(\bar{u}_{z_\nu}) & \leq E(u_\infty) + C \left(\int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{2n} (1+\gamma)} \\
 & \quad + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2} (1+\gamma)}
 \end{aligned}$$

and

$$E(\bar{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}) \leq Y(S^n)$$

for all $1 \leq k \leq m$. Hence, the assertion follows from Proposition 6.15.

7. Proof of Proposition 3.3

Using the identity

$$R_{g_\nu} = -u_\nu^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u_\nu - R_{g_0} u_\nu \right),$$

we obtain

$$\begin{aligned}
 (171) \quad r_{g_\nu} & = \int_M \left(\frac{4(n-1)}{n-2} |du_\nu|_{g_0}^2 + R_{g_0} u_\nu^2 \right) dvol_{g_0} \\
 & = \int_M \left(\frac{4(n-1)}{n-2} |dv_\nu|_{g_0}^2 + R_{g_0} v_\nu^2 \right) dvol_{g_0} \\
 & \quad + 2 \int_M u_\nu^{\frac{n+2}{n-2}} R_{g_\nu} w_\nu dvol_{g_0} \\
 & \quad - \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0}.
 \end{aligned}$$

This implies

(172)

$$\begin{aligned}
r_{g_\nu} &= E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
&\quad + 2 \int_M u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty) w_\nu d\text{vol}_{g_0} \\
&\quad - \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) d\text{vol}_{g_0} \\
&\quad + r_\infty \int_M \left(-\frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu \right) d\text{vol}_{g_0}.
\end{aligned}$$

In view of the volume normalization, we have

$$(173) \quad \int_M (v_\nu + w_\nu)^{\frac{2n}{n-2}} d\text{vol}_{g_0} = 1.$$

Furthermore, it is not difficult to show that

$$\begin{aligned}
(174) \quad &\left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} - 1 \\
&\leq \frac{n-2}{n} \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right) - \frac{n-2}{n},
\end{aligned}$$

hence

$$\begin{aligned}
(175) \quad &\left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} - 1 \\
&\leq \int_M \left(\frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right) d\text{vol}_{g_0}.
\end{aligned}$$

From this, it follows that

(176)

$$\begin{aligned}
r_{g_\nu} &\leq r_\infty + (E(v_\nu) - r_\infty) \left(\int_M v_\nu^{\frac{2n}{n-2}} d\text{vol}_{g_0} \right)^{\frac{n-2}{n}} \\
&\quad + 2 \int_M u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty) w d\text{vol}_{g_0} \\
&\quad - \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) d\text{vol}_{g_0} \\
&\quad + r_\infty \int_M \left(\frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 \right. \\
&\quad \left. + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right) d\text{vol}_{g_0}.
\end{aligned}$$

Using Hölder's inequality, we obtain

$$(177) \quad \int_M u_\nu^{\frac{n+2}{n-2}} (R_{g_\nu} - r_\infty) w_\nu dvol_{g_0} \leq \left(\int_M u_\nu^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{2n}} \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}}.$$

Moreover, it follows from Corollarys 5.5 and 6.10 that

$$(178) \quad \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) dvol_{g_0} \geq c \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 \right) dvol_{g_0},$$

hence

$$(179) \quad \int_M \left(\frac{4(n-1)}{n-2} |dw_\nu|_{g_0}^2 + R_{g_0} w_\nu^2 - \frac{n+2}{n-2} r_\infty v_\nu^{\frac{4}{n-2}} w_\nu^2 \right) dvol_{g_0} \geq c \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}}.$$

Finally, it follows from the pointwise estimate

$$(180) \quad \left| \frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right| \leq C v_\nu^{\max\{0, \frac{4}{n-2}-1\}} |w_\nu|^{\min\{\frac{2n}{n-2}, 3\}} + C |w_\nu|^{\frac{2n}{n-2}}$$

that

$$(181) \quad \int_M \left| \frac{n-2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} v_\nu^{\frac{4}{n-2}} w_\nu^2 + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n-2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right| dvol_{g_0} \leq C \int_M v_\nu^{\max\{0, \frac{2n}{n-2}-3\}} |w_\nu|^{\min\{\frac{2n}{n-2}, 3\}} dvol_{g_0} + C \int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \leq C \left(\int_M |w_\nu|^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n} \min\{\frac{n}{n-2}, \frac{3}{2}\}}.$$

