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# $L^1$-CONVERGENCE RATE OF VISCOSITY METHODS FOR SCALAR CONSERVATION LAWS WITH THE INTERACTION OF ELEMENTARY WAVES AND THE BOUNDARY

By

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**Abstract.** This paper is concerned with global error estimates for viscosity methods to initial-boundary problems for scalar conservation laws $u_t+f(u)_x=0$ on $[0,\infty)\times[0,\infty)$, with the initial data $u(x,0)=u_0(x)$ and the boundary data $u(0,t)=u_-$, where $u_-$ is a constant, $u_0(x)$ is a step function with a discontinuous point, and $f\in C^2$ satisfies $f''>0, f(0)=f'(0)=0$. The structure of global weak entropy solution of the inviscid problem in the sense of Bardos-Leroux-Nedelec [11] is clarified. If the inviscid solution includes the interaction that the central rarefaction wave collides with the boundary $x=0$ and the boundary reflects a shock wave, then the error of the viscosity solution to the inviscid solution is bounded by $O(\varepsilon^{1/2}+\varepsilon|\ln\varepsilon|+\varepsilon)$ in $L^1$-norm. If the inviscid solution includes no interaction of the central rarefaction wave and the boundary or the interaction that the rarefaction wave collides with the boundary and is absorbed completely or partially by the boundary, then the error bound is $O(\varepsilon|\ln\varepsilon|+\varepsilon)$. In particular, if there is no central rarefaction wave included in the inviscid solution, the error bound is improved to $O(\varepsilon)$. The proof is given by a matching method and the traveling wave solutions.

**1. Introduction.** Consider the scalar conservation laws

$$u_t+f(u)_x=0,\quad x>0,t>0 \tag{1.1}$$

with the initial condition

$$u(x,0)=u_0(x),\quad x>0, \tag{1.2}$$

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and the boundary condition

$$u(0,t) = u_b(t), \quad t > 0. \tag{1.3}$$

The viscosity method approximating the initial-boundary problem (1.1)–(1.3) is to solve the parabolic equation on $[0,\infty) \times [0,\infty)$

$$(v_\varepsilon)_t + f(v_\varepsilon)_x = \varepsilon(v_\varepsilon)_{xx} \tag{1.4}$$

with the initial data

$$v_\varepsilon(x,0) = v_0(x), \quad x \geq 0, \tag{1.5}$$

and the boundary data

$$v_\varepsilon(0,t) = v_b(t), \quad t \geq 0, \tag{1.6}$$

where $\varepsilon > 0$ is a small viscosity parameter, and the weak entropy solution $u$ of (1.1)–(1.3) can be constructed as the limit of solutions of the initial-boundary problems (1.4)–(1.6) for the parabolic equation.

Viscosity methods play an important role in both theoretical analysis and practical computation for hyperbolic conservation laws. The accuracy and error bound of viscosity approximation are of much concern from the viewpoint of numerical computation. Goodman-Xin [1] developed a matching method, by which they showed the viscosity method approaching piecewise smooth solutions with a finite number of non-interacting shocks to a system of conservation laws has a local $\varepsilon$ rate of convergence away from shocks. By using the matching method, Teng-Zhang [2] derived an optimal first-order rate of $L^1$-convergence for the viscosity methods to piecewise constant entropy solutions on the Cauchy problem of scalar conservation laws, which is an improvement over the half-order rates of $L^1$-convergence for various approximation methods (see [3–9], etc.). The analysis used in [2] was extended to more general cases for scalar convex conservation laws, such as piecewise smooth solutions with finitely many discontinuities, and an $L^1$-convergence rate of $O(\varepsilon|\ln\varepsilon| + \varepsilon)$ was established (see [10]).

For the initial-boundary problems of scalar conservation laws with several space variables, vanishing viscosity method proving the existence of the global weak solution was established by Bardos-Leroux-Nedelec [11]. The main difficulty for scalar conservation laws with boundary is to have a good formation of the boundary condition. Namely, for a fixed initial value as (1.2), we really cannot impose such a condition at the boundary as (1.3), and the boundary condition is necessarily linked to the entropy condition (see also [11–14]). In other words, the weak entropy solution $u(x,t)$ of (1.1)–(1.3) does not admit a trace at the boundary, namely, $u(0,t) \not\equiv u_b(t)$ for $t > 0$. Whereas, as viscosity approximation of the weak entropy solution $u(x,t)$ of (1.1)–(1.3), the solution of initial-boundary problems of parabolic equation (1.4)–(1.6) does admit a fixed trace at the boundary. Therefore, it is very interesting to consider the error estimates for the viscosity approximation to the initial-boundary problems of scalar conservation laws.

For the Riemann initial-boundary problem of scalar conservation laws, i.e., $u_0(x)$, $u_b(t)$ are constant, by using a matching method used in [1,2,10], we proved that the error of the viscosity solution to the inviscid solution is bounded by $O(\varepsilon|\ln\varepsilon| + \varepsilon)$ in $L^1$-norm for all cases with convex fluxes and some cases with non-convex fluxes, respectively. If there is no central rarefaction wave included in the inviscid solution, the error bound was improved to $O(\varepsilon)$ (see [15]). As a next step, we investigate the cases in which the weak entropy solutions of (1.1)–(1.3) include the interaction of elementary waves and the boundary. In other words, we consider the following initial-boundary problem:

$$\begin{cases} u_t + f(u)_x = 0, \quad x > 0, t > 0 \\ u(0,t) = u_b(t) := u_-, \quad t > 0 \\ u(x,0) = u_0(x) := \begin{cases} u_m, & 0 < x < a \\ u_+, & x > a, \end{cases} \end{cases} \tag{1.7}$$

where $u_\pm, u_m$ are constant, $u_0(x) \not\equiv u_-$ for $x > 0$ and $x \neq a$, $a > 0$ is a constant, and $f \in C^2$ satisfies

$$f'' > 0, \qquad f(0) = f'(0) = 0. \tag{1.8}$$

The viscosity equation with initial-boundary conditions corresponding to (1.7) is denoted by

$$\begin{cases} (v_\varepsilon)_t + f(v_\varepsilon)_x = \varepsilon(v_\varepsilon)_{xx}, \quad x > 0, t > 0 \\ v_\varepsilon(0,t) = u_-, \quad t \geq 0 \\ v_\varepsilon(x,0) = v_0(x)(x \geq 0) \to u_+(x \to \infty). \end{cases} \tag{1.9}$$

In this paper, we establish the global error bounds of $L^1$-convergence for the viscosity methods for the initial-boundary problem (1.7). Because the weak entropy solutions of (1.7) include the interaction of elementary waves and the boundary, there are some different phenomena in the solution structure from the initial value problem, especially for the case when the inviscid solution includes the interaction that the central rarefaction wave collides with the boundary $x = 0$ and the boundary reflects a shock wave. This discussion will help us understand the more general problem with boundary, such as the problem with piecewise smooth solution. The problem with piecewise smooth solution will be investigated in our forthcoming paper.

This paper is organized as follows. The structure of the weak entropy solution of (1.7) is stated in Sec. 2. In Sec. 3, we introduce an $L^1$-stability lemma and a traveling wave solution lemma, which play important roles in obtaining the $L^1$-convergence rate. Using the conclusions obtained in Sec. 2 and Sec. 3, we extend the analysis used in [1,2,10] to our problem and derive the uniform error estimates for the viscosity methods in the final section.

**2. Solution structures.** Following Bardos-Leroux-Nedelec [11], we give the definition of the weak entropy solution to the initial-boundary problems (1.1)–(1.3) (see also [12–14]).

DEFINITION 2.1. A bounded and local bounded variation function $u(x,t)$ on $[0,\infty) \times [0,\infty)$ is called a weak entropy solution of the initial-boundary problems (1.1)–(1.3), if for each $k \in (-\infty,\infty)$, and for any nonnegative test function $\phi \in C_0^\infty([0,\infty) \times [0,\infty))$, it satisfies the following inequality

$$\int_0^\infty \int_0^\infty \{|u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\phi_x\}dxdt + \int_0^\infty |u_0(x)-k|\phi(x,0)dx + \int_0^\infty \operatorname{sgn}(u_b(t)-k)(f(u(0,t))-f(k))\phi(0,t)dt \geq 0. \tag{2.1}$$

For the initial-boundary problems (1.1)–(1.3) with general bounded and local bounded variation initial data and boundary data, the existence and uniqueness of the global weak entropy solution in the sense of (2.1) have been obtained, and the global weak entropy solution satisfies the following boundary entropy condition (2.2) (see also [11–14]).

LEMMA 2.1. If $u(x,t)$ is a weak entropy solution of (1.1)–(1.3), then,

$$\begin{aligned} &u(0,t)=u_b(t) \quad \text{or} \\ &\frac{f(u(0,t))-f(k)}{u(0,t)-k}\le 0,\quad k\in I(u(0,t),u_b(t)),\quad k\neq u(0,t),\quad \text{a.e.}\quad t\ge 0, \end{aligned} \tag{2.2}$$

where $I(u(0,t),u_b(t))=[\min\{u(0,t),u_b(t)\},\max\{u(0,t),u_b(t)\}]$.

The following lemma is easily proved by Definition 2.1 and Lemma 2.1 for the piecewise smooth solution (see also [14,16]).

