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CONVERGENCE OF TIME-STEPPING SCHEMES FOR PASSIVE AND EXTENDED LINEAR COMPLEMENTARITY SYSTEMS[*]

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Abstract. Generalizing recent results in [M. K. Camlibel, *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*, Ph.D. thesis, Center for Economic Research, Tilburg University, Tilburg, The Netherlands, 2001], [M. K. Camlibel, W. P. M. H. Heemels, and J. M. Schumacher, *IEEE Trans. Circuits Systems* I: *Fund. Theory Appl.*, 49 (2002), pp. 349–357], and [J.-S. Pang and D. Stewart, *Math. Program. Ser. A*, 113 (2008), pp. 345–424], this paper provides an in-depth analysis of time-stepping methods for solving initial-value and boundary-value, non-Lipschitz linear complementarity systems (LCSs) under passivity and broader assumptions. The novelty of the methods and their analysis lies in the use of "least-norm solutions" in the discrete-time linear complementarity subproblems arising from the numerical scheme; these subproblems are not necessarily monotone and are not guaranteed to have convex solution sets. Among the principal results, it is shown that, using such least-norm solutions of the discrete-time subproblems, an implicit Euler scheme is convergent for passive initial-value LCSs; generalizations under a strict copositivity assumption and for boundary-value LCSs are also established.

Key words. linear complementarity systems, time stepping, convergence analysis, passivity

AMS subject classifications. 65L80, 65L12, 90C33, 34B10

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1. Introduction. The class of differential variational inequalities (DVIs) is a new mathematical paradigm that bridges the classical domain of ordinary differential equations (ODEs) with the contemporary subject of finite-dimensional constrained optimization and variational problems. An important subclass of the DVIs is the class of linear complementarity systems (LCSs) consisting of a linear ODE parameterized by an algebraic variable that is a solution to a linear complementarity problem (LCP) parameterized in turn by the state variable of the ODE. The LCS provides a mathematical framework that extends classical linear system theory to allow for unilateral constraints and disjunctions modeled by the complementarity conditions. Due to its wide application in many areas such as linear-quadratic differential Nash games, nonsmooth mechanics, robotics, traffic flow theory, biological systems, and circuit systems, the LCS has attracted growing interest from the mathematical programming community and the systems and control community; see the recent surveys [3, 31] and research articles [4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 28, 32, 33, 34, 35].

A dominant class of computational methods for solving LCSs (and, more generally, DVIs) is the family of time-stepping schemes that have been studied extensively in [4, 6, 29]. In particular, it was proved in [4, 6] that, under passivity, minimality, and certain rank conditions, the implicit Euler method converges to a weak solution

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of an initial-value LCS; see [4, Theorem 6.4.1]. In [29], based on a general theorem [29, Theorem 7.1], several convergence results for initial-value problems (IVPs) and boundary-value problems (BVPs) were established under various conditions, all of which imply a key linear growth condition and yet fail to hold for passive systems. Combined together, these two sets of convergence results do not include the important special case of a passive LCS not satisfying the minimality condition and/or failing the rank condition. This gap has served as the initial impetus for this paper; as we extend the theories in the cited references, we are able to broaden our investigation much beyond passive initial-value systems. In fact, the extension is by no means easy; for one thing, we need to utilize the *least-norm* solutions of nonmonotone LCPs and some advanced fixed-point arguments in establishing the solvability of the time-discretized LCPs.

This paper focuses on the LCS, which is a time-invariant dynamical system; the "linearity" of this system is not in the usual sense of a linear ODE in which the right-hand side of the differential equation is a simple linear function. As a single-valued ODE, the LCS is one in which the right-hand side is at best piecewise linear. This is the "P-case" discussed in [29, subsection 5.4]. More generally, as is the focus of this paper, the LCS belongs to the class of differential inclusions [1] with the right-hand side being a polyhedral multifunction [30]. Consequently, whereas there is an extensive literature on numerical methods, including high-order ones, for solving standard ODEs, both IVPs and BVPs, the convergence theory of these methods is not applicable to the LCS. Even in the favorable P-case, the right-hand side of the equivalent ODE is generally not differentiable, thus jeopardizing the high-order ODE methods whose rate of convergence invariably requires a smooth right-hand side. Furthermore, the piecewise property of the LCS leads to many technical issues (such as the Zeno-phenomenon [32]) that are simply absent in classical ODEs. This paper focuses on the well-known implicit Euler method applied to the LCS and analyzes its convergence under conditions that extend those in the state-of-the-art analysis of time-stepping methods in the references [4, 6, 29].

A critical step in our convergence analysis is to establish a linear growth property of the algebraic variable in terms of the differential variable. In the above cited references, this is accomplished via the introduction of some sufficient conditions under which such a growth property holds for *all* solutions of the LCP; this turns out to be a restrictive approach and is failed by a passive LCS and many other systems lacking an important solution boundedness property. The search for relaxed sufficient conditions is therefore the main goal of this paper. Our contribution herein is to identify classes of LCSs for which *one* solution of the LCP exists that is linearly bounded in norm by the state variable of the LCS; this consideration motivates the use of a least-norm solution of the LCP.

The rest of this paper is organized as follows. In section 2 we present the least-norm time-stepping scheme and a motivating example, and state other preliminary results needed in the rest of the paper. In section 3 we prove the convergence of this scheme for LCSs under two different sets of conditions. We study BVPs in section 4. In section 5 we summarize the entire paper. As mentioned above, the initial impetus for this work is the paper [4] on the passive LCS. While we continue to treat only the time-invariant LCS, we believe that the results in this paper can be extended to nonlinear problems, and possibly to the DVI. Regrettably, due to page limits, such an extension is left for future research. Another omitted but definitely worthwhile topic for investigation is the high-order methods for smooth ODEs extended to the LCS and DVI. Such an investigation will also be part of our future work.

Preceded by various lemmas and auxiliary propositions, the main convergence results of the paper are Theorems 11 and 12 for IVPs and Theorems 22 and 23 for BVPs. Theorem 11 yields the important Corollaries 13 and 14 that identify classes of LCSs, including the passive ones, for which the desired convergence holds. While Theorem 22 is applicable to a passive boundary-value LCS, Theorems 12 and 23 both involve an LCP-range condition that is weaker than those in [29].

2. Preliminaries. In this section, we first present the least-norm time-stepping schemes for both homogeneous LCSs and inhomogeneous LCSs. We then provide some preliminary technical results.

2.1. Time-stepping schemes. Consider the following initial-value linear complementarity system:

$$\dot{x} = Ax + Bu,$$
$$(1) \qquad 0 \le u \perp Cx + Du \ge 0,$$
$$x(0) = x^0, \quad t \in [0, T],$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, and the notation $\perp$ between two vectors means that they are perpendicular. In this case, this orthogonality condition expresses the complementarity condition that for each component i, either $u_i = 0$ or $(Cx + Du)_i = 0$, or, in the aggregate form, $u^T(Cx + Du) = 0$.

By a *weak solution* of system (1), we mean a pair of trajectories $(x(t), u(t))$ such that $x(t)$ is absolutely continuous and $u(t)$ is integrable on $[0, T]$ such that

$$x(t) - x(s) = \int_s^t \left[Ax(\tau) + Bu(\tau) \right] d\tau \quad \forall\, t \ge s \text{ in } [0, T]$$

and

$$0 \le u(t) \perp Cx(t) + Du(t) \ge 0 \quad \text{for almost all } t \in [0, T].$$

It should be pointed out that the latter complementarity condition is required to hold for almost all, but not necessarily all, t. The reason for not insisting that this condition hold for all t is to accommodate possible discontinuities in the algebraic function $u(t)$, as is often the case such as when D is identically equal to zero.

The idea of *time stepping* to compute such a solution is to replace the time derivative $\dot{x} \equiv dx/dt$ by a finite-difference quotient. Specifically, we first divide the time interval $[0, T]$ into N_h subintervals of equal length $h > 0$:

$$0 = t_{h,0} < t_{h,1} < t_{h,2} < \cdots < t_{h,N_h} = T.$$

Therefore, $hN_h = T$ and

$$t_{h,i+1} = t_{h,i} + h \quad \forall\, i = 0, 1, \dots, N_h.$$

We then let $x^{h,0} = x^0$ and compute

$$\{x^{h,1}, x^{h,2}, \dots, x^{h,N_h}\} \subset \mathbb{R}^n \quad \text{and} \quad \{u^{h,1}, u^{h,2}, \dots, u^{h,N_h}\} \subset \mathbb{R}^m_+$$

by the following recursion: for $i = 0, 1, \dots, N_h - 1$,

$$(2) \qquad x^{h,i+1} = x^{h,i} + h \left\{ A \left[\theta x^{h,i} + (1 - \theta) x^{h,i+1} \right] + Bu^{h,i+1} \right\},$$

$$(3) \qquad 0 \le u^{h,i+1} \perp Cx^{h,i+1} + Du^{h,i+1} \ge 0,$$

where $\theta \in [0, 1]$ is a scalar to distinguish an explicit ($\theta = 1$), an implicit ($\theta = 0$), or a semi-implicit ($\theta \in (0, 1)$) discretization of the ODE. In this paper, we focus on the implicit case, namely, $\theta = 0$. Solving for $x^{h,i+1}$ from (2) and substituting it into (3), we get the following LCP:

$$(4) \qquad 0 \le u^{h,i+1} \perp C(I - hA)^{-1}x^{h,i} + D^h u^{h,i+1} \ge 0,$$

where $D^h \equiv D + hC(I - hA)^{-1}B$, from which $x^{h,i+1}$ can be computed by

$$x^{h,i+1} = (I - hA)^{-1}\left(x^{h,i} + hBu^{h,i+1}\right).$$

In general, the LCP (4), if solvable, could have multiple solutions. To deal with the situation of nonunique solutions, we consider the scheme by finding the *least-norm* solution of the LCP (4). Namely, we let $x^{h,0} = x^0$ for $i = 0, 1, \ldots, N_h - 1$, and we compute

$$\{x^{h,1}, x^{h,2}, \ldots, x^{h,N_h}\} \subset \mathbb{R}^n \quad \text{and} \quad \{u^{h,1}, u^{h,2}, \ldots, u^{h,N_h}\} \subset \mathbb{R}^m_+$$

by letting

$$u^{h,i+1} \in \arg\min \ \|u\|^2$$
$$\text{subject to} \quad 0 \le u \perp C(I - hA)^{-1}x^{h,i} + D^h u \ge 0,$$
$$x^{h,i+1} = (I - hA)^{-1}\left(x^{h,i} + hBu^{h,i+1}\right),$$

where $\|\bullet\|$ denotes the Euclidean norm. Let $\widehat{x}^h(t)$ be the continuous piecewise linear interpolant of $\{x^{h,i}\}$ and $\widehat{u}^h(t)$ be the piecewise constant interpolant of $\{u^{h,i}\}$; i.e.,

$$\widehat{x}^h(t) \equiv x^{h,i} + \frac{t - t_{h,i}}{h}(x^{h,i+1} - x^{h,i}) \quad \forall \, t \in [t_{h,i}, t_{h,i+1}],$$
$$\widehat{u}^h(t) \equiv u^{h,i+1} \qquad\qquad\qquad \forall \, t \in (t_{h,i}, t_{h,i+1}].$$

The above scheme is referred to as the *least-norm time-stepping scheme* in the remainder of this paper. We want to study the convergence properties of $\widehat{x}^h(t)$ and $\widehat{u}^h(t)$ as h goes to 0. Before doing so, we should highlight the novelty of this scheme in the context of the existing literature, particularly the references [4, 29]. First of all, the use of a least-norm solution in a time-stepping scheme is a constructive suggestion adding to the literature of numerical methods for solving LCSs. A natural question can be asked about why such a particular solution to the LCP (4) is needed and how to compute such a solution, in particular, whether it is enough to use any solution of this LCP that is bounded in norm by a constant multiple of $\|x^{k,h}\| + 1$. As supported by Lemma 4, which is a restatement of [29, Lemma 7.2], the answer to the latter question is in the affirmative. Nevertheless, there are several reasons for using a least-norm solution. First and foremost is that the cited lemma assumes the existence of such a solution and does not address the question of its existence. Second, if the desired bound is satisfied by some solution, then the same bound is also satisfied by the least-norm solution. More importantly, the least-norm idea provides a constructive way of identifying a needed solution obeying the bound. In the case where the matrix D^h is positive semidefinite, there are several approaches to computing each least-norm iterate $u^{h,i+1}$. One approach is to solve the LCP and get a solution first; then, based on the polyhedral representation of the solution set of a monotone

LCP [17, Theorem 3.1.7], one can obtain the least-norm solution by solving a convex quadratic program on the solution set of the LCP. Another approach is to use the Tikhonov regularization algorithm that is well known for producing the least-norm solution of a solvable monotone LCP [18, Theorem 12.2.3]. Yet a third approach is to use a high-order path-following algorithm as described in the recent paper [27]. As we will see, for a passive LCS, the matrix D^h is always positive semidefinite for all $h > 0$ sufficiently small. Nevertheless, our general convergence results do not require the latter matrix to be positive semidefinite. Such generality comes with a price, namely, that computing a least-norm solution of a nonmonotone LCP is not a trivial task in practice. Indeed, such a computational problem is an instance of a quadratic program with linear complementarity constraints whose solution is currently under investigation by the author Pang; see, for instance, [25, 26], where the global resolution of linear programs with linear complementarity constraints is being studied.