Thus, we conclude that

$$(182) \quad r_{g_\nu} \leq r_\infty + (E(v_\nu) - r_\infty) \left(\int_M v_\nu^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{n}} + C \left(\int_M u^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{n}}.$$

Since

$$E(v_\nu) - r_\infty \leq C \left(\int_M u^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{2n} (1+\gamma)},$$

we obtain

$$(183) \quad r_{g_\nu} \leq r_\infty + C \left(\int_M u^{\frac{2n}{n-2}} |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} dvol_{g_0} \right)^{\frac{n+2}{2n} (1+\gamma)}.$$

This completes the proof of Proposition 3.3.

Appendix A. The interior regularity theorem and the Harnack inequality

Proposition A.1. *Let $q > \frac{n}{2}$. Then, we can find positive constants η_1 and C with the following significance: if $g = u^{\frac{4}{n-2}} g_0$ is a conformal metric such that*

$$(184) \quad \int_{B_r(x)} dvol_g \leq 1$$

and

$$(185) \quad \int_{B_r(x)} |R_g|^q dvol_g \leq \eta_1,$$

then we have

$$(186) \quad u(x) \leq C r^{-\frac{n-2}{2}} \left(\int_{B_r(x)} dvol_g \right)^{\frac{n-2}{2n}}.$$

Proof. Let r_0 be a real number such that $r_0 < r$ and

$$(r - s)^{\frac{n-2}{2}} \sup_{B_s(x)} u \leq (r - r_0)^{\frac{n-2}{2}} \sup_{B_{r_0}(x)} u$$

for all $s < r$. Moreover, we choose a point $x_0 \in B_{r_0}(x)$ such that

$$\sup_{B_{r_0}(x)} u = u(x_0).$$

Using a standard interior estimate for linear elliptic equations (see [8], Theorem 8.17 on p. 194), we obtain

$$(187) \quad s^{\frac{n-2}{2}} u(x_0) \leq C \left(\int_{B_s(x_0)} u^{\frac{2n}{n-2}} dvol_{g_0} \right)^{\frac{n-2}{2n}} + C s^{\frac{n+2}{2} - \frac{n}{q}} \left(\int_{B_s(x_0)} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u \right|^q dvol_{g_0} \right)^{\frac{1}{q}}$$

for $s \leq \frac{r-r_0}{2}$. From this, it follows that

$$(188) \quad s^{\frac{n-2}{2}} u(x_0) \leq C \left(\int_{B_s(x_0)} dvol_g \right)^{\frac{n-2}{2n}} + C s^{\frac{n+2}{2} - \frac{n}{q}} \left(\int_{B_r(x_0)} u^{\frac{n+2}{n-2} q - \frac{2n}{n-2}} |R_g|^q dvol_g \right)^{\frac{1}{q}}$$

for $s \leq \frac{r-r_0}{2}$. By definition of r_0 and x_0 , we have

$$(189) \quad \sup_{B_{\frac{r-r_0}{2}}(x_0)} u \leq \sup_{B_{\frac{r+r_0}{2}}(x)} u \leq 2^{\frac{n-2}{2}} \sup_{B_{r_0}(x)} u = 2^{\frac{n-2}{2}} u(x_0).$$

Hence, we can find a fixed constant K such that

$$(190) \quad s^{\frac{n-2}{2}} u(x_0) \leq K \left(\int_{B_s(x_0)} dvol_g \right)^{\frac{n-2}{2n}} + K (s^{\frac{n-2}{2}} u(x_0))^{\frac{n+2}{n-2} - \frac{2n}{n-2} \frac{1}{q}} \left(\int_{B_s(x_0)} |R_g|^q dvol_g \right)^{\frac{1}{q}}$$

for all $s \leq \frac{r-r_0}{2}$. We now choose $\eta_1 > 0$ such that

$$(2K)^{\frac{n+2}{n-2} - \frac{2n}{n-2} \frac{1}{q}} \eta_1^{\frac{1}{q}} \leq \frac{1}{2}.$$