LEMMA 2.2. Under the assumption of (1.8), a piecewise smooth function $u(x,t)$ with piecewise smooth discontinuity curves is a weak entropy solution of (1.7) in the sense of (2.1), if and only if the following conditions are satisfied:

(1) $u(x,t)$ satisfies Eq. $(1.7)_1$ on its smooth domains.

(2) If $x=x(t)$ is a weak discontinuity of $u(x,t)$, then $\frac{dx(t)}{dt}=f'(u(x(t),t))$. If $x=x(t)$ is a strong discontinuity of $u(x,t)$, then

$$\frac{dx(t)}{dt}=\frac{f(u^-)-f(u^+)}{u^--u^+}, \qquad \text{(Rankine – Hugoniot condition)}$$

and

$$u^->u^+, \qquad \text{(Lax's shock condition)}$$

where $u^-=u(x(t)-0,t), u^+=u(x(t)+0,t)$.

(3) The boundary entropy condition (2.2) holds.

(4) $u(x,0)=u_0(x)$ a.e. $x\ge 0$.

For the scalar conservation laws, Chang-Hsiao [17] discussed the interaction of elementary waves on the upper half of the $x$-$t$ plane $(-\infty,\infty)\times(0,\infty)$, and clarified the structure of solutions of the following Cauchy problem

$$\begin{cases} v_t+f(v)_x=0, & -\infty<x<\infty, t>0 \\ v(x,0)=\begin{cases} u_-, & x<0 \\ u_m, & 0<x<a \\ u_+, & x>a. \end{cases} \end{cases} \tag{2.3}$$

With the aid of the analysis method in [17], we study the interaction among elementary waves and the boundary $x=0$. From this, and by using Lemma 2.2, we clarify the structure of the weak entropy solution and its behavior at boundary $x=0$ to (1.7), which are very important for error estimates for vicosity methods.

When $u_m=u_+\neq u_-$, (1.7) is degenerated into a Riemann initial-boundary problem, which was investigated in [15]. From now on, $u_m\neq u_+$ is supposed. We divide our problem into five cases: (I) $u_-=u_m\neq u_+$; (II) $u_-<u_m<u_+$; (III) $u_+<u_m<u_-$;

(IV) $u_-, u_+ < u_m$; (V) $u_m < u_-, u_+$. For convenience, in this paper we denote

$$s(u_1, u_2) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

2.1 Case (I): $u_- = u_m \neq u_+$.

In this case, (2.3) is degenerated into a Riemann problem. Let $u(x,t)$ be the restriction of weak entropy solution $v(x,t)$ of (2.3) in the first quadrant of the $x$-$t$ plane, i.e., $u(x,t) = v(x,t)|_{(0,\infty)\times(0,\infty)}$. Then it is easy to verify that $u(x,t)$ satisfies conditions (1)-(4) in Lemma 2.2. Thus $u(x,t)$ is the weak entropy solution of (1.7).

2.2 Case (II): $u_- < u_m < u_+$.

When $u_- < u_m < u_+$, the weak entropy solution $v(x,t)$ of (2.3) includes two centered rarefaction waves $R_1$ and $R_2$, centered at point $(0,0)$ and $(a,0)$ of the $x$-$t$ plane, respectively, which are not able to overtake each other since the propagating speed of the wave front in $R_1$ is the same as that of the wave back in $R_2$ (see [17]). Let $u(x,t) = v(x,t)|_{(0,\infty)\times(0,\infty)}$; then it is also easy to verify that $u(x,t)$ satisfies the conditions in Lemma 2.2, hence $u(x,t)$ is the weak entropy solution of (1.7). If $f'(u_m) > 0$, the weak entropy solution of (1.7) includes two centered rarefaction waves $R$ and $R_2$ away from the boundary $x = 0$ (where $R$ is the restriction of $R_1$ on $(0,\infty)\times(0,\infty)$, i.e., $R = R_1|_{(0,\infty)\times(0,\infty)}$). If $f'(u_m) = 0$, the solution only includes the centered rarefaction wave $R_2$ away from the boundary. If $f'(u_m) < 0$, the solution only includes the centered rarefaction wave $R_2$ which will interact with the boundary and be absorbed by the boundary.

2.3 Case (III): $u_- > u_m > u_+$.

In this case, two shock waves $S_1$ with speed $s(u_-, u_m)$, and $S_2$ with speed $s(u_m, u_+)$, starting at point $(0,0)$ and $(a,0)$, respectively, appear in the weak entropy solution of the initial value problem (2.3). By Lax's shock condition, $S_1$ will overtake $S_2$, and the interaction of $S_1$ and $S_2$ generates a new shock wave $S_3$, with speed $s(u_-, u_+)$, starting at point $(a_1, t_1)$, where $t_1 = a/(s(u_-, u_m) - s(u_m, u_+))$, $a_1 = s(u_-, u_m)t_1$.

We divide this case into two sub-cases: (1) $s(u_-, u_m) \leq 0$; (2) $s(u_-, u_m) > 0$.

In sub-case (1), the shock wave $S_1$ is in the second quadrant or lies on the $t$-axis and the shock wave $S_2$ intersects the $t$-axis at time $t_* = -a/s(u_m, u_+)$. We construct a local weak entropy solution $v(x,t)$ on $(-\infty,\infty)\times[0,t_*)$ for (2.3), and let

$$v(x,t_*) = \begin{cases} u_-, & x < 0 \\ v(x, t_* - 0) \equiv u_+, & x > 0. \end{cases} \tag{2.4}$$

We then take $v(x,t_*)$ as the new initial value and solve $(2.3)_1$, (2.4) on $(-\infty,\infty)\times[t_*,\infty)$; thus we can extend the solution $v(x,t)$ to $(-\infty,\infty)\times[0,\infty)$. From Lemma 2.2, $u(x,t) = v(x,t)|_{(0,\infty)\times(0,\infty)}$ is the weak entropy solution of (1.7), whose expression is as follows:

$$u(x,t) = \begin{cases} u_m, & 0 < x < a + s(u_m, u_+)t \\ u_+, & x > a + s(u_m, u_+)t \end{cases}$$

for $0 < t < t_*$, and

$$u(x,t) \equiv u_+ \quad \text{for } x > 0, \quad t \geq t_*.$$

Namely, in sub-case (1), the weak entropy solution of (1.7) includes the shock wave $S_2$ with speed $s(u_m, u_+)$, starting at point $(a, 0)$. The shock wave $S_2$ collides with the boundary $x = 0$ and is absorbed by the boundary at time $t = t_*$.

In sub-case (2), the shock wave $S_1$ is in the first quadrant. If $s(u_-, u_+) \geq 0$, $S_3$ is in the first quadrant. If $s(u_-, u_+) < 0$, $S_3$ crosses the $t$-axis from the first quadrant and enters the second quadrant. We construct a local solution $v(x, t)$ on $(-\infty, \infty) \times [0, t_1)$ for (2.3) and take

$$v(x, t_1) =: \begin{cases} u_-, \ x < 0 \\ v(x, t_1 - 0), \ x > 0 \end{cases}$$

as the new initial value. Using the previous method, we can extend $v(x, t)$ to $(-\infty, \infty) \times [0, \infty)$ for $s(u_-, u_+) \geq 0$, and to $(-\infty, \infty) \times [0, t_2)$ for $s(u_-, u_+) < 0$, where $t_2 = t_1 - a_1/s(u_-, u_+)$ is the time at which the shock $S_3$ intersects the $t$-axis. When $s(u_-, u_+) < 0$, using

$$v(x, t_2) := \begin{cases} u_-, \ x < 0 \\ v(x, t_2 - 0) \equiv u_+, \ x > 0 \end{cases}$$

as the new initial value, as above, we can extend $v(x, t)$ to $(-\infty, \infty) \times [0, \infty)$. From Lemma 2.2, $u(x, t) = v(x, t)|_{(0,\infty)\times(0,\infty)}$ is the weak entropy solution of (1.7), whose expression is as follows.

When $0 < t < t_1$,

$$u(x, t) = \begin{cases} u_-, & 0 < x < s(u_-, u_m)t \\ u_m, & s(u_-, u_m)t < x < a + s(u_m, u_+)t \\ u_+, & x > a + s(u_m, u_+)t, \end{cases} \tag{2.5$_1$}$$

when $t_1 < t < t_2$,

$$u(x, t) = \begin{cases} u_-, & 0 < x < a_1 + s(u_-, u_+) \\ u_+, & x > a_1 + s(u_-, u_+), \end{cases} \tag{2.5$_2$}$$

when $t > t_2$, for $x > 0$,

$$u(x, t) = \begin{cases} (2.5)_2, & \text{if } \ s(u_-, u_+) \geq 0 \\ u_+, & \text{if } \ s(u_-, u_+) < 0. \end{cases} \tag{2.5$_3$}$$

Namely, for sub-case (2), the weak entropy solution of (1.7) includes two shock waves $S_1$ with speed $s(u_-, u_m)$, and $S_2$ with speed $s(u_m, u_+)$, starting at point $(0, 0)$ and $(a, 0)$, respectively, and the interaction of $S_1$ and $S_2$ generates a new shock wave $S_3$, with speed $s(u_-, u_+)$, starting at point $(a_1, t_1)$. If $s(u_-, u_+) \geq 0$, then $S_3$ does not interact with the boundary; if $s(u_-, u_+) < 0$, then $S_3$ interacts with the boundary $x = 0$ and is absorbed by the boundary at time $t = t_2$.

