Chapter 1

Convergence of Vector Subdivision Schemes and Construction of Biorthogonal Multiple Wavelets

Biorthogonal Multiple Wavelets R. Q. Jia
Rong-Qing Jia[1]

[1]Department of Mathematical Sciences, University of Alberta, Edmonton, Canada T6G 2G1. Research was supported in part by NSERC Canada under Grant OGP 121336.

Abstract

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a refinable vector of compactly supported functions in $L_2(\mathbb{R})$. It is shown in this paper that there exists a refinable vector $\tilde{\phi}$ of compactly supported functions in $L_2(\mathbb{R})$ such that $\tilde{\phi}$ is dual to ϕ if and only if the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. This result is established on the basis of a complete characterization of the convergence of vector subdivision schemes associated with exponentially decaying masks. As an application of the general theory, two interesting examples of biorthogonal double wavelets are constructed.

1.1 Introduction

We are interested in multiple refinable functions and multiple wavelets. Suppose $\phi_1, \ldots, \phi_r$ are complex-valued functions on $\mathbb{R}$. Denote by ϕ the vector $(\phi_1, \ldots, \phi_r)^T$, the transpose of $(\phi_1, \ldots, \phi_r)$. We say that ϕ is refinable if it satisfies the following refinement equation:

$$\phi = \sum\nolimits_{\alpha \in \mathbb{Z}} a(\alpha)\phi(2 \cdot - \alpha), \tag{1.1}$$

where each $a(\alpha)$ is an $r \times r$ matrix of complex numbers.

Let $L_2(\mathbb{R})$ denote the linear space of all square integrable complex-valued functions on $\mathbb{R}$. It is well known that $L_2(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx,$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. The norm of $f \in L_2(\mathbb{R})$ is given by $\|f\|_2 := \langle f, f \rangle^{1/2}$. More generally, for $1 \le p < \infty$, we define

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p\, dx \right)^{1/p}.$$

For $p = \infty$, define $\|f\|_\infty$ to be the essential supremum of $|f|$ on $\mathbb{R}$. Let $L_p(\mathbb{R})$ denote the linear space of all functions f for which $\|f\|_p < \infty$. Equipped with the norm $\|\cdot\|_p$, $L_p(\mathbb{R})$ is a Banach space. By $(L_p(\mathbb{R}))^r$ we denote the linear space of all vectors $f = (f_1, \ldots, f_r)^T$ such that $f_1, \ldots, f_r \in L_p(\mathbb{R})$. The norm on $(L_p(\mathbb{R}))^r$ is defined by

$$\|f\|_p := \left(\sum\nolimits_{j=1}^{r} \|f_j\|_p^p \right)^{1/p}, \qquad f = (f_1, \ldots, f_r)^T \in (L_p(\mathbb{R}))^r.$$

Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ and $\tilde\phi = (\tilde\phi_1, \ldots, \tilde\phi_r)^T$ belong to $(L_2(\mathbb{R}))^r$. We say that the shifts of $\phi_1, \ldots, \phi_r$ and the shifts of $\tilde\phi_1, \ldots, \tilde\phi_r$ are biorthogonal, if

$$\langle \phi_j(\cdot - \alpha), \tilde\phi_k(\cdot - \beta) \rangle = \delta_{jk}\delta_{\alpha\beta} \qquad \forall\, j, k = 1, \ldots, r,\ \alpha, \beta \in \mathbb{Z}, \tag{1.2}$$

where δ_{jk} and $\delta_{\alpha\beta}$ stand for the Kronecker sign. If this is the case, then $\tilde\phi$ is said to be a dual to ϕ. If, in addition, ϕ and $\tilde\phi$ are refinable, then ϕ and $\tilde\phi$ are a pair of biorthogonal vectors of multiple refinable functions. Biorthogonal multiple wavelets are generated from biorthogonal multiple refinable functions. In the scalar case ($r = 1$), a basic theory of biorthogonal wavelets was established by Cohen, Daubechies, and Feauveau [6].

Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is a refinable vector of compactly supported functions in $L_2(\mathbb{R})$. Under what conditions does there exist a refinable dual vector of compactly supported functions? The main purpose of this paper is to address this fundamental question.

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ and $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^T$ be dual vectors of compactly supported functions in $L_2(\mathbb{R})$. Suppose $c_1, \ldots, c_r$ are sequences on $\mathbb{Z}$ such that

$$\sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}} c_j(\alpha)\phi_j(\cdot - \alpha) = 0.$$

Taking inner product of both sides of the above equation with $\tilde{\phi}_k(\cdot - \beta)$ and employing the dual relation (1.2), we obtain $c_k(\beta) = 0$ for all $\beta \in \mathbb{Z}$ and all $k = 1, \ldots, r$. In other words, the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. Thus, linear independence is a necessary condition for the existence of a dual vector of compactly supported functions. This fact was observed by Dahmen and Micchelli in [9, Proposition 4].

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a refinable vector of compactly supported functions in $L_2(\mathbb{R})$ such that the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. In Section 3 we shall show that there exists a refinable vector $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^T$ of compactly supported functions in $L_2(\mathbb{R})$ such that the dual relation (1.2) holds true. In other words, linear independence is also a sufficient condition for the existence of a dual refinable vector of compactly supported functions. Thus, we give a complete answer to the aforementioned fundamental question.

For the scalar case ($r = 1$), Lemarié-Rieusset [30] proved that for any minimally supported refinable function, there exists a compactly supported dual refinable function. See Chui and Wang [3] for a discussion on minimally supported refinable functions, and Jia and Wang [26] for a characterization of the linear independence of the shifts of a refinable function in terms of its mask. However, being minimally supported is not an appropriate condition for either the multivariate setting or the multiple setting.

Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is a vector of compactly supported functions in $L_2(\mathbb{R})$ such that the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. If ϕ satisfies the refinement equation (1.1), then the matrix $M := \sum_{\alpha \in \mathbb{Z}} a(\alpha)/2$ must have a simple eigenvalue 1 and its other eigenvalues are less than 1 in modulus (see [9]). Throughout this paper we assume that this condition is satisfied.

The key to our investigation of multiple refinable functions will be a study of vector subdivision schemes, which are of independent interest. Subdivision schemes have been studied mainly for the case in which the mask a is finitely supported. In the scalar case ($r = 1$), the uniform convergence of stationary subdivision schemes was investigated by Cavaretta, Dahmen, and Micchelli [1]. In [18] Jia gave a characterization for the L_p-convergence of a subdivision scheme ($1 \le p \le \infty$). In particular, the L_2-convergence of a subdivision scheme was characterized in terms of the spectral radius of a certain finite matrix associated to the mask. His results were extended by Han and Jia [15] to the multivariate setting. For the vector case ($r > 1$), Cohen, Daubechies, and Plonka [7] obtained some sufficient conditions for L_∞-convergence and L_2-convergence of cascade algorithms, and Shen [35] gave a characterization for the L_2-convergence of cascade algorithms. In [23], Jia, Riemenschneider, and Zhou provided a characterization for the L_p-convergence of subdivision schemes ($1 \le p \le \infty$).

For the reason which will become clear later, we need to consider the case where the mask a is not finitely supported but decays exponentially fast. Let a be such a mask. Let Q_a be the bounded linear operator on $(L_2(\mathbb{R})^r)$ given by

$$Q_a\phi := \sum\nolimits_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\cdot-\alpha), \qquad \phi = (\phi_1,\ldots,\phi_r)^T \in (L_2(\mathbb{R}))^r. \tag{1.3}$$

Let $y = (y_1,\ldots,y_r)$ be a left eigenvector of M corresponding to the eigenvalue 1, that is, $yM = y$ and $y \neq 0$. In the scalar case ($r = 1$), $M = (1)$, so y is chosen to be 1. The vector y will be fixed throughout this paper. By $L_{2,c}(\mathbb{R})$ we denote the linear space of all compactly supported functions in $L_2(\mathbb{R})$. A vector $\phi = (\phi_1,\ldots,\phi_r)^T \in (L_{2,c}(\mathbb{R}))^r$ is said to satisfy the moment conditions of order 1 if

$$y\sum\nolimits_{\alpha\in\mathbb{Z}} \phi(\cdot-\alpha) = 1.$$

We say that the (vector) subdivision scheme associated with a converges in the L_2-norm, if there exists a vector $\phi \in (L_2(\mathbb{R}))^r$ such that for any $\phi_0 \in (L_{2,c}(\mathbb{R}))^r$ satisfying the moment conditions of order 1, the sequence $Q_a^n\phi_0$ converges to ϕ in the L_2-norm. If this is the case, then ϕ is a solution of the refinement equation (1.1).

The Kronecker product of two matrices is a useful tool in our study of vector refinement equations. Let us recall some basic properties of the Kronecker product from [28]. Suppose

$$A = (a_{ij})_{1\le i\le m, 1\le j\le n} \quad \text{and} \quad B = (b_{ij})_{1\le i\le r, 1\le j\le s}$$

are two matrices. The (right) Kronecker product of A and B, written $A\otimes B$, is defined to be the block matrix

$$A\otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

For three matrices A, B, and C of the same type, we have

$$\begin{aligned} (A+B)\otimes C &= (A\otimes C) + (B\otimes C); \\ A\otimes(B+C) &= (A\otimes B) + (A\otimes C). \end{aligned}$$

If A, B, C, D are four matrices such that the products AC and BD are well defined, then

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD).$$

If $\lambda_1,\ldots,\lambda_r$ are the eigenvalues of an $r\times r$ matrix A and $\mu_1,\ldots,\mu_r$ are the eigenvalues of an $r\times r$ matrix B, then the eigenvalues of $A\otimes B$ are $\lambda_j\mu_k$, $j,k = 1,\ldots,r$. See [28, Chap. 12] for a proof of these results.

The Kronecker product was used by Goodman, Jia, and Micchelli [13] in their study of the spectral radius of a bi-infinite periodic and slanted matrix. It was also employed by Jiang [27] in his work on the regularity of matrix refinable functions, and by Zhou [36] in his investigation of the joint spectral radius of a finite collection of matrices.

For $\mu > 0$, let E_μ denote the linear space of all sequences u on $\mathbb{Z}$ for which

$$\|u\|_{E_\mu} := \sum_{\alpha\in\mathbb{Z}} |u(\alpha)|e^{\mu|\alpha|} < \infty.$$

Equipped with the norm $\|\cdot\|_{E_\mu}$, E_μ becomes a Banach space. Note that a similar space was used by Cohen and Daubechies [5]. Let E_μ^r denote the linear space of all mappings u from $\mathbb{Z}$ to $\mathbb{C}^r$ for which there exist $u_1, \ldots, u_r \in E_\mu$ such that $u(\alpha) = (u_1(\alpha), \ldots, u_r(\alpha))^T$ for all $\alpha \in \mathbb{Z}$. The norm on E_μ^r is defined by

$$\|u\|_{E_\mu^r} := \max_{1\le j\le r} \|u_j\|_{E_\mu}.$$

By $E_\mu^{r\times r}$ we denote the linear space of all mappings g from $\mathbb{Z}$ to $\mathbb{C}^{r\times r}$ for which there exist $g_{jk} \in E_\mu$, $j, k = 1, \ldots, r$, such that $g(\alpha) = (g_{jk}(\alpha))_{1\le j,k\le r}$ for all $\alpha \in \mathbb{Z}$. The norm on $E_\mu^{r\times r}$ is defined by

$$\|g\|_{E_\mu^{r\times r}} := \max_{1\le j,k\le r} \|g_{jk}\|_{E_\mu}.$$

Suppose the mask a belongs to $E_\mu^{r\times r}$ for some $\mu > 0$. Let b be defined by

$$b(\alpha) := \sum_{\beta\in\mathbb{Z}} \overline{a(\beta)} \otimes a(\alpha+\beta)/2, \qquad \alpha \in \mathbb{Z}. \tag{1.4}$$

Then b lies in $E_\mu^{r^2\times r^2}$. Let T_b be the transition operator on $E_\mu^{r^2}$ defined by

$$T_b u(\alpha) := \sum_{\beta\in\mathbb{Z}} b(2\alpha-\beta)u(\beta), \qquad \alpha \in \mathbb{Z}, \tag{1.5}$$

where $u \in E_\mu^{r^2}$. It is easily seen that T_b is a bounded operator. The transition operator plays an important role in our study of refinement equations. When $r = 1$ and b is finitely supported, transition matrices were introduced by Deslauriers and Dubuc [10] in their study of interpolatory subdivision schemes. In [14], Goodman, Micchelli, and Ward connected transition operators with subdivision operators in their work on spectral radius formulas.