We claim that $(\frac{r-r_0}{2})^{\frac{n-2}{2}} u(x_0) \leq 2K$. Indeed, if $2K \leq (\frac{r-r_0}{2})^{\frac{n-2}{2}} u(x_0)$, then we may apply inequality (190) with $s = (\frac{2K}{u(x_0)})^{\frac{2}{n-2}} \leq \frac{r-r_0}{2}$. This yields

$$2K \leq K \left(\int_{B_r(x)} dvol_g \right)^{\frac{n-2}{2n}} + K (2K)^{\frac{n+2}{n-2} - \frac{2n}{n-2} \frac{1}{q}} \left(\int_{B_r(x)} |R_g|^q dvol_g \right)^{\frac{1}{q}},$$

hence

$$2K \leq K + K (2K)^{\frac{n+2}{n-2} - \frac{2n}{n-2} \frac{1}{q}} \eta_1^{\frac{1}{q}}.$$

This contradicts the choice of η_1 . Thus, we conclude that

$$(191) \quad \left(\frac{r-r_0}{2}\right)^{\frac{n-2}{2}} u(x_0) \leq 2K.$$

Using (190) with $s = \frac{r-r_0}{2}$, we obtain

$$(192) \quad \left(\frac{r-r_0}{2}\right)^{\frac{n-2}{2}} u(x_0) \leq K \left(\int_{B_r(x)} d\text{vol}_g\right)^{\frac{n-2}{2n}} \\ + K (2K)^{\frac{4}{n-2} - \frac{2n}{n-2} \frac{1}{q}} \left(\int_{B_r(x)} |R_g|^q d\text{vol}_g\right)^{\frac{1}{q}} \\ \cdot \left(\frac{r-r_0}{2}\right)^{\frac{n-2}{2}} u(x_0).$$

This implies

$$(193) \quad \left(\frac{r-r_0}{2}\right)^{\frac{n-2}{2}} u(x_0) \leq K \left(\int_{B_r(x)} d\text{vol}_g\right)^{\frac{n-2}{2n}} \\ + \frac{1}{2} (2K)^{\frac{n+2}{n-2} - \frac{2n}{n-2} \frac{1}{q}} \eta_1^{\frac{1}{q}} \left(\frac{r-r_0}{2}\right)^{\frac{n-2}{2}} u(x_0),$$

hence

$$(194) \quad \left(\frac{r-r_0}{2}\right)^{\frac{n-2}{2}} u(x_0) \leq 2K \left(\int_{B_r(x)} d\text{vol}_g\right)^{\frac{n-2}{2n}}.$$

Thus, we conclude that

$$(195) \quad r^{\frac{n-2}{2}} u(x) \leq (r-r_0)^{\frac{n-2}{2}} u(x_0) \leq 2^{\frac{n}{2}} K \left(\int_{B_r(x)} d\text{vol}_g\right)^{\frac{n-2}{2n}}.$$

This proves the assertion.

Proposition A.2. *Let P be a smooth function on M . Moreover, assume that u is a positive function on M such that*

$$-\frac{4(n-1)}{n-2} \Delta_{g_0} u + P u \geq 0.$$

Then, there exists a constant C , depending only on g_0 and P , such that

$$(196) \quad \int_M u d\text{vol}_{g_0} \leq C \inf_M u.$$

Proof. Fix $r > 0$ sufficiently small. Using the weak Harnack inequality for linear elliptic equations (see [8], Theorem 8.18 on p. 194), we obtain

$$\int_{B_{2r}(x)} u \, d\text{vol}_{g_0} \leq L_0 \inf_{B_r(x)} u$$

for some constant L_0 . In particular, we have

$$\int_{B_{2r}(x)} u \, d\text{vol}_{g_0} \leq L_1 \int_{B_r(x)} u \, d\text{vol}_{g_0}$$

for some constant L_1 . We claim that

$$\int_{B_r(x)} u \, d\text{vol}_{g_0} \leq L_0 L_1^{m-1} u(y)$$

whenever $d(x, y) \leq mr$. Indeed, if $d(x, y) \leq mr$, then we can find a sequence of points $\{x_k : 1 \leq k \leq m\}$ such that $x_1 = x$, $d(x_m, y) \leq r$, and $d(x_k, x_{k+1}) \leq r$ for all $1 \leq k \leq m - 1$. From this, it follows that

$$\int_{B_{2r}(x_m)} u \, d\text{vol}_{g_0} \leq L_0 \inf_{B_r(x_m)} u \leq L_0 u(y).$$