Similar to the homogeneous system (1), we also consider the following inhomogeneous system:

$$
\begin{aligned}
\dot{x} &= Ax + Bu + f(t), \\
0 &\leq u \perp Cx + Du + g(t) \geq 0, \\
x(0) &= x^0,
\end{aligned}
$$
(5)

where f and g are given Lipschitz continuous functions (inputs) of t. The inhomogeneous LCS is of practical interest in circuit and other systems; see the next subsection and examples in [13]. The time-stepping scheme for (5) is as follows. Letting $x^{h,0} = x^0$ for $i = 0, 1, \ldots, N_h - 1$, we compute

$$\{x^{h,1}, x^{h,2}, \ldots, x^{h,N_h}\} \subset \mathbb{R}^n \quad \text{and} \quad \{u^{h,1}, u^{h,2}, \ldots, u^{h,N_h}\} \subset \mathbb{R}^m_+$$

by

$$u^{h,i+1} \in \arg\min \ \|u\|^2$$
$$\text{subject to } \ 0 \leq u \perp C(I - hA)^{-1}[x^{h,i} + hf(t_{h,i+1})] + g(t_{h,i+1}) + D^h u \geq 0,$$
$$x^{h,i+1} = (I - hA)^{-1} \left(x^{h,i} + hBu^{h,i+1} + hf(t_{h,i+1}) \right).$$

2.2. A motivating example. Consider the electrical circuit depicted in Figure 1. By extracting the ideal diodes and using Kirchoff laws, one can derive the
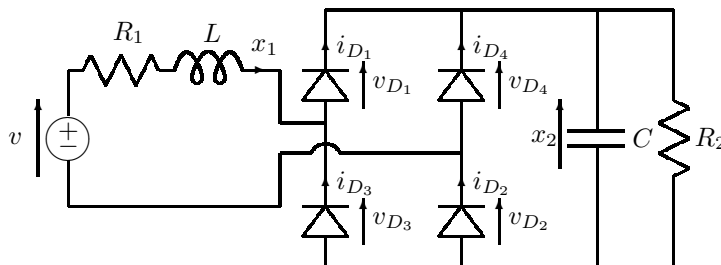

FIG. 1. *Power converter diode bridge.*

governing circuit equations as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -\dfrac{R_1}{L} & 0 \\ 0 & -\dfrac{1}{R_2 C} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(6) \qquad + \begin{bmatrix} 0 & \dfrac{1}{L} & -\dfrac{1}{L} & 0 \\ \dfrac{1}{C} & 0 & 0 & \dfrac{1}{C} \end{bmatrix} \begin{pmatrix} i_{D_1} \\ v_{D_2} \\ v_{D_3} \\ i_{D_4} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{L} \\ 0 \end{pmatrix} v,$$

$$\begin{pmatrix} v_{D_1} \\ i_{D_2} \\ i_{D_3} \\ v_{D_4} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} i_{D_1} \\ v_{D_2} \\ v_{D_3} \\ i_{D_4} \end{pmatrix}.$$

Here x_1 is the current through the inductor L, x_2 is the voltage across the capacitor C, (v_{D_i}, i_{D_i}) is the voltage-current pair associated to the ith diode, and v is a sinusoidal voltage source. Characteristics of ideal diodes can be given in the form of complementarity conditions as

$$0 \leq v_{D_i} \perp i_{D_i} \geq 0 \quad \forall i = 1, 2, 3, 4.$$

In this way, we obtain a linear complementarity system of the form (5). Although the underlying linear system is passive (see the next subsection for the definition of passivity) and minimal, the convergence result of the backward Euler time-stepping scheme in [4, 6] is not applicable in this case, the main culprit being the fact that $\mathrm{col}(B, D + D^T)$ is not of full column rank. The latter condition is used in [4, 6] to guarantee the positive definiteness of the matrix D^h. When this fails to hold, D^h cannot be a positive definite matrix and the arguments employed in the cited references do not work. A main motivation of this paper is to weaken this rank condition, which does not hold in applications where the number of complementarity variables exceeds the number of state variables. Typical instances of such examples include power converters, as in Figure 1, and network/resource problems [13, Example 2]. The convergence results of [29] cannot be applied to the above example in particular and to applications such as power converters and network/resource problems in general as the underlying system is typically passive in all such applications and the linear growth condition of [29] fails to hold for passive systems. We implemented the proposed least-norm time-stepping scheme for this example using MATLAB (R2009a 7.8.0.347 32bit (glnx86)) on a high-end desktop (2.0 GB RAM, 2 Core2 (64 bit) processors at 2.40 GHz) running Ubuntu 9.04 (i386). We ran the algorithm in two ways. First, we used just the well-known Lemke method [17] to find a solution of the LCP subproblems without any norm-minimization. In the second approach, Lemke's method was followed by a norm-minimization procedure using the MATLAB internal quadratic program solver. It turns out that Lemke's method already yields the least-norm solution in this particular example. Without the norm-minimization step, the run time was about 15 seconds for 2^{14} steps. With the norm-minimization step included, the execution time increased to about 85 seconds. $\quad\square$

2.3. Notation and preliminary technical results. Given a matrix M, we use Range(M) and Ker(M) to denote the range and the kernel of M, respectively. Given a matrix $C \in \mathbb{R}^{m \times n}$ and a set $S \subseteq \mathbb{R}^m$, we write $C^{-1}(S)$ for the set $\{v \in \mathbb{R}^n \mid Cv \in S\}$. For a cone $K \subseteq \mathbb{R}^n$, its dual cone K^* is defined as

$$K^* \equiv \{d \in \mathbb{R}^n \mid d^T v \geq 0 \ \ \forall\, v \in K\}.$$

Given a matrix $M \in \mathbb{R}^{m \times n}$ and a vector $q \in \mathbb{R}^n$, we write LCP(q, M) for the linear complementarity problem $0 \leq z \perp q + Mz \geq 0$ and denote its solution set by SOL(q, M). Since such a solution set is the union of finitely many polyhedra, it follows that SOL(q, M) must have a least-norm element, provided that SOL(q, M) is nonempty. If M is not positive semidefinite, such a least-norm LCP-solution is not necessarily unique. Nevertheless, we have the following important lemma whose proof follows from the well-known Hoffman's error bound for polyhedra (see [18, Lemma 3.2.3]).

LEMMA 1. *Given a matrix M, there exists a constant $\eta > 0$, depending only on M, such that for every vector q for which SOL(q, M) $\neq \emptyset$, a least-norm solution of the LCP(q, M) is bounded in norm by $\eta\|q\|$.*

Proof. Let α be a subset of $\{1, 2, \ldots, n\}$ and $\overline{\alpha}$ be the complement of it. Let $M_{\alpha\bullet}$ be the rows of M indexed by α. It is clear that the solution set of LCP(q, M) can be written as SOL(q, M) $= \bigcup_{\alpha \subseteq \{1,2,\ldots,n\}} \mathcal{C}_\alpha$, where

$$\mathcal{C}_\alpha = \left\{ u \ \middle| \ \begin{array}{r} u_\alpha \geq 0 \\ u_{\overline{\alpha}} = 0 \\ q_\alpha + M_{\alpha\bullet} u = 0 \\ q_{\overline{\alpha}} + M_{\overline{\alpha}\bullet} u \geq 0 \end{array} \right\}.$$

Since $\mathcal{C}_\alpha$ is a polyhedron for all α, for each of the nonempty $\mathcal{C}_\alpha$, we can find a least-norm element, say u^α, in C_α; i.e.,

$$u^\alpha = \arg\min_{u \in \mathcal{C}_\alpha} \|u\|.$$

By Hoffman's error bound, we know that there exists η_α, depending only on M, such that $\|u^\alpha\| \leq \eta_\alpha \|q\|$. It suffices to take $\eta = \max_{\alpha \,:\, \mathcal{C}_\alpha \neq \emptyset}\{\eta_\alpha\}$. $\square$

To provide the context and background for our results, we review below some definitions and results from LCP and control theories. Given a matrix M, the *LCP-Range* of M, denoted LCP-Range(M), is the set of all vectors q for which the LCP(q, M) is solvable, i.e., SOL(q, M) $\neq \emptyset$; the *LCP-Kernel of M*, which we denote LCP-Kernel(M), is the solution set SOL($0, M$) of the homogeneous LCP: $0 \leq v \perp Mv \geq 0$. An R_0-matrix is a matrix M for which LCP-Kernel(M) $= \{0\}$. If M is positive semidefinite, we have the following duality relation:

$$\text{LCP-Kernel}(M) = \{\, v \,:\, M^T v \leq 0 \leq v \,\} = [\,\text{LCP-Range}(M)\,]^*,$$

where $[\,\text{LCP-Range}(M)\,]^*$ is the dual cone of LCP-Range(M). Moreover, if $M \in \mathbb{R}^{m \times m}$, then

$$\text{LCP-Range}(M) = \mathbb{R}^m_+ - M\mathbb{R}^m_+.$$

For a given pair of matrices $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$, let $\overline{\mathcal{O}}(C, A)$ denote the unobservability space of the pair of matrices (C, A); i.e., $v \in \overline{\mathcal{O}}(C, A)$ if and only if

$CA^i v = 0$ for all $i = 0, 1, \ldots, n - 1$. Note that $v \in \overline{\mathcal{O}}(C, A)$ if and only if $C(I - hA)^{-1}v = 0$ for all $h > 0$ sufficiently small, and that for any subset β of $\{1, \ldots, m\}$, $\overline{\mathcal{O}}(C_{\beta\bullet}, A)$ is a well-defined superset of $\overline{\mathcal{O}}(C, A)$. A linear system $\Sigma(A, B, C, D)$ given by

$$
(7) \qquad \begin{aligned}
\dot{x}(t) &= Ax(t) + Bz(t), \\
w(t) &= Cx(t) + Dz(t)
\end{aligned}
$$

is *passive* if there exits a nonnegative-valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that, for all $t_0 \leq t_1$ and all trajectories (z, x, w) satisfying system (7), the following inequality holds:

$$
V(x(t_0)) + \int_{t_0}^{t_1} z^T(t)w(t)dt \geq V(x(t_1)).
$$

Passivity is a fundamental property in linear systems theory; see, e.g., [13] and the references therein. In particular, it is well known that the system $\Sigma(A, B, C, D)$ is passive if and only if there exists a symmetric positive semidefinite matrix K such that the symmetric matrix

$$
(8) \qquad \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix}
$$

is negative semidefinite. Checking passivity can be accomplished by solving a linear matrix inequality using methods of semidefinite programming [2]. The following lemma collects several useful results about passive tuples taken from [13, Proposition 2.1, Lemmas 2.1 and 2.2].

LEMMA 2. *Suppose the system $\Sigma(A, B, C, D)$ is passive. Let K be any matrix such that matrix (8) is negative semidefinite. Then the following statements hold.*
  (a) *D is positive semidefinite.*
  (b) *$u^T(D + D^T)u = 0 \Rightarrow C^T u = KBu$.*
  (c) *$u^T(D + D^T)u = 0 \Rightarrow u^T CBu = u^T B^T KBu \geq 0$.*
  (d) *$\mathrm{Ker}(K) \subseteq \overline{\mathcal{O}}(C, A)$.*
  (e) *D^h is positive semidefinite for all $h > 0$ sufficiently small.*
  (f) *$\mathrm{Ker}(D^h + (D^h)^T) = \mathrm{Ker}(\left[\begin{smallmatrix} KB \\ D+D^T \end{smallmatrix}\right])$.*
  (g) *LCP-$\mathrm{Range}(D^h) = LCP$-$\mathrm{Range}(D) + \mathrm{Range}(C)$ for all $h > 0$ sufficiently small.* ☐

We shall need the following theorem and a lemma which follows from [29].