### 2.4 Case (IV): $u_-, u_+ < u_m$.

In this case, a central rarefaction wave $R$ centered at point $(0, 0)$ and a shock wave $S$ with original speed $s(u_m, u_+)$, starting at point $(a, 0)$, appear in the weak entropy solution of the initial problem (2.3). The shock wave $S$ will overtake the central rarefaction wave $R$ at finite time $t_1 = a/(f'(u_m) - s(u_m, u_+))$ by virtue of Lax's shock condition. The shock $S$ will cross $R$ with a varying speed of propagation during the penetration; that is, the shock $S$, also denoted by $x = X(t)$, is no longer a straight line at $t > t_1$, which satisfies the Rankine-Hugoniot condition

$$\frac{dX}{dt} = s(u(X(t)-0,t), u(X(t)+0,t)) \tag{2.6}$$

and Lax's shock condition

$$u(X(t)-0,t) > u(X(t)+0,t). \tag{2.7}$$

The varying speed of propagation can be determined by

$$\begin{cases} \dfrac{dX}{dt} = s(u,u_+), \quad f'(u) = \dfrac{X}{t}, \quad f'(u_-) \le \dfrac{X}{t} \le f'(u_m) \\ X(t_1) = a_1, \end{cases} \tag{2.8}$$

where $a_1 = f'(u_m)t_1$. Because $X = f'(u)t$,

$$dX/dt = tf''(u)du/dt + f'(u),$$

with which (2.8) can be integrated as

$$\int_{u_m}^{u} \frac{f''(u)du}{s(u,u_+) - f'(u)} = \ln\frac{t}{t_1} \quad (u_- \le u \le u_m),$$

from which, if $u_- = u_+$, the shock $x = X(t)$ is able to cross the whole of the rarefaction wave $R$ completely only when $t \to \infty$. If $u_- > u_+$, the shock $x = X(t)$ will cross the whole of $R$ completely at time

$$t_2 = t_1 \exp\left(\int_{u_m}^{u_-} \frac{f''(u)du}{s(u,u_+) - f'(u)}\right)$$

and after $t = t_2$, the shock wave is a straight line; if $u_- < u_+$, it is impossible for the shock wave to cross the whole of $R$ completely (see also [17]). We denote by $s(t)$ the speed function of the shock $x = X(t)$, i.e., $s(t) = dX(t)/dt$. Then from (2.8), $s(t) \in C([0,\infty))$, and is not increasing for $t \in (0,\infty)$.

Next, we construct the weak entropy solution of (1.7) by dividing this case into three sub-cases: (1) $u_-, u_+ < u_m \le 0$; (2) $0 \le u_-, u_+ < u_m$ or $u_- < 0 \le u_+ < u_m$ or $u_+ < 0 < u_- < u_m, f(u_+) \le f(u_-)$; (3) $u_+ < 0 \le u_- < u_m, f(u_+) > f(u_-)$ or $u_-, u_+ < 0 < u_m$.

2.4.1 $u_-, u_+ < u_m \le 0$.

In this sub-case, the centered rarefaction $R$ is in the second quadrant and the initial shock speed $s(u_m, u_+) < 0$. We construct a local solution $v(x,t)$ on $(-\infty,\infty) \times [0,t_0)$ for (2.3), where $t_0 = -a/s(u_m,u_+)$ is the intersection of the shock $S$ and the $t$-axis. Let

$$v(x,t_0) = \begin{cases} u_-, & x < 0 \\ v(x,t_0-0) \equiv u_+, & x > 0, \end{cases}$$

repeat the process using $v(x,t_0)$ as the new initial condition, and solve the corresponding Cauchy problem on $(-\infty,\infty) \times [t_0,\infty)$; then we can extend the solution $v(x,t)$ to $(-\infty,\infty) \times [0,\infty)$. Let $u(x,t) = v(x,t)\mid_{(0,\infty)\times(0,\infty)}$. In view of Lemma 2.2, $u(x,t)$ is the weak entropy solution of (1.7). Consequently, the interaction in the weak entropy solution of (1.7) is that the initial shock $S$ interacts with the boundary $x = 0$ and is absorbed by the boundary at the time $t = t_0$.

2.4.2 $0 \le u_-, u_+ < u_m$ or $u_- < 0 \le u_+ < u_m$ or $u_+ < 0 < u_- < u_m, f(u_+) \le f(u_-)$.

In this sub-case, the central rarefaction wave interacts with the shock wave $S$ in the first quadrant, and the generating shock wave $x = X(t)$ $(t \in (t_1, \infty))$ is also located in the first quadrant in the weak entropy solution of (2.3). From (2.8), we conclude that the shock speed $s(t) > 0$ for $t \in (0, \infty)$. We take $u(x,t) = v(x,t)\mid_{(0,\infty)\times(0,\infty)}$ as the weak entropy solution of (1.7). In fact, by virtue of Lemma 2.2, we can really do this.

We now state the interaction in the weak entropy solution of (1.7). A central rarefaction wave $R_0$ centered at point (0,0) (where $R_0 = R|_{(0,\infty)\times(0,\infty)}$) overtakes the shock S at $t = t_1$ and at $t > t_1$ the shock will cross $R_0$ with a varying speed of propagation during the penetration. The shock before and after interaction is just $x = X(t)$ $(t \in (0, \infty))$. As an example, for the case of $u_- < 0 \le u_+ < u_m$, the weak entropy solution of (1.7) is written as for $t > 0$:

$$u(x,t) = \begin{cases} u_l(x,t), & 0 < x < X(t) \\ u_+, & x > X(t), \end{cases} \tag{2.9}$$

where

$$u_l(x,t) = \begin{cases} (f')^{-1}(\frac{x}{t}), & 0 < \frac{x}{t} < f'(u_m) \\ u_m, & \frac{x}{t} > f'(u_m). \end{cases} \tag{2.10}$$

2.4.3 $u_+ < 0 \le u_- < u_m, f(u_+) > f(u_-)$ or $u_-, u_+ < 0 < u_m$.

In this sub-case, we have for (2.3) that after penetrating the part of the central rarefaction wave located in the first quadrant completely, the generating shock $x = X(t)$ crosses the $t$-axis and enters the second quadrant. From this, using Lemma 2.2, we can construct the weak entropy solution of (1.7). First, construct a local solution $v(x,t)$ to (2.3) on $(-\infty, \infty) \times [0, t_2)$ (where $t_2$ is the time at which the shock $x = X(t)$ intersects the $t$-axis). Next, take

$$v(x,t_2) := \begin{cases} u_-, & x < 0 \\ v(x, t_2 - 0) \equiv u_+, & x > 0 \end{cases}$$

as the new initial value of $(2.3)_1$ and solve the corresponding initial value problem; then $v(x,t)$ can be extended to $(-\infty,\infty)\times[0,\infty)$. The function $u(x,t) := v(x,t)|_{(0,\infty)\times(0,\infty)}$ is just the weak entropy solution of (1.7). We only give the expression of the weak entropy solution of (1.7) for the case of $u_-, u_+ < 0 < u_m$ as follows:

$$u(x,t) = \begin{cases} (2.9), & (x,t) \in (0,\infty) \times (0, t_2) \\ u_+, & (x,t) \in (0,\infty) \times (t_2, \infty). \end{cases} \tag{2.11}$$

The interaction in the weak entropy solution of (1.7) is stated as follows: the shock wave $x = X(t)$ interacts with the central rarefaction wave at $t = t_1$ and crosses the rarefaction wave at $t > t_1$, then collides with the boundary $x = 0$; finally, it is absorbed by the boundary.

2.5 CASE (V): $u_m < u_-, u_+$.

When $u_m < u_-, u_+$, a shock wave $S$ with original speed $s(u_-, u_m)$ starting at point $(0,0)$ and a central rarefaction wave $R$ centered at point $(a, 0)$ appear in the weak entropy solution of the initial problem (2.3). The shock $S$ overtakes the central rarefaction wave $R$ at time $t_0 = a/(s(u_-, u_m) - f'(u_m))$, and at $t > t_0$ the shock $S : x = X(t)$ will cross

$R$ with a varying speed of propagation during the penetration. As in Sec. 2.4, the shock speed function $s(t) := dX(t)/dt$ is determined by

$$\begin{cases} \dfrac{dX}{dt} = s(u_-, u), \quad f'(u) = \dfrac{X-a}{t}, \quad f'(u_m) \le \dfrac{X-a}{t} \le f'(u_+) \\ X(t_0) = a_0 \end{cases} \tag{2.12}$$

for $t \ge t_0$, where $a_0 = s(u_-, u_m)t_0$ and the size of $u_\pm$ determines whether the shock $x = X(t)$ is able to cross the whole of $R$ completely. By (2.12), $s(t) \in C([0,\infty))$ and is not decreasing.

We divide this case into the following three sub-cases: (1) $f(u_m) > f(u_-)$; (2) $f(u_m) = f(u_-)$; (3) $f(u_m) < f(u_-)$.

2.5.1 $f(u_m) > f(u_-)$.