Moreover, we have

$$\int_{B_r(x_k)} u \, d\text{vol}_{g_0} \leq \int_{B_{2r}(x_{k+1})} u \, d\text{vol}_{g_0} \leq L_1 \int_{B_r(x_{k+1})} u \, d\text{vol}_{g_0}$$

for all $1 \leq k \leq m - 1$. Therefore, we obtain

$$\int_{B_r(x)} u \, d\text{vol}_{g_0} \leq L_0 L_1^{m-1} u(y)$$

as claimed. Hence, we can find a constant C such that

$$\int_{B_r(x)} u \, d\text{vol}_{g_0} \leq C u(y)$$

for all $x, y \in M$. From this, the assertion follows.

Corollary A.3. *Let P be a smooth function on M . Moreover, assume that u is a positive function on M such that*

$$-\frac{4(n-1)}{n-2} \Delta_{g_0} u + P u \geq 0.$$

Then, there exists a constant C , depending only on g_0 and P , such that

$$(197) \quad \int_M u^{\frac{2n}{n-2}} \, d\text{vol}_{g_0} \leq C \inf_M u \left(\sup_M u \right)^{\frac{n+2}{n-2}}.$$

Appendix B. Estimates for the functions $\bar{u}_{(x_k, \varepsilon_k)}$

We first review the definition of conformal normal coordinates (see [11], Theorem 5.1 or [16], Theorem 3.1 on p. 208). Given any point $x \in M$, we can find a conformal metric $h_x = \varphi_x^{\frac{4}{n-2}} g_0$ and a map $\Phi_x : TM_x \rightarrow M$ with the following properties:

- (i) The function φ_x satisfies $\varphi_x(x) = 1$ and $2^{-\frac{n-2}{2}} \leq \varphi_x(y) \leq 2^{\frac{n-2}{2}}$ for all $y \in M$.
- (ii) The map $\Phi_x : TM_x \rightarrow M$ is the exponential map relative to the metric h_x . This means that $\Phi_x(0) = x$ and, for every vector $\xi \in TM_x$, the curve $\{\Phi_x(t\xi) : t \in \mathbb{R}\}$ is a geodesic relative to the metric h_x .
- (iii) Let $D\Phi_x(\xi)$ be the differential of the map Φ_x at $\xi \in TM_x$. Then, the determinant of $D\Phi_x(\xi)$ relative to the metric h_x satisfies $\det D\Phi_x(\xi) = 1 + O(|\xi|^N)$. Here, N is a fixed positive integer which can be chosen arbitrary large.
- (iv) The scalar curvature of h_x satisfies $|R_{h_x}(y)| \leq C \rho_x(y)^2$, where $\rho_x(y)$ denotes the Riemannian distance from x to y relative to the metric h_x .

For every point $x \in M$, we denote by G_x the Green's function of the conformal Laplacian relative to the metric h_x with pole at x . This implies

$$(198) \quad \frac{4(n-1)}{n-2} \Delta_{h_x} G_x(y) - R_{h_x} G_x(y) = 0$$

for $y \neq x$. Moreover, the Green's function satisfies the estimates

$$(199) \quad |G_x(y) - \rho_x(y)^{2-n} - A_x| \leq C \rho_x(y)$$

and

$$(200) \quad |d(G_x(y) - \rho_x(y)^{2-n})|_{h_x} \leq C,$$

where A_x is a constant depending on $x \in M$ (see [11], Lemma 6.4 or [16], Theorem 3.5 on p. 213). Observe that $A_x > 0$ for all $x \in M$ by the positive mass theorem. According to results of Habermann and Jost, the function $x \mapsto A_x$ is smooth (see [9], Proposition I.1.3 and [10], Proposition 3.5). In particular, this implies