The transition operator T_b defined by (1.5) is a compact operator. Indeed, if b is finitely supported, then T_b is the limit of a sequence of finite-rank operators, so T_b is a compact operator. In general, we can find a sequence $b^{(N)}$ $(N = 1, 2, \ldots)$ of elements of $E_\mu^{r^2\times r^2}$ with finite support such that $\|b^{(N)} - b\|_{E_\mu^{r^2\times r^2}} \to 0$ as $N \to \infty$. It follows that $\|T_{b^{(N)}} - T_b\| \to 0$ as $N \to \infty$. As the limit of a sequence of compact operators, T_b itself is a compact operator. The reader is referred to [34, Chap. 4] for a basic theory of the spectral properties of compact operators. In particular, if we denote by $\rho(T_b)$ the spectral radius of T_b, then $\rho(T_b) = |\sigma|$ for some eigenvalue σ of T_b.

Recall that M is the matrix $\sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$. By (1.4) we have

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)/2 = \Big(\sum_{\beta\in\mathbb{Z}} \overline{a(\beta)}/2\Big) \otimes \Big(\sum_{\alpha\in\mathbb{Z}} a(\alpha+\beta)/2\Big) = \overline{M} \otimes M.$$

Thus, the matrix $\sum_{\alpha\in\mathbb{Z}} b(\alpha)/2$ has a simple eigenvalue 1 and its other eigenvalues are less than 1 in modulus. Also, recall that y is a left eigenvector of M corresponding to the eigenvalue 1. Hence, we have

$$(\overline{y}\otimes y)(\overline{M}\otimes M) = (\overline{y}\overline{M})\otimes(yM) = \overline{M}\otimes M.$$

In other words, $\overline{y}\otimes y$ is a left eigenvector of $\overline{M}\otimes M$ corresponding to the eigenvalue 1.

In Section 2 we will establish a characterization for the L_2-convergence of a vector subdivision scheme. Suppose $a \in E_\mu^{r\times r}$. Let $b \in E_\mu^{r^2\times r^2}$ be defined by (1.4), and let T_b be the transition operator on $E_\mu^{r^2}$ given by (1.5). Consider the subspace V of $E_\mu^{r^2}$ defined by

$$V := \Big\{ v \in E_\mu^{r^2} : (\overline{y}\otimes y)\sum\nolimits_{\alpha\in\mathbb{Z}} v(\alpha) = 0 \Big\}. \tag{1.6}$$

We will show that the subdivision scheme associated with a converges in the L_2-norm if and only if V is invariant under T_b and $\rho(T_b|_V) < 1$.

Finally, in Section 4 we shall apply the general theory to construction of biorthogonal multiple wavelets. Two examples are given. In the first example, the wavelets are piecewise linear functions with short support. In the second example, the wavelets are almost in C^2, and the dual wavelets are in C^1. All the wavelets and dual wavelets are either symmetric or anti-symmetric about the origin. The approximation and smoothness properties of these wavelets will be analyzed.

1.2 Vector Subdivision Schemes

This section is devoted to a study of vector subdivision schemes. We shall establish a characterization for the L_2-convergence of a vector subdivision scheme in terms of the corresponding transition operator.

Let $\ell(\mathbb{Z})$ denote the linear space of all complex-valued sequences on $\mathbb{Z}$, and let $\ell_0(\mathbb{Z})$ denote the linear space of all finitely supported sequences on $\mathbb{Z}$. The difference operators ∇ and Δ on $\ell(\mathbb{Z})$ are defined by

$$\nabla u := u - u(\cdot - 1) \quad \text{and} \quad \Delta u := -u(\cdot+1) + 2u - u(\cdot - 1), \qquad u \in \ell(\mathbb{Z}).$$

For $\beta \in \mathbb{Z}$, we denote by δ_β the sequence on $\mathbb{Z}$ given by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \in \mathbb{Z}\setminus\{\beta\}. \end{cases}$$

In particular, we write δ for δ_0.

Let $u \in \ell(\mathbb{Z})$. For $1 \le p < \infty$, we define

$$\|u\|_p := \Big(\sum\nolimits_{\alpha\in\mathbb{Z}} |u(\alpha)|^p\Big)^{1/p}.$$

For $p = \infty$, define $\|u\|_\infty$ to be the supremum of $|u|$ on $\mathbb{Z}$. Let $\ell_p(\mathbb{Z})$ denote the linear space of all sequences u for which $\|u\|_p < \infty$. Equipped with the norm $\|\cdot\|_p$, $\ell_p(\mathbb{Z})$ becomes a Banach space. By $\ell_p(\mathbb{Z} \to \mathbb{C}^r)$ we denote the linear space of all sequences u such that $u(\alpha) = (u_1(\alpha), \ldots, u_r(\alpha))^T$ for some $u_1, \ldots, u_r \in \ell_p(\mathbb{Z})$ and for all $\alpha \in \mathbb{Z}$. Obviously, $u \mapsto (u_1, \ldots, u_r)^T$ is a canonical isomorphism between $\ell_p(\mathbb{Z} \to \mathbb{C}^r)$ and $(\ell_p(\mathbb{Z}))^r$. Thus, we may identify $\ell_p(\mathbb{Z} \to \mathbb{C}^r)$ with $(\ell_p(\mathbb{Z}))^r$. The norm of $u = (u_1, \ldots, u_r)^T$ is given by

$$\|u\|_p := \left(\sum_{j=1}^r \|u_j\|_p^p\right)^{1/p}.$$

Equipped with this norm, $(\ell_p(\mathbb{Z}))^r$ becomes a Banach space. We also identify $\ell_p(\mathbb{Z} \to \mathbb{C}^{r\times r})$ with $(\ell_p(\mathbb{Z}))^{r\times r}$. The spaces $(\ell(\mathbb{Z}))^r$, $(\ell_0(\mathbb{Z}))^r$, $(\ell(\mathbb{Z}))^{r\times r}$, and $(\ell_0(\mathbb{Z}))^{r\times r}$ are defined analogously. The difference operators ∇ and Δ can be naturally extended to $(\ell(\mathbb{Z}))^r$ and $(\ell(\mathbb{Z}))^{r\times r}$.

For two functions f, g in $L_2(\mathbb{R})$, $f \odot g$ is defined as follows:

$$f \odot g(x) := \int_{\mathbb{R}} f(x+y)\overline{g(y)}\, dy, \qquad x \in \mathbb{R}.$$

In other words, $f \odot g$ is the convolution of f with the function $y \mapsto \overline{g(-y)}$, $y \in \mathbb{R}$. It is easily seen that $f \odot g$ lies in $C_0(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ which vanish at ∞ (see [12, p. 232]). In particular, $f \odot g$ is uniformly continuous. Clearly,

$$\|f \odot g\|_\infty \le \|f\|_2 \|g\|_2.$$

Moreover, $\|f\|_2^2 = (f \odot f)(0)$.

For a matrix $A = (a_{ij})_{1\le i,j\le r}$, the vector

$$(a_{11}, \ldots, a_{r1}, a_{12}, \ldots, a_{r2}, \ldots, a_{1r}, \ldots, a_{rr})^T$$

is said to be the vec-function of A and written as $\mathrm{vec}\, A$. Suppose A, X, and B are three $r \times r$ matrices. Then we have (see [28, p. 410])

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}\, X. \tag{2.1}$$

For $\phi, \psi \in (L_2(\mathbb{R}))^r$, let $\phi \odot \psi^T$ be defined as follows:

$$\phi \odot \psi^T := \begin{bmatrix} \phi_1 \odot \psi_1 & \phi_1 \odot \psi_2 & \cdots & \phi_1 \odot \psi_r \\ \phi_2 \odot \psi_1 & \phi_2 \odot \psi_2 & \cdots & \phi_2 \odot \psi_r \\ \vdots & \vdots & \ddots & \vdots \\ \phi_r \odot \psi_1 & \phi_r \odot \psi_2 & \cdots & \phi_r \odot \psi_r \end{bmatrix}.$$

Suppose $\phi \in (L_2(\mathbb{R}))^r$ is a solution of the refinement equation (1.1), where the mask a is assumed to be in $(\ell_1(\mathbb{Z}))^{r\times r}$ for the time being. It follows from (1.1) that

$$\phi \odot \phi^T = \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} a(\alpha)\phi(2\cdot - \alpha) \odot \phi^T(2\cdot - \beta)\overline{a(\beta)}^T.$$

Note that

$$\phi(2\cdot-\alpha)\odot\phi^T(2\cdot-\beta)=\frac{1}{2}\phi\odot\phi^T(2\cdot-\alpha+\beta).$$

In light of (2.1) we obtain

$$\begin{aligned}&\operatorname{vec}\left(a(\alpha)\phi(2\cdot-\alpha)\odot\phi^T(2\cdot-\beta)\overline{a(\beta)}^T\right)\\&\quad=\tfrac{1}{2}\overline{a(\beta)}\otimes a(\alpha)\operatorname{vec}(\phi\odot\phi^T)(2\cdot-\alpha+\beta).\end{aligned}$$

Therefore,

$$\operatorname{vec}(\phi\odot\phi^T)=\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}}\frac{1}{2}\overline{a(\beta)}\otimes a(\alpha)\operatorname{vec}(\phi\odot\phi^T)(2\cdot-\alpha+\beta). \tag{2.2}$$

Let $f:=\operatorname{vec}(\phi\odot\phi^T)$. Then f lies in $(C_0(\mathbb{R}))^{r^2}$, the linear space of $r^2\times 1$ vectors of functions in $C_0(\mathbb{R})$. It follows from (2.2) that f satisfies the following refinement equation:

$$f=\sum\nolimits_{\alpha\in\mathbb{Z}}b(\alpha)f(2\cdot-\alpha),$$

where b is given by (1.4). For $c,d\in(\ell_1(\mathbb{Z}))^{r\times r}$, let $c\diamond d$ be defined by

$$(c\diamond d)(\alpha):=\sum\nolimits_{\beta\in\mathbb{Z}}\overline{d(\beta)}\otimes c(\alpha+\beta),\qquad \alpha\in\mathbb{Z}.$$

Then $b=a\diamond a/2$.