In this sub-case, the initial shock wave $S : x = X(t)$ $(t \in (0, t_0])$ is in the second quadrant. If $u_m < u_-, u_+ \le 0$ or $u_m < u_- \le 0 < u_+$ or $u_m < u_+ < 0 < u_-$, $f(u_+) \ge f(u_-)$ or $u_m < u_- = u_+ = 0$, the generating shock wave $x = X(t)$ $(t \in (t_0, \infty))$ in the weak entropy solution of (2.3) is in the second quadrant and its speed $s(t) < 0$. Therefore the weak entropy solution of (1.7) is the following central rarefaction wave, which was absorbed completely or partially by the boundary $x = 0$,

$$u(x,t) = u_r(x,t)|_{(0,\infty)\times(0,\infty)}, \tag{2.13}$$

where

$$u_r(x,t) = \begin{cases} u_m, & x < a + f'(u_m)t \\ (f')^{-1}(\dfrac{x-a}{t}), & a + f'(u_m)t < x < a + f'(u_+)t \\ u_+, & x > a + f'(u_+)t. \end{cases} \tag{2.14}$$

If $u_m < u_+ < 0 < u_-$, $f(u_+) < f(u_-)$, or $u_m < 0 \le u_+, u_-$, $u_- \ne 0$, there exists $u_* \in (u_m, 0)$ such that $f(u_*) = f(u_-)$; furthermore, there is $t_* > t_0$ such that $s(t_*) = s(u_-, u_*) = 0$, where $s(t)$ is the speed function of the shock wave $x = X(t)$ in the weak entropy solution of (2.3). Moreover, after $t = t_*$, $s(t) > 0$ and the shock crosses the $t$-axis at some finite time, it then enters the first quadrant. If we take the restriction of the solution of (2.3) in the first quadrant as $u(x,t)$, then $u(x,t)$ does not satisfy the boundary entropy condition (2.2) at $t > t_1!$, where $t_1 = -a/f'(u_*) < t_*$ is the time at which the characteristic $x = a + f'(u_*)t$ intersects the $t$-axis (see Figures 2.1–2.2). Thus, by virtue of Lemma 2.2, it is not the weak entropy solution of (1.7). So, in the process of constructing a weak entropy solution of (1.7), we must take into account the boundary entropy condition (2.2) in our minds. Now we reconstruct the solution of (1.7). Using the local solution $v(x,t)$ of (2.3) on $(-\infty,\infty) \times [0, t_1)$, by taking

$$v(x, t_1) := \begin{cases} u_-, & x < 0 \\ v(x, t_1 - 0), & x > 0 \end{cases}$$

as the new initial value of $(2.3)_1$, we can extend $v(x,t)$ to $(-\infty,\infty) \times [0,\infty)$. Let $u(x,t) = v(x,t)|_{(0,\infty)\times(0,\infty)}$. Then, from Lemma 2.2, this $u(x,t)$ is the weak entropy solution of (1.7) (see Figures 2.1–2.2), which can be expressed as follows:

$$u(x,t) = u_r(x,t)|_{(0,\infty)\times(0,\infty)} \quad \text{for } 0 < t < t_1, \tag{2.15$_1$}$$

and for $t > t_1$

$$u(x,t) = \begin{cases} u_-, & 0 < x < X^+(t) \\ u_r(x,t), & x > X^+(t), \end{cases} \tag{2.15$_2$}$$

where $u_r(x,t)$ is defined by (2.14), and $x = X^+(t)$ is a new shock wave, starting at point $(0, t_1)$, which is located in the first quadrant and determined by the following problem:

$$\begin{cases} \dfrac{dX^+}{dt} = s(u_-, u), \quad f'(u) = \dfrac{X^+ - a}{t}, \quad f'(u_*) \le \dfrac{X^+ - a}{t} \le f'(u_+) \\ X^+(t_1) = 0. \end{cases} \tag{2.16}$$

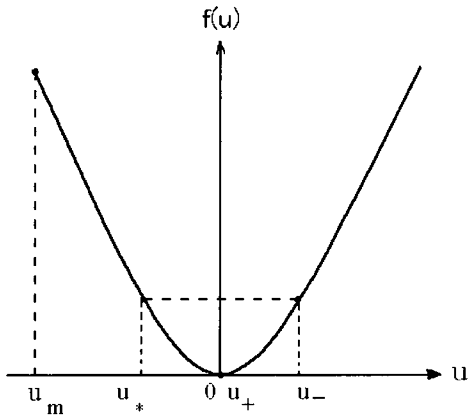

Figure 2.1

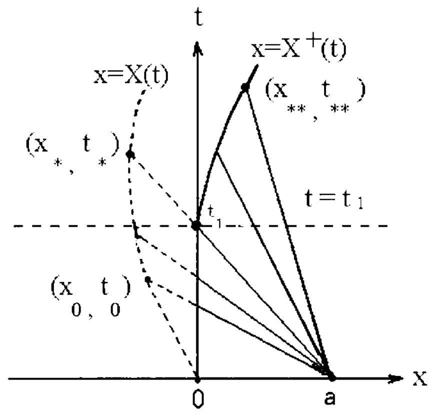

Figure 2.2

In what follows, we give the statement of the interaction in the weak entropy solution of (1.7). One part of the rarefaction wave $R$ collides with the boundary $x = 0$, and then the boundary $x = 0$ reflects a new shock wave $x = X^+(t)$ at time $t = t_1$, which will penetrate another part $R^+$ of the rarefaction wave $R$ with a varying speed of propagation determined by (2.16), where $R^+$ is just the restriction of $R$ at $\frac{x-a}{t} \ge f'(u_*)$. Similar to the analysis on the previous shock wave $x = X(t)$, making use of (2.16), we can derive the following properties for the new shock wave $x = X^+(t)$. If $u_- = u_+$, the shock wave $x = X^+(t)$ is able to cross the whole of $R^+$ completely only when $t \to \infty$. If $u_- > u_+$, the shock wave $x = X^+(t)$ will cross the whole of $R^+$ at finite time. If $u_- < u_+$, it is impossible for the shock wave $x = X^+(t)$ to cross the whole of $R^+$ completely. Moreover, the new shock speed function, still denoted by $s(t)$, with $s(t_1) = 0$ and $s(t) > 0$ for $t > t_1$, is continuous and not decreasing for $t \ge t_1$.

2.5.2 $f(u_m) = f(u_-)$.

In this sub-case, the part (from $t = 0$ to $t_1 = -a/f'(u_m)$) of the shock wave $x = X(t)$ appearing in the weak entropy solution $v(x,t)$ of (2.3) lies on the $t$-axis, and another part (from $t = t_1$ to $\infty$) is in the first quadrant. Thus, by Lemma 2.2, $u(x,t) = v(x,t)|_{(0,\infty)\times(0,\infty)}$ is the weak entropy solution $u(x,t)$ of (1.7), which can be written as

$$u(x,t) = u_r(x,t)|_{x\in(0,\infty)} \quad \text{for} \quad 0 < t < t_1$$

and for $t > t_1$,

$$u(x,t) = \begin{cases} u_-, & 0 < x < X^+(t) \\ u_r(x,t), & x > X^+(t), \end{cases}$$

where $u_r(x,t)$ is defined by (2.14) and $x = X^+(t)$ is the restriction of $x = X(t)$ at $t \geq t_1$.

We now state the structure of the weak entropy solution of (1.7). The rarefaction wave $R$ collides with the boundary $x = 0$ at time $t_1$. The interaction of $R$ and the boundary $x = 0$ generates a shock $x = X^+(t)$, which will penetrate $R$ after $t = t_1$. This shock $x = X^+(t)$ is just the restriction of $x = X(t)$ at $t \geq t_1$. Similar to sub-section 2.5.1, the speed $s(t)$ of the shock wave $x = X^+(t)$ satisfies $s(t_1) = 0$ and $s(t) > 0$ for $t \in (t_1, \infty)$.

2.5.3 $f(u_m) < f(u_-)$.

In this sub-case, for the initial problem (2.3), the shock wave $x = X(t)$ interacts with the central rarefaction wave in the first quadrant, and the shock wave $x = X(t)$ ($t \in (0,\infty)$) is located in the first quadrant. By (2.12), the shock speed $s(t) \geq s(u_-, u_m) > 0$.

Therefore, from Lemma 2.2, $u(x,t) = v(x,t)|_{(0,\infty)\times(0,\infty)}$ is the weak entropy solution of (1.7), where $v(x,t)$ is the weak entropy solution of (2.3). The weak entropy solution $u(x,t)$ of (1.7) does not include the interaction of elementary waves and the boundary, which can be written as

$$u(x,t) = \begin{cases} u_-, & 0 < x < X(t) \\ u_r(x,t), & x > X(t), \end{cases}$$

where $X(t), u_r(x,t)$ is determined by (2.12), (2.14), respectively.

From the discussion in Cases (I)–(V), we can obtain the following lemma, which is necessary for the error analysis.

LEMMA 2.3. Assume that (1.8) holds. Then the weak entropy solution $u(x,t)$ of (1.7) satisfies for $0 < \varepsilon \leq t < T < \infty$,

$$|u_x(0+,t)|, \quad |u_x(z \pm 0, t)| \leq \frac{C}{t}, \quad z > 0$$

and

$$\int_\varepsilon^t \|u_{xx}(\cdot,\tau)\|_{L^1[0,\infty)} d\tau \leq C|\ln\varepsilon| + C,$$

where $C$ is a constant independent of $\varepsilon$.