$$(201) \quad \inf_{x \in M} A_x > 0.$$

Suppose that we are given a set of pairs $(x_k, \varepsilon_k)_{1 \leq k \leq m}$. For every $1 \leq k \leq m$, we define a function $\bar{u}_{(x_k, \varepsilon_k)}$ by

$$(202) \quad \bar{u}_{(x_k, \varepsilon_k)}(y) = \varphi_{x_k}(y) \bar{U}_{(x_k, \varepsilon_k)}(y),$$

where

(203)

$$\begin{aligned} \bar{U}_{(x_k, \varepsilon_k)}(y) = & \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} \cdot \left(\chi_\delta(\rho_{x_k}(y)) (\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \right. \\ & \left. + (1 - \chi_\delta(\rho_{x_k}(y))) G_{x_k}(y) \right). \end{aligned}$$

Here, the function $\chi_\delta : \mathbb{R} \rightarrow [0, 1]$ is defined by $\chi_\delta(s) = \chi(s/\delta)$, where $\chi : \mathbb{R} \rightarrow [0, 1]$ is a cut-off function satisfying $\chi(s) = 1$ for $s \leq 1$ and $\chi(s) = 0$ for $s \geq 2$. Moreover, δ is a positive real number such that $\varepsilon_k \ll \delta$ for all $1 \leq k \leq m$.

Proposition B.1. *We have*

(204)

$$\begin{aligned} & \left| \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}(y) - R_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}(y) + r_\infty \bar{U}_{(x_k, \varepsilon_k)}(y)^{\frac{n+2}{n-2}} \right. \\ & \quad \left. + \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} A_{x_k} \cdot \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \right| \\ & \leq C \left(\frac{\varepsilon_k}{\varepsilon_k^2 + \rho_{x_k}(y)^2} \right)^{\frac{n-2}{2}} \rho_{x_k}(y)^2 1_{\{\rho_{x_k}(y) \leq 2\delta\}} \\ & \quad + C \frac{\varepsilon_k^{\frac{n-2}{2}}}{\delta} 1_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} + C \left(\frac{\varepsilon_k}{\varepsilon_k^2 + \rho_{x_k}(y)^2} \right)^{\frac{n+2}{2}} 1_{\{\rho_{x_k}(y) \geq \delta\}}. \end{aligned}$$

Proof. By definition of $\bar{U}_{(x_k, \varepsilon_k)}$, we have

$$\begin{aligned} & \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}(y) - R_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}(y) + r_\infty \bar{U}_{(x_k, \varepsilon_k)}(y)^{\frac{n+2}{n-2}} \\ & \quad + \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} A_{x_k} \cdot \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \\ & = \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} (I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \chi_\delta(\rho_{x_k}(y)) \left(\frac{4(n-1)}{n-2} \Delta_{h_{x_k}} (\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \right. \\ & \quad \left. + 4n(n-1) \varepsilon_k^2 (\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n+2}{2}} \right) \\ I_2 &= -\chi_\delta(\rho_{x_k}(y)) R_{h_{x_k}}(y) (\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \\ I_3 &= -\frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) (G_{x_k}(y) - \rho_{x_k}(y)^{2-n} - A_{x_k}) \end{aligned}$$

$$\begin{aligned}
 I_4 &= \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \left((\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} - \rho_{x_k}(y)^{2-n} \right) \\
 I_5 &= -\frac{8(n-1)}{n-2} \left\langle d\chi_\delta(\rho_{x_k}(y)), d(G_{x_k}(y) - \rho_{x_k}(y)^{2-n}) \right\rangle_{h_{x_k}} \\
 I_6 &= \frac{8(n-1)}{n-2} \left\langle d\chi_\delta(\rho_{x_k}(y)), d\left((\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} - \rho_{x_k}(y)^{2-n} \right) \right\rangle_{h_{x_k}} \\
 I_7 &= 4n(n-1) \varepsilon_k^2 \left[\left(\chi_\delta(\rho_{x_k}(y)) (\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \right. \right. \\
 &\quad \left. \left. + (1 - \chi_\delta(\rho_{x_k}(y))) G_{x_k}(y) \right)^{\frac{n+2}{n-2}} \right. \\
 &\quad \left. - \chi_\delta(\rho_{x_k}(y)) (\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n+2}{2}} \right].
 \end{aligned}$$

From this, the assertion follows.