Iterating (1.3) n times yields

$$Q_a^n\phi=\sum_{\alpha\in\mathbb{Z}}a_n(\alpha)\phi(2^n\cdot-\alpha),\qquad n=1,2,\ldots, \tag{2.3}$$

where each a_n is independent of the choice of ϕ. In particular, $a_1=a$. Consequently, for $n>1$ we have

$$\begin{aligned}Q_a^n\phi&=Q_a^{n-1}(Q_a\phi)=\sum\nolimits_{\beta\in\mathbb{Z}}a_{n-1}(\beta)(Q_a\phi)(2^{n-1}\cdot-\beta)\\&=\sum\nolimits_{\beta\in\mathbb{Z}}\sum\nolimits_{\alpha\in\mathbb{Z}}a_{n-1}(\beta)a(\alpha)\phi(2^n\cdot-2\beta-\alpha)\\&=\sum\nolimits_{\alpha\in\mathbb{Z}}\left[\sum\nolimits_{\beta\in\mathbb{Z}}a_{n-1}(\beta)a(\alpha-2\beta)\right]\phi(2^n\cdot-\alpha).\end{aligned}$$

This establishes the following iteration relation for a_n $(n=1,2,\ldots)$:

$$a_1=a\qquad\text{and}\qquad a_n(\alpha)=\sum_{\beta\in\mathbb{Z}}a_{n-1}(\beta)a(\alpha-2\beta),\quad \alpha\in\mathbb{Z}. \tag{2.4}$$

Similarly, for $f\in(C_0(\mathbb{R}))^{r^2}$ we have

$$Q_b^n f=\sum\nolimits_{\alpha\in\mathbb{Z}}b_n(\alpha)f(2^n\cdot-\alpha), \tag{2.5}$$

where b_n $(n=1,2,\ldots)$ are given by the following iteration relation:

$$b_1=b\qquad\text{and}\qquad b_n(\alpha)=\sum_{\beta\in\mathbb{Z}}b_{n-1}(\beta)b(\alpha-2\beta),\quad \alpha\in\mathbb{Z}. \tag{2.6}$$

The sequences a_n and b_n are related by the following equation:

$$b_n(\alpha) = \sum\nolimits_{\beta\in\mathbb{Z}} \overline{a_n(\beta)} \otimes a_n(\alpha+\beta)/2^n, \qquad \alpha \in \mathbb{Z}, n = 1, 2 \ldots . \tag{2.7}$$

This will be proved by induction on n. By the definition of b, (2.7) is true for $n = 1$. Suppose $n > 1$ and (2.7) is valid for $n-1$. For $\alpha \in \mathbb{Z}$, by (2.4) we have

$$\begin{aligned}
&\sum\nolimits_{\beta\in\mathbb{Z}} \overline{a_n(\beta)} \otimes a_n(\alpha+\beta) \\
&= \sum\nolimits_{\beta\in\mathbb{Z}} \sum\nolimits_{\gamma\in\mathbb{Z}} \sum\nolimits_{\eta\in\mathbb{Z}} \big(\overline{a_{n-1}(\gamma)a(\beta-2\gamma)}\big) \otimes \big(a_{n-1}(\eta)a(\alpha+\beta-2\eta)\big) \\
&= \sum\nolimits_{\beta\in\mathbb{Z}} \sum\nolimits_{\gamma\in\mathbb{Z}} \sum\nolimits_{\eta\in\mathbb{Z}} \big(\overline{a_{n-1}(\gamma)} \otimes a_{n-1}(\eta+\gamma)\big)\big(\overline{a(\beta)} \otimes a(\alpha-2\eta+\beta)\big) \\
&= \sum\nolimits_{\eta\in\mathbb{Z}} 2^n b_{n-1}(\eta) b(\alpha-2\eta),
\end{aligned}$$

where the induction hypothesis has been used to derive the last equality. This together with (2.6) establishes (2.7).

Let ϕ_0 and ψ_0 be two elements in $(L_2(\mathbb{R}))^r$. By using the same argument as was done in the proof of (2.2), we obtain

$$\begin{aligned}
&\operatorname{vec}\big((Q_a^n\phi_0) \odot (Q_a^n\psi_0)^T\big) \\
&\quad = \sum\nolimits_{\alpha\in\mathbb{Z}} \sum\nolimits_{\beta\in\mathbb{Z}} 2^{-n}\overline{a_n(\beta)} \otimes a_n(\alpha) \operatorname{vec}(\phi_0 \odot \psi_0^T)(2^n\cdot - \alpha + \beta).
\end{aligned}$$

This in connection with (2.7) shows that, for $n = 1, 2, \ldots$,

$$\operatorname{vec}\big((Q_a^n\phi_0) \odot (Q_a^n\psi_0)^T\big) = Q_b^n\big(\operatorname{vec}(\phi_0 \odot \psi_0^T)\big). \tag{2.8}$$

We claim that, for $w \in \mathbb{C}^{r^2}$ and $n = 1, 2, \ldots$,

$$T_b^n(w\delta_\beta)(\alpha) = b_n(2^n\alpha - \beta)w \qquad \forall \alpha, \beta \in \mathbb{Z}. \tag{2.9}$$

This will be proved by induction on n. When $n = 1$, (2.9) follows from (1.5). Suppose $n > 1$ and (2.9) is valid for $n-1$. We have

$$\begin{aligned}
T_b^n(w\delta_\beta) &= T_b^{n-1}\big(T_b(w\delta_\beta)\big) = T_b^{n-1}\Big(\sum\nolimits_{\gamma\in\mathbb{Z}} b(2\gamma-\beta)w\delta_\gamma\Big) \\
&= \sum\nolimits_{\alpha\in\mathbb{Z}} \sum\nolimits_{\gamma\in\mathbb{Z}} b_{n-1}(2^{n-1}\alpha-\gamma) b(2\gamma-\beta) w\delta_\alpha \\
&= \sum\nolimits_{\alpha\in\mathbb{Z}} b_n(2^n\alpha-\beta) w\delta_\alpha,
\end{aligned}$$

where (2.6) has been used to derive the last equality. This completes the induction procedure. It follows from (2.9) that

$$T_b^n(w\nabla\delta_\beta)(\alpha) = \nabla b_n(2^n\alpha-\beta)w, \qquad \forall\, \alpha, \beta \in \mathbb{Z}. \tag{2.10}$$

Recall that y is a left eigenvector of the matrix $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ corresponding to the eigenvalue 1. Let $e_1, \ldots, e_r$ be a basis for $\mathbb{C}^r$ such that $ye_1 = 1$ and $ye_j = 0$ for $j = 2, \ldots, r$. Let $e_{jk} := \overline{e_k} \otimes e_j$ for $j, k = 1, \ldots, r$. Then $\{e_{jk} : j, k = 1, \ldots, r\}$ is a basis for $\mathbb{C}^{r^2}$ such that $(\overline{y} \otimes y)e_{11} = 1$ and $(\overline{y} \otimes y)e_{jk} = 0$ for $(j, k) \neq (1, 1)$.

The following theorem gives a necessary condition for the L_2-convergence of a vector subdivision scheme with an exponentially decaying mask.

Theorem 1.2.1 *Let $a \in E_\mu^{r\times r}$ for some $\mu > 0$ and let b be given by (1.4). If the subdivision scheme associated with a converges in the L_2-norm, then*

$$\lim_{n\to\infty} \|T_b^n v\|_\infty = 0 \qquad \forall\, v \in V, \tag{2.11}$$

where T_b is the transition operator defined in (1.5) and V is the linear space given by (1.6).

Proof. Suppose that the subdivision scheme associated with a converges in the L_2-norm. Let ϕ_0 and ψ_0 be two elements in $(L_{2,c}(\mathbb{R}))^r$ satisfying the moment conditions of order 1. Then both sequences $\phi_n := Q_a^n\phi_0$ and $\psi_n := Q_a^n\psi_0$ converge to the same limit function ϕ in the L_2-norm. For $n = 0, 1, \ldots$, let $f_n := \mathrm{vec}(\phi_n \odot \psi_n^T)$, and let $f := \mathrm{vec}(\phi \odot \phi^T)$. Then

$$\begin{aligned}\|f_n - f\|_\infty &= \|\mathrm{vec}\left(\phi_n \odot \psi_n^T - \phi \odot \phi^T\right)\|_\infty\\ &\le \|\mathrm{vec}\left(\phi_n \odot (\psi_n - \phi)^T\right)\|_\infty + \|\mathrm{vec}\left((\phi_n - \phi) \odot \phi^T\right)\|_\infty\\ &\le \|\phi_n\|_2\|\psi_n - \phi\|_2 + \|\phi_n - \phi\|_2\|\phi\|_2.\end{aligned}$$

This shows that f_n converges to f uniformly.

In particular, choose $\phi_0 = \psi_0 = e_1\chi$, where χ is the characteristic function of the unit interval $[0, 1)$. Then both ϕ_0 and ψ_0 satisfy the moment conditions of order 1. But $\mathrm{vec}\left((e_1\chi) \odot (e_1\chi)^T\right) = e_{11}h$, where h is the hat function given by $h(x) := \max\{1-|x|, 0\}$, $x \in \mathbb{R}$. With the help of (2.8), we see that $Q_b^n(e_{11}h)$ converges to f uniformly. Since f is uniformly continuous, $\|f - f(\cdot - 2^{-n})\|_\infty \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|Q_b^n(e_{11}h) - Q_b^n(e_{11}h)(\cdot - 2^{-n})\|_\infty = 0.$$

But (2.5) implies

$$Q_b^n(e_{11}h) - Q_b^n(e_{11}h)(\cdot - 2^{-n}) = \sum\nolimits_{\alpha\in\mathbb{Z}} \nabla b_n(\alpha) e_{11} h(2^n \cdot - \alpha).$$

It follows that

$$\lim_{n\to\infty} \|\nabla b_n e_{11}\|_\infty = 0. \tag{2.12}$$

Furthermore, we observe that, for $j = 2, \ldots, r$, $e_1\chi$ and $(e_1 + e_j)\chi$ both satisfy the moment conditions of order 1. Hence, $Q_a^n(e_1\chi)$ and $Q_a^n(e_1 + e_j)\chi$ both converge to the same limit ϕ in the L_2-norm. This shows that, for $j = 2, \ldots, r$, $\|Q_a^n(e_j\chi)\|_2 \to 0$ as $n \to \infty$. Choosing $\phi_0 = e_j\chi$ and $\psi_0 = e_k\chi$ in (2.8), we obtain

$$\mathrm{vec}\left((Q_a^n(e_j\chi)) \odot (Q_a^n(e_k\chi))^T\right) = Q_b^n(e_{jk}h), \quad j, k = 1, \ldots, r,\ n = 1, 2, \ldots.$$

Therefore,

$$\lim_{n\to\infty} \|Q_b^n(e_{jk}h)\|_\infty = 0, \qquad \text{for } (j, k) \ne (1, 1).$$

But by (2.5) we have

$$Q_b^n(e_{jk}h) = \sum\nolimits_{\alpha\in\mathbb{Z}} b_n(\alpha) e_{jk} h(2^n \cdot - \alpha).$$

This shows that

$$\lim_{n\to\infty} \|b_n e_{jk}\|_\infty = 0, \qquad (j,k) \neq (1,1). \tag{2.13}$$

To summarize, we have shown that (2.12) and (2.13) are necessary conditions for the subdivision scheme associated with a to converge in the L_2-norm.