According to the arguments in sub-sections 2.1–2.5, we can divide the interaction of elementary waves and the boundary $x = 0$ in the weak entropy solution of (1.7) into the following three cases, which will be respectively investigated for the error bounds for the viscosity methods in Sec. 4.

(1) The central rarefaction wave interacts with the boundary and the boundary reflects a shock wave if $u_m, u_\pm$ satisfy one of the following conditions:

($A_1$) $u_m < u_+ < 0 < u_-$, and $f(u_+) < f(u_-) < f(u_m)$;

($A_2$) $u_m < 0 \leq u_+, u_-, u_- \neq 0$, and $f(u_-) < f(u_m)$;

($A_3$) $u_m < u_-, u_+$, and $f(u_m) = f(u_-)$.

See sub-sections 2.5.1 and 2.5.2.

(2) There is no central rarefaction wave included in the weak entropy solution if $u_m, u_\pm$ satisfy one of the following conditions:

$(A_4)$ $u_- > u_m \geq u_+$;

$(A_5)$ $u_-, u_+ < u_m \leq 0$.

See sub-sections 2.1, 2.3, and 2.4.1.

(3) The central rarefaction wave does not interact with the boundary or the rarefaction wave is absorbed completely or partially by the boundary if conditions $(A_1)$–$(A_5)$ do not hold (see sub-sections 2.1, 2.2, 2.4.2, 2.4.3, 2.5.1, 2.5.3).

**3. Basic Lemmas.** Throughout this paper, the norm $\|\cdot\|$ denotes the standard $L^1$-norm, $\|\cdot\|_{L^1[0,\infty)}$; $C$ or $C(t)$ denotes a positive constant independent of $\varepsilon$, and $c$ denotes a positive constant independent of $t$ and $\varepsilon$, but with different values at different places.

As in [2,10], the error estimates are based on a stability lemma for nonhomogeneous viscous equations with initial-boundary conditions and a traveling wave solution lemma. We first establish an $L^1$-stability lemma.

LEMMA 3.1. Let $v^{(i)}(x,t)$ $(i=1,2)$ be continuous and piecewise smooth solutions of the following equations

$$(v^{(i)})_t + f(v^{(i)})_x - \varepsilon(v^{(i)})_{xx} = g_i(x,t), \quad x>0,\ t>d\geq 0,\ i=1,2. \tag{3.1}$$

The above equations hold for all values of $x>0$ except on some curves $X_m(t)$, $1\leq m\leq M$, where $v_x^{(i)}$ may not exist. If $\omega := v^{(1)} - v^{(2)} \to 0$ as $x\to\infty$, then

$$\begin{aligned}\|\omega(\cdot,t)\| \;&\leq \|\omega(\cdot,d)\| + \int_d^t \|g_1(\cdot,\tau)-g_2(\cdot,\tau)\|d\tau\\ &+\varepsilon\int_d^t |\omega_x(0+,\tau)|d\tau + \int_d^t \mathrm{sgn}\omega(0,\tau)(f(v^{(1)}(0,\tau)) - f(v^{(2)}(0,\tau)))d\tau\\ &+\varepsilon\sum_{m=1}^{M}\int_d^t |\omega_x(X_m(\tau)+0,\tau) - \omega_x(X_m(\tau)-0,\tau)|d\tau.\end{aligned} \tag{3.2}$$

*Proof* (see also [15]). We now prove this Lemma by a similar technique as in [10]. It follows from (3.1) that

$$\omega_t + (f(v^{(1)}) - f(v^{(2)}))_x = \varepsilon\omega_{xx} + g_1(x,t) - g_2(x,t). \tag{3.3}$$

If $\omega\geq 0$ or $\omega\leq 0$ for all $x$, then straightforward integration on the above equation gives (3.2). Let $(0<)p_1(t)<p_2(t)<\cdots$ be the points such that, at those points, $\omega$ changes signs. Let $\alpha_j$ be the sign of $\omega$ in $(p_j,p_{j+1})(j=0,1,2,3,\cdots,p_0=0)$. Multiplying (3.3) by $\alpha_j(j=1,2,\cdots)$ and integrating the resulting equation over $(p_j,p_{j+1})$ yield

$$\begin{aligned}\alpha_j\int_{p_j}^{p_{j+1}}\omega_t dx \;&= \varepsilon(\alpha_j\omega_x(p_{j+1}-0,t) - \alpha_j\omega_x(p_j+0,t)) + \alpha_j\int_{p_j}^{p_{j+1}}(g_1(x,t)-g_2(x,t))dx\\ &+\varepsilon\sum_{p_j<X_m<p_{j+1}}\alpha_j(\omega_x(X_m(t)+0,t) - \omega_x(X_m(t)-0,t)).\end{aligned} \tag{3.4}$$

Since $\omega(p_j,t)=\omega(p_{j+1},t)=0$ and $\alpha_j\omega\geq 0$ for $x\in(p_j,p_{j+1})$, we have

$$\frac{d}{dt}\int_{p_j}^{p_{j+1}}|\omega|dx = \alpha_j\int_{p_j}^{p_{j+1}}\omega_t dx.$$

Moreover, observing that $\alpha_j\omega_x(p_{j+1}-0,t) \le 0$ and $\alpha_j\omega_x(p_j+0,t) \ge 0$ $(j = 1,2,...)$ (because $\alpha_j\omega_x(p_{j+1}-0,t) = \alpha_j \lim_{x\to p_{j+1}^-} \frac{\omega(x,t)-\omega(p_{j+1},t)}{x-p_{j+1}} = \lim_{x\to p_{j+1}^-} \frac{\alpha_j\omega(x,t)}{x-p_{j+1}} \le 0$), we obtain from (3.4) that

$$\begin{aligned}\frac{d}{dt}\int_{p_j}^{p_{j+1}} |\omega|dx \quad &\le \sum_{p_j<X_m<p_{j+1}} \varepsilon\alpha_j|\omega_x(X_m(t)+0,t)-\omega_x(X_m(t)-0,t)| \\ &+\int_{p_j}^{p_{j+1}} \big|g_1(x,t)-g_2(x,t)\big|dx.\end{aligned}$$

Since the above inequality is true for all $j \ge 1$, we have

$$\frac{d}{dt}\int_{p_1}^{p^*} |\omega|dx \le \sum_{p_1<X_m<p^*} \varepsilon|\omega_x(X_m(t)+0,t)-\omega_x(X_m(t)-0,t)|+\int_{p_1}^{p^*} |g_1(x,t)-g_2(x,t)|dx, \tag{3.5}$$

where $p^* = \sup_j p_j$. If $p^* < \infty$, using a similar method as above yields

$$\frac{d}{dt}\int_{p^*}^{\infty} |\omega|dx \le \varepsilon \sum_{X_m>p^*} |\omega_x(X_m(t)+0,t)-\omega_x(X_m(t)-0,t)|+\int_{p^*}^{\infty} |g_1(x,t)-g_2(x,t)|dx. \tag{3.6}$$

In obtaining the last inequality, we have used the fact that $\omega \to 0$ as $x \to \infty$. In order to get (3.2), we need to estimate $\frac{d}{dt}\int_0^{p_1} |\omega|dx$. Multiplying (3.3) by $\alpha_0$ and integrating the resulting equation over $(0,p_1)$ yield

$$\begin{aligned}&\frac{d}{dt}\int_0^{p_1} |\omega|dx = \alpha_0(f(v^{(1)}(0,t)) - f(v^{(2)}(0,t))) + \alpha_0\int_0^{p_1} (g_1(x,t)-g_2(x,t))dx \\ &+\varepsilon(\alpha_0\omega_x(p_1-0,t) - \alpha_0\omega_x(0+,t)) + \sum_{0<X_m<p_1} \alpha_0\varepsilon(\omega_x(X_m(t)+0,t)-\omega_x(X_m(t)-0,t)),\end{aligned}$$

then

$$\begin{aligned}\frac{d}{dt}\int_0^{p_1} |\omega|dx \quad &\le \varepsilon|\omega_x(0+,t)| + \sum_{0<X_m<p_1} \varepsilon|\omega_x(X_m(t)+0,t)-\omega_x(X_m(t)-0,t)| \\ &+\int_0^{p_1} |g_1(x,t)-g_2(x,t)|dx + \mathrm{sgn}\omega(0,t)(f(v^{(1)}(0,t)) - f(v^{(2)}(0,t))).\end{aligned} \tag{3.7}$$

Using (3.5)–(3.7), we obtain the desired result.

Assume that $x{=}X(t)$ is a smooth curve satisfying the R-H jump condition (2.6) and Lax's shock condition (2.7), where $dX/dt$ is the shock speed. Let $u^- := u(X(t)-0,t)$ and $u^+ := u(X(t)+0,t)$. If $u^\pm$ are independent of $t$, it is known that there is a traveling wave solution to scalar viscous conservation laws (1.4), in the form $v_\varepsilon(x,t) = V(x-X(t))$, where the shock speed $dX/dt$ is a constant, satisfying $V(-\infty) = u^-, V(\infty) = u^+$. When $u^\pm$ are functions of $t$, we introduce the following generalization.