Corollary B.2. *We have*

(205)

$$\begin{aligned}
 &\left| \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}(y) - R_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}(y) + r_\infty \bar{U}_{(x_k, \varepsilon_k)}(y)^{\frac{n+2}{n-2}} \right| \\
 &\leq C \left(\frac{\varepsilon_k}{\varepsilon_k^2 + \rho_{x_k}(y)^2} \right)^{\frac{n-2}{2}} \rho_{x_k}(y)^2 1_{\{\rho_{x_k}(y) \leq 2\delta\}} \\
 &\quad + C \frac{\varepsilon_k^{\frac{n-2}{2}}}{\delta^2} 1_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} + C \left(\frac{\varepsilon_k}{\varepsilon_k^2 + \rho_{x_k}(y)^2} \right)^{\frac{n+2}{2}} 1_{\{\rho_{x_k}(y) \geq \delta\}}.
 \end{aligned}$$

We next estimate the Yamabe energy of the test function $\bar{u}_{(x_k, \varepsilon_k)}$.

Proposition B.3. *Suppose that either $3 \leq n \leq 5$ or M is locally conformally flat. Moreover, assume that M is not conformally equivalent to the standard sphere S^n . If δ is sufficiently small, then the Yamabe energy of the test function $\bar{u}_{(x_k, \varepsilon_k)}$ satisfies the estimate*

$$(206) \quad E(\bar{u}_{(x_k, \varepsilon_k)}) \leq Y(S^n) - c\varepsilon_k^{n-2} + C\delta\varepsilon_k^{n-2} + C\delta^{-n}\varepsilon_k^n$$

for some constant $c > 0$.

Proof. We adapt an argument due to Schoen (see [14] or [16], Theorem 4.1 on p. 219). We only consider the case $3 \leq n \leq 5$. Using Proposition B.1, we obtain

$$\begin{aligned}
 &\int_M \left(\frac{4(n-1)}{n-2} |d\bar{U}_{(x_k, \varepsilon_k)}|_{h_{x_k}}^2 + R_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)}^2 - r_\infty \bar{U}_{(x_k, \varepsilon_k)}^{\frac{2n}{n-2}} \right) d\text{vol}_{h_{x_k}} \\
 &= - \int_M \left(\frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)} - R_{h_{x_k}} \bar{U}_{(x_k, \varepsilon_k)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + r_\infty \overline{U}_{(x_k, \varepsilon_k)}^{\frac{n+2}{n-2}} \Big) \overline{U}_{(x_k, \varepsilon_k)} d\text{vol}_{h_{x_k}} \\
 \leq & \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} \\
 & \cdot A_{x_k} \int_M \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \overline{U}_{(x_k, \varepsilon_k)}(y) d\text{vol}_{h_{x_k}} \\
 & + C \delta^{6-n} \varepsilon_k^{n-2} + C \delta \varepsilon_k^{n-2} + C \delta^{-n} \varepsilon_k^n \\
 \leq & \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{2}} \varepsilon_k^{n-2} \\
 & \cdot A_{x_k} \int_M \frac{4(n-1)}{n-2} \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \rho_{x_k}(y)^{2-n} d\text{vol}_{h_{x_k}} \\
 & + C \delta^{6-n} \varepsilon_k^{n-2} + C \delta \varepsilon_k^{n-2} + C \delta^{-n} \varepsilon_k^n \\
 \leq & - \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{2}} \cdot 4(n-1) \omega_{n-1} \varepsilon_k^{n-2} A_{x_k} \\
 & + C \delta^{6-n} \varepsilon_k^{n-2} + C \delta \varepsilon_k^{n-2} + C \delta^{-n} \varepsilon_k^n,
 \end{aligned}$$

where ω_{n-1} denotes the volume of the $(n-1)$ -dimensional unit sphere S^{n-1} . This implies