Let v be an element of V. Then v can be expressed as

$$v = \sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}} c_{jk}(\beta)e_{jk}\delta_\beta,$$

where $c_{jk} \in E_\mu$, $j,k = 1,\ldots,r$. Since $v \in V$, we have $(\overline{y}\otimes y)\sum_{\alpha\in\mathbb{Z}} v(\alpha) = 0$. But $(\overline{y}\otimes y)e_{11} = 1$ and $(\overline{y}\otimes y)e_{jk} = 0$ for $(j,k)\neq(1,1)$. Hence,

$$0 = (\overline{y}\otimes y)\sum_{\alpha\in\mathbb{Z}} v(\alpha) = (\overline{y}\otimes y)\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}} c_{jk}(\beta)e_{jk} = \sum_{\beta\in\mathbb{Z}} c_{11}(\beta).$$

So c_{11} can be written as $\sum_{\beta\in\mathbb{Z}} d(\beta)\nabla\delta_\beta$, where d is given by

$$d(\beta) := \begin{cases} c_{11}(\beta) + c_{11}(\beta-1) + c_{11}(\beta-2) + \cdots & \text{for } \beta \le -1, \\ -c_{11}(\beta+1) - c_{11}(\beta+2) - c_{11}(\beta+3) - \cdots & \text{for } \beta \ge 0. \end{cases}$$

Clearly, d belongs to $\ell_1(\mathbb{Z})$. The relations (2.12) and (2.13) together with (2.9) and (2.10) imply that, for every $\beta\in\mathbb{Z}$ and $(j,k)\neq(1,1)$,

$$\lim_{n\to\infty}\|T_b^n(e_{11}\nabla\delta_\beta)\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty}\|T_b^n(e_{jk}\delta_\beta)\|_\infty = 0.$$

In light of the expression of v, (2.11) follows at once. $\square$

Condition (2.11) is also sufficient for the L_2-convergence of the vector subdivision scheme associated with mask a. To see this, we first show that, for any $\psi \in (L_{2,c}(\mathbb{R}))^r$,

$$\|Q_a^n\psi\|_2^2 \le r\|T_b^n v(0)\|_\infty, \qquad n = 1,2,\ldots, \tag{2.14}$$

where

$$v(\alpha) := \operatorname{vec}(\psi\odot\psi^T)(\alpha), \qquad \alpha\in\mathbb{Z}.$$

Let $g := \operatorname{vec}(\psi\odot\psi^T)$, $\psi_n := Q_a^n\psi$, and $g_n := \operatorname{vec}(\psi_n\odot\psi_n^T)$ $(n = 1,2,\ldots)$. Then we have

$$\|\psi_n\|_2^2 \le r\|g_n(0)\|_\infty. \tag{2.15}$$

But (2.8) tells us that $g_n = Q_b^n g$. Hence,

$$g_n(0) = (Q_b^n g)(0) = \sum_{\alpha\in\mathbb{Z}} b_n(\alpha)g(-\alpha) = \sum_{\beta\in\mathbb{Z}} b_n(-\beta)g(\beta). \tag{2.16}$$

Furthermore, for $n = 1, 2, \ldots$, we have

$$T_b^n v(\alpha) = \sum_{\beta \in \mathbb{Z}} b_n(2^n \alpha - \beta) v(\beta), \qquad \alpha \in \mathbb{Z}. \tag{2.17}$$

This will be proved by induction on n. By the definition of T_b, (2.17) is true for $n = 1$. Suppose (2.17) is valid for $n-1$. For $\alpha \in \mathbb{Z}$, we have

$$\begin{aligned} T_b^n v(\alpha) &= T_b^{n-1}(T_b v)(\alpha) \\ &= \textstyle\sum_{\beta \in \mathbb{Z}} b_{n-1}(2^{n-1}\alpha - \beta)(T_b v)(\beta) \\ &= \textstyle\sum_{\beta \in \mathbb{Z}} \sum_{\gamma \in \mathbb{Z}} b_{n-1}(2^{n-1}\alpha - \beta) b(2\beta - \gamma) v(\gamma) \\ &= \textstyle\sum_{\gamma \in \mathbb{Z}} \Big[\sum_{\beta \in \mathbb{Z}} b_{n-1}(\beta) b(2^n \alpha - \gamma - 2\beta) \Big] v(\gamma) \\ &= \textstyle\sum_{\gamma \in \mathbb{Z}} b_n(2^n \alpha - \gamma) v(\gamma), \end{aligned}$$

where (2.6) has been used to derive the last equality. This completes the induction procedure. It follows from (2.16) and (2.17) that $g_n(0) = T_b^n v(0)$. This togetehr with (2.15) implies (2.14).

If a is finitely supported and the matrix $M := \sum_{\alpha \in \mathbb{Z}} a(\alpha)/2$ has a simple eigenvalue 1 and its other eigenvalues are less than 1 in modulus, then we must have $\rho(T_b) \geq 1$. Indeed, if $\rho(T_b) < 1$, then (2.14) tells us that $Q_a^n(e_1 \chi)$ would converge to 0 in the L_2-norm. On the other hand, it was proved in [23] that the limit of $Q_a^n(e_1\chi)$ must be a nonzero vector of functions in $L_2(\mathbb{R})$. This contradiction demonstrates $\rho(T_b) \geq 1$.

When $a \in E_\mu^{r \times r}$, this conclusion remains valid. To see this, we recall the following fact from functional analysis (see [34, Theorem 10.20]). Let $A, A_1, A_2, \ldots$ be bounded linear operators on a Banach space. Suppose $\|A_n - A\| \to 0$ as $n \to \infty$. Then for a given $\varepsilon > 0$ there exists some n_0 such that $\rho(A_n) < \rho(A) + \varepsilon$ for all $n \geq n_0$. In order to apply this result, we find sequences $a^{(N)}$ $(N = 1, 2, \ldots)$ such that each $a^{(N)}$ is supported on $[-N, N]$, $y \sum_{\alpha \in \mathbb{Z}} a^{(N)}(\alpha)/2 = y$, and $\|a^{(N)} - a\|_{E_\mu^{r\times r}} \to 0$ as $N \to \infty$. Let $b^{(N)} := a^{(N)} \diamond a^{(N)}/2$. Then $\|T_{b^{(N)}} - T_b\| \to 0$ as $N \to \infty$. If $\rho(T_b) < 1$, then $\rho(T_{b^{(N)}}) < 1$ for sufficiently large N. But the latter is impossible. Therefore, we must have $\rho(T_b) \geq 1$.

Let $a \in E_\mu^{r \times r}$ for some $\mu > 0$. We say that a satisfies the basic sum rule if

$$y \sum\nolimits_{\alpha \in \mathbb{Z}} a(2\alpha) = y \sum\nolimits_{\alpha \in \mathbb{Z}} a(2\alpha - 1) = y.$$

Let b be given by (1.4). If a satisfies the basic sum rule, then b satisfies the basic sum rule stated as follows:

$$(\overline{y} \otimes y) \sum\nolimits_{\alpha \in \mathbb{Z}} b(2\alpha) = (\overline{y} \otimes y) \sum\nolimits_{\alpha \in \mathbb{Z}} b(2\alpha - 1) = \overline{y} \otimes y.$$

The converse of this statement is also true. Moreover, if b satisfies the basic sum rule, then the space V given by (1.6) is invariant under T_b.

We are in a position to establish the main result of this section.

Theorem 1.2.2 *Let $a \in E_\mu^{r \times r}$ for some $\mu > 0$ and let b be given by (1.4). Then the subdivision scheme associated with a converges in the L_2-norm if and only if*

(a) *a satisfies the basic sum rule, and*
(b) $\rho(T_b|_V) < 1$.

Proof. Suppose the subdivision scheme associated with a converges in the L_2-norm. Then (2.11) is valid, by Theorem 2.1. If V is not invariant under T_b, then there exists $v \in V$ such that $T_b v \notin V$. Note that the codimension of V in $E_\mu^{r^2}$ is 1. Hence, any $u \in E_\mu^{r^2}$ can be represented as $u = w + c(T_b v)$ for some $w \in V$ and $c \in \mathbb{C}$. It follows from (2.11) that

$$\lim_{n\to\infty} \|T_b^n u\|_\infty = 0 \qquad \forall\, u \in E_\mu^{r^2}.$$

Consequently, $\rho(T_b) < 1$. But we have proved $\rho(T_b) \geq 1$. This contradiction shows that V is invariant under T_b.

Since V is invariant under T_b, we have $T_b(e_{jk}\nabla\delta) \in V$ for $j,k = 1,\ldots,r$. It follows that

$$\sum_{\alpha\in\mathbb{Z}} (\overline{y}\otimes y)\big[b(2\alpha) - b(2\alpha-1)\big]e_{jk} = (\overline{y}\otimes y)\sum_{\alpha\in\mathbb{Z}} T_b(e_{jk}\nabla\delta)(\alpha) = 0.$$

Since the above relation is true for all $j,k = 1,\ldots,r$, we deduce that

$$(\overline{y}\otimes y)\sum\nolimits_{\alpha\in\mathbb{Z}}\big[b(2\alpha) - b(2\alpha-1)\big] = 0.$$

But

$$(\overline{y}\otimes y)\sum\nolimits_{\alpha\in\mathbb{Z}}\big[b(2\alpha) + b(2\alpha-1)\big] = 2(\overline{y}\otimes y).$$

Therefore, b satisfies the basic sum rule. Consequently, a also satisfies the basic sum rule.

Since T_b is a compact operator, $\rho(T_b|_V) = |\tau|$ for some eigenvalue τ of $T_b|_V$. Suppose $T_b v = \tau v$ for some $v \in V$ with $v \neq 0$. It follows that $T_b^n v = \tau^n v$ for $n = 1, 2, \ldots$. By (2.11), $\|T_b^n v\|_\infty$ converges to 0 as $n \to \infty$. Therefore, $|\tau|^n \to 0$ as $n \to \infty$. This shows $\rho(T_b|_V) = |\tau| < 1$, as desired.

It remains to prove the sufficiency of conditions (a) and (b). For this purpose, let ϕ_0 be an $r\times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$ such that ϕ_0 satisfies the moment conditions of order 1. We wish to prove that $Q_a^n\phi_0$ is a Cauchy sequence in $(L_2(\mathbb{R}))^r$. We observe that

$$Q_a^{n+1}\phi_0 - Q_a^n\phi_0 = Q_a^n(Q_a\phi_0 - \phi_0) = Q_a^n\psi, \tag{2.18}$$

where $\psi := Q_a\phi_0 - \phi_0$. Since $y\sum_{\alpha\in\mathbb{Z}}\phi_0(\cdot-\alpha) = 1$ and a satisfies the basic sum rule, we have

$$\begin{aligned} y\textstyle\sum_{\alpha\in\mathbb{Z}}(Q_a\phi_0)(\cdot-\alpha) &= y\textstyle\sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} a(\beta)\phi_0(2\cdot-2\alpha-\beta) \\ &= \textstyle\sum_{\beta\in\mathbb{Z}} y\big[\sum_{\alpha\in\mathbb{Z}} a(\beta-2\alpha)\big]\phi_0(\cdot-\beta) = \sum_{\beta\in\mathbb{Z}} y\phi_0(\cdot-\beta) = 1. \end{aligned}$$

In other words, $Q_a\phi_0$ also satisfies the moment conditions of order 1. Consequently,

$$y\sum\nolimits_{\alpha\in\mathbb{Z}}\psi(\cdot-\alpha) = 0.$$

Let $v(\alpha) := \mathrm{vec}\,(\psi \odot \psi^T)(\alpha)$, $\alpha \in \mathbb{Z}$. Then the above relation and (2.1) imply

$$(\overline{y} \otimes y) \sum_{\alpha \in \mathbb{Z}} v(\alpha) = (\overline{y} \otimes y) \sum_{\alpha \in \mathbb{Z}} \int_{\mathbb{R}} \mathrm{vec}\left(\psi(\alpha + x)\overline{\psi(x)}^T\right) dx = 0.$$

Hence v lies in V. Since $\rho(T_b|_V) < 1$, by (2.14) we see that there exist two constants $C > 0$ and $t \in (0, 1)$ such that

$$\|Q_a^n \psi\|_2 \leq C t^n, \qquad n = 1, 2, \ldots.$$

Since $0 < t < 1$, this together with (2.18) tells us that $Q_a^n \phi_0$ is a Cauchy sequence in $(L_2(\mathbb{R}))^r$. Thus, $Q_a^n \phi_0$ converges in the L_2-norm for every ϕ_0 in $(L_{2,c}(\mathbb{R}))^r$ satisfying the moment conditions of order 1. If ψ_0 is another such vector, then $y \sum_{\alpha \in \mathbb{Z}} (\phi_0 - \psi_0)(\cdot - \alpha) = 0$. By what has been proved, $Q_a^n(\phi_0 - \psi_0)$ converges to 0 in the L_2-norm. In other words, $Q_a^n \phi_0$ and $Q_a^n \psi_0$ converge to the same limit. We conclude that the subdivision scheme associated with a converges in the L_2-norm. □

1.3 Biorthogonal Multiple Refinable Functions

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a refinable vector of compactly supported functions in $L_2(\mathbb{R})$. In this section we show that there exists a dual refinable vector of compactly supported functions in $L_2(\mathbb{R})$ if and only if the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent.