LEMMA 3.2 ([10]). Let $u^-(t) > u^+(t)$ be two given functions, $X(t)$ be defined by (2.6). Then there is a unique traveling wave $V(x-X(t);u^-,u^+)$ of (1.4) taking on $V(-\infty;u^-,u^+) = u^-, V(0;u^-,u^+) = (u^-+u^+)/2, V(\infty;u^-,u^+) = u^+$, which has the following properties:

(1) $V(\xi\,; u^-, u^+)$ *is a decreasing function with respect to* $\xi$.
(2) $|V(\xi\,; u^-, u^+) - H(\xi; u^-, u^+)| \leq (u^- - u^+)\exp\{-\alpha(u^- - u^+)|\xi|/2\varepsilon\}$.
(3) $\|H(\cdot\,; u^-, u^+) - V(\cdot\,; u^-, u^+)\|_{L^1(-\infty,\infty)} \leq c\varepsilon$.
(4) $\|V_{u^-}(\cdot\,; u^-, u^+)\dot{u}^- + V_{u^+}(\cdot\,; u^-, u^+)\dot{u}^+ - H(\cdot\,; \dot{u}^-, \dot{u}^+)\|$
$\leq c\varepsilon(|u_x(X(t)+0,t)| + |u_x(X(t)-0,t)|)$,

where $\dot{u}^{\pm} = d(u^{\pm})/dt$, $H$ is the so-called Heaviside function defined by

$$H(x; u^-, u^+) = \begin{cases} u^-, & x < 0 \\ u^+, & x > 0, \end{cases}$$

and $\alpha$ is a constant defined by

$$\alpha = \inf_{\min\{u_m, u_\pm\} \leq u \leq \max\{u_m, u_\pm\}} f''(u). \tag{3.8}$$

**4. Main results.** In this section, we apply the solution structures, $L^1$-stability lemma, and traveling wave lemma obtained in previous sections to prove the main conclusions of this paper.

In this section, we always suppose that $\|v_0(\cdot) - u_0(\cdot)\| < \infty$.

THEOREM 4.1. Assume that (1.8) and one of the conditions $(A_1)-(A_3)$ hold. If $v_\varepsilon$ is the smooth solution of (1.9) and $u$ is the weak entropy solution of (1.7), then the following error estimate holds for any $T \geq 0$:

$$\sup_{0\leq t\leq T} \|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \leq \|v_0(\cdot) - u_0(\cdot)\| + C(T)(\varepsilon^{1/2} + \varepsilon|\ln\varepsilon| + \varepsilon). \tag{4.1}$$

THEOREM 4.2. Suppose that (1.8) is valid and that $u_m, u_\pm$ are three constants which do not satisfy the conditions $(A_1)-(A_3)$. Let $v_\varepsilon$ be the smooth solution of (1.9) and $u$ be the weak entropy solution of (1.7). Then the following error estimate holds for any $T \geq 0$:

$$\sup_{0\leq t\leq T} \|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \leq \|v_0(\cdot) - u_0(\cdot)\| + C(T)(\varepsilon|\ln\varepsilon| + \varepsilon). \tag{4.2}$$

In particular, if there is no rarefaction wave included in the solution of (1.7), namely, $u_m$ and $u_\pm$ satisfy $(A_4)$ or $(A_5)$, then the following error bound is valid for any $T \geq 0$:

$$\sup_{0\leq t\leq T} \|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \leq \|v_0(\cdot) - u_0(\cdot)\| + C(T)\varepsilon. \tag{4.3}$$

It is well known that under the condition that $v_0 \in C^2([0,\infty))$ satisfies the compatibility conditions $v_0(0) = u_-$ and $v_0'(0) = v_0''(0) = 0$ and $\|v_0(\cdot)\|_{C([0,\infty))}$, $\|v_0(\cdot)\|_{w^{2,1}([0,\infty))}$ are bounded for $\varepsilon$, there exists a unique smooth solution $v_\varepsilon$ to the parabolic equation (1.9) with initial-boundary conditions such that $\|v_\varepsilon\|_{C([0,\infty)\times[0,T])}$ and $\|v_{\varepsilon x}(0,\cdot)\|_{L^1([0,T])}$ ($\forall T > 0$) are bounded for $\varepsilon$ (see, for instance, [18], or [19]).

We will use Lemma 3.1 and Lemma 3.2 to prove Theorem 4.1 and Theorem 4.2. Applying Lemma 3.1 requires the condition that the solutions of (3.1) should be continuous. However, as discussed in Sec. 2, there is rarefaction wave or shock wave discontinuity for the weak entropy solution of (1.7). Across the rarefaction wave region, the solution is continuous, but on the shock curves, the solution is discontinuous. In order to get the desired results, following [2,10], if the weak entropy solution $u(x,t)$ of (1.7) does not contain shock discontinuity for $t \in (\tau_1, \tau_2)$ $(0 < \tau_1 < \tau_2 < T < \infty)$, then in the interval $(\tau_1, \tau_2)$, we can directly apply Lemma 3.1 to $v_\varepsilon(x,t)$ and $u(x,t)$; otherwise, we construct a reasonable approximation $\overline{v}_\varepsilon(x,t)$ to $u(x,t)$ in $(\tau_1,\tau_2)$ to get rid of shock discontinuities and use Lemma 3.1 to $\overline{v}_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$.

*Proof of Theorem* 4.1. By the arguments in sub-sections 2.5.1 and 2.5.2, when $t \in (0, t_1)$, the weak entropy solution of (1.7) contains no shock discontinuity; when $t \in (t_1, \infty)$, the solution of (1.7) contains only one shock $x = X^+(t) (> 0)$, starting at point $(0, t_1)$ (see Fig. 2.2). We replace $v^{(1)}(x,t), v^{(2)}(x,t)$ in Lemma 3.1 by $v_\varepsilon(x,t), u(x,t)$, respectively, then we get for $0 < \varepsilon < t \le t_1$,

$$\begin{aligned}\|v_\varepsilon(\cdot,t) - u(\cdot,t)\| &\le \|v_\varepsilon(\cdot,\varepsilon) - u(\cdot,\varepsilon)\| + \varepsilon\int_\varepsilon^t (|v_{\varepsilon x}(0,\tau)| + |u_x(0,\tau)|)d\tau \\ &\quad + \varepsilon\int_\varepsilon^t \|u_{xx}(\cdot,\tau)\|d\tau + \int_\varepsilon^t \operatorname{sgn}(u_- - u(0,\tau))(f(u_-) - f(u(0,\tau)))d\tau \\ &\quad + \varepsilon\int_\varepsilon^t (|u_x(X_l(\tau)+0,\tau)| + |u_x(X_r(\tau)-0,\tau)|)d\tau \\ &\le \|v_\varepsilon(\cdot,\varepsilon) - u(\cdot,\varepsilon)\| + C(T)(\varepsilon|\ln\varepsilon| + \varepsilon),\end{aligned} \tag{4.4}$$

where $X_l(t) := a + f'(u_m)t$, $X_r(t) := a + f'(u_+)t$. In obtaining (4.4), we have used $(2.15)_1$, (2.2), and Lemma 2.3. Since $v_\varepsilon$ and $u$ satisfy the following stability results

$$\|v_\varepsilon(\cdot,\tau) - v_\varepsilon(\cdot,\tau_0)\| \le C|\tau - \tau_0|, \qquad \|u(\cdot,\tau) - u(\cdot,\tau_0)\| \le C|\tau - \tau_0|, \tag{4.5}$$

from (4.4), we obtain for $0 < t \le t_1$

$$\|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \le \|v_0(\cdot) - u_0(\cdot)\| + C(\varepsilon + \varepsilon|\ln\varepsilon|). \tag{4.6}$$

When $t_1 < t < \infty$, we take the approximation $\overline{v}_\varepsilon(x,t)$ to $u(x,t)$ as

$$\overline{v}_\varepsilon(x,t) = u(x,t) + V(x - X^+(t); u_-, u^+) - H(x - X^+(t); u_-, u^+), \tag{4.7}$$

where $u^+ = u^+(t) := u(X^+(t)+0, t)$ and $u(x,t)$ is determined by $(2.15)_2$. We easily verify that $\overline{v}_\varepsilon(x,t)$ is continuous and piecewise smooth in $(0,\infty)\times(t_1,\infty)$, which satisfies that

$$\overline{v}_\varepsilon(x,t) - v_\varepsilon(x,t) \to 0, \quad \text{as} \quad x \to \infty$$

and $(\overline{v}_\varepsilon)_x, u_x$ are discontinuous on the same curves. By direct computation, $\overline{v}_\varepsilon$ satisfies the equation

$$(\overline{v}_\varepsilon)_t + f(\overline{v}_\varepsilon)_x - \varepsilon(\overline{v}_\varepsilon)_{xx} = \overline{g}(x,t)$$

in its smooth regions, where $\overline{g}(x,t) = I_1 + I_2 - \varepsilon u_{xx}, I_1 = -f(u)_x - f(V)_x + f(\overline{v}_\varepsilon)_x, I_2 = V_{u^+}\dot{u}^+ - H(x - X^+; 0, \dot{u}^+)$. By using the same technique as in [10], from Lemma 2.3 and Lemma 3.2, one has

$$\|\overline{g}(\cdot,t)\| \le \frac{C\varepsilon}{t},$$

from which it follows for $t_1 < t \le T < \infty$,

$$\int_{t_1}^t \|\overline{g}(\cdot,\tau)\|d\tau \le C(T)\varepsilon. \tag{4.8}$$