$$\begin{aligned}
 & \int_M \left(\frac{4(n-1)}{n-2} |d\overline{U}_{(x_k, \varepsilon_k)}|_{h_{x_k}}^2 + R_{h_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}^2 \right) d\text{vol}_{h_{x_k}} \\
 & \leq r_\infty \int_M \overline{U}_{(x_k, \varepsilon_k)}^{\frac{2n}{n-2}} d\text{vol}_{h_{x_k}} \\
 & \quad - \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{2}} \cdot 4(n-1) \omega_{n-1} \varepsilon_k^{n-2} A_{x_k} \\
 & \quad + C \delta^{6-n} \varepsilon_k^{n-2} + C \delta \varepsilon_k^{n-2} + C \delta^{-n} \varepsilon_k^n.
 \end{aligned}$$

Moreover, we have

$$\int_M \overline{U}_{(x_k, \varepsilon_k)}^{\frac{2n}{n-2}} d\text{vol}_{h_{x_k}} \leq \left(\frac{Y(S^n)}{r_\infty} \right)^{\frac{n}{2}} + C \delta^{-n} \varepsilon_k^n.$$

Putting these facts together, we conclude that

$$\begin{aligned}
 & \int_M \left(\frac{4(n-1)}{n-2} |d\overline{U}_{(x_k, \varepsilon_k)}|_{h_{x_k}}^2 + R_{h_{x_k}} \overline{U}_{(x_k, \varepsilon_k)}^2 \right) d\text{vol}_{h_{x_k}} \\
 & \leq Y(S^n) \left(\int_M \overline{U}_{(x_k, \varepsilon_k)}^{\frac{2n}{n-2}} d\text{vol}_{h_{x_k}} \right)^{\frac{n-2}{n}}
 \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{2}} \cdot 4(n-1) \omega_{n-1} \varepsilon_k^{n-2} A_{x_k} \\
& + C \delta^{6-n} \varepsilon_k^{n-2} + C \delta \varepsilon_k^{n-2} + C \delta^{-n} \varepsilon_k^n.
\end{aligned}$$

Since $3 \leq n \leq 5$, the assertion follows.

Lemma B.4. *We have the estimate*

$$(207) \quad \int_M \bar{u}_{(x_i, \varepsilon_i)} \bar{u}_{(x_j, \varepsilon_j)}^{\frac{n+2}{n-2}} d\text{vol}_{g_0} \leq C \left(\frac{\varepsilon_j^2 + d(x_i, x_j)^2}{\varepsilon_i \varepsilon_j} \right)^{-\frac{n-2}{2}}.$$

Proof. On the set $\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\}$, we have

$$\varepsilon_j + d(y, x_j) \geq \varepsilon_j + d(x_i, x_j) - d(x_i, y) \geq \frac{1}{2} (\varepsilon_j + d(x_i, x_j)).$$

Therefore, we obtain

$$\begin{aligned}
& \int_M \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\
& \leq \int_{\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\
& \quad + \int_{\{2d(x_i, y) \geq \varepsilon_j + d(x_i, x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\
& \leq C \int_{\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}} d\text{vol}_{g_0} \\
& \quad + C \int_M \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\
& \leq C \frac{\varepsilon_i^{\frac{n-2}{2}} \varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}}.
\end{aligned}$$

This proves the assertion.

Lemma B.5. *We have the estimate*

$$\begin{aligned}
(208) \quad & \int_M \bar{u}_{(x_i, \varepsilon_i)} \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_j, \varepsilon_j)} - R_{g_0} \bar{u}_{(x_j, \varepsilon_j)} + r_\infty \bar{u}_{(x_j, \varepsilon_j)}^{\frac{n+2}{n-2}} \right| d\text{vol}_{g_0} \\
& \leq C \left(\delta^4 + \delta^{n-2} + \frac{\varepsilon_j^2}{\delta^2} \right) \left(\frac{\varepsilon_j^2 + d(x_i, x_j)^2}{\varepsilon_i \varepsilon_j} \right)^{-\frac{n-2}{2}}.
\end{aligned}$$