The linear independence of the shifts of a finite number of compactly supported functions was characterized by Jia and Micchelli [21] in terms of the Fourier transform of these functions. The Fourier-Laplace transform of a compactly supported integrable function f is defined by

$$\hat{f}(\zeta) := \int_{\mathbb{R}} f(x) e^{-ix\zeta}\, dx, \qquad \zeta \in \mathbb{C}.$$

Let $\phi_1, \ldots, \phi_r$ be compactly supported integrable functions. It was proved in [21] that the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent if and only if, for any $\zeta \in \mathbb{C}$, the sequences $(\hat{\phi}_j(\zeta + 2\beta\pi))_{\beta \in \mathbb{Z}}$ $(j = 1, \ldots, r)$ are linearly independent. This result is also valid if $\phi_1, \ldots, \phi_r$ are compactly supported distributions.

Another important concept is stability. Let $\phi_1, \ldots, \phi_r$ be a finite number of functions in $L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$. We say that the shifts of $\phi_1, \ldots, \phi_r$ are L_p-stable if there exist two positive constants C_1 and C_2 such that, for arbitrary finitely supported sequences $\lambda_1, \ldots, \lambda_r$ on $\mathbb{Z}$,

$$C_1 \sum_{j=1}^r \|\lambda_j\|_p \leq \left\| \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}} \phi_j(\cdot - \alpha)\lambda_j(\alpha) \right\|_p \leq C_2 \sum_{j=1}^r \|\lambda_j\|_p\,.$$

Given a function ϕ on $\mathbb{R}$, set $\phi^\circ := \sum_{\alpha \in \mathbb{Z}} |\phi(\cdot - \alpha)|$. By $\mathcal{L}_p(\mathbb{R})$ we denote the linear space of all functions ϕ for which $(\phi^\circ)^p$ is integrable on the interval $[0, 1]$.

Let $\phi_1, \ldots, \phi_r$ be functions in $\mathcal{L}_p(\mathbb{R})$ $(1 \le p \le \infty)$. It was proved by Jia and Micchelli [21] that the shifts of $\phi_1, \ldots, \phi_r$ are L_p-stable if and only if, for any $\xi \in \mathbb{R}$, the sequences $(\hat{\phi}_j(\xi + 2\beta\pi))_{\beta \in \mathbb{Z}}$ $(j = 1, \ldots, r)$ are linearly independent. In particular, when $\phi_1, \ldots, \phi_r$ are compactly supported, linear independence implies stability.

In what follows, by I_r we denote the $r \times r$ identity matrix. The complex conjugate of a matrix M is denoted by M^*. For $f = (f_1, \ldots, f_r)^T$ and $g = (g_1, \ldots, g_r)^T$ in $(L_2(\mathbb{R}))^r$, we define

$$\langle f, g^T \rangle := \begin{bmatrix} \langle f_1, g_1 \rangle & \langle f_1, g_2 \rangle & \cdots & \langle f_1, g_r \rangle \\ \langle f_2, g_1 \rangle & \langle f_2, g_2 \rangle & \cdots & \langle f_2, g_r \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_r, g_1 \rangle & \langle f_r, g_2 \rangle & \cdots & \langle f_r, g_r \rangle \end{bmatrix}.$$

Thus, f and g are dual to each other if and only if

$$\langle f(\cdot - \gamma), g^T \rangle = \delta_{\gamma,0} I_r \qquad \forall\, \gamma \in \mathbb{Z}.$$

Let a be an element in $(\ell_0(\mathbb{Z}))^{r \times r}$ such that $M := \sum_{\alpha \in \mathbb{Z}} a(\alpha)/2$ has a simple eigenvalue 1 and its other eigenvalues are less than 1 in modulus. Let y be a nonzero $1 \times r$ vector such that $yM = y$. It was proved by Heil and Colella [16] that there exists a unique distributional solution ϕ of the refinement equation (1.1) such that ϕ is compactly supported and $y\hat{\phi}(0) = 1$. Similarly, let $\tilde{a}$ be an element in $(\ell_0(\mathbb{Z}))^{r \times r}$ such that the matrix $\widetilde{M} := \sum_{\alpha \in \mathbb{Z}} \tilde{a}(\alpha)/2$ has a simple eigenvalue 1 and its other eigenvalues are less than 1 in modulus. Let $\tilde{y}$ be a $1 \times r$ vector such that $\tilde{y}\widetilde{M} = \tilde{y}$ and $\tilde{y}y^* = 1$. Then there exists a unique distributional solution $\tilde{\phi}$ of the refinement equation

$$\tilde{\phi} = \sum\nolimits_{\alpha \in \mathbb{Z}} \tilde{a}(\alpha)\tilde{\phi}(2 \cdot - \alpha) \tag{3.1}$$

such that $\tilde{\phi}$ is compactly supported and $\tilde{y}\hat{\tilde{\phi}}(0) = 1$.

Theorem 1.3.1 *The vectors ϕ and $\tilde{\phi}$ belong to $(L_2(\mathbb{R}))^r$ and are dual to each other if and only if*
(a) $\sum_{\alpha \in \mathbb{Z}} a(\alpha)\tilde{a}(\alpha + 2\gamma)^* = 2\delta_{\gamma,0} I_r$ *for all* $\gamma \in \mathbb{Z}$, *and*
(b) *the subdivision schemes associated with both a and $\tilde{a}$ converge in the L_2-norm.*

Proof. Suppose $\phi \in (L_2(\mathbb{R}))^r$ and $\tilde{\phi} \in (L_2(\mathbb{R}))^r$ are dual to each other. Then we have $\langle \phi(\cdot - \gamma), \tilde{\phi}^T \rangle = \delta_{\gamma,0} I_r$ for all $\gamma \in \mathbb{Z}$. By using the refinement equations (1.1) and (3.1) we see that, for each $\gamma \in \mathbb{Z}$,

$$\begin{aligned} \langle \phi(\cdot - \gamma), \tilde{\phi}^T \rangle &= \sum\nolimits_{\alpha \in \mathbb{Z}} \sum\nolimits_{\beta \in \mathbb{Z}} \langle a(\alpha)\phi(2 \cdot - 2\gamma - \alpha), \tilde{\phi}(2 \cdot - \beta)^T \tilde{a}(\beta)^T \rangle \\ &= \sum\nolimits_{\alpha \in \mathbb{Z}} a(\alpha)\tilde{a}(\alpha + 2\gamma)^*/2. \end{aligned}$$

Hence, condition (a) is satisfied. Moreover, ϕ and $\tilde{\phi}$ are dual to each other implies that the shifts of $\phi_1, \ldots, \phi_r$ are stable. By [23, Theorem 3.2], the subdivision scheme associated with a is L_2-convergent. The same reason shows that the subdivision scheme associated with $\tilde{a}$ also converges in the L_2-norm.

Now suppose conditions (a) and (b) are satisfied. If $\phi_0 \in (L_2(\mathbb{R}))^r$ and $\tilde{\phi}_0 \in (L_2(\mathbb{R}))^r$ are dual to each other, then condition (a) tells us that $Q_a\phi_0$ and $Q_{\tilde{a}}\tilde{\phi}_0$ are also dual to each other. We choose the initial vectors ϕ_0 and $\tilde{\phi}_0$ as follows. Let $f_1 := \chi_{[0,1)}$ and, for $j = 2, \ldots, r$, let

$$f_j := \sum_{k=0}^{2^j-1} (-1)^k \chi_{[k/2^j,(k+1)/2^j)},$$

where $\chi_{[s,t)}$ denotes the characteristic function of the interval $[s,t)$. It is easily seen that $f := (f_1, \ldots, f_r)^T$ is dual to itself (see [23, Theorem 8.1] for the construction of f). Since $y\tilde{y}^* = 1$, we can find two $r \times r$ matrices A and $\tilde{A}$ such that the first column of A is $\tilde{y}^*$, the first column of $\tilde{A}$ is y^*, and $\tilde{A}^*A = I_r$. Let $\phi_0 := Af$ and $\tilde{\phi}_0 := \tilde{A}f$. Then ϕ_0 and $\tilde{\phi}_0$ satisfy the moment conditions of order 1 (with respect to y and $\tilde{y}$). Moreover, ϕ_0 and $\tilde{\phi}_0$ are dual to each other. Therefore, for $n = 1, 2, \ldots$, $Q_a^n\phi_0$ and $Q_{\tilde{a}}^n\tilde{\phi}_0$ are dual to each other. In other words,

$$\big\langle (Q_a^n\phi_0)(\cdot - \gamma), (Q_{\tilde{a}}^n\tilde{\phi}_0)^T \big\rangle = \delta_{\gamma,0} I_r \qquad \forall\, \gamma \in \mathbb{Z}. \tag{3.2}$$

But condition (b) tells us that $\|Q_a^n\phi_0 - \phi\|_2 \to 0$ and $\|Q_{\tilde{a}}^n\tilde{\phi}_0 - \tilde{\phi}\|_2 \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (3.2), we obtain

$$\langle \phi(\cdot - \gamma), \tilde{\phi}^T \rangle = \delta_{\gamma,0} I_r \qquad \forall\, \gamma \in \mathbb{Z}.$$

This proves that ϕ and $\tilde{\phi}$ are dual to each other. $\square$

Taking the Fourier transforms of both sides of (1.1) and (3.1), we obtain

$$\hat{\phi}(\xi) = H(\xi/2)\hat{\phi}(\xi/2) \quad \text{and} \quad \hat{\tilde{\phi}}(\xi) = \tilde{H}(\xi/2)\hat{\tilde{\phi}}(\xi/2), \qquad \xi \in \mathbb{R}, \tag{3.3}$$

where

$$H(\xi) := \sum_{\alpha\in\mathbb{Z}} a(\alpha)e^{-i\alpha\xi}/2 \quad \text{and} \quad \tilde{H}(\xi) := \sum_{\alpha\in\mathbb{Z}} \tilde{a}(\alpha)e^{-i\alpha\xi}/2. \tag{3.4}$$

It is easily seen that condition (a) is equivalent to

$$H(\xi)\tilde{H}(\xi)^* + H(\xi+\pi)\tilde{H}(\xi+\pi)^* = I_r \qquad \forall\, \xi \in \mathbb{R}. \tag{3.5}$$

Moreover, Theorem 2.2 tells us that condition (b) is equivalent to

$$\rho(T_b|_V) < 1 \quad \text{and} \quad \rho(T_{\tilde{b}}|_{\tilde{V}}) < 1, \tag{3.6}$$

where $\tilde{b} := \tilde{a} \diamond \tilde{a}/2$, V is the space given by (1.6), and

$$\tilde{V} := \Big\{ v \in E_\mu^{r^2} : (\overline{\tilde{y}} \otimes \tilde{y}) \sum\nolimits_{\alpha\in\mathbb{Z}} v(\alpha) = 0 \Big\}. \tag{3.7}$$

Thus, Theorem 3.1 can be restated as follows: The vectors ϕ and $\tilde{\phi}$ belong to $(L_2(\mathbb{R}))^r$ and are dual to each other if and only if both (3.5) and (3.6) hold true. Since a is finitely supported, [23, Theorem 7.1] tells us that, in (3.6), V can be chosen to be the minimum invariant subspace of T_b generated by $e_{11}(\Delta\delta), e_{22}\delta, \ldots, e_{rr}\delta$, and $\tilde{V}$ can be chosen in a similar way.