THEOREM 3 (see [29, Theorem 7.1]). *Let f and g be Lipschitz continuous functions on $[0, T]$ and let D be a positive semidefinite matrix. Suppose that there exist positive scalars $c_{0,x}$, $c_{1,x}$, $c_{0,u}$, $c_{1,u}$, and $\bar{h}$ such that for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h - 1$,*

$$
(9) \qquad \|x^{h,i+1}\| \leq c_{0,x} + c_{1,x}\|x^0\| \quad \text{and} \quad \|u^{h,i+1}\| \leq c_{0,u} + c_{1,u}\|x^0\|.
$$

Then there is a sequence $\{h_\nu\} \downarrow 0$ such that the following two limits exist: $\hat{x}^{h_\nu} \to \hat{x}$ uniformly on $[0, T]$ and $\hat{u}^{h_\nu} \to \hat{u}$ weakly in $L^2(0, T)$. Furthermore, all such limits $(\hat{x}, \hat{u})$ are weak solutions of the initial-value inhomogeneous LCS (5). ☐

LEMMA 4 (see [29, Lemma 7.2]). *Suppose there are positive constants h_1, ρ_u, and ψ_x such that, for all $h \in (0, h_1]$ and all nonnegative integers $i = 0, 1, \ldots, N_h - 1$,*

$$
(10) \qquad \|u^{h,i+1}\| \leq \rho_u(1 + 2\|x^{h,i}\|),
$$

$$
(11) \qquad \|x^{h,i+1} - x^{h,i}\| \leq h\psi_x(1 + \|x^{h,i}\|).
$$

Then there are constants $c_{0,x}$, $c_{1,x}$, $c_{0,u}$, and $c_{1,u}$ such that (9) holds for all $h \in (0, h_1]$ and all $i = 0, 1, \ldots, N_h - 1$. □

3. Initial-value LCSs. We are now ready to analyze the convergence of the least-norm time-stepping scheme for initial-value LCSs. For the most part, our task is to establish the existence of solutions to the finite-dimensional linear complementarity subproblems that satisfy (10). This turns out to be a nontrivial task.

3.1. Homogeneous systems. We first consider the homogeneous initial-value LCS and establish the existence of a positive scalar γ, independent of h, such that $\|u^{h,i+1}\| \le \gamma\|x^{h,i}\|$ for all $i = 1, \ldots, N_h - 1$ and all h sufficiently small. We first present the following lemma.

LEMMA 5. *Let D be a positive semidefinite matrix. If there exist a sequence of positive scalars $\{h_\nu\} \downarrow 0$, a sequence of vectors $\{x^\nu\} \subseteq C^{-1}(\mathcal{P})$ for some polyhedral cone $\mathcal{P}$, and a sequence of solutions $\{u^\nu\}$ such that $u^\nu \in \mathrm{SOL}(C(I - h_\nu A)^{-1}x^\nu, D^{h_\nu})$ for all ν and such that*

$$\lim_{\nu\to\infty} \|u^\nu\| = \infty, \quad \lim_{\nu\to\infty} \frac{u^\nu}{\|u^\nu\|} = u^\infty \neq 0, \quad and \quad \lim_{\nu\to\infty} \frac{x^\nu}{\|u^\nu\|} = 0,$$

then there exists a vector $s^\infty \in \mathcal{P}$ such that

$$LCP\text{-}Kernel(D) \ni u^\infty \perp r^\infty = s^\infty + CBu^\infty \in [\, LCP\text{-}Kernel(D)\,]^* .$$

Proof. Since $u^\nu \in \mathrm{SOL}(C(I - h_\nu A)^{-1}x^\nu, D^{h_\nu})$, we have

$$0 \le u^\nu \perp w^\nu \equiv C(I - h_\nu A)^{-1}x^\nu + [D + h_\nu C(I - h_\nu A)^{-1}B]u^\nu \ge 0.$$

It follows that

$$C(I - h_\nu A)^{-1}x^\nu + h_\nu C(I - h_\nu A)^{-1}Bu^\nu = w^\nu - Du^\nu \in \text{LCP-Range}(D).$$

Noticing that

$$(I - h_\nu A)^{-1} = I + h_\nu A(I - h_\nu A)^{-1},$$

we have

$$(12) \qquad h_\nu\,[C(I - h_\nu A)^{-1}Bu^\nu + CA(I - h_\nu A)^{-1}x^\nu] = [w^\nu - Du^\nu] - Cx^\nu.$$

Moreover, it follows readily from

$$\lim_{\nu\to\infty} h_\nu = 0, \quad \lim_{\nu\to\infty} \|u^\nu\| = \infty, \quad \lim_{\nu\to\infty} \frac{u^\nu}{\|u^\nu\|} = u^\infty \neq 0, \quad \lim_{\nu\to\infty} \frac{x^\nu}{\|u^\nu\|} = 0$$

that

$$0 \le u^\infty \perp w^\infty \equiv \lim_{\nu\to\infty} \frac{w^\nu}{\|u^\nu\|} = Du^\infty \ge 0;$$

thus $u^\infty \in \text{LCP-Kernel}(D) = [\text{LCP-Range}(D)]^*$, where the last equality is due to the positive semidefiniteness of D. Furthermore, we have $u^\nu \perp w^\infty$ and $w^\nu \perp u^\infty$ for all ν sufficiently large. The former implies $0 = (u^\nu)^T Du^\infty = -(u^\nu)^T D^T u^\infty$, where

the second equality holds by the positive semidefiniteness of D and the fact that $(u^\infty)^T D u^\infty = 0$. Thus $D u^\nu \in (u^\infty)^\perp$. From (12), we deduce

$$(13) \qquad C(I - h_\nu A)^{-1} B \frac{u^\nu}{\|u^\nu\|} + CA(I - h_\nu A)^{-1} \frac{x^\nu}{\|u^\nu\|} = r^\nu - s^\nu,$$

where

$$r^\nu \equiv \frac{1}{h_\nu \|u^\nu\|} [w^\nu - D u^\nu] \quad \text{and} \quad s^\nu \equiv \frac{1}{h_\nu \|u^\nu\|} C x^\nu.$$

The vector r^ν belongs to LCP-Range(D) and $(u^\infty)^\perp$. Moreover, since the vector Cx^ν belongs to a polyhedral cone $\mathcal{P}$, so does s^ν. Thus the vector $r^\nu - s^\nu$, from the right-hand side of (13), is in the set

$$[(u^\infty)^\perp \cap \text{LCP-Range}(D)] - \mathcal{P},$$

which is the difference of two polyhedral sets and hence is again a polyhedron. Therefore, passing the limit as $\nu \to \infty$ in (13), and using the closedness of the polyhedral set $[(u^\infty)^\perp \cap \text{LCP-Range}(D)] - \mathcal{P}$, we deduce that there exist vectors r^∞ and s^∞, with r^∞ belonging to $(u^\infty)^\perp \cap \text{LCP-Range}(D)$ and s^∞ belonging to $\mathcal{P}$ such that

$$CB u^\infty = r^\infty - s^\infty.$$

Thus, recalling that $u^\infty \in \text{LCP-Kernel}(D)$ and $r^\infty \in \text{LCP-Range}(D) = [\text{LCP-Kernel}(D)]^*$, we have the desired result.   □

The following is a linear growth result for solutions to LCPs under perturbation. Note that this theorem does not require the matrix D^h to be positive semidefinite; another noteworthy point of the theorem is that the constant γ applies to all sufficiently small scalars h and all specified vectors x; in other words, we obtain a uniform bound on the least-norm solutions of the LCPs $(C(I - hA)^{-1}x, D^h)$ for all such pairs (h, x). This is a main result establishing the desired bounding property of the least-norm iterates.

THEOREM 6. *Let D be a positive semidefinite matrix and let $\mathcal{P}$ be a polyhedral cone. Suppose that the implication below holds:*

$$\left. \begin{array}{r} LCP\text{-Kernel}(D) \ni u^\infty \perp s^\infty + CB u^\infty \in [LCP\text{-Kernel}(D)]^* \\ for \ some \ s^\infty \in \mathcal{P} \end{array} \right\}$$

$$(14) \qquad \Rightarrow \ B u^\infty \in \overline{\mathcal{O}}(C_{\beta\bullet}, A), \quad where \ \beta \equiv \{i : (D u^\infty)_i = 0\}.$$

Then positive scalars $\bar{h}$ and γ exist such that, for every $h \in (0, \bar{h}]$ and every $x \in C^{-1}(\mathcal{P})$ with $C(I - hA)^{-1}x \in LCP\text{-Range}(D^h)$, a solution $u^h \in SOL(C(I - hA)^{-1}x, D^h)$ exists such that $\|u^h\| \leq \gamma \|x\|$.

Proof. We claim that the conclusion of the theorem holds with u^h being a least-norm solution of the LCP$(C(I - hA)^{-1}x, D^h)$. If this is false, then there exist a sequence of positive scalars $\{h_\nu\} \downarrow 0$, a sequence of vectors $\{x^\nu\} \subset C^{-1}(\mathcal{P})$, and a sequence of solutions $\{u^\nu\}$ such that u^ν belongs to SOL$(C(I - h_\nu A)^{-1}x^\nu, D^{h_\nu})$ for every ν, and the following limits hold:

$$\lim_{\nu \to \infty} \|u^\nu\| = \infty, \quad \lim_{\nu \to \infty} \frac{u^\nu}{\|u^\nu\|} = u^\infty \neq 0, \quad \text{and} \quad \lim_{\nu \to \infty} \frac{x^\nu}{\|u^\nu\|} = 0.$$

By Lemma 5 and the implication (14) we get a vector $u^\infty \in$ LCP-Kernel(D) such that $Bu^\infty \in \overline{\mathcal{O}}(C_{\beta\bullet}, A)$. We claim that $u^\nu - \frac{1}{2}\|u^\nu\|u^\infty \in \mathrm{SOL}(C(I - h_\nu A)^{-1}x^\nu, D^{h_\nu})$ for all ν sufficiently large. Once this claim is established, since $0 \le u^\nu - \frac{1}{2}\|u^\nu\|u^\infty \le u^\nu$ and $u^\infty \ne 0$, we obtain a contradiction to the least-norm property of u^ν. Since $Bu^\infty \in \overline{\mathcal{O}}(C_{\beta\bullet}, A)$, it follows that $C_{\beta\bullet}(I - hA)^{-1}Bu^\infty = 0$ for all $h > 0$ sufficiently small, implying that, for all such h, $(D^h u^\infty)_\beta = (Du^\infty)_\beta$. Therefore, writing $w^\nu \equiv C(I - h_\nu A)^{-1}x^\nu + D^{h_\nu}u^\nu$, we deduce that

$$\left[C(I - h_\nu A)^{-1}x^\nu + D^{h_\nu}(u^\nu - \tfrac{1}{2}\|u^\nu\|u^\infty) \right]_\beta = \left[w^\nu - \tfrac{1}{2}\|u^\nu\|Du^\infty \right]_\beta.$$

The claim will hold if the following two complementarity conditions hold for all ν sufficiently large:

(a) $0 \le \left[u^\nu - \tfrac{1}{2}\|u^\nu\|u^\infty \right]_\beta \perp \left[w^\nu - \tfrac{1}{2}\|u^\nu\|Du^\infty \right]_\beta \ge 0$,

(b) $0 \le \left[u^\nu - \tfrac{1}{2}\|u^\nu\|u^\infty \right]_{\bar\beta} \perp \left[w^\nu - \tfrac{1}{2}\|u^\nu\|D^{h_\nu}u^\infty \right]_{\bar\beta} \ge 0$,

where $\bar\beta$ is the complement of β in $\{1, \ldots, m\}$. Clearly, if $u_i^\infty > 0$ for some component $i \in \beta$, then $u_i^\nu/\|u^\nu\| > \frac{1}{2}u_i^\infty$ for all ν sufficiently large. Hence $[u_i^\nu - \frac{1}{2}\|u^\nu\|u^\infty]_\beta$ is nonnegative for all such ν; similarly, the same is true for $[w^\nu - \frac{1}{2}\|u^\nu\|Du^\infty]_\beta$. The complementarity in (a) holds because $u^\nu \perp Du^\infty$ and $w^\nu \perp u^\infty$, as we have shown above. To show (b), note that if $i \in \bar\beta$, then $(Du^\infty)_i > 0$, implying, via the fact that $0 \le u^\nu \perp Du^\infty \ge 0$, that $u_i^\nu = (u^\infty)_i = 0$ for all ν sufficiently large. This in turn yields $\left[u^\nu - \frac{1}{2}\|u^\nu\|u^\infty \right]_{\bar\beta} = 0$ for all ν sufficiently large. Moreover,

$$\left[w^\nu - \tfrac{1}{2}\|u^\nu\|D^{h_\nu}u^\infty \right]_i$$
$$= \|u^\nu\| \left\{ \frac{w_i^\nu}{\|u^\nu\|} - \tfrac{1}{2}(Du^\infty)_i - \frac{h_\nu}{2\|u^\nu\|}\left[C(I - h_\nu A)^{-1}Bu^\infty \right]_i \right\},$$

which is nonnegative for all ν sufficiently large. Therefore conditions (a) and (b) both hold. $\square$

The following corollary pertains to a special case of Theorem 6, where the implication (14), and thus the conclusion of the theorem, holds for all polyhedral cones $\mathcal{P}$.