Proof. The inequality $2^{-\frac{n-2}{2}} \leq \varphi_x \leq 2^{\frac{n-2}{2}}$ implies $\frac{1}{2}d(x, y) \leq \rho_x(y) \leq 2d(x, y)$ for all $x, y \in M$. With the aid of Corollary B.2, we obtain

$$\begin{aligned} & \left| \frac{4(n-1)}{n-2} \Delta_{g_0} \bar{u}_{(x_j, \varepsilon_j)} - R_{g_0} \bar{u}_{(x_j, \varepsilon_j)} + r_\infty \bar{u}_{(x_j, \varepsilon_j)}^{\frac{n+2}{n-2}} \right| \\ & \leq C(\delta^2 + \delta^{n-4}) \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d(y, x_j)^2} \right)^{\frac{n-2}{2}} 1_{\{d(y, x_j) \leq 4\delta\}} \\ & \quad + C \left(\frac{\varepsilon_j}{\varepsilon_j^2 + d(y, x_j)^2} \right)^{\frac{n+2}{2}} 1_{\{d(y, x_j) \geq \frac{\delta}{2}\}}. \end{aligned}$$

On the set $\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\} \cap \{d(y, x_j) \leq 4\delta\}$, we have

$$\varepsilon_j + d(y, x_j) \geq \varepsilon_j + d(x_i, x_j) - d(x_i, y) \geq \frac{1}{2}(\varepsilon_j + d(x_i, x_j)),$$

hence

$$d(x_i, y) \leq \frac{1}{2}(\varepsilon_j + d(x_i, x_j)) \leq \varepsilon_j + d(y, x_j) \leq 8\delta.$$

From this, it follows that

$$\begin{aligned} & \int_{\{d(y, x_j) \leq 4\delta\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n-2}{2}}} d\text{vol}_{g_0} \\ & \leq \int_{\substack{\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\} \\ \cap \{d(y, x_j) \leq 4\delta\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \\ & \quad \cdot \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n-2}{2}}} d\text{vol}_{g_0} \\ & \quad + \int_{\substack{\{2d(x_i, y) \geq \varepsilon_j + d(x_i, x_j)\} \\ \cap \{d(y, x_j) \leq 4\delta\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \\ & \quad \cdot \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n-2}{2}}} d\text{vol}_{g_0} \\ & \leq C \int_{\{d(x_i, y) \leq 8\delta\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}} d\text{vol}_{g_0} \\ & \quad + C \int_{\{d(y, x_j) \leq 4\delta\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n-2}{2}}} d\text{vol}_{g_0} \end{aligned}$$

$$\leq C \delta^2 \frac{\varepsilon_i^{\frac{n-2}{2}} \varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}}.$$

Similarly, on the set $\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\} \cap \{d(y, x_j) \geq \frac{\delta}{2}\}$, we have

$$\varepsilon_j + d(y, x_j) \geq \varepsilon_j + d(x_i, x_j) - d(x_i, y) \geq \frac{1}{2}(\varepsilon_j + d(x_i, x_j)).$$

This implies

$$\begin{aligned} & \int_{\{d(y, x_j) \geq \frac{\delta}{2}\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\ & \leq \int_{\substack{\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\} \\ \cap \{d(y, x_j) \geq \frac{\delta}{2}\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \\ & \quad \cdot \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\ & \quad + \int_{\substack{\{2d(x_i, y) \geq \varepsilon_j + d(x_i, x_j)\} \\ \cap \{d(y, x_j) \geq \frac{\delta}{2}\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \\ & \quad \cdot \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\ & \leq C \int_{\{2d(x_i, y) \leq \varepsilon_j + d(x_i, x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, y)^2)^{\frac{n-2}{2}}} \\ & \quad \cdot \frac{\varepsilon_j^{\frac{n+2}{2}}}{\delta^2 (\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n}{2}}} d\text{vol}_{g_0} \\ & \quad + C \int_{\{d(y, x_j) \geq \frac{\delta}{2}\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}} \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y, x_j)^2)^{\frac{n+2}{2}}} d\text{vol}_{g_0} \\ & \leq C \frac{\varepsilon_j^2}{\delta^2} \frac{\varepsilon_i^{\frac{n-2}{2}} \varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i, x_j)^2)^{\frac{n-2}{2}}}. \end{aligned}$$

From this, the assertion follows.

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