For the scalar case ($r = 1$), Lawton [29] first gave a characterization for orthogonality of the shifts of a refinable function in terms of the spectral radius of the corresponding transition matrix. Cohen and Daubechies [4] established the above form of characterization of biorthogonality of a pair of refinable functions. In [31], Long and Chen extended the results in [4] to the multivariate setting. Note that an essential ingredient of the proof of [4, Theorem 4.3] is the fact that a univariate trigonometric polynomial has only finitely many zeros in the interval $[-\pi, \pi]$. So the extension to the multivariate setting is not trivial. For the vector case ($r > 1$), on the basis of the work of Long, Chen, and Yuan [32], Shen [35] proved that ϕ and $\tilde{\phi}$ are dual to each other is equivalent to conditions (3.5), (3.6), and an additional condition on the eigenvectors of T_b and $T_{\tilde{b}}$ corresponding to the eigenvalue 1. However, as was demonstrated above, the condition on the eigenvectors of T_b and $T_{\tilde{b}}$ is redundant.

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported functions in $L_2(\mathbb{R})$ such that the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. Suppose ϕ satisfies the refinement equation (1.1) with a finitely supported mask a. Then there exists $\tilde{a} \in (\ell_0(\mathbb{Z}))^r$ such that the $r \times r$ matrices $H(\xi)$ and $\tilde{H}(\xi)$ given by (3.4) satisfy (3.5). To see this, we first show that the $r \times (2r)$ matrix

$$\begin{bmatrix} H(\xi) & H(\xi + \pi) \end{bmatrix} \tag{3.8}$$

has full rank r for every $\xi \in \mathbb{C}$. Suppose to the contrary that there exists some $\xi \in \mathbb{C}$ such that this matrix has rank less than r. Then there exists a nonzero $1 \times r$ vector $t = (t_1, \ldots, t_r)$ such that $tH(\xi) = 0$ and $tH(\xi + \pi) = 0$. By (3.3) we have

$$t\hat{\phi}(2\xi + 4\beta\pi) = tH(\xi)\hat{\phi}(\xi + 2\beta\pi) = 0 \qquad \forall\, \beta \in \mathbb{Z}$$

and

$$t\hat{\phi}(2\xi + 2\pi + 4\beta\pi) = tH(\xi + \pi)\hat{\phi}(\xi + \pi + 2\beta\pi) = 0 \qquad \forall\, \beta \in \mathbb{Z}.$$

It follows that $t\hat{\phi}(2\xi + 2\beta\pi) = 0$ for all $\beta \in \mathbb{Z}$. Thus, the shifts of $\phi_1, \ldots, \phi_r$ would be linearly dependent. This contradiction shows that the matrix in (3.8) has full rank r for every $\xi \in \mathbb{C}$. Let

$$P(z) := \sum\nolimits_{\alpha \in \mathbb{Z}} a(\alpha) z^\alpha / 2, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Then $P(z)$ is an $r \times r$ matrix of Laurent polynomials and $H(\xi) = P(e^{-i\xi})$ for $\xi \in \mathbb{C}$. By what has been proved, the matrix

$$\begin{bmatrix} P(z) & P(-z) \end{bmatrix}$$

has full rank r for every $z \in \mathbb{C} \setminus \{0\}$. Hence, by a well-known result from algebra, there exist two $r \times r$ matrices $U(z)$ and $V(z)$ of Laurent polynomials such that

$$\begin{bmatrix} P(z) & P(-z) \end{bmatrix} \begin{bmatrix} U(z) \\ V(z) \end{bmatrix} = I_r, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Let $Q(z) := (U(z) + V(-z))/2$. Then we have

$$[\,P(z) \quad P(-z)\,] \begin{bmatrix} Q(z) \\ Q(-z) \end{bmatrix} = I_r, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Let $K(\xi) := Q(e^{-i\xi})^*$, $\xi \in \mathbb{C}$. Then

$$H(\xi)K(\xi)^* + H(\xi+\pi)K(\xi+\pi)^* = I_r \qquad \forall\, \xi \in \mathbb{C}. \tag{3.9}$$

We may express $K(\xi)$ as $\sum_{\alpha\in\mathbb{Z}} c(\alpha)e^{-i\alpha\xi}/2$ for some $c \in (\ell_0(\mathbb{Z}))^{r\times r}$. However, there is no guarantee that the subdivision scheme associated with c will converge in the L_2-norm. To overcome this difficulty, we first construct an exponentially decaying mask $\tilde{a}$ such that the subdivision scheme associated with $\tilde{a}$ converges in the L_2-norm and the $r \times r$ matrices $H(\xi)$ and $\tilde{H}(\xi)$ given by (3.4) satisfy (3.5). In the process, the bracket product of two functions in $\mathcal{L}_2(\mathbb{R})$ introduced by Jia and Micchelli [20, Theorem 3.2] will be used. For $f, g \in \mathcal{L}_2(\mathbb{R})$, their bracket product is defined by

$$[f, g](e^{-i\xi}) := \sum_{\beta\in\mathbb{Z}} \hat{f}(\xi + 2\beta\pi)\overline{\hat{g}(\xi + 2\beta\pi)}, \qquad \xi \in \mathbb{R}.$$

We are in a position to establish the main result of this section.

Theorem 1.3.2 *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be an $r\times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$ with linearly independent shifts. Suppose that ϕ satisfies the refinement equation (1.1) with a finitely supported mask a. Then there exists a refinable vector $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^T$ of compactly supported functions in $L_2(\mathbb{R})$ such that $\tilde{\phi}$ is dual to ϕ.*

Proof. Let

$$G(\xi) := \big([\phi_j, \phi_k](e^{-i\xi})\big)_{1\le j,k\le r}, \qquad \xi \in \mathbb{R}.$$

Then $G(\xi)$ is 2π-periodic. Since the shifts of $\phi_1, \ldots, \phi_r$ are stable, the Gram matrix $G(\xi)$ is positive definite for every $\xi \in \mathbb{R}$ (see [20, Theorem 4.1]). Let $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^T$ be given by

$$\widehat{\tilde{\phi}}(\xi) := G(\xi)^{-1}\hat{\phi}(\xi), \qquad \xi \in \mathbb{R}.$$

Then $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ decay exponentially fast and have stable shifts. In particular, $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ belong to $\mathcal{L}_2(\mathbb{R})$. For every $\xi \in \mathbb{R}$ we have

$$\begin{aligned} \textstyle\sum_{\beta\in\mathbb{Z}} \widehat{\tilde{\phi}}(\xi + 2\beta\pi)\hat{\phi}(\xi + 2\beta\pi)^* &= G(\xi)^{-1} \textstyle\sum_{\beta\in\mathbb{Z}} \hat{\phi}(\xi + 2\beta\pi)\hat{\phi}(\xi + 2\beta\pi)^* \\ &= G(\xi)^{-1}G(\xi) = I_r. \end{aligned}$$

This shows that $\tilde{\phi}$ is dual to ϕ. Moreover, $\tilde{\phi}$ is refinable. Indeed, we have

$$\begin{aligned} \widehat{\tilde{\phi}}(\xi) &= G(\xi)^{-1}\hat{\phi}(\xi) = G(\xi)^{-1}H(\xi/2)\hat{\phi}(\xi/2) \\ &= G(\xi)^{-1}H(\xi/2)G(\xi/2)\widehat{\tilde{\phi}}(\xi/2). \end{aligned}$$

Consequently,

$$\hat{\tilde{\phi}}(\xi) = \tilde{H}(\xi/2)\hat{\tilde{\phi}}(\xi/2),$$

where

$$\tilde{H}(\xi) = G(2\xi)^{-1}H(\xi)G(\xi), \qquad \xi \in \mathbb{R}.$$

Clearly, $\tilde{H}$ is 2π-periodic. Since $\tilde{\phi}$ is dual to ϕ, H and $\tilde{H}$ satisfy the relation (3.5). Suppose $\tilde{H}(\xi) = (\tilde{h}_{jk}(\xi))_{1\le j,k\le r}$, where

$$\tilde{h}_{jk}(\xi) = \sum\nolimits_{\alpha\in\mathbb{Z}} \tilde{a}_{jk}(\alpha)e^{-i\alpha\xi}/2, \qquad \xi \in \mathbb{R}.$$

Each sequence $\tilde{a}_{jk}$ decays exponentially fast. That is, there exists some $\mu > 0$ such that $\tilde{a}_{jk} \in E_\mu$ for all $j,k = 1,\ldots,r$. Let $\tilde{a} := (\tilde{a}_{jk})_{1\le j,k\le r}$. Then $\tilde{a} \in E_\mu^{r\times r}$. Since the shifts of $\tilde{\phi}_1,\ldots,\tilde{\phi}_r$ are stable, from the proof of [23, Theorem 3.1] we see that the subdivision scheme associated with $\tilde{a}$ is L_2-convergent. Let $\tilde{b} := \tilde{a} \diamond \tilde{a}/2$ and let $\tilde{V}$ be the linear space given in (3.7). By Theorem 2.2 we have $\rho(T_{\tilde{b}}|_{\tilde{V}}) < 1$.