COROLLARY 7. *Let D be a positive semidefinite matrix. If $BLCP$-Kernel$(D) \subseteq \overline{\mathcal{O}}(C, A)$, then for every polyhedral cone $\mathcal{P}$, there exist positive scalars $\bar h$ and γ such that for every scalar $h \in (0, \bar h]$ and every vector $x \in C^{-1}\mathcal{P}$ with $C(I - hA)^{-1}x \in LCP$-Range(D^h), a solution $u^h \in SOL(C(I - hA)^{-1}x, D^h)$ exists such that $\|u^h\| \le \gamma\|x\|$.*

Proof. If $BLCP$-Kernel$(D) \subseteq \overline{\mathcal{O}}(C, A)$, then clearly (14) holds for any polyhedral cone $\mathcal{P}$. Thus the conclusion of Theorem 6 follows. $\square$

Pertaining to the case where $\mathcal{P} = $ LCP-Range(D), the next corollary postulates a copositivity condition on the matrix CB with respect to the LCP-Kernel(D).

COROLLARY 8. *Let D be a positive semidefinite matrix. If CB is copositive on the LCP-Kernel(D) and*

(15) $$\left\{ LCP\text{-Kernel}(D) \ni u \perp \left[CB + (CB)^T \right] u \in [LCP\text{-Kernel}(D)]^* \right\}$$
$$\Rightarrow Bu \in \overline{\mathcal{O}}(C, A),$$

then the conclusion of Theorem 6 holds with $\mathcal{P} = LCP$-Range(D).

Proof. We verify that the implication (14) holds with $\mathcal{P} = $ LCP-Range(D). Let u satisfy

$$\text{LCP-Kernel}(D) \ni u^\infty \perp s^\infty + CBu^\infty \in [\text{LCP-Kernel}(D)]^*,$$

where $s^\infty \in \text{LCP-Range}(D) = [\text{LCP-Kernel}(D)]^*$. By the copositivity of CB on the LCP-Kernel(D), it follows that $u^\infty \perp CBu^\infty$, which implies that u^∞ satisfies the left-hand side of (15). Thus $Bu \in \overline{\mathcal{O}}(C, A)$ by (15), establishing that the implication (14) holds. $\quad\square$

The following example shows that copositivity and the implication (15) together are not enough to ensure the solvability of the $\text{LCP}(C(I - hA)^{-1}q, D^h)$ for $q \in \mathbb{R}^n$.

Example 1. Let

$$A = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and } q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have $Cq = 0$ and $CAq = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Since $C(I - hA)^{-1}q = Cq + hCAq + O(h^2)$, we know that $C(I - hA)^{-1}q < 0$ for all $h > 0$ sufficiently small. As $B = 0$ and $D^h = D$, it follows that $\text{LCP}(C(I - hA)^{-1}q, D^h)$ is not solvable for all $h > 0$ sufficiently small.

While the implication (15) and the copositivity of CB on LCP-Kernel(D) together are not sufficient to ensure the solvability of the LCP $(C(I - hA)^{-1}q, D^h)$, the strict copositivity of CB on LCP-Kernel(D) is sufficient. The following proposition establishes this claim.

PROPOSITION 9. *Let D be a positive semidefinite matrix. If CB is strictly copositive on the LCP-Kernel(D), then there exists an $\bar{h}$ such that $\text{LCP}(Cq, D^h)$ is solvable for all $q \in \mathbb{R}^n$ and all $h \in (0, \bar{h}]$.*

Proof. Based on [18, Theorem 2.6.1], it suffices to show that there exists an $\bar{h}$ such that for all $h \in (0, \bar{h}]$, for all $q \in \mathbb{R}^n$ with $\|q\| = 1$ the set $\bigcup_{\tau > 0} \text{SOL}(Cq, D^h + \tau I)$ is bounded. Assume the contrary. Then there exists a sequence of positive scalars $\{h_\nu\} \downarrow 0$ such that for each ν, there exist a vector q^ν with $\|q^\nu\| = 1$, a sequence of positive scalars $\{\tau_{\nu,\mu}\}$, and a sequence of solutions $u^{\nu,\mu} \in \text{SOL}(Cq^\nu, D^{h_\nu} + \tau_{\nu,\mu}I)$ satisfying $\lim_{\mu \to \infty} \|u^{\nu,\mu}\| = \infty$. That is,

$$0 \le u^{\nu,\mu} \perp Cq^\nu + \left[D + h_\nu C(I - h_\nu A)^{-1}B + \tau_{\nu,\mu}I\right] u^{\nu,\mu} \ge 0.$$

We claim that, for each ν, $\limsup_{\mu \to \infty} \tau_{\nu,\mu} < \infty$. For otherwise, we have

$$0 = (u^{\nu,\mu})^T \left(Cq^\nu + \left[D + h_\nu C(I - h_\nu A)^{-1}B + \tau_{\nu,\mu}I\right] u^{\nu,\mu}\right)$$

$$\ge \tau_{\nu,\mu}\|u^{\nu,\mu}\|^2 - \|u^{\nu,\mu}\|\|Cq^\nu\| - h_\nu\|C(I - h_\nu A)^{-1}B\|\|u^{\nu,\mu}\|^2 > 0$$

for μ sufficiently large since ν is fixed. This is a contradiction. From the complementarity condition, we have

$$(16) \qquad 0 \le \frac{u^{\nu,\mu}}{\|u^{\nu,\mu}\|} \perp \frac{Cq^\nu}{\|u^{\nu,\mu}\|} + \left[D + h_\nu C(I - h_\nu A)^{-1}B + \tau_{\nu,\mu}I\right] \frac{u^{\nu,\mu}}{\|u^{\nu,\mu}\|} \ge 0.$$

For each fixed ν, by taking a proper subsequence, we may assume, without loss of generality, that

$$\lim_{\mu \to \infty} \frac{u^{\nu,\mu}}{\|u^{\nu,\mu}\|} = \widehat{u}^\nu \quad \text{and} \quad \lim_{\mu \to \infty} \tau_{\nu,\mu} = \widehat{\tau}_\nu.$$

Now, letting μ go to ∞, we see from (16) that

$$0 \le \widehat{u}^\nu \perp \left[D + h_\nu C(I - h_\nu A)^{-1}B + \widehat{\tau}_\nu I\right] \widehat{u}^\nu \ge 0.$$

Thus,

$$0 = (\widehat{u}^\nu)^T \left[D + h_\nu C(I - h_\nu A)^{-1} B + \widehat{\tau}_\nu I \right] \widehat{u}^\nu$$

(17)
$$= (\widehat{u}^\nu)^T D\widehat{u}^\nu + h_\nu (\widehat{u}^\nu)^T C(I - h_\nu A)^{-1} B\widehat{u}^\nu + \widehat{\tau}_\nu.$$

We assume, without loss of generality, that

$$\lim_{\nu \to \infty} \widehat{u}^\nu = u^\infty.$$

Letting ν go to ∞, we get, by the positive semidefiniteness of D, that

$$(u^\infty)^T Du^\infty = 0 \quad \text{and} \quad \lim_{\nu \to \infty} \widehat{\tau}_\nu = 0.$$

From (17), we obtain that

$$h_\nu (\widehat{u}^\nu)^T C(I - h_\nu A)^{-1} B\widehat{u}^\nu = -\widehat{\tau}_\nu - (\widehat{u}^\nu)^T D\widehat{u}^\nu \leq 0,$$

which in turn implies that

$$(\widehat{u}^\nu)^T C(I - h_\nu A)^{-1} B\widehat{u}^\nu \leq 0.$$

Now, letting ν go to ∞, we have

$$(u^\infty)^T CBu^\infty \leq 0,$$

which contradicts the assumption that CB is strictly copositive on LCP-Kernel(D). This completes the proof. □

Parallel to the above development, we can also ensure the desired uniform bound under another assumption. In the following theorem, the first assertion establishes the solvability of the LCPs that arise after discretization; the second assertion shows that one such solution exists satisfying the desired bound.

THEOREM 10. *Let D be a positive semidefinite matrix. Suppose that*

(18)
$$C \left[\text{Range}(A) + B\mathbb{R}^m_+ \right] \subseteq LCP\text{-Range}(D).$$

Positive constants $\bar{h}$ and γ exist such that for every scalar $h \in (0, \bar{h}]$, the following two statements are valid:
(a) *for every $x \in C^{-1}LCP\text{-Range}(D)$, it holds that $C(I - hA)^{-1}x \in LCP\text{-Range}(D^h)$;*
(b) *for every $x \in C^{-1}LCP\text{-Range}(D)$, a solution $u^h \in SOL(C(I - hA)^{-1}x, D^h)$ exists such that $\|u^h\| \leq \gamma\|x\|$.*

Proof. We establish (a) by applying Kakutani's fixed-point theorem to a certain LCP solution map; from this proof, part (b) will follow readily. To begin, based on Lemma 1, let $\eta > 0$ be a scalar such that for all vectors $r \in \text{LCP-Range}(D)$, the (necessarily unique) least-norm solution of the monotone LCP (r, D), denoted $\Psi(r)$, satisfies $\|\Psi(r)\| \leq \eta\|r\|$. Let $\bar{h} > 0$ be such that for all $h \in (0, \bar{h}]$, $h\eta\|C(I - hA)^{-1}B\| \leq \frac{1}{2}$. Fix such a scalar h and a vector $x \in C^{-1}\text{LCP-Range}(D)$ with unit norm. (A simple scaling easily extends the argument below to any such vector x of arbitrary norm.) Let $R \equiv 2\eta\|C(I - hA)^{-1}\|$. For every vector $u \geq 0$ satisfying $\|u\| \leq R$, let $\Phi(u) \equiv \{v \in \text{SOL}(\widehat{u}, D) : \|v\| \leq \eta\|\widehat{u}\|\}$, where $\widehat{u} \equiv C(I - hA)^{-1}[x + hBu]$. Since

$$\widehat{u} = Cx + C\left[(I - hA)^{-1} - I\right]x + hC(I - hA)^{-1}Bu$$
$$= Cx + hCA(I - hA)^{-1}x + hC(I - hA)^{-1}Bu$$
$$= Cx + hC(Ax + Bu) + h^2 CA(I - hA)^{-1}(Ax + Bu),$$

it follows that $\widehat{u} \in$ LCP-Range(D) because it is the sum of three vectors all of which belong to this LCP-Range that is a polyhedral cone. Consequently, we deduce that $\widehat{u} \in$ LCP-Range(D); hence the least-norm solution $\Psi(\widehat{u})$ is well defined and belongs to $\Phi(u)$. Thus $\Phi(u)$ is nonempty. We claim that $\|v\| \leq R$ for every $v \in \Phi(u)$. Indeed,

$$\| v \| \leq \eta \left\| C \left(I - hA \right)^{-1} [\, x + h \, Bu \,] \right\|$$
$$\leq \eta \left\| C \left(I - hA \right)^{-1} \right\| + \tfrac{1}{2} \| u \| \leq R.$$

Since D is positive semidefinite, $\Phi(u)$ is a convex set; it is clearly closed. Moreover, by a simple limiting argument, it can be shown that the graph of Φ, i.e., the set $\{(u,v)\,|\,u \geq 0, \|u\| \leq R \text{ and } v \in \Phi(u)\}$, is also closed. Therefore, Φ is a closed, nonempty-valued, closed-valued, and convex-valued, point-to-set self-map from the compact convex semiball $\{u\,|\,u \geq 0, \|u\| \leq R\}$ into itself. Hence, by Kakutani's fixed-point theorem, Φ has a fixed point, which we denote u^h. It is easy to see that u^h is a solution of the LCP $(C(I - hA)^{-1}x, D^h)$. Furthermore, we have

$$\| u^h \| \leq \eta \left\| C \left(I - hA \right)^{-1} [\, x + h \, Bu^h \,] \right\|,$$

which yields, since $\|x\| = 1$ and by the choice of h, the inequality $\|u^h\| \leq 2\eta \left\| C \left(I - hA \right)^{-1} \right\|$. $\square$

Remark 1. Clearly, the condition (18) is implied by the inclusion Range$(C) \subseteq$ LCP-Range(D), which holds if and only if the implication $D^T v \leq 0 \leq v \Rightarrow C^T v = 0$ holds. Under this more restrictive range condition, we have $C(I - hA)^{-1}\mathbb{R}^n \subseteq$ LCP-Range(D^h), which is a stronger consequence than part (a) of Theorem 10. Nevertheless, unlike (18), the range condition Range$(C) \subseteq$ LCP-Range(D) imposes no restriction on the pair (A, B), thus making it a restrictive assumption on the pair (C, D).