Using Lemma 3.2 yields

$$\|\overline{v}_\varepsilon(\cdot,t)-u(\cdot,t)\|\le c\varepsilon. \tag{4.9}$$

Therefore, Lemma 3.1 can be applied to $\overline{v}_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$, and by (4.7)–(4.9), Lemma 2.3, and Lemma 3.2, it can be concluded that for $t_1<t\le T<\infty$,

$$\begin{aligned}
&\|v_\varepsilon(\cdot,t)-\overline{v}_\varepsilon(\cdot,t)\|\le \|v_\varepsilon(\cdot,t_1)-\overline{v}_\varepsilon(\cdot,t_1)\|+\int_{t_1}^t\|\overline{g}(\cdot,\tau)\|d\tau\\
&+\varepsilon\int_{t_1}^t(|v_{\varepsilon x}(0,\tau)|+|\overline{v}_{\varepsilon x}(0,\tau)|)d\tau+\int_{t_1}^t \operatorname{sgn}(u_--\overline{v}_\varepsilon(0,\tau))(f(u_-)-f(\overline{v}_\varepsilon(0,\tau)))d\tau\\
&+\varepsilon\int_{t_1}^t(|u_x(X^+(\tau)\pm 0,\tau)|+|u_x(X_r(\tau)\pm 0,\tau)|)d\tau\\
&\le \|v_\varepsilon(\cdot,t_1)-u(\cdot,t_1)\|+C(T)\varepsilon+c\int_{t_1}^t|V(-X^+(\tau);u_-,u^+)-u_-|d\tau\\
&\le \|v_\varepsilon(\cdot,t_1)-u(\cdot,t_1)\|+C(T)\varepsilon+c\int_{t_1}^t(u_--u^+)\exp\{-\alpha(u_--u^+)X^+(\tau)/2\varepsilon\}.
\end{aligned} \tag{4.10}$$

Let $I_0:=\int_{t_1}^t(u_--u^+)\exp\{-\alpha(u_--u^+)X^+(\tau)/2\varepsilon\}d\tau$. We now estimate the boundary integral $I_0$. The key point is to estimate $X^+(t)$ for $t>t_1$. Since $X^+(t)(t>t_1)$ is smooth and increasing, there exists $t_2>t_1$ such that $X^+(t_2)=\varepsilon_0$, where $\varepsilon_0$ is a given constant which satisfies that $0<\varepsilon_0<a$ for the case when $(A_2)$ or $(A_3)$ holds and $0<\varepsilon_0\le a+f'(u_+)t_3$ for the case when $(A_1)$ is valid, here

$$t_3=t_1\exp\Big(\int_{u_*}^{u_+}\frac{f''(u)du}{s(u_-,u)-f'(u)}\Big).$$

Furthermore, there exists $u_{**}\in(u_*,0)$ such that $u^+(t_2)=u_{**}$ and $f'(u_{**})=(\varepsilon_0-a)/t_2<0<f'(u_+)$ (if $(A_3)$ holds, then $u_*=u_m$). Thus when $t\in[t_1,t_2]$, $u^+\in[u_*,u_{**}]$ and $0\le s(t)\le s(t_2)$, where $u^+(t_1)=u_*$, $s(t):=\dot{X}^+(t)=s(u_-,u^+(t))$. Consequently, by direct computation and Taylor's formula, we have for $t\in[t_1,t_2]$

$$\begin{aligned}
\ddot{X}^+(t)&=\frac{f(u_-)-f(u^+)-f'(u^+)(u_--u^+)}{(u_--u^+)^2}\cdot\frac{du^+}{dt}\\
&=\frac{f''(\eta)}{2}\cdot\frac{s(t)t+a-X^+(t)}{f''(u^+)t^2}\ge\frac{(a-\varepsilon_0)\alpha}{2\beta t_2^2}>0,
\end{aligned}$$

where $u^+<\eta<u_-$, $\alpha$ is defined by (3.8), $\beta=\sup_{\min\{u_m,u_\pm\}\le u\le\max\{u_m,u_\pm\}}f''(u)$. Then, by applying Taylor's formula again, we conclude that

$$\begin{aligned}
X^+(t)&=X^+(t_1)+\dot{X}^+(t_1)(t-t_1)+\ddot{X}^+(l)(t-t_1)^2/2\quad(l\in(t_1,t))\\
&\ge\frac{(a-\varepsilon_0)\alpha}{4\beta t_2^2}(t-t_1)^2\quad(t\in[t_1,t_2]),
\end{aligned}$$

from which we get for $t\in[t_1,t_2]$,

$$I_0\le\int_{t_1}^t(u_--u_m)\exp\{-\frac{(a-\varepsilon_0)\alpha^2u_-}{8\beta t_2^2}(\tau-t_1)^2/\varepsilon\}d\tau\le c\varepsilon^{1/2}. \tag{4.11}$$

When $t\in[t_2,\infty)$, since $s(t)$ is not decreasing,

$$X^+(t)\ge X^+(t_2)+s(t_2)(t-t_2)=\varepsilon_0+s(t_2)(t-t_2).$$

Then, making use of the above inequality, one can derive that for $t \in [t_2, T]$ $(t_2 < T < \infty)$,

$$\begin{aligned}&\int_{t_2}^{t}(u_- - u^+)\exp\{-\alpha(u_- - u^+)X^+(\tau)/2\varepsilon\}d\tau\\ &\le \int_{t_2}^{t}(u_- - u_m)\exp\{-\alpha(u_- - u^+(T))(\varepsilon_0 + s(t_2)(\tau - t_2))/2\varepsilon\}d\tau\\ &\le C(T)\varepsilon. \qquad (4.12)\end{aligned}$$

Combining (4.9)–(4.12) and (4.6) gives (4.1).

*Proof of Theorem* 4.2. We prove this theorem only for the following cases: (1) $u_+ < u_m < u_-$, $f(u_m) < f(u_-)$; (2) $u_m < u_- \le 0 < u_+$; (3) $u_m < u_-, u_+$, $f(u_m) < f(u_-)$, or $u_- < 0 \le u_+ < u_m$; (4) $u_-, u_+ < 0 < u_m$.

For the other cases, we have an analogous discussion.

(1) When $u_+ < u_m < u_-$ and $f(u_m) < f(u_-)$, there is no rarefaction wave contained in the weak entropy solution $u(x,t)$ of (1.7) and the structure of this solution was obtained in sub-section 2.3.

We take the approximation to $u(x,t)$ for $t \in [0, t_1]$ as

$$\begin{aligned}\overline{v}_\varepsilon(x,t) = \;& u(x,t) + V(x - s(u_-, u_m)t; u_-, u_m) - H(x - s(u_-, u_m)t; u_-, u_m)\\ &+V(x - s(u_m, u_+)t - a; u_m, u_+) - H(x - s(u_m, u_+)t - a; u_m, u_+).\end{aligned} \qquad (4.13)$$

Then, $\overline{v}_\varepsilon(x,t)$ is a smooth function on $(0,\infty) \times [0, t_1]$. Similar to [10], by Lemma 3.2, we can verify that $\overline{v}_\varepsilon(x,t)$ satisfies

$$(\overline{v}_\varepsilon)_t + f(\overline{v}_\varepsilon)_x - \varepsilon(\overline{v}_\varepsilon)_{xx} = \overline{h}(x,t), \quad 0 \le t \le t_1,$$

where $\overline{h}(x,t)$ satisfies

$$\int_0^t \|\overline{h}(\cdot,\tau)\| d\tau \le c\varepsilon.$$

Therefore, using Lemma 3.1 yields

$$\begin{aligned}\|v_\varepsilon(\cdot,t) - \overline{v}_\varepsilon(\cdot,t)\| \;&\le \|v_0(\cdot) - \overline{v}_\varepsilon(\cdot,0)\| + c\varepsilon + c\int_0^t |u_- - \overline{v}_\varepsilon(0,\tau)|d\tau\\ &\le \|v_0(\cdot) - u_0(\cdot)\| + \|V(\cdot; u_-, u_m) - H(\cdot; u_-, u_m)\|_{L^1(-\infty,\infty)}\\ &+\|V(\cdot; u_m, u_+) - H(\cdot; u_m, u_+)\|_{L^1(-\infty,\infty)} + c\varepsilon\\ &+c\int_0^t (|V(-s(u_-, u_m)\tau; u_-, u_m) - H(-s(u_-, u_m)\tau; u_-, u_m)|\\ &+|V(-s(u_m, u_+)\tau - a; u_m, u_+) - H(-s(u_m,u_+)\tau - a; u_m,u_+)|)d\tau.\end{aligned}$$

Applying Lemma 3.2 to (4.13) and the above inequality yields for $t \in [0, t_1]$,

$$\|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \le \|v_\varepsilon(\cdot,t) - \overline{v}_\varepsilon(\cdot,t)\| + \|\overline{v}_\varepsilon(\cdot,t) - u(\cdot,t)\| \le \|v_0(\cdot) - u_0(\cdot)\| + C\varepsilon. \quad (4.14)$$

If $f(u_-) \ge f(u_+)$, the approximation to $u(x,t)$ on $[t_1,\infty)$ is taken as $\overline{v}_\varepsilon(x,t) = V(x - s(u_-, u_+)t - a_1; u_-, u_+)$; if $f(u_-) < f(u_+)$, the approximation to $u(x,t)$ on $[t_1, \infty)$ is taken as follows:

$$\overline{v}_\varepsilon(x,t) = \begin{cases} V(x - s(u_-, u_+)t - a_1; u_-, u_+), & t \in (t_1, t_2]\\ u_+, \quad t \in (t_2, \infty).\end{cases}$$

Repeating the above process, we can prove that (4.14) holds for $t \in [t_1, T] (t_1 < T < \infty)$. Thus (4.3) is valid.