Let

$$M := \sum\nolimits_{\alpha\in\mathbb{Z}} a(\alpha)/2 \quad \text{and} \quad \tilde{M} := \sum\nolimits_{\alpha\in\mathbb{Z}} \tilde{a}(\alpha)/2.$$

There exists a unique $1\times r$ vector y such that $yM = y$ and $y\hat{\phi}(0) = 1$. Similarly, there exists a unique $1\times r$ vector $\tilde{y}$ such that $\tilde{y}\tilde{M} = \tilde{y}$ and $\tilde{y}\hat{\tilde{\phi}}(0) = 1$. The duality of ϕ and $\tilde{\phi}$ implies $y\tilde{y}^* = 1$. Since the subdivision scheme associated with $\tilde{a}$ converges in the L_2-norm, Theorem 2.2 tells us that $\tilde{a}$ satisfies the basic sum rule:

$$\tilde{y}\sum\nolimits_{\alpha\in\mathbb{Z}} \tilde{a}(2\alpha) = \tilde{y}\sum\nolimits_{\alpha\in\mathbb{Z}} \tilde{a}(2\alpha-1) = \tilde{y}.$$

For $N = 1,2,\ldots$, we can find $\tilde{a}^{(N)} \in (\ell_0(\mathbb{Z}))^{r\times r}$ such that each $\tilde{a}^{(N)}$ is finitely supported and $\|\tilde{a}^{(N)} - \tilde{a}\|_{E_\mu^{r\times r}} \to 0$ as $N\to\infty$. For $\xi\in\mathbb{R}$, let

$$\tilde{H}_N(\xi) := \sum\nolimits_{\alpha\in\mathbb{Z}} \tilde{a}^{(N)}(\alpha)e^{-i\alpha\xi}/2, \qquad \xi\in\mathbb{R},$$

and

$$\varepsilon_N(\xi) := I_r - \big[H(\xi)\tilde{H}_N(\xi)^* + H(\xi+\pi)\tilde{H}_N(\xi+\pi)^*\big], \qquad \xi\in\mathbb{R}. \tag{3.10}$$

Then ε_N is π-periodic: $\varepsilon_N(\xi) = \varepsilon_N(\xi+\pi)$ for all $\xi\in\mathbb{R}$. Let

$$F_N(\xi) := \tilde{H}_N(\xi) + \varepsilon_N(\xi)^*K(\xi), \qquad \xi\in\mathbb{R},$$

where K is an $r\times r$ matrix of trigonometric polynomials satisfying (3.9). Thus, by (3.9) and (3.10) we have

$$\begin{aligned} & H(\xi)F_N(\xi)^* + H(\xi+\pi)F_N(\xi+\pi)^* \\ = \; & \big[H(\xi)\tilde{H}_N(\xi)^* + H(\xi+\pi)\tilde{H}_N(\xi+\pi)^*\big] \\ & \quad + \big[H(\xi)K(\xi)^* + H(\xi+\pi)K(\xi+\pi)^*\big]\varepsilon_N(\xi) \\ = \; & (I_r - \varepsilon_N(\xi)) + \varepsilon_N(\xi) = I_r. \end{aligned}$$

Write

$$F_N(\xi) = \sum\nolimits_{\alpha\in\mathbb{Z}} c^{(N)}(\alpha)e^{-i\alpha\xi}/2, \qquad \xi \in \mathbb{R},$$

where each $c^{(N)} \in (\ell_0(\mathbb{Z}))^{r\times r}$. Since $\|\tilde{a}^{(N)} - \tilde{a}\|_{E_\mu^{r\times r}} \to 0$ as $N \to \infty$, by the construction of F_N we also have $\|c^{(N)} - \tilde{a}\|_{E_\mu^{r\times r}} \to 0$ as $N \to \infty$. Observe that $\tilde{a}$ satisfies the basic sum rule. Hence, we may choose $\tilde{a}^{(N)}$ $(N = 1, 2, \ldots)$ in such a way that each $c^{(N)}$ satisfies the basic sum rule (with respect to $\tilde{y}$):

$$\tilde{y}\sum\nolimits_{\alpha\in\mathbb{Z}} c^{(N)}(2\alpha) = \tilde{y}\sum\nolimits_{\alpha\in\mathbb{Z}} c^{(N)}(2\alpha-1) = \tilde{y}.$$

Let $\widetilde{M}^{(N)} := \sum_{\alpha\in\mathbb{Z}} c^{(N)}(\alpha)/2$. For sufficiently large N, 1 is a simple eigenvalue of $\widetilde{M}^{(N)}$ and its other eigenvalues are less than 1 in modulus. Let $\tilde{b}^{(N)} := c^{(N)} \diamond c^{(N)}/2$. Then $\tilde{b}^{(N)} \to \tilde{b}$ in the space $E_\mu^{r^2\times r^2}$ as $N \to \infty$. But $\rho(T_{\tilde{b}}|_{\tilde{V}}) < 1$, where $\tilde{V}$ is the linear space given in (3.7); hence $\rho(T_{\tilde{b}^{(N)}}|_{\tilde{V}}) < 1$ for sufficiently large N. Therefore, by Theorem 2.2, the subdivision scheme associated with $c^{(N)}$ converges in the L_2-norm. By Theorem 3.1, the limit f is an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$ and f is dual to ϕ. The proof of the theorem is complete. $\square$

1.4 Biorthogonal Multiple Wavelets

In this section we apply the general theory developed so far to the construction of biorthogonal multiple wavelets.

The first nontrivial example of continuous symmetric orthogonal double wavelets was constructed by Donovan, Geronimo, Hardin, and Massopust in [11] by means of fractal interpolation. In [2], Chui and Lian constructed orthogonal double wavelets with symmetry by using refinement equations. However, they did not prove that the double refinable functions they constructed are functions in $L_2(\mathbb{R})$ with orthogonal shifts. In [23], Jia, Riemenschneider, and Zhou did that and constructed an entire family of orthogonal double wavelets that are continuous and have symmetry.

Biorthogonal wavelets have advantages over orthogonal wavelets in several aspects. In particular, biorthogonal wavelets can be constructed from spline functions, and the coefficients in the corresponding filters can be chosen to be rational numbers. For Hermite cubic splines, Dahmen, Han, Jia, and Kunoth [8] found a refinable dual vector of continuous functions and constructed biorthogonal double wavelets on the interval.

In this section, we will give two examples of biorthogonal double wavelets. In the first example, the wavelets are piecewise linear functions with short support. In the second example, the wavelets are almost in C^2, and the dual wavelets are in C^1. All the wavelets and dual wavelets are either symmetric or anti-symmetric about the origin.

Let us start with multiresolution of $L_2(\mathbb{R})$. Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is a refinable vector of compactly supported functions in $L_2(\mathbb{R})$. Let S denote the closed linear subspace of $L_2(\mathbb{R})$ generated by $\phi_1, \ldots, \phi_r$. For $k \in \mathbb{Z}$, let S_k be the 2^k-dilate of S:

$$S_k := \{g(2^k \cdot) : g \in S\}.$$

It was proved by Jia and Shen [25] that $(S_k)_{k\in\mathbb{Z}}$ forms a multiresolution of $L_2(\mathbb{R})$. In other words, $S_k \subset S_{k+1}$ for $k \in \mathbb{Z}$, $\cup_{k\in\mathbb{Z}} S_k$ is dense in $L_2(\mathbb{R})$, and $\cap_{k\in\mathbb{Z}} S_k = \{0\}$. Suppose the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. By Theorem 3.2, there exists a refinable vector $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^T$ of compactly supported functions in $L_2(\mathbb{R})$ such that $\tilde{\phi}$ and ϕ are dual to each other. Let $\tilde{S}$ denote the closed linear subspace of $L_2(\mathbb{R})$ generated by $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$. For $k \in \mathbb{Z}$, let $\tilde{S}_k$ be the 2^k-dilate of $\tilde{S}$. Then $(\tilde{S}_k)_{k\in\mathbb{Z}}$ also forms a multiresolution of $L_2(\mathbb{R})$. We wish to find $\psi_1, \ldots, \psi_r \in S_1$ and $\tilde{\psi}_1, \ldots, \tilde{\psi}_r \in \tilde{S}_1$ such that

$$\langle \psi_j, \tilde{\phi}_k(\cdot - \gamma)\rangle = 0 \qquad \forall\, j, k = 1, \ldots, r \quad \text{and} \quad \gamma \in \mathbb{Z}, \tag{4.1}$$

$$\langle \phi_j, \tilde{\psi}_k(\cdot - \gamma)\rangle = 0 \qquad \forall\, j, k = 1, \ldots, r \quad \text{and} \quad \gamma \in \mathbb{Z}, \tag{4.2}$$

and

$$\langle \psi_j, \tilde{\psi}_k(\cdot - \gamma)\rangle = \delta_{\gamma,0}\delta_{jk} \qquad \forall\, j, k = 1, \ldots, r \quad \text{and} \quad \gamma \in \mathbb{Z}. \tag{4.3}$$

Let W be the closed linear space of $L_2(\mathbb{R})$ generated by the shifts of $\psi_1, \ldots, \psi_r$, and let $\tilde{W}$ be the closed linear space of $L_2(\mathbb{R})$ generated by the shifts of $\tilde{\psi}_1, \ldots, \tilde{\psi}_r$. If (4.1), (4.2), and (4.3) are true, then S_1 is the direct sum of S_0 and W, and $\tilde{S}_1$ is the direct sum of $\tilde{S}_0$ and $\tilde{W}$. As was done in [4, Theorem 5.1], it can be proved that

$$\left\{2^{k/2}\psi_j(2^k \cdot - \alpha) : j = 1, \ldots, r,\, k \in \mathbb{Z},\, \alpha \in \mathbb{Z}\right\}$$

forms a Riesz basis for $L_2(\mathbb{R})$, and

$$\left\{2^{k/2}\tilde{\psi}_j(2^k \cdot - \alpha) : j = 1, \ldots, r,\, k \in \mathbb{Z},\, \alpha \in \mathbb{Z}\right\}$$

forms the dual basis.

Suppose

$$\psi = \sum_{\alpha\in\mathbb{Z}} c(\alpha)\phi(2\cdot - \alpha) \quad \text{and} \quad \tilde{\psi} = \sum_{\alpha\in\mathbb{Z}} \tilde{c}(\alpha)\tilde{\phi}(2\cdot - \alpha). \tag{4.4}$$

Then (4.1), (4.2), and (4.3) are respectively equivalent to the following equations:

$$\sum\nolimits_{\beta\in\mathbb{Z}} c(\beta)\tilde{a}(2\gamma + \beta)^* = 0 \qquad \forall\, \gamma \in \mathbb{Z}, \tag{4.5}$$

$$\sum\nolimits_{\beta\in\mathbb{Z}} \tilde{c}(\beta)a(2\gamma + \beta)^* = 0 \qquad \forall\, \gamma \in \mathbb{Z}, \tag{4.6}$$

$$\sum\nolimits_{\beta\in\mathbb{Z}} c(\beta)\tilde{c}(2\gamma + \beta)^* = 2\delta_{\gamma,0}I_r \qquad \forall\, \gamma \in \mathbb{Z}. \tag{4.7}$$

Before giving two examples of biorthogonal double wavelets, we take a brief review of the approximation and smoothness properties of multiple refinable functions and multiple wavelets.

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$. We say that ϕ has accuracy k if the shift-invariant space generated by $\phi_1, \ldots, \phi_r$ contains all polynomials of degree less than k. If, in addition, $\phi_1, \ldots, \phi_r$ are functions in $L_p(\mathbb{R})$ $(1 \le p \le \infty)$, then ϕ has accuracy k if and only if the shift-invariant space generated by $\phi_1, \ldots, \phi_r$ provides approximation order k (see [19]).

Now suppose ϕ satisfies the refinement equation (1.1) with mask a. The optimal accuracy of ϕ was characterized in terms of the mask by Heil, Strang, Strela [17], and by Plonka [33] under the condition that the shifts of $\phi_1, \ldots, \phi_r$ be linearly independent. In [22], Jia, Riemenschneider, and Zhou gave a characterization for the accuracy without the assumption of linear independence.

In the following we give a characterization for the accuracy of ϕ in a form slightly different from that of [17]. For $m = 0, 1, 2, \ldots$, set

$$E_m := \frac{1}{m!} \sum_{\alpha \in \mathbb{Z}} (2\alpha)^m a(2\alpha) \quad \text{and} \quad O_m := \frac{1}{m!} \sum_{\alpha \in \mathbb{Z}} (2\alpha - 1)^m a(2\alpha - 1).$$

Theorem 1.4.1 *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions. Suppose ϕ satisfies the refinement equation (1.1) with mask a. Let k be a positive integer. If there exist $1 \times r$ vectors $c_m = (c_{m1}, \ldots, c_{mr})$ $(m = 0, 1, \ldots, k-1)$ such that*

$$\sum_{m=0}^{j} (-1)^m 2^{j-m} c_{j-m} E_m = c_j \quad \text{and} \quad \sum_{m=0}^{j} (-1)^m 2^{j-m} c_{j-m} O_m = c_j \qquad (4.8)$$

are true for $j = 0, 1, \ldots, k-1$, and if $c_0 \neq 0$, then ϕ has accuracy k. Moreover, under the condition $c_0 \hat{\phi}(0) = 1$, we have

$$\frac{x^j}{j!} = \sum_{\alpha \in \mathbb{Z}} \sum_{m=0}^{j} \frac{\alpha^m}{m!} c_{j-m} \phi(x - \alpha), \qquad j = 0, 1, \ldots, k-1, \ x \in \mathbb{R}.$$

Conversely, if ϕ has accuracy k, and if the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent, then there exist $1 \times r$ vectors c_m $(m = 0, 1, \ldots, k-1)$ satisfying $c_0 \neq 0$ and the conditions in (4.8).