Remark 2. The strict copositivity of CB on the LCP-kernel(D) and condition (18) are different, with each one not implying the other. With $D = 0$ and CB a strictly copositive matrix, it is clear that the former holds, but the latter does not. Conversely, the following counterexample shows that the (more restrictive) condition Range$(C) \subseteq$ LCP-Range(D), which implies (18) (see Remark 1), can hold without CB being strictly copositive on the LCP-Kernel(D).

Example 2. Let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad C = D.$$

Since D is symmetric positive semidefinite, we have Range$(C) \subseteq$ LCP-Range(D). The vector $v \equiv (0, 1)$ satisfies $Dv = 0$ and

$$\text{LCP-Kernel}(D) \ni v \perp CBv \in \text{LCP-Range}(D);$$

thus CB is not strictly copositive on the LCP-Kernel(D).

Summarizing the above preliminary results, we present our two main results about the convergence of the least-norm time-stepping scheme.

THEOREM 11. *Suppose the following statements hold:*
(A) *D is positive semidefinite;*
(B) Range$(C) \subseteq$ *LCP*-Range(D^h) *for all $h > 0$ sufficiently small; and*
(C) *implication (14) holds with $\mathcal{P} = $ LCP*-Range(D).

Then there exists an $\bar{h} > 0$ such that, for every x^0 satisfying $Cx^0 \in LCP\text{-}Range(D)$, the two trajectories $\widehat{x}^h(t)$ and $\widehat{u}^h(t)$ generated by the least-norm time-stepping scheme are well defined for all $h \in (0, \bar{h}]$, and there is a sequence $\{h_\nu\} \downarrow 0$ such that the following two limits exist: $\widehat{x}^{h_\nu}(\cdot) \to \widehat{x}(\cdot)$ uniformly on $[0, T]$ and $\widehat{u}^{h_\nu}(\cdot) \to \widehat{u}(\cdot)$ weakly in $L^2(0, T)$. Moreover, all such limits $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ are weak solutions of the initial-value LCS (1).

Proof. From assumption (B), there exists an $\bar{h}_1$ such that $Range(C) \subseteq LCP\text{-}Range$ (D^h) for all h in $(0, \bar{h}_1]$. Take an $\bar{h} < \min\{\bar{h}_1, \frac{1}{\|A\|}\}$. The existence of $x^{h,i}$ for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h$ is guaranteed by assumption (B). Notice also that $x^{h,i+1} \in LCP\text{-}Range(D)$ for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h - 1$ since $0 \le u^{h,i+1} \perp Cx^{h,i+1} + Du^{h,i+1} \ge 0$. Recalling that $u^{h,i+1}$ is a least-norm solution of $LCP(C(I - hA)^{-1}x^{h,i}, D^h)$, by Lemma 5 and Theorem 6, we deduce that there exists a γ such that $\|u^{h,i+1}\| \le \gamma \|x^{h,i}\|$ for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h - 1$. Hence, we have

$$\|x^{h,i+1} - x^{h,i}\| = h\|Ax^{h,i+1} + Bu^{h,i+1}\| \le h\|Ax^{h,i+1}\| + h\|B^{h,i+1}\|$$

$$\le h\|A(x^{h,i+1} - x^{h,i})\| + h\|Ax^{h,i}\| + h\|Bu^{h,i}\|$$

$$\le h\|A\|\|x^{h,i+1} - x^{h,i}\| + h(\|A\| + \gamma\|B\|)\|x^{h,i}\|.$$

Since $h \le \bar{h} \le \frac{1}{\|A\|}$, we have

$$\|x^{h,i+1} - x^{h,i}\| \le \frac{h(\|A\| + \gamma\|B\|)\|x^{h,i}\|}{1 - h\|A\|} \le h\psi_x\|x^{h,i}\|$$

for some $\psi_x > 0$. By Lemma 4 and Theorem 3 we have the desired result. $\square$

THEOREM 12. *Suppose the following statements hold:*

(A) *D is positive semidefinite;*

(B) *$C\left[Range(A) + B\mathbb{R}_+^m\right] \subseteq LCP\text{-}Range(D)$.*

Then the conclusion of Theorem 11 *holds.*

Proof. The proof follows from Theorem 10 and is similar to the proof of Theorem 11. $\square$

Remark 3. Assumption (B) in Theorem 11, which involves the tuple (A, B, C, D), does not imply, nor is implied by, the inclusion $Range(C) \subseteq LCP\text{-}Range(D)$, which involves only the pair (C, D).

Remark 4. Theorem 11 extends the previous result [29, Theorem 7.4] which shows that if D is positive semidefinite and is also an R_0-matrix, then one can ascertain the convergence of the time-stepping scheme. In fact, under the assumption of D being an R_0-matrix, by [29, Proposition 7.1] we can ensure that assumption (B) in Theorem 11 holds. Also by the definition of an R_0-matrix, one has $u^\infty \in LCP\text{-}Kernel(D) \Rightarrow u^\infty = 0$, and hence the implication (14) holds.

Applying Corollary 8 and Proposition 9, one can easily prove the following corollary.

COROLLARY 13. *Under conditions* (A) + (B), *or* (A) + (B$'$), *where*

(A) *D is positive semidefinite,*

(B) *the inclusion* (18) *holds, and*

(B$'$) *CB is strictly copositive on $LCP\text{-}Kernel(D)$,*

the conclusion of Theorem 11 *holds.* $\square$

Remark 5. The assertion of Corollary 13 under conditions (A) + (B) extends the previous result [29, Theorem 7.4] which assumes the interiority inclusion $C\mathbb{R}^n \subseteq int(LCP\text{-}Range(D))$. The inclusion (18) is much less restrictive.

Another important corollary of Theorem 11 is the following result regarding passive LCSs.

COROLLARY 14. *Given an LCS* (1), *if* $\Sigma(A, B, C, D)$ *is passive, then the conclusion of Theorem* 11 *holds.*

Proof. We need only verify assumptions (A), (B), and (C) in Theorem 11. The first two, (A) and (B), follow directly from Lemma 2. From part (c) of Lemma 2, we know that CB is copositive on LCP-Kernel(D). Moreover, if

$$LCP\text{-Kernel}(D) \ni u \perp \left[CB + (CB)^T \right] u \in [\, LCP\text{-Kernel}(D) \,]^*,$$

then $u^T C B u = u^T B^T K B u = 0$ for a matrix K which makes the matrix (8) negative semidefinite. Therefore, $Bu \in \ker(K) \subseteq \bigcap_{i=0}^{n-1} \ker(CA^i)$; i.e., $CBu = CABu = \cdots = CA^{n-1}Bu = 0$, or, equivalently, $Bu \in \overline{O}(C, A)$. $\quad\square$

Remark 6. Corollary 14 extends the result obtained in [4], which states that under passivity and minimality and a rank condition, one can ensure the convergence of the time-stepping scheme. In Corollary 14, we show that the minimality assumption and the rank condition can be dropped if we use the least-norm time-stepping scheme. To reiterate a remark made earlier, for a passive LCS, the matrix D^h is positive semidefinite for all $h > 0$ sufficiently small; thus the least-norm iterates are not difficult to obtain.

Apart from establishing the convergence of time-stepping methods, Theorem 11 and its corollaries assert the existence of weak solutions to the LCS under assumptions that are less restrictive than some previous results [29, Theorem 6.1(c) and (d)], which require either an R_0-property on D or an interiority condition.

3.2. Inhomogeneous systems. We can extend the above analysis to the inhomogeneous case. In contrast to Lemma 5 for homogeneous systems, we can prove the following lemma whose proof is very similar to Lemma 5 and hence is omitted.

LEMMA 15. *Suppose D is a positive semidefinite matrix, and f and g are Lipschitz continuous on $[0, T]$. If there exist a sequence of positive scalars $\{h_\nu\} \downarrow 0$, a sequence of vectors $\{x^\nu\}$, a sequence of scalars $\{t_\nu\} \subseteq [0, T]$, and a sequence of solutions $\{u^\nu\}$ satisfying $\{Cx^\nu + g(t_\nu)\} \subseteq \mathcal{P}$ for some polyhedral cone $\mathcal{P}$ such that*

$$u^\nu \in SOL(C(I - h_\nu A)^{-1} [x^\nu + h_\nu f(t_\nu)] + g(t_\nu), D^{h_\nu})$$

and

$$\lim_{\nu \to \infty} \|u^\nu\| = \infty, \quad \lim_{\nu \to \infty} \frac{u^\nu}{\|u^\nu\|} = u^\infty \neq 0, \quad \lim_{\nu \to \infty} \frac{x^\nu}{\|u^\nu\|} = 0,$$

then there exists a vector $s^\infty \in \mathcal{P}$ such that

$$LCP\text{-}Kernel(D) \ni u^\infty \perp r^\infty = s^\infty + CBu^\infty \in [\, LCP\text{-}Kernel(D) \,]^*. \quad \square$$

Similarly to the homogeneous case, we can prove the following linear growth result whose proof is the same as that of Theorem 6 and hence is omitted.

THEOREM 16. *Let D be a positive semidefinite matrix, let $\mathcal{P}$ be a polyhedral cone, and let f and g be two Lipschitz continuous functions satisfying $g(t) \in \mathcal{P}$ for all $t \in [0, T]$. Suppose that the following implication holds:*

(19)
$$\left.\begin{array}{r} LCP\text{-}Kernel(D) \ni u^\infty \perp s^\infty + CBu^\infty \in [\, LCP\text{-}Kernel(D) \,]^* \\[4pt] \text{for some } s^\infty \in \mathcal{P} \end{array}\right\}$$
$$\Rightarrow \ Bu^\infty \in \overline{O}(C_{\beta\bullet}, A), \quad \text{where } \beta \equiv \{i : (Du^\infty)_i = 0\}.$$

Then, there exist positive scalars $\bar{h}$ and γ such that for every $h \in (0, \bar{h}]$ and for every pair (x, t) satisfying $C(I - hA)^{-1}[x + hf(t)] + g(t) \in LCP\text{-}Range(D^h)$, a solution $u^h \in SOL(C(I - hA)^{-1}[x + hf(t)] + g(t), D^h)$ exists such that $\|u^h\| \leq \gamma(1 + \|x\|)$.

Analogous to Theorem 11, we have the following main convergence result for the inhomogeneous case.

THEOREM 17. *Suppose the following statements hold:*
(A) *D is positive semidefinite;*
(B) *f and g are Lipschitz continuous;*
(C) *implication (19) holds;*
(D) *$Range(C) \subseteq LCP\text{-}Range(D^h)$ for all $h > 0$ sufficiently small; and*
(E) *$g(t) \in LCP\text{-}Range(D^h)$ for all $h > 0$ sufficiently small and all $t \in [0, T]$.*
Then there exists an $\bar{h} > 0$ such that, for every x^0 satisfying $Cx^0 + g(t_0) \in LCP\text{-}Range (D)$, the two trajectories $\hat{x}^h(t)$ and $\hat{u}^h(t)$ generated by the least-norm time-stepping scheme are well defined for all $h \in (0, \bar{h}]$, and there is a sequence $\{h_\nu\} \downarrow 0$ such that the following two limits exist: $\hat{x}^{h_\nu}(\cdot) \to \hat{x}(\cdot)$ uniformly on $[0, T]$ and $\hat{u}^{h_\nu}(\cdot) \to \hat{u}(\cdot)$ weakly in $L^2(0, T)$. Moreover, all such limits $(\hat{x}(\cdot), \hat{u}(\cdot))$ are weak solutions of the initial-value LCS (5).