(2) When $u_m < u_- \le 0 < u_+$, by the argument in sub-section 2.5.1, there is no shock appearing in the weak entropy solution of (1.7). We can directly apply Lemma 3.1 to $v_\varepsilon(x,t)$ and $u(x,t)$ to get (4.2).

(3) When $u_m < u_-, u_+$, $f(u_m) < f(u_-)$ or $u_- < 0 \le u_+ < u_m$, the weak entropy solution of (1.7) includes one shock $x = X(t), t \in [0, \infty)$ and we have for $0 < t \le T < \infty$, respectively,

$$X(t) \ge s(u_-, u_m)t \quad \text{or} \quad X(t) > \frac{X(T)}{T}t. \tag{4.15}$$

See sub-section 2.5.3 or 2.4.2. The approximation to $u(x,t)$ is taken as, respectively,

$$\overline{v}_\varepsilon(x,t) = u(x,t) + V(x - X(t); u_-, u^+) - H(x - X(t); u_-, u^+),$$

where $u^+ = u^+(t) := u(X(t)+0, t)$, or

$$\overline{v}_\varepsilon(x,t) = u(x,t) + V(x - X(t); u^-, u_+) - H(x - X(t); u^-, u_+),$$

where $u^- = u^-(t) := u(X(t)-0, t)$.

In what follows, we estimate the boundary integral

$$\begin{aligned} \overline{I}_0 \quad &:= \int_\varepsilon^t \operatorname{sgn}(v_\varepsilon(0,\tau) - \overline{v}_\varepsilon(0,\tau))(f(v_\varepsilon(0,\tau)) - f(\overline{v}_\varepsilon(0,\tau)))d\tau \\ &= \int_\varepsilon^t \operatorname{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-)(f(\overline{v}_\varepsilon(0,\tau)) - f(u_-))d\tau. \end{aligned}$$

For the case of $u_m < u_-, u_+$, $f(u_m) < f(u_-)$, using (4.15) and Lemma 3.2, one gets for $\varepsilon < t \le T < \infty$,

$$\begin{aligned} \overline{I}_0 \quad &\le c\int_\varepsilon^t |\overline{v}_\varepsilon(0,\tau) - u_-|d\tau \\ &\le c\int_\varepsilon^t (u_- - u^+)\exp\{-\alpha(u_- - u^+)X(\tau)/2\varepsilon\}d\tau \\ &\le c\int_\varepsilon^t (u_- - u_m)\exp\{-\alpha(u_- - u^+(T))X(\tau)/2\varepsilon\}d\tau \le C(T)\varepsilon. \end{aligned} \tag{4.16}$$

We now estimate $\overline{I}_0$ for the case of $u_- < 0 \le u_+ < u_m$. By Lemma 3.2 and (4.15), one gets

$$\begin{aligned} 0 \quad &< H(-X(t); u^-(t), u_+) - V(-X(t); u^-(t), u_+) \\ &\le c(u_m - u_+)\exp\{-\alpha(u_m - u^-(T))X(T)t/2T\varepsilon\} := e(t), \quad t > 0. \end{aligned} \tag{4.17}$$

Since the function $e(t)$ defined by (4.17) is strictly decreasing for all $t > 0$, there exists at most one point such that, at this point, the function $-u_- - e(t)$ changes sign; that is to say, the function $-u_- - e(t)$ satisfies one of the following for $t \in (0, T]$: (i) $-u_- - e(t) > 0$; (ii) $-u_- - e(t) \le 0$; (iii) There is a unique point $\tau_\varepsilon \in (0, T]$ such that, at this point, $-u_- - e(t)$ changes sign. In case (i),

$$\overline{v}_\varepsilon(0,\tau) - u_- = -u_- - (H(-X(t); u^-(t), u_+) - V(-X(t); u^-(t), u_+)) \ge -u_- - e(t) > 0$$

for $\varepsilon < t \le T < \infty$; thus by $f(u_-) > 0$, (1.8), and (4.17),

$$\begin{aligned}\overline{I}_0 &= \int_\varepsilon^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-) f(\overline{v}_\varepsilon(0,\tau)) d\tau - \int_\varepsilon^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-) f(u_-) d\tau \\ &\le c\int_\varepsilon^t |\overline{v}_\varepsilon(0,\tau)| d\tau = c\int_\varepsilon^t |V(-X(\tau); u^-(\tau), u_+) - H(-X(\tau); u^-(\tau), u_+)| d\tau \\ &\le c\int_\varepsilon^t e(\tau) d\tau \le C(T)\varepsilon. \end{aligned} \tag{4.18$_1$}$$

In case (ii), for $\varepsilon < t \le T < \infty$, it follows from (4.17)

$$\begin{aligned}\overline{I}_0 &\le c\int_\varepsilon^t |\overline{v}_\varepsilon(0,\tau) - u_-| d\tau \\ &\le c\int_\varepsilon^t |V(-X(t); u^-(t), u_+) - H(-X(t); u^-(t), u_+) - u_-| d\tau \\ &\le c\int_\varepsilon^t (-u_- + H(-X(t); u^-(t), u_+) - V(-X(t); u^-(t), u_+)) d\tau \\ &\le c\int_\varepsilon^t 2e(t) d\tau \le C(T)\varepsilon. \end{aligned} \tag{4.18$_2$}$$

In case (iii), by the techniques used in case (i) and case (ii), we can obtain for $\varepsilon < t \le T < \infty$

$$\begin{aligned}\overline{I}_0 &= \int_\varepsilon^{\tau_\varepsilon} \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-)(f(\overline{v}_\varepsilon(0,\tau)) - f(u_-)) d\tau \\ &+ \int_{\tau_\varepsilon}^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-)(f(\overline{v}_\varepsilon(0,\tau)) - f(u_-)) d\tau \\ &\le c\int_\varepsilon^t 2e(t) d\tau \le C(T)\varepsilon. \end{aligned} \tag{4.18$_3$}$$

Following the proof of Theorem 4.1, we can verify (4.2) by means of Lemma 3.1, (4.16), (4.18), and (4.5).

(4) When $u_-, u_+ < 0 < u_m$, from the discussion in sub-section 2.4.3, there is only one shock, starting at point $(a, 0)$ and terminating at point $(0, t_2)$, included in the weak entropy solution $u(x,t)$ of (1.7), and for $t \in [0, t_2]$,

$$X(t) \ge a - \frac{a}{t_2} t \ge 0. \tag{4.19}$$

We take the approximation to $u(x,t)$ as

$$\overline{v}_\varepsilon(x,t) = \begin{cases} u(x,t) + V(x - X(t); u^-, u_+) - H(x - X(t); u^-, u_+), & \text{if } \ t \in (0, t_2] \\ u_+, \ \text{if } \ t \in (t_2, \infty), \end{cases}$$

where $u^- = u^-(t) := u(X(t) - 0, t)$. Similar to (4.18), for $t \in (\varepsilon, t_2]$, we can conclude that the boundary integral

$$\int_\varepsilon^t \mathrm{sgn}(v_\varepsilon(0,\tau) - \overline{v}_\varepsilon(0,\tau))(f(v_\varepsilon(0,\tau)) - f(\overline{v}_\varepsilon(0,\tau))) d\tau \le C(T)\varepsilon. \tag{4.20}$$

As in the proof of Theorem 4.1, the desired error estimate (4.2) can be derived by using Lemma 3.1, Lemma 3.2, (4.19), (4.20), and (4.5). The details are omitted here.

REMARK. If we take $v_0(x) \in C^2([0,\infty))$ as follows:

$$v_0(x) = \begin{cases} v_{1\varepsilon}(x), & x \in [0,\varepsilon) \\ u_m, & x \in [\varepsilon, a) \\ v_{2\varepsilon}(x), & x \in [a, a+\varepsilon) \\ u_+, & x \in [a+\varepsilon, \infty), \end{cases}$$

where $v_{1\varepsilon}(\cdot) \in C^2([0,\varepsilon])$, $v_{2\varepsilon}(\cdot) \in C^2([a, a+\varepsilon])$ satisfies the corresponding compatibility conditions, and $\|v_{1\varepsilon}(\cdot)\|_{C([0,\varepsilon])}$, $\|v_{2\varepsilon}(\cdot)\|_{C([a,a+\varepsilon])}$, and $\|v_1(\cdot)\|_{w^{2,1}([0,\varepsilon])}$, $\|v_2(\cdot)\|_{w^{2,1}([a,a+\varepsilon])}$ are bounded for $\varepsilon$, then (4.1), (4.2), and (4.3) are respectively replaced by the following:

$$\sup_{0\le t\le T} \|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \le C(T)(\varepsilon^{1/2} + \varepsilon|\ln\varepsilon| + \varepsilon),$$

$$\sup_{0\le t\le T} \|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \le C(T)(\varepsilon|\ln\varepsilon| + \varepsilon),$$

and

$$\sup_{0\le t\le T} \|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \le C(T)\varepsilon.$$

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