In fact, Theorem 4.1 remains true if, for $\xi = 0$ and $\xi = \pi$, the sequences $(\hat{\phi}_j(\xi + 2\beta\pi))_{\beta \in \mathbb{Z}}$ $(j = 1, \ldots, r)$ are linearly independent.

We use the generalized Lipschitz space to measure smoothness of a given function. By $(\mathrm{Lip}^*(\nu, L_p(\mathbb{R})))^r$ we denote the linear space of all vectors $f = (f_1, \ldots, f_r)^T$ such that $f_1, \ldots, f_r \in \mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$. The optimal smoothness of a vector $f \in (L_p(\mathbb{R}))^r$ in the L_p-norm is described by its critical exponent $\nu_p(f)$ defined by

$$\nu_p(f) := \sup\Big\{ \nu : f \in \big(\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))\big)^r \Big\}.$$

It is easily seen that

$$\nu_\infty(f) \ge \nu_2(f) - 1/2.$$

In [24], Jia, Riemenschneider, and Zhou gave a characterization for the smoothness order of a refinable vector of functions in terms of the p-norm joint spectral radius of two matrices associated with the mask. In particular, for the case $p = 2$, Theorem 3.4 of [24] can be restated as follows.

Theorem 1.4.2 *Suppose $\phi = (\phi_1, \ldots, \phi_r)^T \in (L_2(\mathbb{R}))^r$ is a compactly supported solution of the refinement equation (1.1) with mask a. Let $b := a \diamond a/2$ and let T_b be the corresponding transition operator. Then, for any positive integer k,*

$$\nu_2(\phi) \geq -\log_2 \sqrt{\rho(T_b|_W)},$$

where W is the minimal invariant subspace of T_b generated by $e_{jj}(\Delta^k \delta)$, $j = 1, \ldots, r$. Moreover, $\nu_2(\phi) = -\log_2 \sqrt{\rho(T_b|_W)}$, provided

$$k > -\log_2 \sqrt{\rho(T_b|_W)}$$

and the shifts of $\phi_1, \ldots, \phi_r$ are stable.

We are ready to provide two examples of biorthogonal double wavelets.

Example 4.3 Let ϕ_1 and ϕ_2 be two functions on $\mathbb{R}$ given by

$$\phi_1(x) := \begin{cases} 3x + 2 & \text{for } x \in [-2/3, -1/3], \\ 1 & \text{for } x \in [-1/3, 1/3], \\ -3x + 2 & \text{for } x \in [1/3, 2/3], \\ 0 & \text{for } x \in \mathbb{R} \setminus [-2/3, 2/3], \end{cases}$$

and

$$\phi_2(x) := \begin{cases} -3x - 2 & \text{for } x \in [-2/3, -1/3], \\ 3x & \text{for } x \in [-1/3, 1/3], \\ -3x + 2 & \text{for } x \in [1/3, 2/3], \\ 0 & \text{for } x \in \mathbb{R} \setminus [-2/3, 2/3]. \end{cases}$$

Then ϕ_1 is symmetric about the origin, and ϕ_2 is anti-symmetric about the origin. It can be directly verified that the shifts of ϕ_1 and ϕ_2 are linearly independent.

The vector $\phi = (\phi_1, \phi_2)^T$ is refinable:

$$\phi(x) = a(-1)\phi(2x + 1) + a(0)\phi(2x) + a(1)\phi(2x - 1), \qquad x \in \mathbb{R},$$

where the mask a is given by

$$a(-1) = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}, \quad a(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Since the shifts of ϕ_1 and ϕ_2 are linearly independent, Theorem 3.2 tells us that there exists a dual refinable vector of compactly supported functions in $L_2(\mathbb{R})$. The corresponding mask $\tilde{a}$ must satisfy condition (a) of Theorem 3.1. We choose $\tilde{a}$ such that $\tilde{a}(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \{-1, 0, 1\}$, and

$$\tilde{a}(-1) = \begin{bmatrix} 1/2 & 1/2 \\ -7/8 & -7/8 \end{bmatrix}, \quad \tilde{a}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \tilde{a}(1) = \begin{bmatrix} 1/2 & -1/2 \\ 7/8 & -7/8 \end{bmatrix}.$$

Then $\tilde{a}$ possesses the desired property. Moreover, the subdivision schemes associated with a and $\tilde{a}$ converge in the L_2-norm (see [24, Example 5.2]). Let $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)^T$ be the solution of the refinement equation with mask $\tilde{a}$ such that $\widehat{\tilde{\phi}_1}(0) = 1$ and $\widehat{\tilde{\phi}_2}(0) = 0$. Then $\tilde{\phi} \in (L_2(\mathbb{R}))^2$ is dual to ϕ. It was proved in [24] that $\nu_\infty(\tilde{\phi}) = 0.375$ and the optimal accuracy of $\tilde{\phi}$ is 2.

In order to construct biorthogonal wavelets ψ and $\tilde{\psi}$ we need to find $c \in (\ell_0(\mathbb{Z}))^2$ and $\tilde{c} \in (\ell_0(\mathbb{Z}))^2$ such that they satisfy (4.5), (4.6), and (4.7). We choose c and $\tilde{c}$ such that $c(\alpha) = \tilde{c}(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \{-1, 0, 1\}$,

$$c(-1) = \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad c(0) = \begin{bmatrix} 1 & 0 \\ 0 & 7/2 \end{bmatrix}, \quad c(1) = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix},$$

and

$$\tilde{c}(-1) = \begin{bmatrix} -1/2 & -1/2 \\ 1/8 & 1/8 \end{bmatrix}, \quad \tilde{c}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \tilde{c}(1) = \begin{bmatrix} -1/2 & 1/2 \\ -1/8 & 1/8 \end{bmatrix}.$$

If $\psi = (\psi_1, \psi_2)^T$ and $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^T$ are given by (4.4), then ψ_1, ψ_2 and $\tilde{\psi}_1, \tilde{\psi}_2$ are biorthogonal double wavelets. $\square$

Example 4.4 Consider the following refinement equation:

$$\phi = \sum\nolimits_{\alpha \in \mathbb{Z}} a(\alpha) \phi(2\cdot - \alpha),$$

where the refinement mask a is supported on $\{-1, 0, 1\}$ and

$$a(-1) = \begin{bmatrix} 1/2 & 3/2 \\ -1/8 & -1/2 \end{bmatrix}, \quad a(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1/2 & -3/2 \\ 1/8 & -1/2 \end{bmatrix}.$$

It was proved in [24, Example 4.2] that the shifts of ϕ_1 and ϕ_2 are linearly independent. Moreover, $\nu_\infty(\phi) = 2$ and the optimal accuracy of ϕ is 3. By Theorem 3.2, there exists a dual refinable vector of compactly supported functions in $L_2(\mathbb{R})$. We choose $\tilde{a}$ to be the sequence supported on $\{-2, -1, 0, 1, 2\}$ and given by

$$\begin{aligned}
\tilde{a}(-2) &= \begin{bmatrix} -53/512 & -7/256 \\ 359/512 & 99/512 \end{bmatrix}, \quad \tilde{a}(-1) = \begin{bmatrix} 1/2 & 25/256 \\ -421/128 & -161/256 \end{bmatrix}, \\
\tilde{a}(0) &= \begin{bmatrix} 309/256 & 0 \\ 0 & 281/256 \end{bmatrix}, \quad \tilde{a}(1) = \begin{bmatrix} 1/2 & -25/256 \\ 421/128 & -161/256 \end{bmatrix}, \\
\tilde{a}(2) &= \begin{bmatrix} -53/512 & 7/256 \\ -359/512 & 99/512 \end{bmatrix}.
\end{aligned}$$

It is easy to verify that a and $\tilde{a}$ satisfy condition (a) of Theorem 3.1. Moreover, the subdivision schemes associated with a and $\tilde{a}$ converge in the L_2-norm. Let $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)^T$ be the solution of the refinement equation with mask $\tilde{a}$ such that $\widehat{\tilde{\phi}_1}(0) = 1$ and $\widehat{\tilde{\phi}_2}(0) = 0$. By Theorem 4.1 we find that the optimal accuracy of

$\tilde\phi$ is 2. The smoothness order of $\tilde\phi$ can be computed by using Theorem 4.2 and the result is $\nu_2(\tilde\phi) \approx 1.5510$. It follows that

$$\nu_\infty(\tilde\phi) \geq \nu_2(\tilde\phi) - 0.5 > 1.05.$$

So $\tilde\phi$ is a vector of C^1 functions.

Finally, let us construct wavelets and dual wavelets associated with ϕ and $\tilde\phi$. We choose c and $\tilde c$ as follows:

$$\begin{aligned}
c(-3) &= \begin{bmatrix} 43/512 & 57/128 \\ 43/512 & 57/128 \end{bmatrix}, & c(-2) &= \begin{bmatrix} -7/32 & -99/128 \\ -7/32 & -99/128 \end{bmatrix}, \\
c(-1) &= \begin{bmatrix} 981/512 & 1505/128 \\ 407/512 & 703/128 \end{bmatrix}, & c(0) &= \begin{bmatrix} -57/16 & 0 \\ 0 & 743/64 \end{bmatrix}, \\
c(1) &= \begin{bmatrix} 981/512 & -1505/128 \\ -407/512 & 703/128 \end{bmatrix}, & c(2) &= \begin{bmatrix} -7/32 & 99/128 \\ 7/32 & -99/128 \end{bmatrix}, \\
c(3) &= \begin{bmatrix} 43/512 & -57/128 \\ -43/512 & 57/128 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\tilde c(-2) &= \begin{bmatrix} -1/64 & 0 \\ -1/64 & 0 \end{bmatrix}, & \tilde c(-1) &= \begin{bmatrix} 1/8 & 1/32 \\ 1/8 & 1/32 \end{bmatrix}, \\
\tilde c(0) &= \begin{bmatrix} -7/32 & 0 \\ 0 & 1/8 \end{bmatrix}, & \tilde c(1) &= \begin{bmatrix} 1/8 & -1/32 \\ -1/8 & 1/32 \end{bmatrix}, \\
\tilde c(2) &= \begin{bmatrix} -1/64 & 0 \\ 1/64 & 0 \end{bmatrix}.
\end{aligned}$$

If $\psi = (\psi_1, \psi_2)^T$ and $\tilde\psi = (\tilde\psi_1, \tilde\psi_2)^T$ are given by (4.4), then ψ_1, ψ_2 and $\tilde\psi_1, \tilde\psi_2$ are biorthogonal double wavelets. Note that both ψ and $\tilde\psi$ are supported on $[-2, 2]$. All $\phi_1, \tilde\phi_1, \psi_1, \tilde\psi_1$ are symmetric about the origin, and all $\phi_2, \tilde\phi_2, \psi_2, \tilde\psi_2$ are anti-symmetric about the origin. $\square$

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