Corollaries similar to those for the homogeneous case can readily be obtained. Below, we just state the specialization of Theorem 17 to an inhomogeneous passive LCS.

COROLLARY 18. *Given an LCS (5), if $\Sigma(A, B, C, D)$ is passive, f and g are Lipschitz continuous, and $g(t) \in LCP\text{-}Range(D^h)$ for all $h > 0$ sufficiently small, then the conclusion of Theorem 17 holds.*

4. Boundary-value LCSs. In this section, we consider the following LCS with a two-point boundary condition:

$$
\begin{aligned}
\dot{x} &= Ax + Bu, \\
0 &\leq u \perp Cx + Du \geq 0, \\
b &= Mx(0) + Nx(T),
\end{aligned}
\tag{20}
$$

where b is an n-vector and M and N are $n \times n$ matrices. For this BVP, we compute

$$\{x^{h,0}, x^{h,1}, \dots, x^{h,N_h}\} \subset \mathbb{R}^n \quad \text{and} \quad \{u^{h,1}, u^{h,2}, \dots, u^{h,N_h}\} \subset \mathbb{R}_+^m$$

by solving an aggregated mixed LCP that is defined by

$$
\begin{aligned}
\left.
\begin{array}{l}
x^{h,i+1} - x^{h,i} = h \left[Ax^{h,i+1} + Bu^{h,i+1} \right] \\
0 \leq u^{h,i+1} \perp Cx^{h,i+1} + Du^{h,i+1} \geq 0
\end{array}
\right\}, \quad i = 0, 1, \dots, N_h - 1, \\
Mx^{h,0} + Nx^{h,N_h} = b.
\end{aligned}
\tag{21}
$$

It should be noted that, unlike the IVP, the mixed LCP (21) is not readily decomposable into independent subproblems. The issue of how to solve such a mixed LCP efficiently in practice is outside the scope of this paper.

We first study the passive case. Before we present our results, we follow the treatment in [29] (for more details see also the references therein) and review some definitions and results in topological fixed-point theory [19]. An *acyclic* set is a topological space X where the rational homology groups of X are isormorphic to those of a singleton. An *absolute retract* (AR) is a topological space X such that if X is embedded as a closed subset X' of a space Y, then X is a retract of Y. Every compact convex set is an AR; every homeomorphic image of a compact convex set is acyclic.

An *acyclic map* is an upper semicontinuous set-valued map which has compact acyclic values. A single-valued map $p : X \mapsto Y$ is a *Vietoris map* if it is continuous and, for each $y \in Y$, $p^{-1}(y)$ is a nonempty compact acyclic set. In Górniewicz's terminology an *admissible map* is an upper semicontinuous set-valued map $F : X \mapsto Y$ with compact values which can be represented as $F = r \circ p^{-1}$, where $r : Z \mapsto Y$ is a continuous single-valued map and $p : Z \mapsto X$ is a Vietoris map. It can be easily shown that acyclic maps are necessarily admissible maps; furthermore, compositions of admissible maps are admissible. The following fixed-point theorem is the key of our derivation.

THEOREM 19 (Górniewicz [19]). *Every admissible multifunction $F : X \mapsto X$ on a compact AR X has a fixed point $x \in F(x)$ for some $x \in X$.* □

For a passive LCS BVP, letting γ be the constant whose existence is ascertained by Theorem 6, we define, for any fixed $h > 0$ sufficiently small and any x^{ref} satisfying $Cx^{\mathrm{ref}} \in \mathrm{LCP\text{-}Range}(D)$, the following set-valued maps:

$$(22) \qquad \Phi(x^{\mathrm{ref}}) \equiv \left\{ u \,\Big|\, u \in \mathrm{SOL}(C(I - hA)^{-1}x^{\mathrm{ref}}, D^h), \; \|u\| \leq \gamma \|x^{\mathrm{ref}}\| \right\},$$

$$(23) \qquad \Psi(x^{\mathrm{ref}}) \equiv (I - hA)^{-1}[x^{\mathrm{ref}} + hB\Phi(x^{\mathrm{ref}})],$$

$$(24) \qquad \Gamma^k(x^{\mathrm{ref}}) \equiv \underbrace{\Psi \circ \Psi \circ \cdots \circ \Psi}_{k \text{ fold}}(x^{\mathrm{ref}}), \quad k = 0, 1, \ldots, N_h.$$

Notice that all these maps depend on h. For a passive LCS, if x^{ref} satisfies $Cx^{\mathrm{ref}} \in \mathrm{LCP\text{-}Range}(D)$, then $\Phi(x^{\mathrm{ref}})$ is nonempty for all $h > 0$ sufficiently small, by Lemma 2; hence Γ^k is well defined on $\mathcal{K}_D \equiv C^{-1}\mathrm{LCP\text{-}Range}(D)$. Moreover, Γ^k maps $\mathcal{K}_D$ into itself. Indeed, if $x^{\mathrm{ref}} \in \mathcal{K}_D$, then, for any $v \in \Psi(x^{\mathrm{ref}})$, $v = (I - hA)^{-1}[x^{\mathrm{ref}} + hBu]$ for some $u \in \Phi(x^{\mathrm{ref}})$. It is easy to see that $u \in \mathrm{SOL}(Cv, D)$; thus $Cv \in \mathrm{LCP\text{-}Range}(D)$ or, equivalently, $v \in \mathcal{K}_D$. Inductively, it follows that Γ^k is a multivalued self-map from $\mathcal{K}_D$ into itself. The lemma below collects properties about the above set-valued maps for a passive system.

LEMMA 20. *Given an LCS with $\Sigma(A, B, C, D)$ being passive, positive scalars $\bar{h}$ and κ exist such that for any $h \in (0, \bar{h}]$ and any x^{ref} satisfying $Cx^{\mathrm{ref}} \in LCP\text{-}Range(D)$, the following statements hold:*
  (a) *$\Phi(x^{\mathrm{ref}})$, $\Psi(x^{\mathrm{ref}})$, and $\Gamma^k(x^{\mathrm{ref}})$ are well defined with $\Phi(x^{\mathrm{ref}}) \neq \emptyset$;*
  (b) *for all $x^h \in \Psi(x^{\mathrm{ref}})$, $\|x - x^{ref}\| \leq \kappa h \|x^{\mathrm{ref}}\|$;*
  (c) *for each $k = 0, 1, \ldots, N_h$, Γ^k is an admissible map in the sense of Górniewicz;*
  (d) *for all $x \in \Gamma^{N_h}(x^{\mathrm{ref}})$, $\|x - x^{\mathrm{ref}}\| \leq (e^{T\kappa} - 1)\|x^{\mathrm{ref}}\|$.*

Proof. Take $\bar{h}$ small enough so that Lemma 2(e) and (g) and Theorem 6 hold. The nonemptiness of Φ holds readily for every $h \in (0, \bar{h}]$. Notice that, for any $v \in \Psi(x^{\mathrm{ref}})$, $v = (I - hA)^{-1}[x^{\mathrm{ref}} + hBu]$ for some $u \in \Phi(x^{\mathrm{ref}})$; therefore

$$\|v - x^{\mathrm{ref}}\| = \left\| (I - hA)^{-1} \left(x^{\mathrm{ref}} + hBu \right) - x^{\mathrm{ref}} \right\|$$

$$= \left\| \left[I + hA(I - hA)^{-1} \right] x^{\mathrm{ref}} + h(I - hA)^{-1}Bu - x^{\mathrm{ref}} \right\|$$

$$= \left\| hA(I - hA)^{-1}x^{\mathrm{ref}} + h(I - hA)^{-1}Bu \right\|$$

$$\leq h \left\| A(I - hA)^{-1} \right\| \left\| x^{\mathrm{ref}} \right\| + h \left\| (I - hA)^{-1}B \right\| \gamma \|x^{\mathrm{ref}}\|.$$

Since $\|(I - hA)^{-1}\|$ is bounded for all $h \in (0, \bar{h}]$, part (b) holds. To show part (c), we notice that $\Phi(x^{\mathrm{ref}})$ is a compact- and convex-valued map with a closed graph.

Clearly, it is also locally bounded and hence is upper semicontinuous. Therefore, Ψ is also an upper semicontinuous compact- and convex-valued map and therefore is an acyclic map. Since all acyclic maps are admissible maps and the composition of admissible maps is still admissible, (c) follows readily. For part (d), notice that, for all $x \in \Gamma^{N_h}(x^{\mathrm{ref}})$, there exists a finite sequence $\{x^k\}_{k=0}^{N_h}$, such that $x^0 = x^{\mathrm{ref}}$; $x^{k+1} \in \Psi(x^k)$ for all $k = 1, \ldots, N_h - 1$; and $x = x^{N_h}$. By part (b), we know that for $k = 0, 1, \ldots, N_h - 1$

$$\|x^k\| \leq \rho_k, \quad \|x^{k+1} - x^k\| \leq \rho_{k+1} - \rho_k,$$

where $\|x^{\mathrm{ref}}\| = \rho_0$ and $\rho_{k+1} \equiv (1 + \kappa h)\rho_k$ for $k = 0, \ldots, N_h - 1$. Therefore, for any $x \in \Gamma^{N_h}(x^{\mathrm{ref}})$,

$$\|x - x^{\mathrm{ref}}\| \leq \sum_{k=0}^{N_h-1} \|x^{k+1} - x^k\| \leq \rho_{N_h} - \rho_0.$$

Notice that

$$\rho_{N_h} = (1 + \kappa h)^{N_h} \rho_0 \leq e^{T\kappa} \rho_0,$$

where the last inequality follows from the fact that $N_h = T/h$. $\qquad\square$

Under the passivity assumption and a condition on the boundary value matrices M and N that involves the time T and a constant γ whose existence is guaranteed by Theorem 6, we can prove the existence of a solution to the mixed LCP (21).

LEMMA 21. *Given an LCS BVP* (20) *with* $\Sigma(A, B, C, D)$ *being passive, suppose* M *and* N *are such that* $M + N$ *is nonsingular and*

$$(25) \qquad e^{T\kappa} < 1 + \frac{1}{\|(M+N)^{-1}N\|},$$

where $\kappa > 0$ *and* $\bar{h} > 0$ *are as in Lemma* 20. *Then positive scalars* ρ_0 *and* γ *exist such that, for all* b *in* $NA\mathcal{K}_D + NB\mathbb{R}_+^m + (M+N)\mathcal{K}_D$ *and all* $h \in (0, \bar{h}]$, *a solution to* (21) *exists satisfying*
   (a) $\|x^{h,0}\| \leq \rho_0$,
   (b) $\|x^{h,i+1} - x^{h,i}\| \leq \kappa h \|x^{h,i}\|$,
   (c) $\|u^{h,i}\| \leq \gamma \|x^{h,i}\|$.
   Proof. Since $Mx + Ny = b$ if and only if $x = (M+N)^{-1}b - (M+N)^{-1}N(y - x)$, we see that a sufficient condition for the existence of a solution to (21) is for the map

$$H \equiv (M+N)^{-1}b - (M+N)^{-1}N(\Gamma^{N_h} - \Gamma^0)$$

to have a fixed point. Hence, it suffices to show that there exists a scalar ρ_0 such that H is a self-map on $\rho_0 \mathcal{B} \cap \mathcal{K}_D$, where $\rho_0 \mathcal{B}$ is the Euclidean ball centered at the origin and having radius ρ_0. Notice that, for any $x \in \Gamma^{N_h}(x^{\mathrm{ref}})$, there exists a finite sequence $\{x^k\}_{k=0}^{N_h}$, such that $x^0 = x^{\mathrm{ref}}$; $x^{k+1} \in \Psi(x^k)$ for all $k = 1, \ldots, N_h - 1$; and $x = x^{N_h}$. As noted above, if $x^{\mathrm{ref}} \in \mathcal{K}_D$, then $\Psi^k(x^{\mathrm{ref}}) \in \mathcal{K}_D$. Therefore,

$$x^{N_h} - x^{\mathrm{ref}} = \sum_{k=0}^{N_h-1} [x^{k+1} - x^k]$$
$$= h \sum_{k=0}^{N_h-1} [Ax^{k+1} + Bu^{k+1}] \in A\mathcal{K}_D + B\mathbb{R}_+^m,$$

where $u^{k+1} \in \Phi(x^k)$ and the membership is due to the fact that $\mathcal{K}_D$ is a cone. By the assumption that $b \in NA\mathcal{K}_D + NB\mathbb{R}_+^m + (M+N)\mathcal{K}_D$, it follows that $H(x^{\mathrm{ref}}) \subseteq \mathcal{K}$. Now, assume that $\|x^{\mathrm{ref}}\| \leq \rho_0$; it follows that, for all $y \in H(x^{\mathrm{ref}})$, there exists an $x \in \Gamma^{N_h}(x^{\mathrm{ref}})$ such that

$$\|y\| \leq \|(M+N)^{-1}b\| + \|(M+N)^{-1}N\|\|x - x^{\mathrm{ref}}\|$$

$$\text{(26)} \qquad \leq \|(M+N)^{-1}b\| + \|(M+N)^{-1}N\| \left[(1+\kappa h)^{N_h} - 1\right] \|x^{\mathrm{ref}}\|$$

$$\text{(27)} \qquad \leq \|(M+N)^{-1}b\| + \|(M+N)^{-1}N\| \left(e^{T\kappa} - 1\right) \rho_0.$$

If ρ_0 satisfies

$$\|(M+N)^{-1}b\| + \|(M+N)^{-1}N\| \left(e^{T\kappa} - 1\right) \rho_0 < \rho_0,$$

then we have the desired self-map. By assumption, we have $1 - \|(M+N)^{-1}N\|(e^{T\kappa} - 1) > 0$; thus, with the choice of

$$\rho_0 > \frac{\|(M+N)^{-1}b\|}{1 - \|(M+N)^{-1}N\|(e^{T\kappa} - 1)},$$

we get a solution to (21) satisfying part (a). Parts (b) and (c) follow readily from the definition of Φ and Ψ. $\qquad\square$

Remark 7. The assumption on b in Lemma 21 reduces to $b = x^0 \in \mathcal{K}_D$ for the IVP. This is consistent with the assumption we made in Theorem 11.

With the above lemma available, we adapt the time-stepping scheme as follows. We first calculate

$$\{ x^{h,0}, x^{h,1}, \ldots, x^{h,N_h} \} \subset \mathbb{R}^n \quad \text{and} \quad \{ u^{h,1}, u^{h,2}, \ldots, u^{h,N_h} \} \subset \mathbb{R}_+^m$$

by solving the minimization problem

(28)
$$\begin{aligned}
&\text{minimize} \quad \|x^{h,0}\| \\
&\text{such that} \quad Mx^{h,0} + Nx^{h,N_h} = b, \\
&\qquad \left.\begin{array}{l}
x^{h,i+1} - x^{h,i} = h \left[Ax^{h,i+1} + Bu^{h,i+1} \right] \\
0 \leq u^{h,i+1} \perp Cx^{h,i+1} + Du^{h,i+1} \geq 0 \\
u^{h,i+1} \leq \gamma \|x^{h,i}\|
\end{array}\right\}, \quad i = 0, 1, \ldots, N_h - 1,
\end{aligned}$$

and then we use the generated iterates to define the discrete-time trajectories $\widehat{x}^h(t)$ and $\widehat{u}^h(t)$ as in the initial-value case. We refer to this time-stepping scheme as the *BVP least-norm scheme*-I. Lemma 21 guarantees that, for a sufficiently small $h > 0$, (28) is feasible with an optimal solution $x^{*h,0}$ satisfying $\|x^{*h,0}\| \leq \rho_0$ for some ρ_0 independent of h; the two trajectories $\widehat{x}^h(t)$ and $\widehat{u}^h(t)$ are therefore well defined. Applying Lemma 21 and following the argument in [29, Theorem 9.1], we can establish the following convergence theorem whose proof we omit.

THEOREM 22. *Assume the hypotheses of Lemma* 21. *Then there exists an* $\bar{h} > 0$ *such that the two trajectories* $\widehat{x}^h(t)$ *and* $\widehat{u}^h(t)$ *generated by the least-norm BVP scheme*-I *are well defined for all* $h \in (0, \bar{h}]$, *and there is a sequence* $\{h_\nu\} \downarrow 0$ *such that the following two limits exist:* $\widehat{x}^{h_\nu}(\cdot) \to \widehat{x}(\cdot)$ *uniformly on* $[0, T]$ *and* $\widehat{u}^{h_\nu}(\cdot) \to \widehat{u}(\cdot)$ *weakly in* $L^2[0, T]$. *Moreover, all such limits* $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ *are weak solutions of the boundary value LCS* (20). $\qquad\square$

Going beyond the passive case, we can derive a result for LCS BVP (20) under a range condition similar to (18). Let

$$
x^h \equiv \begin{bmatrix} x^{h,0} \\ x^{h,1} \\ \vdots \\ x^{h,N_h} \end{bmatrix} \in \mathbb{R}^{n(N_h+1)} \quad \text{and} \quad u^h \equiv \begin{bmatrix} u^{h,1} \\ u^{h,2} \\ \vdots \\ u^{h,N_h} \end{bmatrix} \in \mathbb{R}^{mN_h};
$$

note the variable dimensions of the vectors x^h and u^h as h varies. The system (21) can be written as the following mixed LCP:

(29)
$$
0 = p^h + P^h x^h - hQ^h u^h,
$$
$$
0 \le u^h \perp R^h x^h + S^h u^h \ge 0,
$$

where

$$
P^h = \left[\begin{array}{c|cccccc} M & & & & & N \\ \hline -I & I-hA & & & & \\ & -I & I-hA & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & -I & I-hA \end{array}\right] \in \mathbb{R}^{n(N_h+1)\times n(N_h+1)},
$$

$$
Q^h = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ B & 0 & 0 & \cdots & \cdots & 0 \\ & B & 0 & \cdots & \cdots & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & B & 0 \\ & & & & & B \end{bmatrix} \in \mathbb{R}^{n(N_h+1)\times mN_h},
$$

$$
R^h = \begin{bmatrix} 0 & C & & & \\ 0 & 0 & C & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 & C \end{bmatrix} \in \mathbb{R}^{mN_h\times n(N_h+1)},
$$

$$
S^h = \begin{bmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{bmatrix} \in \mathbb{R}^{mN_h\times mN_h}, \quad \text{and} \quad p^h = \begin{bmatrix} -b \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n(N_h+1)}.
$$

Our analysis in this case proceeds as follows. We will show that the matrix P^h is invertible and that its inverse is of order $O(N_h)$ in a certain norm and that the matrices

Q^h, R^h, and S^h are bounded in the same norm. We then apply Kakutani's fixed-point theorem to a certain LCP defined by S^h from which we deduce the existence of a solution to (29). It will also be shown that these (discrete-time) solutions remain bounded and that the x-trajectories constructed by a linear interpolation of these points are equicontinuous. We can thus conclude using the arguments in [29] that a sequence $\{h_\nu\} \downarrow 0$ exists for which the constructed continuous-time trajectories converge, in an appropriate sense, to a solution of (20).

The matrix P^h can be written as

$$P^h = \begin{bmatrix} M & Y^h \\ Z^h & W^h \end{bmatrix},$$

where $Y^h = \begin{bmatrix} 0 & \cdots & 0 & N \end{bmatrix} \in \mathbb{R}^{n \times nN_h}$,

$$Z^h = \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{nN_h \times n},$$

$$W^h = \begin{bmatrix} I - hA & & & \\ -I & I - hA & & \\ & \ddots & \ddots & \\ & & -I & I - hA \end{bmatrix} \in \mathbb{R}^{nN_h \times nN_h}.$$

Clearly, W^h is invertible for all $h > 0$ sufficiently small; its inverse is

$$(W^h)^{-1} = \begin{bmatrix} (I - hA)^{-1} & & & \\ (I - hA)^{-2} & (I - hA)^{-1} & & \\ \vdots & \vdots & \ddots & \\ (I - hA)^{-N_h} & (I - hA)^{-N_h+1} & \cdots & (I - hA)^{-1} \end{bmatrix}.$$

The matrix P^h is invertible if and only if the Schur complement of W^h in P^h is, i.e., if the matrix $S_W^h \equiv M - Y^h(W^h)^{-1}Z^h \in \mathbb{R}^{n \times n}$ is invertible. We have

$$S_W^h = M - \begin{bmatrix} 0 & 0 & \cdots & N \end{bmatrix}$$
$$\cdot \begin{bmatrix} (I - hA)^{-1} & & & \\ (I - hA)^{-2} & (I - hA)^{-1} & & \\ \vdots & \vdots & \ddots & \\ (I - hA)^{-N_h} & (I - hA)^{-N_h+1} & \cdots & (I - hA)^{-1} \end{bmatrix} \begin{bmatrix} -I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$= M + N(I - hA)^{-N_h} \rightarrow M + N \exp(TA) \text{ (as } h \to 0).$$

So if $M + N \exp(TA)$ is invertible, then for all sufficiently small $h > 0$, S_W^h is invertible; furthermore, $\|(S_W^h)^{-1}\| \leq \|(M + N \exp(TA))^{-1}\| + 1$. Note that the latter bound is

independent of h. The inverse of P^h can be written as

$$(P^h)^{-1} = \begin{bmatrix} (S_W^h)^{-1} & -(S_W^h)^{-1} Y^h (W^h)^{-1} \\ -(W^h)^{-1} Z^h (S_W^h)^{-1} & (W^h)^{-1} + (W^h)^{-1} Z^h (S_W^h)^{-1} Y^h (W^h)^{-1} \end{bmatrix}.$$

Many matrices displayed above are in block partitioned form with each block being of a fixed order independent of h and the number of blocks growing with h. We define the $||| \bullet |||$-norm of such a partitioned matrix to be the maximum of the norm of each of the entry blocks of fixed order. Thus,

$$||| (P^h)^{-1} ||| \equiv \max \{ || (S_W^h)^{-1} ||, \; ||| (S_W^h)^{-1} Y^h (W^h)^{-1} |||, \\ ||| (W^h)^{-1} Z^h (S_W^h)^{-1} |||, \\ ||| (W^h)^{-1} + (W^h)^{-1} Z^h (S_W^h)^{-1} Y^h (W^h)^{-1} ||| \},$$

and $|||(W^h)^{-1}||| = \max_{1 \le i \le N_h} ||(I - hA)^{-i}||$, etc. Clearly $|||Z^h||| = 1$, $|||Q^h||| = ||B||$, and $|||R^h||| = ||C||$. Similarly, for a vector $v \equiv (v^i)_{i=1}^{N_h}$ in partitioned form, $||| v |||$ is defined as $\max_{1 \le i \le N_h} ||v^i||$.

With an invertible P^h, the mixed LCP (29) decomposes into a standard LCP

$$(30) \qquad 0 \le u^h \perp -R^h (P^h)^{-1} p^h + \left[S^h + h R^h (P^h)^{-1} Q^h \right] u^h \ge 0$$

and the linear equation

$$(31) \qquad x^h = (P^h)^{-1} \left[h Q^h u^h - p^h \right].$$

The least-norm time-stepping scheme for the boundary value LCS is similar to the one for the IVP. We find the least-norm solution u^h to the LCP (30) and then calculate x^h using (31). We then let $\widehat{x}^h(t)$ be the linear interpolant of $\{x^{h,i}\}$ and $\widehat{u}^h(t)$ be the piecewise constant interpolant of $\{u^{h,i}\}$. We call this the *BVP least-norm scheme*-II. This scheme is different from the BVP least-norm scheme-I because we first obtain u^h by solving a least-norm LCP involving the u-variable only in scheme-II, whereas a least-norm solution on $x^{h,0}$ is obtained by solving an aggregated mixed LCP in scheme I.

To derive the desired bound for $|||(P^h)^{-1}|||$, let $h > 0$ be such that $||(I - hA)^{-1}|| \le 2$. We derive a bound for $||(I - hA)^{-i}||$ for all $i = 1, 2, \ldots, N_h$. By the triangle inequality, we have

$$\left| ||(I - hA)^{-i+1}|| - ||(I - hA)^{-i}|| \right| \le ||(I - hA)^{-i+1}|| \; ||I - (I - hA)^{-1}|| \\ \le ||(I - hA)^{-i+1}|| \cdot h ||A||_\infty ||(I - hA)^{-1}|| \\ \le 2 h || A || || (I - hA)^{-i+1} ||.$$

By letting $\alpha \equiv 2||A||$, it follows that

$$(32) \qquad ||(I - hA)^{-i}|| \le (1 + h\alpha)||(I - hA)^{-i+1}|| \le \cdots \le (1 + h\alpha)^i \le e^{\alpha T}.$$

Thus each $n \times n$ block in $(W^h)^{-1}$ is bounded in norm by $e^{\alpha T}$; thus so is $||| (W^h)^{-1} |||$. Likewise, each $n \times n$ block in

$$-(S_W^h)^{-1} Y^h (W^h)^{-1} = \begin{bmatrix} -(S_W^h)^{-1} N (I - hA)^{-N_h} & \cdots & -(S_W^h)^{-1} N (I - hA)^{-1} \end{bmatrix}$$

and each $n \times n$ block in

$$(33) \qquad (W^h)^{-1} Z^h (S_W^h)^{-1} = \begin{pmatrix} -(I - hA)^{-1}(S_W^h)^{-1} \\ -(I - hA)^{-2}(S_W^h)^{-1} \\ \vdots \\ -(I - hA)^{-N_h}(S_W^h)^{-1} \end{pmatrix}$$

are bounded in norm by $\left(\|[M + N \exp(TA)]^{-1}\| + 1 \right) (\|N\| + 1) e^{\alpha T}$. Hence,

$$\| (W^h)^{-1} + (W^h)^{-1} Z^h (S_W^h)^{-1} Y^h (W^h)^{-1} \|$$
$$\leq e^{\alpha T} + \left(\|[M + N \exp(TA)]^{-1}\| + 1 \right) (\|N\| + 1) e^{2\alpha T}.$$

Consequently, we have

$$\| (P^h)^{-1} \| \leq \left[\| (M + N \exp(TA))^{-1} \| + 1 \right] \left[\| N \|_\infty + 2 \right] e^{2\alpha T},$$

which implies, for each partitioned vector $u = (u^i)_{i=1}^{N_h}$,

$$\| R^h (P^h)^{-1} Q^h v \| \leq \| B \| \| C \| \frac{T}{h} \left[\| (M + N \exp(TA))^{-1} \|_\infty + 1 \right]$$
$$\cdot \left[\| N \|_\infty + 2 \right] e^{2\alpha T} \| u \|$$

or, equivalently,

$$h \| R^h (P^h)^{-1} Q^h u \| \leq T \| B \| \| C \| \left[\| (M + N \exp(TA))^{-1} \|_\infty + 1 \right]$$
$$\cdot \left[\| N \|_\infty + 2 \right] e^{2\alpha T} \| u \|.$$

Let $\eta > 0$ be a constant such that for all $r \in \text{LCP-Range}(D)$, the least-norm solution of the LCP (r, D) is bounded above in norm by $\eta \|r\|$.

THEOREM 23. *Let D be positive semidefinite, and let (M, N, A, B, C, T) be such that*

(A) $M + N \exp(TA)$ *is invertible,*

(B) $\gamma(T) \equiv T \eta \|B\| \|C\| \left[\|(M + N \exp(TA))^{-1}\| + 1 \right] \left[\|N\| + 2 \right] e^{2\alpha T} < 1$,

(C) $C \left[A\mathbb{R}^n + (M + N)^{-1} N A \mathbb{R}^n + B\mathbb{R}_+^m \right] \subseteq \text{LCP-Range}(D)$.

Then there exists $\bar{h}$ such that for every $h \in (0, \bar{h}]$ and for every b satisfying $C(M + N)^{-1}b \in \text{LCP-Range}(D)$, the trajectories $(\widehat{x}^h(\cdot), \widehat{u}^h(\cdot))$ obtained by the BVP least-norm scheme-II are well defined and satisfy the same asymptotic properties as in Theorem 22.

Proof. Let $\bar{h} > 0$ be such that, for all $h \in (0, \bar{h}]$, $(S_W^h)^{-1}$ exists and $\|(S_W^h)^{-1}\| \leq 1 + \|(M + N \exp(TA))^{-1}\|$. For any such scalar h and vector b such that $C(M+N)^{-1}b \in \text{LCP-Range}(D)$, we follow the proof of Theorem 10 and use Kakutani's fixed-point theorem to show the existence of a solution to (21). By (33), we have

$$-R^h (P^h)^{-1} p^h = \begin{pmatrix} C(I - hA)^{-1}(S_W^h)^{-1}b \\ C(I - hA)^{-2}(S_W^h)^{-1}b \\ \vdots \\ C(I - hA)^{-N_h}(S_W^h)^{-1}b \end{pmatrix}.$$

Thus, by (32), we have

$$\left\| \left\| R^h (P^h)^{-1} p^h \right\| \right\| \leq \|C\| \left[\left\| (M + N \exp(TA))^{-1} \right\| + 1 \right] e^{\alpha T} \|b\|.$$

Let $\beta(b)$ denote the constant on the right-hand side of the above bound; note that this constant depends on b. We claim that $C(I - hA)^{-\ell}(S_W^h)^{-1} b \in C(M + N)^{-1} b + C\left[A\mathbb{R}^n + (M + N)^{-1} NA\mathbb{R}^n \right]$ for all $\ell = 1, \ldots, N_h$. Since

$$\begin{aligned}
(I - hA)^{-\ell} &= \left[(I - hA)^{-1} \right]^\ell \\
&= \left[I + hA(I - hA)^{-1} \right]^\ell \\
&= I + hA \sum_{k=1}^{\ell} \binom{l}{k} (hA)^{k-1} (I - hA)^{-\ell+k},
\end{aligned}$$

we deduce that

$$\begin{aligned}
& (S_W^h)^{-1} \\
&= \left[M + N(I - hA)^{-N_h} \right]^{-1} \\
&= \left[M + N + hNA \sum_{k=1}^{N_h} \binom{N_h}{k} (hA)^{k-1} (I - hA)^{-N_h+k} \right]^{-1} \\
&= \left\{ \left[I + hNA \sum_{k=1}^{N_h} \binom{N_h}{k} (hA)^{k-1} (I - hA)^{-N_h+k} (M + N)^{-1} \right] (M + N) \right\}^{-1} \\
&= (M + N)^{-1} \left[I + hNA \sum_{k=1}^{N_h} \binom{N_h}{k} (hA)^{k-1} (I - hA)^{-N_h+k} (M + N)^{-1} \right]^{-1} \\
&= (M + N)^{-1} - h(M + N)^{-1} NA\Theta \left[I + hNA\Theta \right]^{-1},
\end{aligned}$$

where $\Theta \equiv \left(\sum_{k=1}^{N_h} \binom{N_h}{k} (hA)^{k-1} (I - hA)^{-N_h+k} \right) (M + N)^{-1}$. Thus,

$$\begin{aligned}
& C(I - hA)^{-\ell} (S_W^h)^{-1} b \\
&\quad = C(M + N)^{-1} b + hC \left[A\Gamma_\ell^h b - (M + N)^{-1} NA\Theta \left[I + hNA\Theta \right]^{-1} b \right] \\
&\quad \in C(M + N)^{-1} b + C \left[A\mathbb{R}^n + (M + N)^{-1} NA\mathbb{R}^n \right],
\end{aligned}$$

where $\Gamma_\ell^h \equiv \sum_{k=1}^{\ell} \binom{l}{k} (hA)^{k-1} (I - hA)^{-\ell+k} (S_W^h)^{-1}$. By a similar calculation, we can

show that, for each partitioned vector $u = \left(u^i\right)_{i=1}^{N_h} \geq 0$,

$$R^h(P^h)^{-1}Q^h u = \begin{pmatrix} C\,(I-hA)^{-1}Bu^1 \\ C\,(I-hA)^{-2}Bu^1 + C\,(I-hA)^{-1}Bu^2 \\ \vdots \\ C\displaystyle\sum_{k=0}^{N_h-1}(I-hA)^{-N_h+k}Bu^{k+1} \end{pmatrix}$$

$$+ \begin{pmatrix} C(I-hA)^{-1}(S_W^h)^{-1}f \\ C(I-hA)^{-2}(S_W^h)^{-1}f \\ \vdots \\ C(I-hA)^{-N_h}(S_W^h)^{-1}f \end{pmatrix},$$

where $f \equiv N \sum_{k=0}^{N_h-1}(I-hA)^{-N_h+k}Bu^{k+1}$. Since

$$C\,(I-hA)^{-\ell}Bu^k = CBu^k + h\,C\,A\sum_{k=1}^{\ell}\binom{l}{k}(hA)^{k-1}(I-hA)^{-\ell+k}Bu^k$$

$$\in C\left[A\mathbb{R}^n + B\mathbb{R}_+^m\right],$$

it follows that each component block of $R^h(P^h)^{-1}Q^h u$ belongs to $C[A\mathbb{R}^n + (M + N)^{-1}NA\mathbb{R}^n + B\mathbb{R}_+^m]$. Consequently, it follows from assumption (C) and the condition on b that, for any such partitioned vector u, $\mathrm{SOL}(\widehat{u}, S^h) \neq \emptyset$, where $\widehat{u} \equiv R^h(P^h)^{-1}[-p^h + hQ^h u]$. Moreover, a solution $v \in \mathrm{SOL}(\widehat{u}, S^h)$ exists such that

$$\||\,v\,\|| \leq \eta\,\left\|\left|\,R^h(P^h)^{-1}(-p^h) + hR^h(P^h)^{-1}Q^h u\,\right|\right\|$$

$$\leq \eta\,\beta(b) + \gamma(T)\,\||\,u\,\||.$$

Consequently, if $\varrho = \frac{\eta\,\beta(b)}{1-\gamma(T)}$ and u satisfies $\||u\|| \leq \varrho$, then

$$\||\,v\,\|| \equiv \eta\,\beta(b) + \frac{\eta\,\beta(b)\,\gamma(T)}{1-\gamma(T)} = \varrho.$$

Hence, defining $\Phi(u) \equiv \{v \in \mathrm{SOL}(\widehat{u}, S^h) : \||v\|| \leq \varrho\}$, where $\widehat{u} \equiv R^h(P^h)^{-1}[-p^h + hQ^h u]$, we conclude that Φ is a set-valued self-map from the compact convex set $\{u \geq 0 : \||u\|| \leq \varrho\}$ into itself; moreover, $\Phi(u)$ is a nonempty closed convex set with a closed graph. By Kakutani's fixed-point theorem, Φ has a fixed point, which we denote u^h. It is evident that u^h is a solution of the LCP (30). Furthermore, we have

$$\||\,u^h\,\|| \leq \eta\,\left\|\left|\,R^h(P^h)^{-1}(-p^h) + hR^h(P^h)^{-1}Q^h u^h\,\right|\right\|,$$

yielding

$$\||\,u^h\|| \leq \frac{\eta\,\beta(b)}{1-\gamma(T)} \equiv \rho\,\|\,b\,\|$$

for some constant $\rho > 0$. Also, since

$$(I-hA)\,x^{h,i+1} - x^{h,i} = hBu^{h,i+1},$$

we deduce that

$$(I - hA) \frac{x^{h,i+1} - x^{h,i}}{h} = Bu^{h,i+1} + Ax^{h,i},$$

yielding

$$\| x^{h,i+1} - x^{h,i} \| \le h \, \rho' \, (1 + \|x^{h,i}\|)$$

for some constant $\rho' > 0$. Hence by Lemma 4 and Theorem 3 we have the desired result. $\quad \square$

The range condition (C) in Theorem 23 reduces to the condition

$$C \left[A\mathbb{R}^n + B\mathbb{R}_+^m \right] \subseteq \text{LCP-Range}(D)$$

for an IVP which has $M = I$ and $N = 0$.

Both Theorems 22 and 23 extend the results in [29]; see Theorems 9.1 and 9.3 therein. In particular, Theorem 9.1 in [29] requires that either D be an R_0-matrix or Range(C) be contained in the interior of LCP-Range(D). These two assumptions are relaxed in Theorems 22 and 23, respectively. Similarly to the results for IVPs, we can also extend Theorems 22 and 23 to the inhomogeneous case. Since the proofs are very similar, we omit these extensions.

5. Conclusion. In this paper, we have studied the convergence of the least-norm implicit backward Euler time-stepping method for solving a passive LCS and its extensions. We have obtained two sets of conditions under which the convergence of the time-stepping scheme is guaranteed. One set of conditions includes an implication related to the unobservable space of the pair (C, A) that is satisfied by passive LCSs. The other set of conditions includes a range condition. Both sets apply to broader classes of LCSs compared to previous results in the literature. In addition to the IVP, we have also identified two sets of conditions under which we can show the convergence of our schemes for the two-point BVP.

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